REPRESENTATIONS OF NODAL ALGEBRAS OF TYPE A

YURIY A. DROZD AND VASYL V. ZEMBYK

Abstract. We define nodal finite dimensional algebras and describe their structure over an algebraically closed field. For a special class of such algebras (*type* A) we find a criterion of tameness.

Contents

Introduction

Nodal (infinite dimensional) algebras first appeared (without this name) in the paper [4] as *pure noetherian*[1] algebras that are tame with respect to the classification of finite length modules. In [3] their derived categories of modules were described showing that such algebras are also derived tame. Voloshyn [13] described their structure. The definition of nodal algebras can easily be applied to finite dimensional algebras too. The simplest examples show that in finite dimensional case the above mentioned results are no more true: most nodal algebras are wild. It is not so strange, since they are obtained from hereditary algebras, most of which are also wild, in contrast to pure noetherian case, where the only hereditary algebras are those of type $\widetilde{\mathrm{A}}$. Moreover, even if we start from hereditary algebras of type A, we often obtain wild nodal algebras. So the natural question arise, which nodal algebras are indeed tame, at least if we start from a hereditary algebra of type A or $\widetilde{\mathrm{A}}$. In this paper we give an answer to this question (Theorem 5.2).

1991 *Mathematics Subject Classification.* 16G60, 16G10.

Key words and phrases. representations of finite dimensional algebras, nodal algebras, gentle algebras, skewed-gentle algebras.

[1] Recall that *pure noetherian* means noetherian without minimal submodules.

The paper is organized as follows. In Section 1 we give the definition of nodal algebras and their description when the base field is algebraically closed. This description is alike that of [13]. Namely, a nodal algebra is obtained from a hereditary one by two operations called *gluing* and *blowing up*. Equivalently, it can be given by a quiver and a certain symmetric relation on its vertices. In Section 2 we consider a special sort of gluings which do not imply representation types. In Section 3 *gentle* and *skewed-gentle* nodal algebras are described. Section 4 is devoted to a class of nodal algebras called *exceptional*. We determine their representation types. At last, in Section 5 we summarize the obtained results and determine representation types of all nodal algebras of type A.

1. Nodal algebras

We fix an algebraically closed field $\mathbb{k}$ and consider algebras over $\mathbb{k}$. Moreover, if converse is not mentioned, all algebras are supposed to be finite dimensional. For such an algebra $\mathbf{A}$ we denote by $\mathbf{A}$-mod the category of (left) finitely generated $\mathbf{A}$-modules. If an algebra $\mathbf{A}$ is *basic* (i.e. $\mathbf{A}/\operatorname{rad}\mathbf{A} \simeq \mathbb{k}^s$), it can be given by a *quiver* (oriented graph) and relations (see [1] or [6]). Namely, for a quiver Γ we denote by $\mathbb{k}\Gamma$ the *algebra of paths* of Γ and J_Γ be its ideal generated by all arrows. Then every basic algebra is isomorphic to $\mathbb{k}\Gamma/I$, where Γ is a quiver and I is an ideal of $\mathbb{k}\Gamma$ such that $J_\Gamma^2 \supseteq I \supseteq J_\Gamma^k$ for some k. Moreover, the quiver Γ is uniquely defined; it is called the *quiver of the algebra* $\mathbf{A}$. For a vertex i of this quiver we denote by $\operatorname{Ar}(i)$ the set of arrows incident to i. Under this presentation $\operatorname{rad}\mathbf{A} = J_\Gamma/I$, so $\mathbf{A}/\operatorname{rad}\mathbf{A}$ can be identified with the vector space generated by the "empty paths" ε_i, where i runs the vertices of Γ. Note that $1 = \sum_i \varepsilon_i$ is a decomposition of the unit of $\mathbf{A}$ into a sum of primitive orthogonal idempotents. Hence simple $\mathbf{A}$-modules, as well as indecomposable projective $\mathbf{A}$-modules are in one-to-one correspondence with the vertices of the quiver Γ. We denote by $\bar{\mathbf{A}}_i$ the simple module corresponding to the vertex i and by $\mathbf{A}_i = \varepsilon_i\mathbf{A}$ the *right* projective $\mathbf{A}$-module corresponding to this vertex. We also write $i = \alpha^+$ ($i = \alpha^-$) if the arrow α ends (respectively starts) at the vertex i. Usually the ideal I is given by a set of generators R which is then called the *relations of the algebra* $\mathbf{A}$. Certainly, the set of relations (even a minimal one) is far from being unique. An arbitrary algebra can be given by a quiver Γ with relations and *multiplicities* m_i of the vertices $i \in \Gamma$. Namely, it is isomorphic to $\operatorname{End}_{\mathbf{A}} P$, where $\mathbf{A}$ is the basic algebra of $\mathbf{A}$ and $P = \bigoplus_i m_i\mathbf{A}_i$. (We denote by mM the direct sum of m copies of module M.) Recall also that path algebras of quivers without (oriented) cycles are just all *hereditary* basic algebras (up to isomorphism) [1, 6].

Definition 1.1. A (finite dimensional) algebra $\mathbf{A}$ is said to be *nodal* if there is a hereditary algebra $\mathbf{H}$ such that

(1) $\operatorname{rad}\mathbf{H} \subset \mathbf{A} \subseteq \mathbf{H}$,
(2) $\operatorname{rad}\mathbf{A} = \operatorname{rad}\mathbf{H}$,

(3) $\operatorname{length}_{\mathbf{A}}(\mathbf{H}\otimes_{\mathbf{A}} U)\le 2$ for each simple $\mathbf{A}$-module U.

We say that the nodal algebra $\mathbf{A}$ *is related to the hereditary algebra* $\mathbf{H}$.

Remark 1.2. From the description of nodal algebras it follows that the condition (3) may be replaced by the opposite one:

(3′) $\operatorname{length}_{\mathbf{A}}(U\otimes_{\mathbf{A}}\mathbf{H})\le 2$ for each simple right $\mathbf{A}$-module U

(see Corollary 1.10 below).

Proposition 1.3. *If an algebra $\mathbf{A}'$ is Morita equivalent to a nodal algebra $\mathbf{A}$ related to a hereditary algebra $\mathbf{H}$, then $\mathbf{A}'$ is a nodal algebra related to a hereditary algebra $\mathbf{H}'$ that is Morita equivalent to $\mathbf{H}$.*

Proof. Denote $J=\operatorname{rad}\mathbf{H}=\operatorname{rad}\mathbf{A}$. Let P be a projective generator of the category mod-$\mathbf{A}$ of right $\mathbf{A}$-modules such that $\mathbf{A}'\simeq\operatorname{End}_{\mathbf{A}}P$. Then also $\mathbf{A}\simeq\operatorname{End}_{\mathbf{A}'}P\simeq P^\vee\otimes_{\mathbf{A}'}P$, where $P^\vee=\operatorname{Hom}_{\mathbf{A}'}(P,\mathbf{A}')\simeq\operatorname{Hom}_{\mathbf{A}}(P,\mathbf{A})$. Let $P'=P\otimes_{\mathbf{A}}\mathbf{H}$. Then P' is a projective generator of the category mod-$\mathbf{H}$. Set $\mathbf{H}'=\operatorname{End}_{\mathbf{H}}P'$. Note that $\operatorname{Hom}_{\mathbf{H}}(P',M)\simeq\operatorname{Hom}_{\mathbf{A}}(P,M)$ for every right $\mathbf{H}$-module M. In particular, $\operatorname{End}_{\mathbf{H}}P'\simeq\operatorname{Hom}_{\mathbf{A}}(P,P')$. Hence, the natural map $\mathbf{A}'\to\mathbf{H}'$ is a monomorphism. Moreover, since $P'J=PJ$,

$$\begin{aligned}\operatorname{rad}\operatorname{End}_{\mathbf{H}}P'&=\operatorname{Hom}_{\mathbf{H}}(P',P'J)\simeq\\&\simeq\operatorname{Hom}_{\mathbf{A}}(P,P'J)=\operatorname{Hom}_{\mathbf{A}}(P,PJ)=\operatorname{rad}\operatorname{End}_{\mathbf{A}}P\end{aligned}$$

