Vlasov scaling for stochastic dynamics of continuous systems

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September 12, 2010

Abstract

We describe a general derivation scheme for the Vlasov-type equations for Markov evolutions of particle systems in continuum. This scheme is based on a proper scaling of corresponding Markov generators and has an algorithmic realization in terms of related hierarchical chains of correlation functions equations. Several examples of realization of the proposed approach in particular models are presented.

Keywords Continuous systems, Vlasov scaling, Vlasov equation, Markov evolution, spatial birth-and-death processes, spatial hopping processes, correlation functions, scaling limits

Mathematics Subject Classification (2000) 82C22, 60K35, 82C21

1 Introduction

Dynamical processes in many-body systems are often approximately described by kinetic equations, see, e.g., the excellent reviews by H.Spohn [35], [36]. A famous example of such equations is the Vlasov equation for a plasma, see e.g. [33], [34]. The Vlasov equation in physics describes the Hamiltonian motion of an infinite particle system in the mean-field scaling limit, thereby taking into account the influence of weak long-range forces. The convergence in the Vlasov scaling limit was shown by W.Braun and K.Hepp [4] (for the Hamiltonian dynamics) and by R.L.Dobrushin [6] (for more general deterministic dynamical systems). Note that the resulting Vlasov-type equations for particle densities are considered in classes of finite measures (in the weak form) or integrable functions (in the strong form). The latter means, in fact, that we are restricted to the case of finite-volume systems or systems with zero mean density in an infinite

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volume. A detailed analysis of Vlasov-type equations for integrable functions presented in the recent paper by V.V.Kozlov [30].

The main aim of this paper is to study Vlasov-type scaling for some classes of stochastic evolutions in continuum. Here we have in mind, first of all, spatial birth-and-death Markov processes (e.g., continuous Glauber dynamics) and hopping particles Markov evolutions (e.g., Kawasaki dynamics in continuum). Note that the approaches to the Vlasov scaling mentioned above seems to be quite difficult to apply to stochastic dynamics considered here (even in a finite volume) due to some essential reasons. For these processes, the possibility of their descriptions in terms of proper stochastic evolutional equations for particle motion is generally speaking absent. This, together with a possible variation of the particle number during the evolution, is an essential trouble in the application of the general Dobrushin's method.

Therefore, we shall look for an alternative approach to derive the kinetic Vlasov-type equations from stochastic dynamics. Contrary to the classical derivation of the Vlasov-type kinetic equations from the Hamiltonian dynamics, we do not prove the law of large numbers for the corresponding processes. We do not even need to show the existence of the corresponding microscopic rescaled processes. Our main idea is to study the evolution of states (distributions) of the system in terms of the corresponding chain of hierarchical equations. As pointed out by H.Spohn [35], the correct Vlasov limit can be easily guessed from the BBGKY hierarchy for the Hamiltonian system. Such heuristic derivation does not assume the integrability condition for the density, but until now, it could not be made rigorous due to the lack of detailed information about the properties of solutions to the BBGKY hierarchy. We would like to stress that different classes of initial data are not only mathematical tools for the rigorous study of the problem. They describe different physical situations in related microscopic models. The zero average density systems were considered in [1] by means of heuristic limit transition in the corresponding hierarchical equations for correlation functions. The framework we are working in is nonzero average density which is related to the case of bounded correlation functions. Our approach is based on Spohn's observation applied in a new dynamical framework. More precisely, we already know that many stochastic evolutions in continuum admit effective descriptions in terms of hierarchical equations for correlation functions which generalize the BBGKY hierarchy from Hamiltonian to Markov setting, see, e.g., [17] and the references therein. Moreover, these hierarchical equations are often the only available technical tools for the construction of corresponding dynamics in several models [21], [23], [14].

In Section 3 we propose a general scheme for the Vlasov scaling of stochastic dynamics for interacting particle systems in continuum. This scaling is actually of mean-field type which is adopted to preserve the spatial structure. Additionally, we scale the class of initial distributions at the level of the corresponding correlation functions. The scheme we use has also a clear interpretation in terms of scaled Markov generators. An application of the considered scaling leads to the limiting hierarchy which possesses a chaos preservation property. Namely, if we start from a Poissonian (non-homogeneous) initial state of the system, then this property will be preserved during the time evolution. The main observation which appears at this point is the following. A special structure of the interaction in the resulting virtual Vlasov system gives a non-linear evolutional equation for the density of the evolving Poisson state. It is for the first time that macroscopic Vlasov-type equations are obtained from the microscopic infinite-particle systems in an unbounded region of non-zero average density using the corresponding system of hierarchical equations.

Section 4 is devoted to the application of the general scheme to a wide class of birth-and-death and hopping particles processes. We state conditions on structural coefficients in the corresponding Markov generators which give a weak convergence of the rescaled generators to the limiting generators of the related Vlasov hierarchies. As a result, we may compute the limiting Vlasov-type equations for the considered processes leaving the question about the strong convergence of the hierarchy solutions open. In Section 5 we present a collection of particular examples of the resulting Vlasov equations for several concrete models. Note that each of the examples considered creates its own non-linear equation for the density in the discussed scaling. These equations include convolution operators as a common point of their structure. To our knowledge, any general results concerning properties of solutions to such kind of non-linear evolutional equation are absent. This is an exiting mathematical problem strongly motivated by concrete models of interacting particle dynamics.

Many problems of the (mathematical) population biology concerns interactions between populations of different types. Our technic, in fact, covers this case. In particular, one can derive spatially inhomogeneous non-linear equations of the Lotka–Volterra type in the Vlasov-type scaling. On the other hand, one may apply this approach to the so-called continuous Ising model (Potts model). We explain these results in forthcoming papers [10], [13].

Note that control of convergence of the Vlasov scalings for the solutions to considered hierarchies is a difficult technical problem which shall be analyzed for every particular model separately. Our results in this direction concern two classes of models: Glauber dynamics in continuum and a spatial ecological model (so-called Bolker–Dieckmann–Law–Pacala model). Due to their technically complicated character, these results will be published in separated works [12], [11].

2 Basic facts and notation

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$; $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ denotes the system of all bounded sets in $\mathcal{B}(\mathbb{R}^d)$.

The configuration space over space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$, namely,

$$\Gamma = \Gamma_{\mathbb{R}^d} := \bigl\{ \gamma \subset \mathbb{R}^d \bigm| |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d) \bigr\}. \tag{2.1}$$

The space Γ is equipped with the vague topology, i.e., the minimal topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support; note that the summation in $\sum_{x\in\gamma} f(x)$ is taken over finitely many points of γ which belong to the support of f. In [20], it was shown that Γ with the vague topology may be metrizable and becomes a Polish space (i.e., complete separable metric space). Corresponding to this topology, the Borel σ-algebra $\mathcal{B}(\Gamma)$ is the smallest σ-algebra for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are measurable for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. Here $\gamma_\Lambda := \gamma \cap \Lambda$, and $|\cdot|$ means the cardinality of a finite set.

The space of n-point configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_Y^{(n)} := \bigl\{\eta \subset Y \bigm| |\eta| = n\bigr\}, \quad n \in \mathbb{N}.$$

We set also $\Gamma_Y^{(0)} := \{\emptyset\}$. As a set, $\Gamma_Y^{(n)}$ may be identified with the symmetrization of

$$\widetilde{Y^n} = \bigl\{(x_1, \ldots, x_n) \in Y^n \bigm| x_k \neq x_l \text{ if } k \neq l\bigr\}.$$

Hence, one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}(\Gamma_Y^{(n)})$. The space of finite configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_{0,Y} := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_Y^{(n)}.$$

This space is equipped with the topology of disjoint unions. Therefore, one can introduce the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_{0,Y})$. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the notation, namely, $\Gamma_0 := \Gamma_{0,\mathbb{R}^d}$, $\Gamma^{(n)} := \Gamma_{\mathbb{R}^d}^{(n)}$.