(see [6, Chapter III, Exercise 6]). Thus $\operatorname{rad}\mathbf{A}'=\operatorname{rad}\mathbf{H}'\subset\mathbf{A}'\subseteq\mathbf{H}'$. Every simple $\mathbf{A}'$-module is isomorphic to $U'=P\otimes_{\mathbf{A}}U$ for some simple $\mathbf{A}$-module U. Therefore

$$\begin{aligned}\mathbf{H}'\otimes_{\mathbf{A}'}U'&=\mathbf{H}'\otimes_{\mathbf{A}'}(P\otimes_{\mathbf{A}}U)\simeq(\mathbf{H}'\otimes_{\mathbf{A}'}P)\otimes_{\mathbf{A}}U\simeq\\&\simeq(P\otimes_{\mathbf{A}}\mathbf{H})\otimes_{\mathbf{A}}U\simeq P\otimes_{\mathbf{A}}(\mathbf{H}\otimes_{\mathbf{A}}U),\end{aligned}$$

since

$$\begin{aligned}\mathbf{H}'\otimes_{\mathbf{A}'}P&\simeq\operatorname{Hom}_{\mathbf{A}}(P,P\otimes_{\mathbf{A}}\mathbf{H})\otimes_{\mathbf{A}'}P\simeq((P\otimes_{\mathbf{A}}\mathbf{H})\otimes_{\mathbf{A}}P^\vee)\otimes_{\mathbf{A}'}P\simeq\\&\simeq(P\otimes_{\mathbf{A}}\mathbf{H})\otimes_{\mathbf{A}}(P^\vee\otimes_{\mathbf{A}'}P)\simeq P\otimes_{\mathbf{A}}(\mathbf{H}\otimes_{\mathbf{A}}\mathbf{A})\simeq P\otimes_{\mathbf{A}}\mathbf{H}.\end{aligned}$$

Hence $\operatorname{length}_{\mathbf{A}'}(\mathbf{H}'\otimes_{\mathbf{A}'}U')=\operatorname{length}_{\mathbf{A}}(\mathbf{H}\otimes_{\mathbf{A}}U)\le 2$, so $\mathbf{A}'$ is nodal. □

This proposition allows to consider only *basic* nodal algebras $\mathbf{A}$, i.e. such that $\mathbf{A}/\operatorname{rad}\mathbf{A}\simeq\Bbbk^m$ for some m. We are going to present a construction that gives all basic nodal algebras.

Definition 1.4. Let $\mathbf{B}$ be a basic algebra, $\bar{\mathbf{B}}=\mathbf{B}/\operatorname{rad}\mathbf{B}=\bigoplus_{i=1}^m\bar{\mathbf{B}}_i$, where $\bar{\mathbf{B}}_i\simeq\Bbbk$ are simple $\mathbf{B}$-modules.

(1) Fix two indices i,j. Let $\bar{\mathbf{A}}$ be the subalgebra of $\bar{\mathbf{B}}$ consisting of all m-tuples $(\lambda_1,\lambda_2,\dots,\lambda_m)$ such that $\lambda_i=\lambda_j$, $\mathbf{A}$ be the preimage of $\bar{\mathbf{A}}$ in $\mathbf{B}$. We say that $\mathbf{A}$ *is obtained from* $\mathbf{B}$ *by gluing the components* $\bar{\mathbf{B}}_i$ *and* $\bar{\mathbf{B}}_j$ (or the corresponding vertices of the quiver of $\mathbf{B}$).

(2) Fix an index i. Let $P = 2\mathbf{B}_i \oplus \bigoplus_{k\neq i}\mathbf{B}_k$, $\mathbf{B}' = \mathrm{End}_{\mathbf{B}} P$, $\bar{\mathbf{B}}' = \bar{\mathbf{B}}/\operatorname{rad}\bar{\mathbf{B}} = \prod_{k=1}^m \bar{\mathbf{B}}'_i$, where $\bar{\mathbf{B}}'_i \simeq \mathrm{Mat}(2,\Bbbk)$ and $\bar{\mathbf{B}}'_k \simeq \Bbbk$ for $k \neq i$. Let $\bar{\mathbf{A}}'$ be the subalgebra of $\bar{\mathbf{B}}'$ consisting of all m-tuples $(b_1, b_2, \dots, b_m)$ such that b_i is a diagonal matrix, and $\mathbf{A}$ be the preimage of $\bar{\mathbf{A}}$ in $\mathbf{B}'$. We say that $\bar{\mathbf{A}}$ *is obtained from* $\mathbf{B}$ *by blowing up the component* $\bar{\mathbf{B}}_i$ (or the corresponding vertex of the quiver $\mathbf{B}$).

This definition immediately implies the following properties.

Proposition 1.5. *We keep the notations of Definition 1.4.*

(1) *If* $\mathbf{A}$ *is obtained from* $\mathbf{B}$ *by gluing components* $\bar{\mathbf{B}}_i$ *and* $\bar{\mathbf{B}}_j$*, then it is basic and* $\mathbf{A}/\operatorname{rad}\mathbf{A} = \bar{\mathbf{A}}_{ij}\times\prod_{k\notin\{i,j\}}\bar{\mathbf{B}}_k$*, where* $\bar{\mathbf{A}}_{ij} = \{\,(\lambda,\lambda) \mid \lambda\in\Bbbk\,\} \subset \bar{\mathbf{B}}_i \times \bar{\mathbf{B}}_j$*. Moreover,* $\operatorname{rad}\mathbf{A} = \operatorname{rad}\mathbf{B}$ *and* $\mathbf{B}'\otimes_{\mathbf{A}}\bar{\mathbf{A}}_{ij} \simeq \bar{\mathbf{B}}_i\times\bar{\mathbf{B}}_j$*.*

(2) *If* $\mathbf{A}$ *is obtained from* $\mathbf{B}$ *by blowing up a component* $\bar{\mathbf{B}}_i$*, then it is basic and* $\mathbf{A}/\operatorname{rad}\mathbf{A} = \bar{\mathbf{A}}_{i1}\times\bar{\mathbf{A}}_{i2}\times\prod_{k\neq i}\bar{\mathbf{B}}_k$*, where* $\bar{\mathbf{A}}_{is} = \{\,\lambda e_{ss} \mid \lambda\in\Bbbk\,\}$ *and* e_{ss} $(s \in \{1,2\})$ *denote the diagonal matrix units in* $\bar{\mathbf{B}}_i \simeq \mathrm{Mat}(2,\Bbbk)$*. Moreover,* $\operatorname{rad}\mathbf{A} = \operatorname{rad}\mathbf{B}'$ *and* $\mathbf{B}\otimes_{\mathbf{A}}\bar{\mathbf{A}}_{is} \simeq V$*, where* V *is the simple* $\bar{\mathbf{B}}'_i$*-module.*

We call the component $\bar{\mathbf{A}}_{ij}$ in the former case and the components $\bar{\mathbf{A}}_{is}$ in the latter case *the new components* of $\mathbf{A}$. We also identify all other simple components of $\bar{\mathbf{A}}$ with those of $\bar{\mathbf{B}}$.

Proposition 1.6. *Under the notations of Definition 1.4 suppose that the algebra* $\mathbf{B}$ *is given by a quiver* Γ *with a set of relations* R*.*

(1) *Let* $\mathbf{A}$ *be obtained from* $\mathbf{B}$ *by gluing the components corresponding to vertices* i *and* j*. Then the quiver of* $\mathbf{A}$ *is obtained from* Γ *by identifying the vertices* i *and* j*, while the set of relations for* $\mathbf{A}$ *is* $R\cup R'$*, where* R' *is the set of all products* $\alpha\beta$*, where* α *starts at* i *(or at* j*) and* β *ends at* j *(respectively, at* i*).*

(2) *Let* $\mathbf{A}$ *be obtained from* $\mathbf{B}$ *by blowing up the component corresponding to a vertex* i *and there are no loops at this vertex.*[2] *Then the quiver of* $\mathbf{A}$ *and the set of relations for* $\mathbf{A}$ *are obtained as follows:*
   - *replace the vertex* i *by two vertices* i' *and* i''*;*
   - *replace every arrow* $\alpha : j \to i$ *by two arrows* $\alpha' : j \to i'$ *and* $\alpha'' : j \to i''$*;*
   - *replace every arrow* $\beta : i \to j$ *by two arrows* $\beta' : i' \to j$ *and* $\beta'' : i'' \to j$*;*
   - *replace every relation* r *containing arrows from* $\mathrm{Ar}(i)$ *by two relations* r' *and* r''*, where* r' (r'') *is obtained from* r *by replacing each arrow* $\alpha\in\mathrm{Ar}(i)$ *by* α' *(respectively, by* α''*);*
   - *keep all other relations;*
   - *for every pair of arrows* α *starting at* i *and* β *ending at* i *add a relation* $\alpha'\beta' = \alpha''\beta''$*.*

[2] One can modify the proposed procedure to include such loops, but this modification looks rather cumbersome and we do not need it.

Definition 1.7. We keep the notations of Definition 1.4 and choose pairwise different indices $i_1, i_2, \dots, i_{r+s}$ and $j_1, j_2, \dots, j_r$ from $\{1, 2, \dots, m\}$. We construct the algebras $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{r+s}$ recursively:

$\mathbf{A}_0 = \mathbf{B}$;
for $1 \le k \le r$ the algebra $\mathbf{A}_k$ is obtained from $\mathbf{A}_{k-1}$ by gluing the components $\bar{\mathbf{B}}_{i_k}$ and $\bar{\mathbf{B}}_{j_k}$;
for $r < k \le r+s$ the algebra $\mathbf{A}_k$ is obtained from $\mathbf{A}_{k-1}$ by blowing up the component $\bar{\mathbf{B}}_{i_k}$.

In this case we say that the algebra $\mathbf{A} = \mathbf{A}_{r+s}$ is obtained from $\mathbf{B}$ by the *suitable sequence of gluings and blowings up* defined by the sequence of indices $(i_1, i_2, \dots, i_{r+s}, j_1, j_2, \dots, j_r)$. Note that the order of these gluings and blowings up does not imply the resulting algebra $\mathbf{A}$.

Usually such sequence of gluings and blowings up is given by a symmetric relation $\sim$ (not an equivalence!) on the vertices of the quiver of $\mathbf{B}$ or, the same, on the set of simple $\mathbf{B}$-modules $\bar{\mathbf{B}}_i$: we set $i_k \sim j_k$ for $1 \le k \le r$ and $i_k \sim i_k$ for $r < k \le r+s$. Note that $\#\{\, j \mid i \sim j \,\} \le 1$ for each vertex i.

Theorem 1.8. *A basic algebra $\mathbf{A}$ is nodal if and only if it is isomorphic to an algebra obtained from a basic hereditary algebra $\mathbf{H}$ by a suitable sequence of gluings and blowings up components.*

In other words, a basic nodal algebra can be given by a quiver and a symmetric relation $\sim$ on the set of its vertices such that $\#\{\, j \mid i \sim j \,\} \le 1$ for each vertex i.

Proof. Proposition 1.5 implies that if an algebra $\mathbf{A}$ is obtained from a basic hereditary algebra $\mathbf{H}$ by a suitable sequence of gluings and blowings up, then it is nodal. To prove the converse, we use a lemma about semisimple algebras.

Lemma 1.9. *Let $\widetilde{\mathbf{S}} = \prod_{i=1}^m \widetilde{\mathbf{S}}_i$ be a semisimple algebra, where $\widetilde{\mathbf{S}}_i \simeq \mathrm{Mat}(d_i, \Bbbk)$ are its simple components, $\mathbf{S} = \prod_{k=1}^r \mathbf{S}_k$ be its subalgebra such that $\mathbf{S}_k \simeq \Bbbk$ and $\mathrm{length}_{\mathbf{S}}(\widetilde{\mathbf{S}} \otimes_{\mathbf{S}} \mathbf{S}_k) \le 2$ for all $1 \le k \le r$. Then, for each $1 \le k \le r$*

1. *either $\mathbf{S}_k = \widetilde{\mathbf{S}}_i$ for some i,*
2. *or $\mathbf{S}_k \subset \widetilde{\mathbf{S}}_i \times \widetilde{\mathbf{S}}_j$ for some $i \ne j$ such that $\widetilde{\mathbf{S}}_i \simeq \widetilde{\mathbf{S}}_j \simeq \Bbbk$ and $\mathbf{S}_k \simeq \Bbbk$ embeds into $\widetilde{\mathbf{S}}_i \times \widetilde{\mathbf{S}}_j \simeq \Bbbk \times \Bbbk$ diagonally,*
3. *or there is another index $q \ne k$ such that $\mathbf{S}_k \times \mathbf{S}_q \subset \widetilde{\mathbf{S}}_i$ for some i, $\widetilde{\mathbf{S}}_i \simeq \mathrm{Mat}(2, \Bbbk)$ and this isomorphism can be so chosen that $\mathbf{S}_k \times \mathbf{S}_q$ embeds into $\widetilde{\mathbf{S}}_i$ as the subalgebra of diagonal matrices.*

Proof. Denote $L_{ik} = \widetilde{\mathbf{S}}_i \otimes_{\mathbf{S}} \mathbf{S}_k$. Certainly $L_{ik} \ne 0$ if and only if the projection of $\mathbf{S}_k$ onto $\widetilde{\mathbf{S}}_i$ is non-zero. Since $L_k = \widetilde{\mathbf{S}} \otimes_{\mathbf{S}} \mathbf{S}_k = \bigoplus_{i=1}^m L_{ik}$, there are at most two indices i such that $L_{ik} \ne 0$. Therefore either $\mathbf{S}_k \subseteq \widetilde{\mathbf{S}}_i$ for some i or $\mathbf{S}_k \subseteq \widetilde{\mathbf{S}}_i \times \widetilde{\mathbf{S}}_j$ for some $i \ne j$ and both L_{ik} and L_{jk} are non-zero. Note that $\dim_{\Bbbk} L_{ik} \ge d_i$ and $\dim_{\Bbbk} L_k \le 2$. So in the latter case $\widetilde{\mathbf{S}}_i \simeq \widetilde{\mathbf{S}}_j \simeq \Bbbk$. Obviously, $\Bbbk$ can embed into $\Bbbk \times \Bbbk$ only diagonally.

Suppose that $\mathbf{S}_k \subseteq \widetilde{\mathbf{S}}_i$ but $\mathbf{S}_k \neq \widetilde{\mathbf{S}}_i$. Then $d_i = 2$, so $\widetilde{\mathbf{S}}_i \simeq \mathrm{Mat}(2, \Bbbk)$. Then the unique simple $\widetilde{\mathbf{S}}_i$-module is 2-dimensional. If $\mathbf{S}_k$ is the only simple $\mathbf{S}$-module such that $\mathbf{S}_k \subset \widetilde{\mathbf{S}}_i$, then it embeds into $\widetilde{\mathbf{S}}_i$ as the subalgebra of scalar matrices, thus $L_{ik} \simeq \widetilde{\mathbf{S}}_i$ is of dimension 4, which is impossible. Hence there is another index $q \neq k$ such that $\mathbf{S}_q \subset \widetilde{\mathbf{S}}_i$. Then the image of $\mathbf{S}_k \times \mathbf{S}_q \simeq \Bbbk^2$ in $\mathrm{Mat}(2, \Bbbk)$ is conjugate to the subalgebra of diagonal matrices [6, Chapter II, Exercise 2]. $\square$

Let now $\mathbf{A}$ be a nodal algebra related to a hereditary algebra $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{S}} = \widetilde{\mathbf{H}}/\operatorname{rad}\widetilde{\mathbf{H}}$ and $\bar{\mathbf{A}} = \mathbf{A}/\operatorname{rad}\mathbf{A}$. We denote by $\mathbf{H}$ the basic algebra of $\widetilde{\mathbf{H}}$ [6, Section III.5] and for each simple component $\widetilde{\mathbf{S}}_i$ of $\widetilde{\mathbf{S}}$ we denote by $\bar{\mathbf{H}}_i$ the corresponding simple components of $\bar{\mathbf{H}} = \mathbf{H}/\operatorname{rad}\mathbf{H}$. We can apply Lemma 1.9 to the algebra $\widetilde{\mathbf{S}} = \widetilde{\mathbf{H}}/\operatorname{rad}\widetilde{\mathbf{H}}$ and its subalgebra $\bar{\mathbf{A}} = \mathbf{A}/\operatorname{rad}\mathbf{A}$. Let $(i_1, j_1), \dots, (i_r, j_r)$ be all indices such that the products $\widetilde{\mathbf{S}}_{i_k} \times \widetilde{\mathbf{S}}_{j_k}$ occur as in the case (2) of the Lemma, while $i_{r+1}, \dots, i_{r+s}$ be all indices such that $\widetilde{\mathbf{S}}_{i_k}$ occur in the case (3). Then it is evident that $\mathbf{A}$ is obtained from $\mathbf{H}$ by the suitable sequence of gluings and blowings up defined by the sequence of indices $(i_1, i_2, \dots, i_{r+s}, j_1, j_2, \dots, j_r)$. $\square$

Since the construction of gluing and blowing up is left–right symmetric, we get the following corollary.