The restriction of the Lebesgue product measure $(dx)^n$ to $\bigl(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\bigr)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2.2}$$

For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the restriction of λ to $\Gamma_\Lambda := \Gamma_{0,\Lambda}$ will be also denoted by λ. The space $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ is the projective limit of the family of spaces $\bigl\{(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))\bigr\}_{\Lambda\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$. The Poisson measure π on $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ is given as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is the probability measure on $\bigl(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)\bigr)$. Here $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define a *Lebesgue–Poisson exponent*

$$e_\lambda(f, \eta) := \prod_{x\in\eta} f(x), \quad \eta \in \Gamma_0; \qquad e_\lambda(f, \emptyset) := 1. \tag{2.3}$$

Then, by (2.2), for $f \in L^1(\mathbb{R}^d, dx)$ we obtain $e_\lambda(f) \in L^1(\Gamma_0, d\lambda)$ and

$$\int_{\Gamma_0} e_\lambda(f, \eta) d\lambda(\eta) = \exp\biggl\{\int_{\mathbb{R}^d} f(x) dx\biggr\}. \tag{2.4}$$

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}$. The set of bounded measurable functions with bounded support we denote by $B_{\mathrm{bs}}(\Gamma_0)$, i.e., $G \in B_{\mathrm{bs}}(\Gamma_0)$ if $G \upharpoonright_{\Gamma_0 \setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0, in fact, is a sequence of functions $\left\{G^{(n)}\right\}_{n\in\mathbb{N}_0}$ where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$. We consider also the set $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ of *cylinder functions* on Γ. Each $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ is characterized by the following relation: $F(\gamma) = F \upharpoonright_{\Gamma_\Lambda} (\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$.

There is the following mapping from $B_{\mathrm{bs}}(\Gamma_0)$ into $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$, which plays the key role in our further considerations:

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.5}$$

where $G \in B_{\mathrm{bs}}(\Gamma_0)$, see, e.g., [19, 31, 32]. The summation in (2.5) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol, $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.6}$$

Here and in the sequel inclusions like $\xi \subset \eta$ hold for $\xi = \emptyset$ as well as for $\xi = \eta$. We denote the restriction of K onto functions on Γ_0 by K_0.

For any fixed $C > 1$ we consider the following Banach space of $\mathcal{B}(\Gamma_0)$-measurable functions

$$\mathcal{L}_C := \left\{ G : \Gamma_0 \to \mathbb{R} \;\middle|\; \|G\|_C := \int_{\Gamma_0} |G(\eta)| C^{|\eta|} d\lambda(\eta) < \infty \right\}. \tag{2.7}$$

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to (w.r.t. for short) the Poisson measure π if for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the projection of μ onto Γ_Λ is absolutely continuous w.r.t. the projection of π onto Γ_Λ. By [19], in this case, there exists a *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ such that for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the following equality holds

$$\int_\Gamma (KG)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta). \tag{2.8}$$

The restrictions $k_\mu^{(n)}$ of this functional on $\Gamma_0^{(n)}$, $n \in \mathbb{N}_0$ are called *correlation functions* of the measure μ. Note that $k_\mu^{(0)} = 1$.

We recall now without a proof the special case of the well-known technical lemma (cf., [28]) which plays very important role in our calculations.

Lemma 2.1. *For any measurable function $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \to \mathbb{R}$*

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta) \, d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi) \, d\lambda(\xi) \, d\lambda(\eta) \tag{2.9}$$

only if both sides of the equality make sense.

3 General scheme

In this section we introduce the notion of the Vlasov scaling for Markov dynamics of IPS on configuration spaces.

We assume that our system evolves in time due to some mechanism whose details will be specified for concrete models. Suppose that the initial distribution of particles in our system is a measure $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, with correlation function k_0. Let $\mu_t \in \mathcal{M}^1(\Gamma)$ be the distribution of particles at time $t > 0$ and k_t be its correlation function. One should note that if evolution $(\mu_t)_{t\geq 0}$ is ruled by an *à priori* given Markov process on Γ (i.e. if such a Markov process exists), then μ_t is a solution to the following Kolmogorov equation:

$$\begin{cases} \dfrac{d\mu_t}{dt} = L^* \mu_t \\ \mu_t\big|_{t=0} = \mu_0, \end{cases}$$

where L^* is the operator adjoint to the generator of functional evolution, i.e.,

$$\begin{cases} \dfrac{dF_t}{dt} = LF_t \\ F_t\big|_{t=0} = F_0. \end{cases}$$

Of course, one should be careful about the functional and measure spaces to have all the above-introduced operators properly defined. We postpone careful definitions of all these objects until the introduction of concrete models.

Now, assume that the evolution of correlation functions $(k_t)_{t\geq 0}$, corresponding to $(\mu_t)_{t\geq 0}$, first of all exists, and is the solution of the following evolutional equation

$$\begin{cases} \dfrac{dk_t}{dt} = L^{\triangle} k_t \\ k_t\big|_{t=0} = k_0 \end{cases} \tag{3.1}$$

where $L^{\triangle}$ is the generator of a semigroup $T^{\triangle}_t$ on some functional space which includes all bounded functions (or bounded with some weight) almost everywhere (a.e.) w.r.t. the Lebesgue–Poisson measure λ. In many applications this space may be taken to be $\mathcal{K}_C := \big\{ k : \Gamma_0 \to \mathbb{R} \;\big|\; k \cdot C^{-|\eta|} \in L^\infty(\lambda) \big\}$ for some fixed $C > 1$. Let us stress that (3.1) is nothing else but a hierarchical system of equations corresponding to the Markov generator considered. This system has the same meaning as the BBGKY hierarchy in the case of Hamiltonian dynamics.

The first important step on the way to construct the Vlasov scaling concerns the proper rescaling of the initial state of the system. Or, equivalently, in the language of correlation functions it means the proper rescaling of the initial conditions of (3.1).

More precisely, at the beginning we rescale k_0 with parameter $\varepsilon > 0$ in such a way that the resulting functions $k_0^{(\varepsilon)}$ as $\varepsilon \to 0$ behave as follows:

$$k^{(\varepsilon)}_{0,\,\mathrm{ren}}(\eta) := \varepsilon^{|\eta|} k^{(\varepsilon)}_0(\eta) \to r_0(\eta), \quad \varepsilon \to 0,\ \eta \in \Gamma_0, \tag{3.2}$$

where the function r_0 is a subject of choice for concrete examples and aims. In general, it has to be a bounded function also (or bounded with some weight) a.e. w.r.t. the Lebesgue–Poisson measure.

Remark 3.1. In the case of $r_0(\eta) = e_\lambda(\rho_0, \eta)$, $\eta \in \Gamma_0$, $\rho_0 : \mathbb{R}^d \to (0, +\infty)$ the assumption about the rescaling of the initial condition means heuristically the following: $\mu^{(\varepsilon)}_{0,\,\mathrm{ren}} \to \pi_{\rho_0}$, where $\mu^{(\varepsilon)}_{0,\,\mathrm{ren}}$ has a correlation function $\varepsilon^{|\eta|} k^{(\varepsilon)}_0(\eta)$.

It is clear that such a rescaling of the initial solution for (3.1) leads to a singular function w.r.t. $\varepsilon > 0$. In applications, this fact can be interpreted as the growth of density of the system with $\varepsilon \to 0$.

We have to consider (and it is our second step) some proper scaling of the generator in (3.1):

$$L^\triangle \longmapsto L^\triangle_\varepsilon. \tag{3.3}$$

The concrete type of this scaling will depend on $L^\triangle$. In the next sections we consider several types of generators and corresponding scalings. Suppose that there exists a solution of the functional evolution

$$\begin{cases} \dfrac{dk^{(\varepsilon)}_t}{dt} = L^\triangle_\varepsilon k^{(\varepsilon)}_t \\ k^{(\varepsilon)}_t \big|_{t=0} = k^{(\varepsilon)}_0 \end{cases} \tag{3.4}$$

We expect (and this will be shown in the concrete models for the concrete scalings in forthcoming papers) that this solution will be also singular w.r.t. $\varepsilon > 0$, hence, this solutions will be in functional spaces depending on ε.