Corollary 1.10. *If an algebra* $\mathbf{A}$ *is nodal, so is its opposite algebra. In particular, in the Definition 1.1 one can replace the condition* (3) *by the condition* (3′) *from Remark 1.2.*

Thus, to define a basic nodal algebra, we have to define a quiver Γ and a sequence of its vertices $(i_1, i_2, \dots, i_{r+s}, j_1, j_2, \dots, j_r)$. Actually, one can easily describe the resulting algebra $\mathbf{A}$ by its quiver and relations. Namely, we must proceed as follows:

(1) For each $1 \leq k \leq r$
  (a) we glue the vertices i_k and j_k keeping all arrows starting and ending at these vertices;
  (b) if an arrow α starts at the vertex i_k (or j_k) and an arrow β ends at the vertex j_k (respectively i_k), we impose the relation $\alpha\beta = 0$.

(2) For each $r < k \leq r + s$
  (a) we replace each vertex i_k by two vertices i'_k and i''_k;
  (b) we replace each arrow $\alpha : j \to i_k$ by two arrows $\alpha' : j \to i'_k$ and $\alpha'' : j \to i''_k$;
  (c) we replace each arrow $\beta : i_k \to j$ by two arrows $\beta' : i'_k \to j$ and $\beta'' : i''_k \to j$;
  (d) if an arrow β starts at the vertex i_k and an arrow α ends at this vertex, we impose the relation $\beta'\alpha' = \beta''\alpha''$.

We say that $\mathbf{A}$ is a *nodal algebra of type* Γ. In particular, if Γ is a Dynkin quiver of type A or a Euclidean quiver of type $\tilde{\mathrm{A}}$, we say that $\mathbf{A}$ is a *nodal algebra of type* A.

To define a nodal algebra which is not necessarily basic, we also have to prescribe the multiples l_i for each vertex i so that $l_{i_k} = l_{j_k}$ for $1 \le k \le r$.

In what follows we often present a basic nodal algebra by the quiver Γ, just marking the vertices $i_1, i_2, \dots, i_r, j_1, j_2, \dots, j_r$ by bullets, with the indices $1, 2, \dots, r$, and marking the vertices $i_{r+1}, \dots, i_{r+s}$ by circles. For instance:

$$\begin{tikzcd}
\cdot \arrow[r, "\alpha_1"] & \bullet 1 \arrow[r, "\alpha_2"] & \cdot \arrow[r, "\alpha_3"] \arrow[d, "\alpha_5"] & \bullet 1 & \bullet 2 \arrow[l, "\alpha_4"'] \\
 & & \circ 3 \arrow[d, "\alpha_6"] & & \\
 & & \bullet 2 & &
\end{tikzcd}$$

In this example the resulting nodal algebra $\mathbf{A}$ is given by the quiver with relations

$$\begin{tikzcd}
 & & \circ 3' \arrow[dr, "\alpha_6'"] & \\
\cdot \arrow[r, "\alpha_1"] & \bullet 1 \arrow[r, shift left, "\alpha_2"] & \cdot \arrow[l, shift left, "\alpha_3"] \arrow[u, "\alpha_5'"] \arrow[dr, "\alpha_5''"'] & \bullet 2 \arrow[ll, bend left, "\alpha_4"'] \\
 & & & \circ 3'' \arrow[u, "\alpha_6''"']
\end{tikzcd}
\qquad
\begin{aligned}
\alpha_2\alpha_3 &= \alpha_2\alpha_4 = 0\\
\alpha_4\alpha_6' &= \alpha_4\alpha_6'' = 0\\
\alpha_6'\alpha_5' &= \alpha_6''\alpha_5''
\end{aligned}$$

2. Inessential gluings

In this section we study one type of gluing which never implies the representation type.

Definition 2.1. Let a basic algebra $\mathbf{B}$ is given by a quiver Γ with relations and an algebra $\mathbf{A}$ is obtained from $\mathbf{B}$ by gluing the components corresponding to the vertices i and j such that there are no arrows ending at i and no arrows starting at j. Then we say that this gluing is *inessential.*

It turns out that the categories $\mathbf{A}$-mod and $\mathbf{B}$-mod are "almost the same."

Theorem 2.2. *Under the conditions of Definition 2.1, there is an equivalence of the categories* $\mathbf{B}\text{-mod}/\langle\, \bar{\mathbf{B}}_i, \bar{\mathbf{B}}_j \,\rangle$ *and* $\mathbf{A}\text{-mod}/\langle\, \bar{\mathbf{A}}_{ij} \,\rangle$*, where* $\mathcal{C}/\langle\, \mathfrak{M} \,\rangle$ *denotes the quotient category of* $\mathcal{C}$ *modulo the ideal of morphisms that factor through direct sums of objects from the set* $\mathfrak{M}$.

Proof. We identify $\mathbf{B}$-modules and $\mathbf{A}$-modules with the representations of the corresponding quivers with relations. Recall that the quiver of $\mathbf{A}$ is obtained from that of $\mathbf{B}$ by gluing the vertices i and j into one vertex (ij). For a $\mathbf{B}$-module M denote by $\mathbf{F}M$ the $\mathbf{A}$-module such that

$$\begin{aligned}
\mathbf{F}M(k) &= M(k) \text{ for any vertex } k \neq (ij),\\
\mathbf{F}M(ij) &= M(i) \oplus M(j),\\
\mathbf{F}M(\gamma) &= M(\gamma) \text{ if } \gamma \notin \mathrm{Ar}(ij),\\
\mathbf{F}M(\alpha) &= \begin{pmatrix} M(\alpha) & 0 \end{pmatrix} \text{ if } \alpha \in \mathrm{Ar}(i) \setminus \mathrm{Ar}(j),\\
\mathbf{F}M(\beta) &= \begin{pmatrix} 0 \\ M(\beta) \end{pmatrix} \text{ if } \beta \in \mathrm{Ar}(j) \setminus \mathrm{Ar}(i)\\
\mathbf{F}M(\alpha) &= \begin{pmatrix} 0 & 0 \\ M(\alpha) & 0 \end{pmatrix} \text{ if } \alpha : i \to j.
\end{aligned} \tag{2.1}$$

If $f : M \to M'$ is a homomorphism of $\mathbf{B}$-modules, we define the homomorphism $\mathbf{F}f : \mathbf{F}M \to \mathbf{F}M'$ setting

$$\begin{aligned}
\mathbf{F}f(k) &= f(k) \text{ if } k \neq (ij),\\
\mathbf{F}f(ij) &= \begin{pmatrix} f(i) & 0 \\ 0 & f(j) \end{pmatrix}
\end{aligned}$$

Thus we obtain a functor $\mathbf{F} : \mathbf{B}\text{-mod} \to \mathbf{A}\text{-mod}$. Obviously, $\mathbf{F}\bar{\mathbf{B}}_i = \mathbf{F}\bar{\mathbf{B}}_j = \bar{\mathbf{A}}_{ij}$, so $\mathbf{F}$ induced a functor $\mathbf{f} : \mathbf{B}\text{-mod}/\langle \bar{\mathbf{B}}_i, \bar{\mathbf{B}}_j \rangle \to \mathbf{A}\text{-mod}/\langle \bar{\mathbf{A}}_{ij} \rangle$.

Let now N be an $\mathbf{A}$-module. We define the $\mathbf{B}$-module $\mathbf{G}N$ as follows:

$$\begin{aligned}
&\mathbf{G}N(k) = N(k) \text{ if } k \notin \{i, j\},\\
&\mathbf{G}N(i) = N(ij)/N_0(ij), \text{ where } N_0(ij) = \bigcap_{\alpha^- = (ij)} \operatorname{Ker} N(\alpha),\\
&\mathbf{G}N(j) = \sum_{\beta^+ = (ij)} \operatorname{Im} N(\beta),\\
&\mathbf{G}N(\beta) = N(\beta) \text{ if } \beta \notin \mathrm{Ar}(i),\\
&\mathbf{G}N(\alpha) \text{ is the induced map } \mathbf{G}N(i) \to \mathbf{G}N(k) \text{ if } \alpha : i \to k.
\end{aligned}$$

Note that if $\beta^+ = j$, then $\operatorname{Im} N(\beta) \subseteq \mathbf{G}N(j)$. If $g : N \to N'$ is a homomorphism of $\mathbf{A}$-modules, then $g(ij)(\mathbf{G}N(j)) \subseteq \mathbf{G}N'(j)$ and $g(ij)(N_0(ij)) \subseteq N'_0(ij)$. So we define the homomorphism $\mathbf{G}g : \mathbf{G}N \to \mathbf{G}N'$ setting

$$\begin{aligned}
&\mathbf{G}g(k) = g(k) \text{ if } k \neq i,\\
&\mathbf{G}g(i) \text{ is the map } \mathbf{G}N(i) \to \mathbf{G}N'(i) \text{ induced by } g(ij),\\
&\mathbf{G}g(j) \text{ is the resriction of } g(ij) \text{ onto } \mathbf{G}N(j).
\end{aligned}$$

Thus we obtain a functor $\mathbf{G} : \mathbf{A}\text{-mod} \to \mathbf{B}\text{-mod}$. Since $\mathbf{G}\bar{\mathbf{A}}_{ij} = 0$, it induces a functor $\mathbf{g} : \mathbf{A}\text{-mod}/\langle \bar{\mathbf{A}}_{ij} \rangle \to \mathbf{B}\text{-mod}/\langle \bar{\mathbf{B}}_i, \bar{\mathbf{B}}_j \rangle$. Suppose that $\mathbf{G}g = 0$. It means that $g(k) = 0$ for $k \neq (ij)$, $\operatorname{Im} g(ij) \subseteq \bigcap_{\alpha^- = (ij)} \operatorname{Ker} N'(\alpha)$

and $\operatorname{Ker} g(ij) \supseteq \sum_{\beta^+=(ij)} \operatorname{Im} N(\beta)$. So $g(ij)$ induces the map

$$\bar{g} : N(j)/\sum_{\beta^+=j} \operatorname{Im} N(\beta) \to N'(ij)$$

with $\operatorname{Im} \bar{g} \subseteq \bigcap_{\alpha^-=(ij)} \operatorname{Ker} N'(\alpha)$. So $g = g''g'$, where

$$\begin{aligned}
&g' : N \to \overline{N} \text{ and } g'' : \overline{N} \to N', \\
&\overline{N}(k) = 0 \text{ if } k \neq (ij), \\
&\overline{N}(ij) = N(j)/\sum_{\beta^+=j} \operatorname{Im} N(\beta), \\
&g'(k) = g''(k) = 0 \text{ if } k \neq (ij), \\
&g'(ij) \text{ is the natural surjection } N(ij) \to \overline{N}(ij), \\
&g''(ij) = \bar{g}.
\end{aligned}$$

Obviously, $\overline{N} \simeq m\bar{\mathbf{A}}_{ij}$ for some m, so $\operatorname{Ker} \mathbf{G}$ is just the ideal $\langle\, \bar{\mathbf{A}}_{ij}\, \rangle$.

By the construction,

$$\begin{aligned}
\mathbf{GF}M(i) &= M(i)/\bigcap_{\alpha^-=i} \operatorname{Ker} \alpha, \\
\mathbf{GF}M(j) &= \sum_{\beta^+=j} \operatorname{Im} \beta, \\
\mathbf{FG}N(ij) &= N(ij)/\bigcap_{\alpha^-=i} \operatorname{Ker} \alpha \oplus \sum_{\beta^+=(ij)} \operatorname{Im} N(\beta).
\end{aligned}$$

So we fix

for every $\mathbf{B}$-module M a retraction $\rho_M : M(j) \to \sum_{\beta^+=j} \operatorname{Im} \beta$,
for every $\mathbf{A}$-module N a retraction $\rho_N : N(ij) \to \sum_{\beta^+=(ij)} \operatorname{Im} \beta$

and define the morphisms of functors

$$\begin{aligned}
&\phi : \mathrm{Id}_{\mathbf{B}\text{-mod}} \to \mathbf{GF} \text{ such that} \\
&\quad \phi_M(k) = \mathrm{Id}_{M(k)} \text{ if } k \notin \{i, j\}, \\
&\quad \phi_M(j) = \rho_M, \\
&\quad \phi_M(i) \text{ is the natural surjection } M(i) \to \mathbf{GF}M(i),
\end{aligned}$$

and

$$\begin{aligned}
&\psi : \mathrm{Id}_{\mathbf{A}\text{-mod}} \to \mathbf{FG} \text{ such that} \\
&\quad \psi_N(k) = \mathrm{Id}_{N(k)} \text{ if } k \neq (ij), \\
&\quad \psi_N(ij) = \rho_N.
\end{aligned}$$

Evidently, if M has no direct summands $\bar{\mathbf{B}}_i$ and $\bar{\mathbf{B}}_j$, then ϕ_M is an isomorphism. Also if N has no direct summands $\bar{\mathbf{A}}_{ij}$, then ψ_N is an isomorphism. Therefore, $\mathbf{g}$ and $\mathbf{f}$ are mutually quasi-inverse, defining an equivalence of the categories $\mathbf{B}\text{-mod}/\langle\, \bar{\mathbf{B}}_i, \bar{\mathbf{B}}_j\, \rangle$ and $\mathbf{A}\text{-mod}/\langle\, \bar{\mathbf{A}}_{ij}\, \rangle$. □

3. Gentle and skewed-gentle case

In what follows we only consider non-hereditary nodal algebras, since the representation types of hereditary algebras are well-known. Evidently, blowing up a vertex i such that there are no arrows starting at i or no arrows ending at i, applied to a hereditary algebra, gives a hereditary algebra. The same happens if we glue vertices i and j such that there are no arrows starting at these vertices or no arrows ending at them.

Recall that a basic (finite dimensional) algebra $\mathbf{A}$ is said to be *gentle* if it is given by a quiver Γ with relations R such that

(1) for every vertex $i \in \Gamma$, there are at most two arrows starting at i and at most two arrows ending at i;
(2) all relations in R are of the form $\alpha\beta$ for some arrows α, β;
(3) if there are two arrows α_1, α_2 starting at i, then, for each arrow β ending at i, either $\alpha_1\beta \in R$ or $\alpha_2\beta \in R$, but not both;
(4) if there are two arrows β_1, β_2 ending at i, then, for each arrow α starting at i, either $\alpha\beta_1 \in R$ or $\alpha\beta_2 \in R$, but not both.

A basic algebra $\mathbf{A}$ is said to be *skewed-gentle* if it can be obtained from a gentle algebra $\mathbf{B}$ by blowing up some vertices i such that there is at most one arrow α starting at i, at most one arrow β ending at i and if both exist then $\alpha\beta \notin R$.[3]

It is well-known that gentle and skewed-gentle algebras are tame, and even derived tame (i.e. their derived categories of finite dimensional modules are also tame). Skowronski and Waschbüsch [12] proved a criterion of representation finiteness for biserial algebras, the class containing, in particular, all gentle algebras. We give a complete description of nodal algebras which are gentle or skewed-gentle.

Theorem 3.1. *A nodal algebra* $\mathbf{A}$ *is skewed-gentle if and only if it is obtained from a direct product of hereditary algebras of type* A *or* $\widetilde{\mathrm{A}}$ *by a suitable sequence of gluings and blowings up defined by a sequence of vertices such that, for each of them, there is at most one arrow starting and at most one arrow ending at this vertex. It is gentle if and only if, moreover, it is obtained using only gluings.*

Proof. If $\mathbf{A}$ is related to a hereditary algebra $\mathbf{H}$ such that its quiver is not a disjoint union of quivers of type A or $\widetilde{\mathrm{A}}$, there is a vertex i in the quiver of $\mathbf{H}$ such that $\mathrm{Ar}(i)$ has at least 3 arrows. The same is then true for the quiver of $\mathbf{A}$. Moreover, there are no relations containing more that one of these arrows, which is impossible in a gentle or skewed-gentle algebra.

So we can suppose that the quiver of $\mathbf{H}$ is a disjoint union of quivers of type A or $\widetilde{\mathrm{A}}$. Let $\mathbf{A}$ is obtained from $\mathbf{H}$ by a suitable sequence of gluings and blowings up defined by a sequence of vertices $i_1, i_2, \dots, i_{r+s}, j_1, j_2, \dots, j_r$. Suppose that there is an index $1 \le k \le r+s$ such that there are two arrows

[3] The original definition of skew-gentle algebras in [7] as well as that in [2] differ from ours, but one can easily see that all of them are equivalent.

α_1, α_2 ending at i_k (the case of two starting arrows is analogous). If $k \leq r$ and there is an arrow ending at j_k, there are 3 vertices ending at the vertex (ij) in the quiver of $\mathbf{A}$, neither two of them occurring in the same relation, which is impossible in gentle or skewed-gentle case. If there is an arrow β starting at j_k, it occurs in two zero relations $\beta\alpha_1 = \beta\alpha_2 = 0$, which is also impossible.

Finally, if we apply blowing up, we obtain three arrows incident to a vertex without zero relations between them which is impossible in a gentle algebra. Thus the conditions of the theorem are necessary.