Moreover, we should choose the type of scaling (3.3) which guarantees that the order of this singularity will be the same for the initial function $k^{(\varepsilon)}_0$. Namely (and it is our third step on the way to realize the Vlasov scaling) we consider, cf. (3.2),

$$k^{(\varepsilon)}_{t,\,\mathrm{ren}}(\eta) := \varepsilon^{|\eta|} k^{(\varepsilon)}_t(\eta), \quad \eta \in \Gamma_0, \tag{3.5}$$

and we want to show that

$$k^{(\varepsilon)}_{t,\,\mathrm{ren}}(\eta) \to r_t(\eta), \quad \varepsilon \to 0,\ \eta \in \Gamma_0. \tag{3.6}$$

In fact, (3.5) means that we consider a renormalized version of the evolution equation (3.4):

$$\begin{cases} \dfrac{dk^{(\varepsilon)}_{t,\,\mathrm{ren}}}{dt} = L^\triangle_{\varepsilon,\mathrm{ren}} k^{(\varepsilon)}_{t,\,\mathrm{ren}} \\ k^{(\varepsilon)}_{t,\,\mathrm{ren}} \big|_{t=0} = k^{(\varepsilon)}_{0,\,\mathrm{ren}} \end{cases} \tag{3.7}$$

where

$$L^\triangle_{\varepsilon,\mathrm{ren}} = \varepsilon^{|\eta|} L^\triangle_\varepsilon \varepsilon^{-|\eta|}. \tag{3.8}$$

Therefore, informally, we want to show that the solution of the evolution equation (3.7) converges (in a proper sense) to some function r_t which satisfies the *Vlasov hierarchy*

$$\begin{cases} \dfrac{dr_t}{dt} = V^{\triangle} r_t \\ r_t\big|_{t=0} = r_0 \end{cases} \tag{3.9}$$

Recall again that the choice of the scaling (3.3) is prescribed by the model. Having applications in mind, it is important to consider the case of $r_0(\eta) = e_\lambda(\rho_0, \eta)$ and the scaling (3.3) which leads to r_t of the same type, i.e.,

$$r_t(\eta) = e_\lambda(\rho_t, \eta), \quad \eta \in \Gamma_0.$$

The latter means the so-called chaos preservation property of the Vlasov hierarchy. Equation (3.9) in this case implies, in general, a non-linear equation for ρ_t:

$$\frac{\partial}{\partial t}\rho_t(x) = \upsilon(\rho_t)(x), \quad x \in \mathbb{R}^d, \tag{3.10}$$

which we will call the *Vlasov-type equation.*

To describe this scheme in a more analytical way, we use the language of semigroups. Suppose that we know the mechanism of the evolution of our system given by the Markov pre-generator L. Let L be defined at least on functions from $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ and $\widehat{L} = K^{-1}LK$ be the corresponding descend mapping on functions from $B_{\mathrm{bs}}(\Gamma_0)$. Let us fix the duality between functions on Γ_0

$$\langle\!\langle G, k \rangle\!\rangle = \int_{\Gamma_0} G(\eta)\, k(\eta)\, d\lambda(\eta), \tag{3.11}$$

and consider the mapping $L^{\triangle}$ being the dual to $\widehat{L}$ w.r.t. (3.11).

Assume that L can be extended to a generator L. We want to construct a scaling of the generator L, say, L_ε, $\varepsilon > 0$, such that the scheme described above will be covered. Assume that we have a semigroup $\widehat{T}_\varepsilon(t)$ with a generator $\widehat{L}_\varepsilon = K^{-1}L_\varepsilon K$ in some functional space over Γ_0. Consider the dual semigroup $T_\varepsilon^{\triangle}(t)$ which corresponds (in a proper sense) to $L_\varepsilon^{\triangle}$. As we said before, we consider an initial condition of (3.4) with a singularity in ε, namely, $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\varepsilon \to 0$, $\eta \in \Gamma_0$ with some function r_0, independent of ε. First of all, we have to choose such a scaling $L \mapsto L_\varepsilon$ for which $T_\varepsilon^{\triangle}(t)$ preserves the order of the singularity:

$$(T_\varepsilon^{\triangle}(t) k_0^{(\varepsilon)})(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta), \quad \varepsilon \to 0,\ \eta \in \Gamma_0. \tag{3.12}$$

And the most important is that the dynamics $r_0 \mapsto r_t$ should preserve the Lebesgue–Poisson exponents: if $r_0(\eta) = e_\lambda(\rho_0, \eta)$ then $r_t(\eta) = e_\lambda(\rho_t, \eta)$, where ρ_t is satisfied (3.10).

Now let us close our construction with the evolution of states in this scheme. Let us consider for any $\varepsilon > 0$ the following mapping of functions on Γ_0

$$(R_\varepsilon r)(\eta) := \varepsilon^{|\eta|} r(\eta). \tag{3.13}$$

This mapping is "self-dual" w.r.t. duality (3.11), moreover, $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$. Then we have $k_0^{(\varepsilon)} \sim R_{\varepsilon^{-1}} r_0$, and we need $r_t \sim R_\varepsilon T_\varepsilon^{\triangle}(t) k_0^{(\varepsilon)} \sim R_\varepsilon T_\varepsilon^{\triangle}(t) R_{\varepsilon^{-1}} r_0$.

Therefore, we have to show that for any $t \geq 0$ the operator family $R_\varepsilon T_\varepsilon^\triangle(t) R_{\varepsilon^{-1}}$, $\varepsilon > 0$ has a limiting (in a proper sense) operator $U(t)$ as $\varepsilon \to 0$ and

$$U(t) e_\lambda(\rho_0) = e_\lambda(\rho_t). \tag{3.14}$$

But, informally, $T_\varepsilon^\triangle(t) = \exp\{tL_\varepsilon^\triangle\}$ and $R_\varepsilon T_\varepsilon^\triangle(t) R_{\varepsilon^{-1}} = \exp\{tR_\varepsilon L_\varepsilon^\triangle R_{\varepsilon^{-1}}\}$. In fact, we need the existence of an operator $V^\triangle$ such that $\exp\{tR_\varepsilon L_\varepsilon^\triangle R_{\varepsilon^{-1}}\} \to \exp\{tV^\triangle\} =: U(t)$ for which (3.14) holds. Therefore, a heuristic way to produce the scaling $L \mapsto L_\varepsilon$ is to demand that

$$\lim_{\varepsilon \to 0} \left(\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) - L_{\varepsilon,\mathrm{ren}}^\triangle e_\lambda(\rho_t, \eta) \right) = 0, \quad \eta \in \Gamma_0 \tag{3.15}$$

if ρ_t is satisfied (3.10). The point-wise limit of $L_{\varepsilon,\mathrm{ren}}^\triangle$ will be the natural candidate for $V^\triangle$.

Sometimes, to show convergence of solutions of evolutional equations in some functional spaces it is much simpler to work with the operators $\widehat{L}_{\varepsilon,\mathrm{ren}}$ and $\widehat{V}$ which are pre-dual to $L_{\varepsilon,\mathrm{ren}}^\triangle$ and $V^\triangle$ w.r.t. the duality (3.11). Note that (3.8) implies

$$\widehat{L}_{\varepsilon,\mathrm{ren}} = R_{\varepsilon^{-1}} \widehat{L}_\varepsilon R_\varepsilon, \tag{3.16}$$

and $\widehat{V}$ should be the point-wise limit of $\widehat{L}_{\varepsilon,\mathrm{ren}}$.

4 Generators of birth, death, and hopping

Through out this section we consider generators of two types for continuous models: the birth-and-death generator $L_{\mathrm{bad}} = L^- + L^+$ and the hopping generator L_{hop}, where for any $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$

$$\left(L^- F\right)(\gamma) := \sum_{x \in \gamma} d\left(x, \gamma \setminus x\right) \left[F\left(\gamma \setminus x\right) - F\left(\gamma\right)\right], \tag{4.1}$$

$$\left(L^+ F\right)(\gamma) := \int_{\mathbb{R}^d} b\left(x, \gamma\right) \left[F\left(\gamma \cup x\right) - F\left(\gamma\right)\right] dx, \tag{4.2}$$

$$\left(L_{\mathrm{hop}} F\right)(\gamma) := \sum_{x \in \gamma} \int_{\mathbb{R}^d} c\left(x, y, \gamma\right) \left[F\left(\gamma \setminus x \cup y\right) - F\left(\gamma\right)\right] dy. \tag{4.3}$$

Here b, d, c are measurable functions of their variables and, additionally, b and c are locally integrable function of the first and second variables, correspondingly. These conditions guarantee that (4.1)–(4.3) are well-defined on $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ since for any $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ there exists some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ such that $F(\gamma \setminus x) = F(\gamma)$ for any $x \in \gamma_{\Lambda^c}$, $F(\gamma \cup x) = F(\gamma)$ for any $x \in \Lambda^c$, and $F(\gamma \setminus x \cup y) = F(\gamma)$ for any $x \in \gamma_{\Lambda^c}$, $y \in \Lambda^c$; as result the sums in (4.1) and (4.3) are over finite set γ_Λ and the integrals in (4.2) and (4.3) are over bounded set Λ.

We may denote $L^- = L^-(d)$, $L^+ = L^+(b)$, $L_{\mathrm{hop}} = L_{\mathrm{hop}}(c)$. Assume that we have some scaling of rates b, d, c, say, $b_\varepsilon, d_\varepsilon, c_\varepsilon$, correspondingly; $\varepsilon > 0$. Then, let us consider the following scaling of $L_{\rm bad}$ and $L_{\rm hop}$:

$$L_{\text{bad},\,\varepsilon} = L^{-}(d_\varepsilon) + \varepsilon^{-1} L^{+}(b_\varepsilon), \tag{4.4}$$

$$L_{\text{hop},\,\varepsilon} = L_{\rm hop}(c_\varepsilon). \tag{4.5}$$

Remark 4.1. In a conservative system with a generator like (4.3) which preserves the "number of particles" during an evolution the Vlasov-type scaling usually means decreasing of the intensity of the interactions between elements of a system together with increasing of correlations in the initial state. However, in a non-conservative birth-and-death dynamics with a generator $L_{\rm bad}$ we need an additional increasing of the birth intensity to preserve the influence of the birth part in the limiting Vlasov hierarchy. Note that the necessity of the concrete factor ε^{-1} in (4.4) is clear *a posteriori* only (see Proposition 4.5).