On the contrary, let $\mathbf{H}$ be a hereditary algebra and its quiver is a disjoint union of quivers of type A or $\widetilde{\mathrm{A}}$, $i_1 \neq i_2$ be vertices of this quiver such that there is a unique arrow α_k starting at i_k as well as a unique arrow β_k ending at i_k $(k = 1, 2)$. Then gluing of vertices i_1, i_2 gives a vertex $i = (i_1 i_2)$ in the quiver of the obtained algebra, two arrows α_k starting at i and two arrows β_k ending at i $(i = 1, 2)$ satisfying relations $\alpha_1\beta_2 = \alpha_2\beta_1 = 0$. Therefore, gluing such vertices give a gentle algebra. Afterwards, blowing up vertices j such that there is one arrow α starting at j and one arrow β ending at it gives a skewed-gentle algebra since $\alpha\beta \neq 0$ in $\mathbf{H}$. □

4. Exceptional algebras

We consider some more algebras obtained from hereditary algebras of type A.

Definition 4.1. Let $\mathbf{H}$ be a basic hereditary algebra with a quiver Γ.

(1) We call a pair of vertices (i, j) of the quiver Γ *exceptional* if they are contained in a full subquiver of the shape

$$\cdot \xrightarrow{\beta} \overset{i}{\cdot} \xleftarrow{\alpha_1} \cdot \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} \cdot \xleftarrow{\alpha_n} \overset{j}{\cdot} \xleftarrow{\gamma} \cdot \tag{4.1}$$

or

$$\cdot \xleftarrow{\beta} \overset{i}{\cdot} \xrightarrow{\alpha_1} \cdot \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} \cdot \xrightarrow{\alpha_n} \overset{j}{\cdot} \xrightarrow{\gamma} \cdot \tag{4.2}$$

where the orientation of the arrows $\alpha_2, \dots, \alpha_{n-1}$ is arbitrary. Possibly $n = 1$, then $\alpha_1 = \alpha_n : j \to i$ in case (4.1) and $\alpha_1 = \alpha_n : i \to j$ in case (4.2).

(2) We call gluing of an exceptional pair of vertices *exceptional gluing.*

(3) A nodal algebra is said to be *exceptional* if it is obtained from a hereditary algebra of type A by a suitable sequence of gluings consisting of one exceptional gluing and, maybe, several inessential gluings.

Recall that inessential gluing does not imply the representations type of an algebra. So we need not take them into account only considering exceptional algebras obtained by a unique exceptional gluing. Note that such gluing results in the algebra $\mathbf{A}$ given by the quiver with relations

(4.3)
$$\begin{tikzcd} & & \bullet \arrow[rr, dotted, no head] \arrow[dr, "\alpha_1"] & & \bullet \\ \bullet \arrow[r, no head, "\beta_m"] & \dots \arrow[r, no head, "\beta_1"] & \bullet \arrow[r, "\beta"] & \underset{(ij)}{\bullet} \arrow[ur, "\alpha_n"'] & \bullet \arrow[l, "\gamma"'] \arrow[r, no head, "\gamma_1"] & \dots \arrow[r, no head, "\gamma_l"] & \bullet \end{tikzcd} \qquad \alpha_n\alpha_1 = \alpha_n\beta = 0$$

in case (4.1) or

(4.4)
$$\begin{tikzcd} & & \bullet \arrow[rr, dotted, no head] & & \bullet \arrow[dl, "\alpha_n"] \\ \bullet \arrow[r, no head, "\beta_m"] & \dots \arrow[r, no head, "\beta_1"] & \bullet & \underset{(ij)}{\bullet} \arrow[l, "\beta"'] \arrow[r, "\gamma"] \arrow[ul, "\alpha_1"'] & \bullet \arrow[r, no head, "\gamma_1"] & \dots \arrow[r, no head, "\gamma_l"] & \bullet \end{tikzcd} \qquad \alpha_1\alpha_n = \beta\alpha_n = 0$$

in case (4.2). The dotted line consists of the arrows $\alpha_2, \dots, \alpha_{n-1}$; if $n = 1$, we get a loop α at the vertex (ij) with the relations $\alpha^2 = 0$ and, respectively, $\alpha\beta = 0$ or $\beta\alpha = 0$. We say that $\mathbf{A}$ is an (n, m, l)-*exceptional algebra.*

We determine representation types of exceptional algebras.

Theorem 4.2. *An (n, m, l)-exceptional algebra is*

(1) *representation finite in cases:*
   (a) $m = l = 0$,
   (b) $l = 0$, $m = 1$, $n \leq 3$,
   (c) $l = 0, 2 \leq m \leq 3$, $n = 1$,
   (d) $m = 0$, $l = 1$, $n \leq 2$.
(2) *tame in cases:*
   (a) $l = 0$, $m = 1$, $n = 4$,
   (b) $l = 0$, $m = 2$, $n = 2$,
   (c) $l = 0$, $m = 4$, $n = 1$,
   (d) $m = 0$, $l = 1$, $n = 3$.
(3) *wild in all other cases.*

Proof. We consider an algebra $\mathbf{A}$ given by the quiver with relations (4.4). Denote by $\mathbf{C}$ the algebra given by the quiver with relations

$$\begin{tikzcd} \cdot \arrow[rr, dotted, no head] & & \cdot \arrow[dl, "\alpha_n"] \\ & \bullet \arrow[ul, "\alpha_1"] & \end{tikzcd} \qquad \alpha_1\alpha_n = 0$$

(the bullet shows the vertex (ij)). It is obtained from the path algebra of the quiver

$$\Gamma_n = \overset{0}{\cdot} \xrightarrow{\alpha_1} \overset{1}{\cdot} \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} \overset{n-1}{\cdot} \xrightarrow{\alpha_n} \overset{n}{\cdot}$$

by gluing vertices 0 and n. This gluing is inessential, so we can use Theorem 2.2. We are interested in the indecomposable representations of $\mathbf{C}$ that are non-zero at the vertex $\bullet$. We denote by $\mathfrak{L}$ the set of such representations. They arise from the representations of the quiver Γ_n that are non-zero at the vertex 0 or n. Such representations are $\tilde{L}_i$ and $\tilde{L}'_i$ $(0 \le i \le n)$, where

$$\tilde{L}_i(k) \begin{cases} \Bbbk & \text{if } k \le i, \\ 0 & \text{if } k > i; \end{cases}$$

$$\tilde{L}'_i(k) = \begin{cases} \Bbbk & \text{if } k \ge i, \\ 0 & \text{if } k < i. \end{cases}$$

We denote by L_i and L'_i respectively the representations $\mathbf{F}\tilde{L}_i$ and $\mathbf{F}\tilde{L}'_i$ (see page 8, formulae (2.1)). Obviously $L_n = L'_0$, $L_0 = L'_n = \overline{\mathbf{C}}_\bullet$ and $\dim_\Bbbk L_i(\bullet) = 1$ for $i \ne n$ as well as $\dim_\Bbbk L'_i(\bullet) = 1$ for $i \ne 0$, while $\dim_\Bbbk L_n(\bullet) = 2$. We denote by e_i $(0 \le i < n)$ and e'_j $(0 < i < n)$ basic vectors respectively of L_i and L'_j, and by e_n, e'_n basic vectors of $L_n = L'_0$ such that $e'_n \in \operatorname{Im} L_n(\alpha_n)$; then $L_n(\alpha_1)(e'_n) = 0$, while $L_n(\alpha_1)(e_n) \ne 0$. We consider the set $\mathbf{E} = \{ e_0, e_1, \dots, e_n \}$ and the relation $\prec$ on $\mathbf{E}$, where $u \prec v$ means that there is a homomorphism f such that $f(u) = v$. From the well-known (and elementary) description of representations of the quiver Γ and Theorem 2.2 it follows that $\prec$ is a linear order and $e_i \prec e_0 \prec e'_j$ for all i, j. Let $u_0, u_1, \dots, u_{2n}$ be such a numeration of the elements of $\mathbf{E}$ that $u_i \prec u_j$ if and only if $i \le j$ (then $u_n = e_0$), and $e_n = u_k, e'_n = u_l$ $(k < n < l)$. Note also that if $f \in \operatorname{End} L_n$, the matrix $f(\bullet)$ in the basis e_n, e'_n is of the shape $\begin{pmatrix} \lambda & 0 \\ \mu & \lambda \end{pmatrix}$.