Suppose that there exists three families of measurable functions on Γ_0: $D_x^{(\varepsilon)}$, $B_x^{(\varepsilon)}$, $C_{x,y}^{(\varepsilon)}$, $\varepsilon > 0$, $\{x, y\} \subset \mathbb{R}^d$, such that

$$d_\varepsilon\,(x,\gamma) = (KD_x^{(\varepsilon)})(\gamma), \qquad b_\varepsilon\,(x,\gamma) = (KB_x^{(\varepsilon)})(\gamma),$$
$$c_\varepsilon\,(x,y,\gamma) = (KC_{x,y}^{(\varepsilon)})(\gamma\setminus x).$$

Note that, in general, $C_{x,y} \neq C_{y,x}$.

Proposition 4.2. *The following formulas hold for any* $k \in B_{\rm bs}(\Gamma_0)$

$$\begin{aligned}\left(L^{\triangle}_{\text{bad},\varepsilon,\text{ren}}k\right)(\eta) = &-\int_{\Gamma_0} k\,(\xi\cup\eta)\sum_{x\in\eta}\sum_{\omega\subset\eta\setminus x}\varepsilon^{-|\xi|}D_x^{(\varepsilon)}\,(\omega\cup\xi)\,d\lambda\,(\xi)\\ &+\int_{\Gamma_0}\sum_{x\in\eta}k\,(\xi\cup(\eta\setminus x))\sum_{\omega\subset\eta\setminus x}\varepsilon^{-|\xi|}B_x^{(\varepsilon)}\,(\omega\cup\xi)\,d\lambda\,(\xi)\,;\end{aligned} \tag{4.6}$$

$$\begin{aligned}\left(L^{\triangle}_{\text{hop},\varepsilon,\text{ren}}k\right)(\eta) = &\sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\Gamma_0} k\,(\xi\cup(\eta\setminus x)\cup y)\sum_{\omega\subset\eta\setminus x}\varepsilon^{-|\xi|}C_{y,x}^{(\varepsilon)}\,(\omega\cup\xi)\,d\lambda\,(\xi)\,dy\\ &-\sum_{x\in\eta}\int_{\Gamma_0} k\,(\xi\cup\eta)\int_{\mathbb{R}^d}\sum_{\omega\subset\eta\setminus x}\varepsilon^{-|\xi|}C_{x,y}^{(\varepsilon)}\,(\omega\cup\xi)\,dy d\lambda\,(\xi)\,.\end{aligned} \tag{4.7}$$

Proof. The proof is straightforward. By [17], from (4.4) and (4.5) we have

$$\begin{aligned}\left(L^{\triangle}_{\text{bad},\varepsilon}k\right)(\eta) = &-\int_{\Gamma_0} k\,(\xi\cup\eta)\sum_{x\in\eta}\sum_{\omega\subset\eta\setminus x}D_x^{(\varepsilon)}\,(\omega\cup\xi)\,d\lambda\,(\xi)\\ &+\varepsilon^{-1}\int_{\Gamma_0}\sum_{x\in\eta}k\,(\xi\cup(\eta\setminus x))\sum_{\omega\subset\eta\setminus x}B_x^{(\varepsilon)}\,(\omega\cup\xi)\,d\lambda\,(\xi)\,;\end{aligned}$$

$$\begin{aligned}\left(L^{\triangle}_{\text{hop},\varepsilon}k\right)(\eta) = &\sum_{y\in\eta}\int_{\mathbb{R}^d}\int_{\Gamma_0} k\,(\xi\cup(\eta\setminus y)\cup x)\sum_{\omega\subset\eta\setminus y}C_{x,y}^{(\varepsilon)}\,(\omega\cup\xi)\,d\lambda\,(\xi)\,dx\\ &-\int_{\Gamma_0} k\,(\xi\cup\eta)\sum_{y\in\eta}\sum_{\omega\subset\eta\setminus y}\int_{\mathbb{R}^d}C_{y,x}^{(\varepsilon)}\,(\omega\cup\xi)\,dxd\lambda\,(\xi)\,.\end{aligned}$$

Then, (4.6) and (4.7) follow directly from (3.8). $\square$

Let ρ_t, $t \geq 0$ be measurable functions on $\mathbb{R}^d$. The explicit formula

$$\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) = \sum_{x \in \eta} e_\lambda(\rho_t, \eta \setminus x) \frac{\partial}{\partial t} \rho_t(x) \tag{4.8}$$

together with our "demand" (3.15) induce us to state the following corollary.

Corollary 4.3. *Let ρ be a measurable function on $\mathbb{R}^d$. Then*

$$\begin{aligned}
&\left(L^{\triangle}_{\mathrm{bad},\varepsilon,\mathrm{ren}} e_\lambda(\rho)\right)(\eta) \qquad (4.9)\\
&= -\sum_{x \in \eta} e_\lambda(\rho, \eta \setminus x)\, \rho(x) \int_{\Gamma_0} e_\lambda(\rho, \xi) \sum_{\omega \subset \eta \setminus x} \varepsilon^{-|\xi|} D_x^{(\varepsilon)}(\omega \cup \xi)\, d\lambda(\xi)\\
&\quad + \sum_{x \in \eta} e_\lambda(\rho, \eta \setminus x) \int_{\Gamma_0} e_\lambda(\rho, \xi) \sum_{\omega \subset \eta \setminus x} \varepsilon^{-|\xi|} B_x^{(\varepsilon)}(\omega \cup \xi)\, d\lambda(\xi);
\end{aligned}$$

and

$$\begin{aligned}
&\left(L^{\triangle}_{\mathrm{hop},\varepsilon,\mathrm{ren}} e_\lambda(\rho)\right)(\eta) \qquad (4.10)\\
&= \sum_{x \in \eta} e_\lambda(\rho, \eta \setminus x) \int_{\mathbb{R}^d} \rho(y) \int_{\Gamma_0} e_\lambda(\rho, \xi) \sum_{\omega \subset \eta \setminus x} \varepsilon^{-|\xi|} C_{y,x}^{(\varepsilon)}(\omega \cup \xi)\, d\lambda(\xi)\, dy\\
&\quad - \sum_{x \in \eta} e_\lambda(\rho, \eta \setminus x)\, \rho(x) \int_{\mathbb{R}^d} \int_{\Gamma_0} e_\lambda(\rho, \xi) \sum_{\omega \subset \eta \setminus x} \varepsilon^{-|\xi|} C_{x,y}^{(\varepsilon)}(\omega \cup \xi)\, d\lambda(\xi)\, dy.
\end{aligned}$$

Proposition 4.4. *Suppose that for any $\{x, y\} \subset \mathbb{R}^d$, $\{\xi, \eta\} \subset \Gamma_0$*

$$\exists \lim_{\varepsilon \to 0} \sum_{\omega \subset \eta} \varepsilon^{-|\xi|} D_x^{(\varepsilon)}(\omega \cup \xi) = \lim_{\varepsilon \to 0} \varepsilon^{-|\xi|} D_x^{(\varepsilon)}(\xi) =: D_x^V(\xi), \tag{4.11}$$

$$\exists \lim_{\varepsilon \to 0} \sum_{\omega \subset \eta} \varepsilon^{-|\xi|} B_x^{(\varepsilon)}(\omega \cup \xi) = \lim_{\varepsilon \to 0} \varepsilon^{-|\xi|} B_x^{(\varepsilon)}(\xi) =: B_x^V(\xi), \tag{4.12}$$

$$\exists \lim_{\varepsilon \to 0} \sum_{\omega \subset \eta} \varepsilon^{-|\xi|} C_{x,y}^{(\varepsilon)}(\omega \cup \xi) = \lim_{\varepsilon \to 0} \varepsilon^{-|\xi|} C_{x,y}^{(\varepsilon)}(\xi) =: C_{x,y}^V(\xi). \tag{4.13}$$

Then, our "demand" (3.15) *holds. More precisely,*

$$\begin{aligned}
\left(V^{\triangle}_{\mathrm{bad}} k\right)(\eta) &:= \lim_{\varepsilon \to 0} \left(L^{\triangle}_{\mathrm{bad},\varepsilon,\mathrm{ren}} k\right)(\eta) \qquad (4.14)\\
&= -\int_{\Gamma_0} k(\xi \cup \eta) \sum_{x \in \eta} D_x^V(\xi)\, d\lambda(\xi)\\
&\quad + \int_{\Gamma_0} \sum_{x \in \eta} k(\xi \cup (\eta \setminus x))\, B_x^V(\xi)\, d\lambda(\xi),
\end{aligned}$$

and if ρ_t is the solution of the equation (3.10) *with*

$$\upsilon(\rho)(x) = \upsilon_{\mathrm{bad}}(\rho)(x) = -\,\rho\,(x)\int_{\Gamma_0} e_\lambda\,(\rho,\xi)\,D_x^V\,(\xi)\,d\lambda\,(\xi) + \int_{\Gamma_0} e_\lambda\,(\rho,\xi)\,B_x^V\,(\xi)\,d\lambda\,(\xi)\,, \tag{4.15}$$

then, $\dfrac{\partial}{\partial t}e_\lambda(\rho_t,\eta) = \big(V_{\mathrm{bad}}^{\triangle}e_\lambda(\rho_t)\big)\,(\eta)$*. Analogously,*

$$\begin{aligned}(V_{\mathrm{hop}}^{\triangle}k)\,(\eta) &:= \lim_{\varepsilon\to 0}\big(L_{\mathrm{hop},\varepsilon,\mathrm{ren}}^{\triangle}k\big)\,(\eta) \qquad (4.16)\\ &= \sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\Gamma_0} k\,(\xi\cup(\eta\setminus x)\cup y)\,C_{y,x}^V\,(\xi)\,d\lambda\,(\xi)\,dy\\ &\quad -\sum_{x\in\eta}\int_{\Gamma_0} k\,(\xi\cup\eta)\int_{\mathbb{R}^d} C_{x,y}^V\,(\xi)\,dyd\lambda\,(\xi)\,,\end{aligned}$$

and if ρ_t is the solution of the equation (3.10) *with*

$$\upsilon(\rho)(x) = \upsilon_{\mathrm{hop}}(\rho)(x) = \int_{\mathbb{R}^d}\rho\,(y)\int_{\Gamma_0} e_\lambda\,(\rho,\xi)\,C_{y,x}^V\,(\xi)\;\,d\lambda\,(\xi)\,dy - \rho\,(x)\int_{\Gamma_0} e_\lambda\,(\rho,\xi)\int_{\mathbb{R}^d} C_{x,y}^V\,(\xi)\,dyd\lambda\,(\xi)\,, \tag{4.17}$$

then, $\dfrac{\partial}{\partial t}e_\lambda(\rho_t,\eta) = \big(V_{\mathrm{hop}}^{\triangle}e_\lambda(\rho_t)\big)\,(\eta)$*.*

Proof. The equalities (4.14) and (4.16) are direct consequences of the Proposition 4.2 and the conditions (4.11)–(4.13). Taking the limit in (4.9) and (4.10) as $\varepsilon\to 0$ and using (4.8) we obtain the statement. □

And now we present the explicit expressions for the corresponding operators $\widehat{L}_{\varepsilon,\mathrm{ren}}$ and $\widehat{V}$.

Proposition 4.5. *For any $G\in B_{\mathrm{bs}}(\Gamma_0)$ the following formulas hold*

$$\begin{aligned}\big(\widehat{L}_{\mathrm{bad},\,\varepsilon,\mathrm{ren}}G\big)(\eta) &= -\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} G\,(\xi\cup x)\sum_{\omega\subset\xi}\varepsilon^{-|(\eta\setminus x)\setminus\xi|}D_x^{(\varepsilon)}\,(\omega\cup(\eta\setminus x)\setminus\xi)\\ &\quad +\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G\,(\xi\cup x)\sum_{\omega\subset\xi}\varepsilon^{-|\eta\setminus\xi|}B_x^{(\varepsilon)}\,(\omega\cup\eta\setminus\xi)\,dx; \qquad (4.18)\\ \big(\widehat{L}_{\mathrm{hop},\,\varepsilon,\mathrm{ren}}G\big)\,(\eta) &= \sum_{y\in\eta}\sum_{\xi\subset\eta\setminus y}\int_{\mathbb{R}^d} G\,(\xi\cup x)\sum_{\omega\subset\xi}\varepsilon^{-|(\eta\setminus y)\setminus\xi|}C_{y,x}^{(\varepsilon)}\,(\omega\cup(\eta\setminus y)\setminus\xi)\,dx\\ &\quad -\sum_{\xi\subset\eta}G\,(\xi)\sum_{x\in\xi}\sum_{\omega\subset\xi\setminus x}\int_{\mathbb{R}^d}\varepsilon^{-|\eta\setminus\xi|}C_{x,y}^{(\varepsilon)}\,(\omega\cup\eta\setminus\xi)\,dy. \qquad (4.19)\end{aligned}$$

If, additionally, (4.11)–(4.13) *hold, then,*

$$
\begin{aligned}
\left(\widehat{V}_{\mathrm{bad}}G\right)(\eta) = &-\sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi} D_x^V(\eta\setminus\xi)\\
&+\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x)\,B_x^V(\eta\setminus\xi)dx; \qquad (4.20)\\
\left(\widehat{V}_{\mathrm{hop}}G\right)(\eta) = &\sum_{y\in\eta}\sum_{\xi\subset\eta\setminus y}\int_{\mathbb{R}^d} G(\xi\cup x)\,C_{y,x}^V((\eta\setminus y)\setminus\xi)\,dx\\
&-\sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi}\int_{\mathbb{R}^d} C_{x,y}^V(\eta\setminus\xi)\,dy. \qquad (4.21)
\end{aligned}
$$

Proof. We may obtain these formulas directly from the duality (3.11) and the Lemma 2.1. Namely, for any $G\in B_{\mathrm{bs}}(\Gamma_0)$ we have

$$
\begin{aligned}
&\int_{\Gamma_0} G(\eta)\left(\widehat{L}_{\mathrm{bad},\,\varepsilon,\mathrm{ren}}k\right)(\eta)\,d\lambda(\eta)\\
=&-\int_{\Gamma_0} G(\eta)\int_{\Gamma_0} k(\xi\cup\eta)\sum_{x\in\eta}\sum_{\omega\subset\eta\setminus x}\varepsilon^{-|\xi|}D_x^{(\varepsilon)}(\omega\cup\xi)\,d\lambda(\xi)\,d\lambda(\eta)\\
&+\int_{\Gamma_0} G(\eta)\int_{\Gamma_0}\sum_{x\in\eta} k(\xi\cup(\eta\setminus x))\sum_{\omega\subset\eta\setminus x}\varepsilon^{-|\xi|}B_x^{(\varepsilon)}(\omega\cup\xi)\,d\lambda(\xi)\,d\lambda(\eta)\\
=&-\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\,k(\xi\cup\eta\cup x)\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}D_x^{(\varepsilon)}(\omega\cup\xi)\,dxd\lambda(\xi)\,d\lambda(\eta)\\
&+\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\,k(\xi\cup\eta)\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}B_x^{(\varepsilon)}(\omega\cup\xi)\,dxd\lambda(\xi)\,d\lambda(\eta)\\
=&-\int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{\eta\subset\xi} G(\eta\cup x)\,k(\xi\cup x)\sum_{\omega\subset\eta}\varepsilon^{-|\xi\setminus\eta|}D_x^{(\varepsilon)}(\omega\cup\xi\setminus\eta)\,dxd\lambda(\xi)\\
&+\int_{\Gamma_0}\sum_{\eta\subset\xi}\int_{\mathbb{R}^d} G(\eta\cup x)\,k(\xi)\sum_{\omega\subset\eta}\varepsilon^{-|\xi\setminus\eta|}B_x^{(\varepsilon)}(\omega\cup\xi\setminus\eta)\,dxd\lambda(\xi)\\
=&-\int_{\Gamma_0}\sum_{x\in\xi}\sum_{\eta\subset\xi\setminus x} G(\eta\cup x)\,k(\xi)\sum_{\omega\subset\eta}\varepsilon^{-|\xi\setminus x\setminus\eta|}D_x^{(\varepsilon)}(\omega\cup\xi\setminus x\setminus\eta)\,d\lambda(\xi)\\
&+\int_{\Gamma_0}\sum_{\eta\subset\xi}\int_{\mathbb{R}^d} G(\eta\cup x)\,k(\xi)\sum_{\omega\subset\eta}\varepsilon^{-|\xi\setminus\eta|}B_x^{(\varepsilon)}(\omega\cup\xi\setminus\eta)\,dxd\lambda(\xi),
\end{aligned}
$$

which implies (4.18). To get (4.20) we may proceed in the same way or just let $\varepsilon\to 0$ in (4.18). Then (4.11)–(4.12) together with equality

$$
\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} G(\xi\cup x)\,D_x^V((\eta\setminus x)\setminus\xi) = \sum_{\xi\subset\eta} G(\xi)\sum_{x\in\xi} D_x^V(\eta\setminus\xi)
$$

provide (4.20).