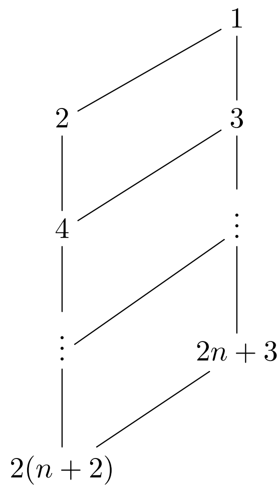
Let M be an $\mathbf{A}$-module, $\bar{M}$ be its restriction onto $\mathbf{C}$. Then $\bar{M} \simeq \bigoplus_i m_i L_i \oplus \bigoplus_j m'_j L'_j$, where $M_i \simeq$ and $M'_j \simeq$. Respectively $M(\bullet) = \bigoplus_{i=1}^{2n+1} U_i$, where U_i is generated by the images of the vectors u_i. Note that, for $i > n$, $u_i = e'_j$ for some j, so $M(\beta)U_i = 0$. Therefore, the matrices $M(\beta)$ and $M(\gamma)$ shall be considered as block matrices

$$\begin{aligned} M(\beta) &= \begin{pmatrix} B_0 & B_1 & \dots & B_n & 0 & \dots & 0 \end{pmatrix}, \\ M(\gamma) &= \begin{pmatrix} C_0 & C_1 & \dots & C_n & C_{n+1} & \dots & C_{2n} \end{pmatrix}, \end{aligned} \tag{4.5}$$

where the matrices B_i, C_i correspond to the summands U_i. If $f \in \operatorname{Hom}_{\mathbf{A}}(M, N)$, then $f(\bullet)$ is a block lower triangular matrix (f_{ij}), where $f_{ij} : U_j \to U_i$ and $f_{ij} = 0$ if $i < j$. Moreover, the non-zero blocks can be arbitrary with the only condition that $f_{kk} = f_{ll}$. Hence given any matrix $f(\bullet)$ with this condition and invertible diagonal blocks f_{ii}, we can construct a module N isomorphic to M just setting $N(\beta) = M(\beta)f(\bullet)$, $N(\gamma) = M(\gamma)f(\bullet)$. Then one can easily transform the matrix $M(\gamma)$ so that there is at most one non-zero element (equal 1) in every row and in every column, if $i \notin \{k, l\}$, the non-zero rows of C_i have the form $\begin{pmatrix} I & 0 \end{pmatrix}$ and the non-zero rows of the matrix $(C_k \mid C_l)$ are of the form

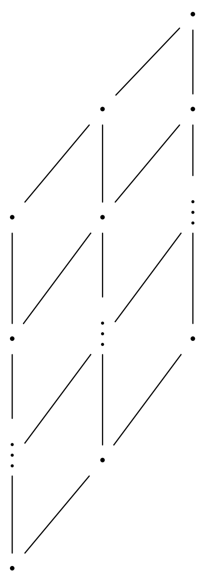
$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We subdivide the columns of the matrices B_i respectively to this subdivision of C_i. It gives $2(n+2)$ new blocks $\widetilde{B}_s (1 \le s \le 2(n+2))$. Namely, the blocks $\widetilde{B}_s$ with s odd correspond to the non-zero blocks of the matrices C_i and those with s even correspond to zero columns of C_i. Two extra blocks come from the subdivision of C_k into 4 vertical stripes. We also subdivide the blocks f_{ij} of the matrix $f(\bullet)$ in the analogous way. From now on we only consider such representations that the matrix $M(\gamma)$ is of the form reduced in this way. One can easily check that it imposes the restrictions on the matrix $f(\bullet)$ so that the new blocks $\tilde{f}_{st}$ obtained from f_{ij} with $0 \le i, j \le n$ can only be non-zero (and then arbitrary) if and only if $s > t$ and, moreover, either t is odd or s is even. It means that these new blocks can be considered as a representation of the *poset* (partially ordered set) $\mathfrak{S}_{n+2}$:



(in the sense of [11]). It is well-known [11] that $\mathfrak{S}_{n+2}$ has finitely many indecomposable representations. It implies that $\mathbf{A}$ is representation finite if $m = l = 0$.

If $l = 1$, let $\gamma : j \to j_1$, $\gamma_1 : j_2 \to j_1$ (the case $\gamma_1 : j_1 \to j_2$ is analogous). We can suppose that the matrix $M(\gamma_1)$ is of the form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Then the rows of all matrices C_i shall be subdivided respectively to this division of $M(\gamma_1)$: $C_i = \begin{pmatrix} C_{i1} \\ C_{i0} \end{pmatrix}$. Moreover, if f is a homomorphism of such representations, then $f(j_1) = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix}$. Quite analogously to the previous considerations one can see that, reducing $M(\gamma)$ to a canonical form, we subdivide the columns of B_i so that resulting problem becomes that of representations of the poset $\mathfrak{C}'_{n+3}$:



($n+3$ points in each column). The results of [8, 10] imply that this problem is finite if $n \leq 2$, tame if $n = 3$ and wild if $n > 3$. Therefore, so is the algebra $\mathbf{A}$ if $m = 0$ and $l = 1$.

If $l > 1$ then after a reduction of the matrices $M(\gamma_2), M(\gamma_1)$ and $M(\gamma)$ we obtain for $M(\beta)$ the problem of the representations of the poset $\mathfrak{S}''_{n+4}$ analogous to $\mathfrak{S}$ and $\mathfrak{S}'$ but with 4 columns and $n+4$ point in every column. This problem is wild [10], hence the algebra $\mathbf{A}$ is also wild. If both $l > 0$ and $m > 0$, analogous consideration shows that if we reduce the matrix $M(\beta_1)$ to the form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, the rows of the matrix $M(\beta)$ will also be subdivided, so that we obtain the problem on representations of a *pair of posets* [8], one of them being $\mathfrak{S}'_{n+3}$ and the other being linear ordered with 2 elements. It is known from [9] that this problem is wild, so the algebra $\mathbf{A}$ is wild as well.

Let now $l = 0$ and fix the subdivision of columns of $M(\beta)$ described by the poset $\mathfrak{S}_{n+2}$ as above. If we reduce the matrices $M(\beta_i)$, which form representations of the quiver of type A_m, the rows of $M(\beta)$ will be subdivided so that as a result we obtain representations of the pair of posets, one of them being $\mathfrak{S}_{n+2}$ and the other being linear ordered with $m+1$ elements. The results of [8, 9] imply that this problem is representation finite if either $m = 1, n \leq 3$ or $2 \leq m \leq 3, n = 1$, tame if either $m = 1, n = 4$, or $m = 2, n = 2$, or $m = 4, n = 1$. In all other cases it is wild. Therefore, the same is true for the algebra $\mathbf{A}$, which accomplishes the proof. □

We use one more class of algebras.

Definition 4.3. A nodal algebra is said to be *super-exceptional* if it is obtained from an algebra of the form (4.3) or (4.4) with $n = 3$ by gluing the ends of the arrow α_2 in the case when such gluing is not inessential, and, maybe, several inessential gluings.

Obviously, we only have to consider super-exceptional algebras obtained without inessential gluings. Using [9, Theorem 2.3], one easily gets the following result.

Proposition 4.4. *A super-exceptional algebra is*

(1) *representation finite if* $m = l = 0$,
(2) *tame if* $m + l = 1$,
(3) *wild if* $m + l > 1$.

5. Final result

Now we can completely describe representation types of nodal algebras of type A.

Definition 5.1. (1) We call an algebra $\mathbf{A}$ *quasi-gentle* if it can be obtained from a gentle or skewed-gentle algebra by a suitable sequence of inessential gluings.
(2) We call an algebra *good exceptional* (*good super-exceptional*) if it is exceptional (respectively, super-exceptional) and not wild.

Theorem 4.2 and Proposition 4.4 give a description of good exceptional and super-exceptional algebras.

Theorem 5.2. *A non-hereditary nodal algebra of type* A *is representation finite or tame if and only if it is either quasi-gentle, or good exceptional, or good super-exceptional. In other cases it is wild.*

Before proving this theorem, we show that gluing or blowing up cannot "improve" representation type.

Proposition 5.3. *Let an algebra* $\mathbf{A}$ *be obtained from* $\mathbf{B}$ *by gluing or blowing up. Then there is an exact functor* $\mathbf{F} : \mathbf{B}\text{-mod} \to \mathbf{A}\text{-mod}$ *such that* $\mathbf{F}M \simeq \mathbf{F}M'$ *if and only if* $M \simeq M'$ *or, in case of gluing vertices* i *and* j, M *and* M' *only differ by trivial direct summands at these vertices.*

Proof. Let $\mathbf{A}$ is obtained by blowing up a vertex i. We suppose that there are no loops at this vertex. The case when there are such loops can be treated analogously but the formulae become more cumbersome. Note that in the further consideration we do not need this case. For a $\mathbf{B}$-module M set $\mathbf{F}M(k) = M(k)$ if $k \neq i$, $\mathbf{F}M(i') = \mathbf{F}M(i'') = M(i)$, $\mathbf{F}M(\alpha) = M(\alpha)$ if $\alpha \notin \mathrm{Ar}(i)$ and $\mathbf{F}M(\alpha') = \mathbf{F}M(\alpha'') = M(\alpha)$ if $\alpha \in \mathrm{Ar}(i)$. If $f : M \to M'$, set $\mathbf{F}f(k) = f(k)$ if $k \neq i$ and $\mathbf{F}f(i') = \mathbf{F}f(i'') = f(i)$. It gives an exact functor $\mathbf{F} : \mathbf{B}\text{-mod} \to \mathbf{A}\text{-mod}$. Conversely, if N is an $\mathbf{A}$-module, set $\mathbf{G}N(k) = N(k)$ if $k \neq i$ and $\mathbf{G}N(i) = N(i')$. It gives a functor $\mathbf{G} : \mathbf{A}\text{-mod} \to \mathbf{B}\text{-mod}$. Obviously $\mathbf{GF}M \simeq M$, hence $\mathbf{F}M \simeq \mathbf{F}M'$ implies that $M \simeq M'$.