Analogously, for any $G \in B_{\rm bs}(\Gamma_0)$ we have

$$
\begin{aligned}
&\int_{\Gamma_0} G(\eta)\left(L^*_{\mathrm{hop},\varepsilon,\mathrm{ren}}k\right)(\eta)\,d\lambda(\eta)\\
=&\int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\int_{\mathbb{R}^d}\int_{\Gamma_0} k(\xi\cup\eta\cup y)\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}C^{(\varepsilon)}_{y,x}(\omega\cup\xi)\,d\lambda(\xi)\,dxdyd\lambda(\eta)\\
&-\int_{\Gamma_0} G(\eta)\int_{\Gamma_0} k(\xi\cup\eta)\sum_{x\in\eta}\sum_{\omega\subset\eta\setminus x}\int_{\mathbb{R}^d}\varepsilon^{-|\xi|}C^{(\varepsilon)}_{x,y}(\omega\cup\xi)\,dyd\lambda(\xi)\,d\lambda(\eta)\\
=&\int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{\eta\subset\xi} G(\eta\cup x)\,k(\xi\cup y)\sum_{\omega\subset\eta}\varepsilon^{-|\xi\setminus\eta|}C^{(\varepsilon)}_{y,x}(\omega\cup\xi\setminus\eta)\,dxdyd\lambda(\xi)\\
&-\int_{\Gamma_0}\sum_{\eta\subset\xi} G(\eta)\,k(\xi)\sum_{x\in\eta}\sum_{\omega\subset\eta\setminus x}\int_{\mathbb{R}^d}\varepsilon^{-|\xi\setminus\eta|}C^{(\varepsilon)}_{x,y}(\omega\cup\xi\setminus\eta)\,dyd\lambda(\xi)\\
=&\int_{\Gamma_0} k(\xi)\int_{\mathbb{R}^d}\sum_{y\in\xi}\sum_{\eta\subset\xi\setminus y} G(\eta\cup x)\sum_{\omega\subset\eta}\varepsilon^{-|\xi\setminus y\setminus\eta|}C^{(\varepsilon)}_{y,x}(\omega\cup\xi\setminus y\setminus\eta)\,dxd\lambda(\xi)\\
&-\int_{\Gamma_0} k(\xi)\sum_{\eta\subset\xi} G(\eta)\sum_{x\in\eta}\sum_{\omega\subset\eta\setminus x}\int_{\mathbb{R}^d}\varepsilon^{-|\xi\setminus\eta|}C^{(\varepsilon)}_{x,y}(\omega\cup\xi\setminus\eta)\,dyd\lambda(\xi),
\end{aligned}
$$

which implies (4.19). To get (4.21) we may proceed again in the same way or just let $\varepsilon\to 0$ in (4.19) and use (4.13). □

In the next Section we consider concrete examples for the operator L.

5 Examples

As we have seen in the previous section, the sufficient conditions (4.11)–(4.13) have identical structure for death, birth and hopping parts. Therefore, to present explicit expressions for $L^{\triangle}_{\varepsilon,\mathrm{ren}}$, $V^{\triangle}$ and others we may proceed in the following manner. Let $a(\gamma)=(KA)(\gamma)$, where A is a measurable function on Γ_0; let $a_\varepsilon = KA_\varepsilon$ be some scaling of a and A, $\varepsilon>0$. Below we consider different types of the function a (linear, exponential etc.) and present possible scalings such that for any $\{\eta,\xi\}\subset\Gamma_0$

$$
\exists \lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}A_\varepsilon(\omega\cup\xi)=\lim_{\varepsilon\to 0}\varepsilon^{-|\xi|}A_\varepsilon(\xi)=:A^V(\xi). \tag{5.1}
$$

And after that we may apply this results to the our situation when A_ε depends additionally on $x,y\in\mathbb{R}^d$.

1. Let $a(\gamma)\equiv\alpha\in\mathbb{R}$. Then $A(\eta)=\alpha\cdot 0^{|\eta|}$ and we don't need scaling at all: if $a_\varepsilon=a$ then $A_\varepsilon(\eta)=\alpha\cdot 0^{|\eta|}$ and (5.1) holds with $A^V(\xi)=\alpha\cdot 0^{|\xi|}$.

2. Let $a(\gamma) = \sum\limits_{x\in\gamma} f(x)$ with some $f : \mathbb{R}^d \mapsto \mathbb{R}$. Then $A(\eta) = \chi_{\{\eta=\{x\}\}} f(x)$. We consider the scaling $f \mapsto \varepsilon f$ for which

$$\lim_{\varepsilon\to 0} \sum_{\omega\subset\eta} \varepsilon^{-|\xi|} A_\varepsilon(\omega \cup \xi) = \lim_{\varepsilon\to 0} \sum_{\omega\subset\eta} \varepsilon^{-|\xi|} \chi_{\{\omega\cup\xi=\{x\}\}} \varepsilon f(x)$$
$$= \lim_{\varepsilon\to 0} \varepsilon^{-|\xi|} \chi_{\{\xi=\{x\}\}} \varepsilon f(x) + \lim_{\varepsilon\to 0} \sum_{x\in\eta} \varepsilon f(x) = \chi_{\{\xi=\{x\}\}} f(x) =: A^V(\xi).$$

3. Let $a(\gamma) = \exp\big\{\sum\limits_{x\in\gamma} f(x)\big\}$, $f : \mathbb{R}^d \mapsto \mathbb{R}$. Then $A(\eta) = e_\lambda(e^f - 1, \eta)$. We consider the same scaling $f \mapsto \varepsilon f$ for which

$$\lim_{\varepsilon\to 0} \sum_{\omega\subset\eta} \varepsilon^{-|\xi|} A_\varepsilon(\omega \cup \xi) = \lim_{\varepsilon\to 0} \sum_{\omega\subset\eta} \varepsilon^{-|\xi|} e_\lambda\left(e^{\varepsilon f} - 1, \omega \cup \xi\right)$$
$$= \lim_{\varepsilon\to 0} e_\lambda\left(\frac{e^{\varepsilon f} - 1}{\varepsilon}, \xi\right) \sum_{\omega\subset\eta} e_\lambda\left(e^{\varepsilon f} - 1, \omega\right)$$
$$= \lim_{\varepsilon\to 0} e_\lambda\left(\frac{e^{\varepsilon f} - 1}{\varepsilon}, \xi\right) \lim_{\varepsilon\to 0} e_\lambda\left(e^{\varepsilon f}, \eta\right) = e_\lambda(f, \xi) =: A^V(\xi).$$

4. Let $a(\gamma) = \sum\limits_{x\in\gamma} \sum\limits_{y\in\gamma\setminus x} g(x,y)$ for some (non-symmetric, in general) function g on $\mathbb{R}^d \times \mathbb{R}^d$. Then

$$A(\eta) = \sum_{x\in\eta} K_0^{-1}\Big(\sum_{y\in\cdot} g(x,y)\Big)(\eta \setminus x) = \sum_{x\in\eta} \chi_{\{\eta\setminus x=\{y\}\}} g(x,y)$$
$$= \chi_{\{\eta=\{x,y\}\}}[g(x,y) + g(y,x)].$$

We consider the scaling $g \mapsto \varepsilon^2 g$. Then

$$\lim_{\varepsilon\to 0} \sum_{\omega\subset\eta} \varepsilon^{-|\xi|} A_\varepsilon(\omega \cup \xi)$$
$$= \lim_{\varepsilon\to 0} \sum_{\omega\subset\eta} \varepsilon^{-|\xi|} \chi_{\{\omega\cup\xi=\{x,y\}\}} \varepsilon^2 [g(x,y) + g(y,x)]$$
$$= \lim_{\varepsilon\to 0} \varepsilon^{-2} \chi_{\{\xi=\{x,y\}\}} \varepsilon^2 [g(x,y) + g(y,x)]$$
$$= \chi_{\{\xi=\{x,y\}\}} [g(x,y) + g(y,x)] =: A^V(\xi).$$

5. Let $a(\gamma) = \sum\limits_{x\in\gamma} f(x) \exp\big\{\sum\limits_{y\in\gamma\setminus x} g(x,y)\big\}$, $f : \mathbb{R}^d \mapsto \mathbb{R}$, $g : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$. Then

$$A(\eta) = \sum_{x\in\eta} f(x)\, e_\lambda\big(e^{g(x,\cdot)} - 1, \eta \setminus x\big).$$

Let us consider the scaling $f \mapsto \varepsilon f$, $g \mapsto \varepsilon g$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}A_\varepsilon\left(\omega\cup\xi\right)\\
=&\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}\sum_{x\in\omega\cup\xi}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g(x,\cdot)}-1,\omega\cup\xi\setminus x\right)\\
=&\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\sum_{x\in\omega}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g(x,\cdot)}-1,\omega\setminus x\right)e_\lambda\left(\frac{e^{\varepsilon g(x,\cdot)}-1}{\varepsilon},\xi\right)\\
&+\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\sum_{x\in\xi}\varepsilon^{-1}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g(x,\cdot)}-1,\omega\right)e_\lambda\left(\frac{e^{\varepsilon g(x,\cdot)}-1}{\varepsilon},\xi\setminus y\right)\\
=&\lim_{\varepsilon\to 0}\sum_{x\in\xi}f(x)e_\lambda\left(e^{\varepsilon g(x,\cdot)},\eta\right)e_\lambda\left(\frac{e^{\varepsilon g(x,\cdot)}-1}{\varepsilon},\xi\setminus y\right)\\
=&\sum_{x\in\xi}f\left(x\right)e_\lambda\left(g\left(x,\cdot\right),\xi\setminus x\right)=:A^V(\xi).
\end{aligned}$$

6. Let $a(\gamma)=\big(\sum_{x\in\gamma}f(x)\big)\exp\big\{\sum_{y\in\gamma}g(y)\big\}$, $f,g:\mathbb{R}^d\mapsto\mathbb{R}$. Then

$$\begin{aligned}
A\left(\eta\right)&=\left(\chi_{\{\cdot=\{x\}\}}f(x)\star e_\lambda\left(e^g-1,\cdot\right)\right)\left(\eta\right)\\
&=\sum_{x\in\eta}f(x)e_\lambda\left(e^g-1,\eta\right)+\sum_{x\in\eta}f(x)e_\lambda\left(e^g-1,\eta\setminus x\right).
\end{aligned}$$

Let us consider the scaling $f \mapsto \varepsilon f$, $g \mapsto \varepsilon g$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}A_\varepsilon\left(\omega\cup\xi\right)\\
=&\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}\sum_{x\in\omega\cup\xi}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g}-1,\omega\cup\xi\right)\\
&+\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\varepsilon^{-|\xi|}\sum_{x\in\omega\cup\xi}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g}-1,\omega\cup\xi\setminus x\right)\\
=&\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\sum_{x\in\omega}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g}-1,\omega\right)e_\lambda\left(\frac{e^{\varepsilon g}-1}{\varepsilon},\xi\right)\\
&+\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\sum_{x\in\xi}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g}-1,\omega\right)e_\lambda\left(\frac{e^{\varepsilon g}-1}{\varepsilon},\xi\right)\\
&+\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\sum_{x\in\omega}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g}-1,\omega\setminus x\right)e_\lambda\left(\frac{e^{\varepsilon g}-1}{\varepsilon},\xi\right)\\
&+\lim_{\varepsilon\to 0}\sum_{\omega\subset\eta}\sum_{x\in\xi}\varepsilon^{-1}\varepsilon f(x)e_\lambda\left(e^{\varepsilon g}-1,\omega\right)e_\lambda\left(\frac{e^{\varepsilon g}-1}{\varepsilon},\xi\setminus x\right)\\
=&\sum_{x\in\xi}f(x)e_\lambda\left(g,\xi\setminus x\right)=:A^V(\xi).
\end{aligned}$$

Now we consider different types of birth-and-death and hopping models with rates which have one of the forms considered above. Using explicit expressions for A and scaling for each concrete model we have the expression for A_ε (which is $D_x^{(\varepsilon)}$, $B_x^{(\varepsilon)}$ or $C_{x,y}^{(\varepsilon)}$) and may easily obtain expressions for $\widehat{L}_{\varepsilon,\mathrm{ren}}$ and $L^{\triangle}_{\varepsilon,\mathrm{ren}}$ from (4.6) or (4.7). Using expressions for A^V (which is D_x^V, B_x^V or $C_{x,y}^V$) we may obtain expression for $\widehat{V}$ and $V^{\triangle}$ as well as the form of υ also from the Propositions 4.4 and 4.5. Let us turn to these concrete examples. We present the Vlasov-type equations only.

Example 5.1 (Surgailis model). This birth-and-death model describes independent appearing and disappearing points from a configuration after exponentially distributed random times. The corresponding dynamics was considered in [37], [38]; the generator may be given for $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$

$$(LF)(\gamma) = m\sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)] + \sigma\int_{\mathbb{R}^d}[F(\gamma\cup x) - F(\gamma)]dx.$$

The scaling $m \mapsto m$, $\sigma \mapsto \varepsilon^{-1}\sigma$ leads us to the following Vlasov-type linear equation

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + \sigma.$$

Example 5.2 (Contact model). This model was considered in [29] (for further investigations see [22], [14]). The model describes independent death of the members of a configuration, and, on the other hand, production of new members of the configuration by the existing ones. This is the simplest model for ecological population dynamics. Note that a similar model was considered already in [7] as a particular case of a spatial branching process in continuum. The generator is given on $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ by the expression

$$\begin{aligned}(LF)(\gamma) =& m\sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)]\\ &+ \lambda\sum_{x\in\gamma}\int_{\mathbb{R}^d}a(x-y)[F(\gamma\cup y) - F(\gamma)]dy.\end{aligned}$$

The described scaling $m \mapsto m$, $\lambda \mapsto \varepsilon^{-1}\lambda$, $a \mapsto \varepsilon a$ (that means that $L = L_\varepsilon$) provides the linear Vlasov-type equation also

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + \lambda(\rho_t * a)(x).$$

Here and below $*$ denotes the usual convolution in $\mathbb{R}^d$.

Example 5.3 (Social model). This model was considered in [9]. It describes birth-and-death process with migration from some “reservoir” and competition between members of a configuration. The generator is given on $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ by the expression

$$(LF)(\gamma) = \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} a(x-y)[F(\gamma\setminus x) - F(\gamma)]$$
$$+ \sigma \int_{\mathbb{R}^d} [F(\gamma \cup x) - F(\gamma)]dx.$$

The described scaling $a \mapsto \varepsilon a$, $\sigma \mapsto \varepsilon^{-1}\sigma$ provides the non-linear Vlasov-type equation:

$$\frac{\partial}{\partial t}\rho_t(x) = -\rho_t(x)(\rho_t * a)(x) + \sigma.$$

Example 5.4 (Bolker–Dieckmann–Law–Pacala model). This model of population ecology was considered in [2], [3], [5]. Rigorous mathematical studying of this model was done in [14]. The individual of a population may die independently as well as due to competition for resources; any individual may produce a new one also. The generator is given on $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ by the expression

$$(LF)(\gamma) = \sum_{x\in\gamma}\Big(m + \sum_{y\in\gamma\setminus x} a^-(x-y)\Big)[F(\gamma\setminus x) - F(\gamma)]$$
$$+ \lambda \sum_{x\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)[F(\gamma\cup y) - F(\gamma)]dy.$$

The scaling $a^\pm \mapsto \varepsilon a^\pm$, $m \mapsto m$, $\lambda \mapsto \varepsilon^{-1}\lambda$ gives the following non-linear Vlasov-type equation:

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) - \rho_t(x)(\rho_t * a^-)(x) + \lambda(\rho_t * a^+)(x).$$

Note that in the space-homogeneous case we obtain the logistic-type equation

$$\frac{d}{dt}\rho_t = \big(\lambda\langle a^+\rangle - m - \langle a^-\rangle\rho_t\big)\rho_t,$$

where $\langle a^\pm\rangle = \int_{\mathbb{R}^d} a^\pm(x)dx$. For a rigorous proof of convergence in this scaling see [11].

Example 5.5 (Contact model with establishment). In this model the above described contact dynamics is improved by taking into account the depressive role of the establishment. Namely, the probability for a newborn member to survive in a new place is smaller if there are more particles near this new place. In the language of a generator we describe this by the following expression

$$(LF)(\gamma) = m\sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)]$$
$$+ \lambda\sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)e^{-\sum_{u\in\gamma}\phi(y-u)}[F(\gamma\cup y) - F(\gamma)]dy.$$

The scaling $m \mapsto m$, $\lambda \mapsto \varepsilon^{-1}\lambda$, $a \mapsto \varepsilon a$, $\phi \mapsto \varepsilon\phi$ provides the following non-linear Vlasov-type equation

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + \lambda(a * \rho_t)(x)e^{-(\phi*\rho_t)(x)}.$$

Example 5.6 (Contact model with fecundity). This model describes influence of competition for resources on birth intensity. Namely, if there are many existing members near a "parent", the probability to sent offspring for it is smaller. We consider the following expression for the generator

$$\begin{aligned}(LF)(\gamma) &= m\sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)] \\ &\quad + \lambda\sum_{x\in\gamma} e^{-\sum_{u\in\gamma\setminus x}\phi(x-u)}\int_{\mathbb{R}^d} a(x-y)[F(\gamma\cup y) - F(\gamma)]dy.\end{aligned}$$

The previous scaling $m \mapsto m$, $\lambda \mapsto \varepsilon^{-1}\lambda$, $a \mapsto \varepsilon a$, $\phi \mapsto \varepsilon\phi$ yields another non-linear Vlasov-type equations

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + \lambda(a * (\rho_t e^{-(\phi*\rho_t)}))(x).$$

Example 5.7 (Dieckmann–Law model). This model, as well as the model from Example 5.4 describes ecological population evolution. However, appearing of new offsprings is proportional to the number of existing members of a population. The generator is given by the following expression

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma}\Big(m + \sum_{y\in\gamma\setminus x} a^-(x-y)\Big)[F(\gamma\setminus x) - F(\gamma)] \\ &\quad + \sum_{x\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)\Big(\lambda + \sum_{u\in\gamma\setminus x} b(x-u)\Big)[F(\gamma\cup y) - F(\gamma)]dy.\end{aligned}$$

Note that without competition ($a^- = 0$) this model explodes, namely, the mean value of the number of members in any bounded region becomes infinite after finite time; otherwise, if the competition kernel a^- is "stronger" than the kernel b this effect is absent (for details see [8]). After scaling $a^\pm \mapsto \varepsilon a^\pm$, $m \mapsto m$, $b \mapsto \varepsilon b$ and $1 \mapsto \varepsilon^{-1}$ (before the whole birth term) we obtain the following non-linear Vlasov-type equation

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) - \rho_t(x)(\rho_t * a^-)(x) + \lambda(\rho_t * a^+)(x) + (((b*\rho_t)\rho_t) * a^+)(x).$$

Example 5.8 (Glauber G^+ dynamics). This model is a continuous analog of the Glauber dynamics on a lattice. It was considered in a couple of works, see, e.g., [24], [26], [23], [28], [15], [16]. The generator of this model is given by

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)] \\ &\quad + z\int_{\mathbb{R}^d} e^{-\sum_{u\in\gamma}\phi(y-u)}[F(\gamma\cup y) - F(\gamma)]dy.\end{aligned}$$

Here $z > 0$ is an activity parameter and ϕ is a pair potential. This generator has a reversible measure, namely, the Gibbs measure with parameters z and ϕ (see, e.g., [24], [16] for details). The scaling $m \mapsto m$, $z \mapsto \varepsilon^{-1} z$, $\phi \mapsto \varepsilon\phi$ yields the following non-linear Vlasov-type equation

$$\frac{\partial}{\partial t}\rho_t(x) = -\rho_t(x) + z e^{-(\rho_t * \phi)(x)}.$$

For a rigorous proof of the convergence in this scaling see [12].

Example 5.9 (Glauber G^- dynamics). This model is similar to the previous one, see, e.g., [21], [26].

$$\begin{aligned}(LF)(\gamma) = &\sum_{x\in\gamma} e^{\sum_{u\in\gamma}\phi(x-u)}[F(\gamma \setminus x) - F(\gamma)] \\ &+ z\int_{\mathbb{R}^d}[F(\gamma \cup y) - F(\gamma)]dy.\end{aligned}$$

The same scaling as before yields the similar non-linear Vlasov-type equation

$$\frac{\partial}{\partial t}\rho_t(x) = -\rho_t(x) e^{(\rho_t * \phi)(x)} + z.$$

Example 5.10 (Free Kawasaki). This simplest exactly solvable hopping model was considered in [27]. It describes independent jumps of particles in the system. The generator is the following

$$(LF)(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)\left[F(\gamma\setminus x \cup y) - F(\gamma)\right] dy.$$

We do not need scaling at all to obtain the linear Vlasov-type equation

$$\frac{\partial}{\partial t}\rho_t(x) = (\rho_t * a)(x) - \langle a\rangle \rho_t(x).$$

Example 5.11 (Density dependent Kawasaki). In this model the intensity of a jump is linearly proportional to the existing population. The generator is given on $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ by the expression

$$(LF)(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} a(x-y)\sum_{u\in\gamma} b(x,y,u)\left[F(\gamma\setminus x\cup y) - F(\gamma)\right] dy.$$

The scaling $b \mapsto \varepsilon b$ provides the following non-linear Vlasov-type equation

$$\begin{aligned}\frac{\partial}{\partial t}\rho_t(x) = &\int_{\mathbb{R}^d}\rho_t(y)\, a(x-y)\int_{\mathbb{R}^d}\rho_t(u)\, b(y,x,u)\, du dy \\ &- \rho_t(x)\int_{\mathbb{R}^d} a(x-y)\int_{\mathbb{R}^d}\rho_t(u)\, b(x,y,u)\, du dy.\end{aligned}$$

In particular, if $b(x,y,u) = b(x-u)$ then

$$\frac{\partial}{\partial t}\rho_t\left(x\right) = \left(\left(\rho_t(\rho_t * b)\right) * a\right)(x) - \langle a\rangle \rho_t\left(x\right)\left(\rho_t * b\right)(x).$$

If $b(x,y,u) = b(y-u)$ then

$$\frac{\partial}{\partial t}\rho_t\left(x\right) = \left(\rho_t * b\right)\left(x\right)\left(\rho_t * a\right)\left(x\right) - \rho_t\left(x\right)\left(\rho_t * a * b\right)(x).$$

Example 5.12 (Gibbs–Kawasaki). This hopping particles model was considered, e.g., in [26]. The generator is given by the expression

$$\left(LF\right)\left(\gamma\right) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} a\left(x-y\right)e^{-E^{\phi}(y,\gamma)}\left[F\left(\gamma\setminus x\cup y\right) - F\left(\gamma\right)\right]dy$$

It has a family of reversible Gibbs measures with the potential ϕ and any activity $z > 0$. The scaling $\phi \mapsto \varepsilon\phi$ gives the non-linear Vlasov-type equation of the form

$$\frac{\partial}{\partial t}\rho_t\left(x\right) = \left(\rho_t * a\right)\left(x\right)\exp\left\{-\left(\rho_t * \phi\right)\left(x\right)\right\} - \rho_t\left(x\right)\left(a * \exp\left\{-\rho * \phi\right\}\right)\left(x\right).$$

Example 5.13. In the last example we consider another type of dynamics. Let L describe the generator of the diffusion dynamics (see, e.g., [25], [18]), namely, for any smooth cylindrical function

$$(LF)(\gamma) = \sum_{x\in\gamma}\Delta_x F(\gamma) - \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}\left\langle\nabla\phi(x-y), \nabla_x F\right\rangle,$$

where Δ is a classical Laplace operator in $\mathbb{R}^d$ and ∇ is a gradient in $\mathbb{R}^d$. Our approach covers this case also. It can be shown that the scaling $\phi \mapsto \varepsilon\phi$ provides the following non-linear partial differential Vlasov-type equation

$$\begin{aligned}\frac{\partial}{\partial t}\rho_t\left(x\right) =& \Delta\rho_t(x) - \int_{\mathbb{R}^d}\phi(x-y)\left\langle\nabla\rho_t(x), \nabla\rho_t(y)\right\rangle dy\\ &- \rho_t(x)\int_{\mathbb{R}^d}\left\langle\nabla\phi(x-y), \nabla\rho_t(y)\right\rangle dy.\end{aligned}$$

Acknowledgement Yu.K. is very thankful to Errico Presutti for several instructive discussions of the scaling limit problems and his permanent friendly support. It is a pleasure to express many thanks for fruitful discussions to Carlo Boldrighini, Tobias Kuna, Joel Lebowitz, and Sandro Pellegrinotti during the visit in Rome in October 2009.

The financial support of DFG through the SFB 701 (Bielefeld University) and German-Ukrainian Project 436 UKR 113/94, 436 UKR 113/97 is gratefully acknowledged.

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