Let now $\mathbf{A}$ be obtained from $\mathbf{B}$ by gluing vertices i and j. As above, we suppose that there are no loops at these vertices. For a $\mathbf{B}$-module M

set $\mathbf{F}M(k) = M(k)$ if $k \neq (ij)$, $\mathbf{F}M(ij) = M(i) \oplus M(j)$, $\mathbf{F}M(\alpha) = M(\alpha))$ if $\alpha \notin \mathrm{Ar}(i) \cup \mathrm{Ar}(j)$, $\mathbf{F}M(\alpha) = \begin{pmatrix} M(\alpha) & 0 \end{pmatrix} \Big(\text{or } \begin{pmatrix} 0 & M(\alpha) \end{pmatrix}\Big)$ if $\alpha^- = i$ (respectively $\alpha^- = j$) and $\mathbf{F}M(\beta) = \begin{pmatrix} M(\beta) \\ 0 \end{pmatrix} \left(\text{or } M(\beta) = \begin{pmatrix} 0 \\ M(\beta) \end{pmatrix}\right)$ if $\beta^+ = i$ (respectively $\beta^+ = j$). If $f : M \to M'$, set $\mathbf{F}f(k) = f(k)$ if $k \neq (ij)$ and $f(ij) = f(i) \oplus f(j)$. It gives an exact functor $\mathbf{F} : \mathbf{B}\text{-mod} \to \mathbf{A}\text{-mod}$. Suppose that $\phi : \mathbf{F}M \xrightarrow{\sim} \mathbf{F}M'$,

$$\phi(ij) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix},$$
$$\phi^{-1}(ij) = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}.$$

Then

$$\begin{aligned}
\phi_{11}M(\beta) = M'(\beta)\phi(k) &\text{ and } \phi_{21}M(\beta) = 0 \text{ if } \beta : k \to i, \\
\phi_{22}M(\beta) = M'(\beta)\phi(k) &\text{ and } \phi_{12}M(\beta) = 0 \text{ if } \beta : k \to j, \\
M'(\alpha)\phi_{11} = \phi(k)M(\alpha) &\text{ and } M'(\alpha)\phi_{12} = 0 \text{ if } \alpha : i \to k, \\
M'(\alpha)\phi_{22} = \phi(k)M(\alpha) &\text{ and } M'(\alpha)\phi_{21} = 0 \text{ if } \alpha : j \to k.
\end{aligned}$$

and analogous relations hold for the components of $\phi^{-1}(ij)$ with interchange of M and M'. We suppose that M has no direct summands $\bar{\mathbf{B}}_i$ and $\bar{\mathbf{B}}_j$. It immediately implies that $\bigcap_{\alpha^-=i} \operatorname{Ker} M(\alpha) \subseteq \sum_{\beta^+=i} \operatorname{Im} M(\beta)$ and $\bigcap_{\alpha^-=j} \operatorname{Ker} M(\alpha) \subseteq \sum_{\beta^+=j} \operatorname{Im} M(\beta)$. If M' also contains no direct summands $\bar{\mathbf{B}}_i$ and $\bar{\mathbf{B}}_j$, it satisfies the same conditions. Therefore

$$\operatorname{Im} \psi_{21} \subseteq \textstyle\bigcap_{\alpha^-=j} \operatorname{Ker} M(\alpha) \subseteq \sum_{\beta^+=j} \operatorname{Im} M(\beta),$$

whence $\phi_{12}\psi_{21} = 0$ and $\phi_{11}\psi_{11} = 1$. Quite analogously, $\phi_{22}\psi_{22} = 1$ and the same holds if we interchange ϕ and ψ. Therefore we obtain an isomorphism $\tilde{\phi} : M \xrightarrow{\sim} M'$ setting $\tilde{\phi}(i) = \phi_{11}$, $\tilde{\phi}(j) = \phi_{22}$ and $\tilde{\phi}(k) = \phi(k)$ if $k \notin \{i, j\}$. $\square$

Corollary 5.4. *If an algebra* $\mathbf{A}$ *is obtained from* $\mathbf{B}$ *by gluing or blowing up and* $\mathbf{B}$ *is representation infinite or wild, then so is also* $\mathbf{A}$.

Proof of Theorem 5.2. We have already proved the "if" part of the theorem. So we only have to show that all other nodal algebras are wild. Moreover, we can suppose that there were no inessential gluings during the construction of a nodal algebra $\mathbf{A}$. As $\mathbf{A}$ is neither gentle nor quasi-gentle, there must be at least one exceptional gluing. Hence $\mathbf{A}$ is obtained from an algebra $\mathbf{B}$ of the form (4.3) or (4.4) by some additional gluings (not inessential) or blowings up. One easily sees that any blowing up of $\mathbf{B}$ gives a wild algebra. Indeed, the crucial case is when $n = 1$, $m = l = 0$ and we blow up the end of the arrow β. Then, after reducing α_1 and γ, just as in the proof of Theorem 4.2, we obtain for the non-zero parts of the two arrows obtained from β the problem of the pair of posets $(1, 1)$ and $\mathfrak{S}_1$ (see page 14), which is wild by [9, Theorem 2.3]. The other cases are even easier.

Thus no blowing up has been used. Suppose that we glue the ends of β (or some β_k) and γ (or some γ_k). Then, even for $n = 1$, $m = l = 0$, we obtain the algebra

$$\begin{tikzcd} \cdot \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\gamma"'] & \cdot \end{tikzcd} \qquad \alpha^2 = \beta\alpha = 0$$

(or its dual). Reducing α, we obtain two matrices of the forms

$$\beta = \begin{pmatrix} 0 & B_2 & B_3 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} G_1 & G_2 & G_3 \end{pmatrix}.$$

Given another pair (β', γ') of the same kind, its defines an isomorphic representation if and only if there are invertible matrices X and Y such that $X\beta = \beta' Y$ and $X\gamma = \gamma' Y$, and T is of the form

$$Y = \begin{pmatrix} Y_1 & Y_3 & Y_4 \\ 0 & Y_2 & Y_5 \\ 0 & 0 & Y_1 \end{pmatrix},$$

where the subdivision of Y corresponds to that of β, γ. The Tits form of this matrix problem (see [5]) is $Q = x^2 + 2y_1^2 + y_2^2 + 2y_1y_2 - 3xy_1 - 2xy_2$. As $Q(2,1,1) = -1$, this matrix problem is wild. Hence the algebra $\mathbf{A}$ is also wild. Analogously, one can see that if we glue ends of some of β_i or γ_i, we get a wild algebra (whenever this gluing is not inessential). Gluing of an end of some α_i with an end of β or γ gives a wild quiver algebra as a subalgebra (again if it is not inessential). Just the same is in the case when we glue ends of some α_i so that this gluing is not inessential and $n > 3$. It accomplishes the proof. $\square$

References

[1] M. Auslander, I. Reiten and S. O. Smalø. Representation theory of Artin algebras. Cambridge University Press, 1995.

[2] V. Bekkert, E. N. Marcos and H. Merklen. Indecomposables in derived categories of skewed-gentle algebras. Commun. Algebra 31, No. 6 (2003), 2615–2654.

[3] I. Burban and Y. Drozd. Derived categories of nodal algebras. J. Algebra 272 (2004), 46–94.

[4] Y. A. Drozd. Finite modules over pure Noetherian algebras. Trudy Mat. Inst. Steklov Acad. Sci. USSR, 183 (1990), 56–68.

[5] Y. Drozd. Reduction algorithm and representations of boxes and algebras. Comtes Rendue Math. Acad. Sci. Canada, 23 (2001) 97–125.

[6] Y. A. Drozd and V. V. Kirichenko. Finite Dimensional Algebras. Vyshcha Shkola, Kiev, 1980. (English translation: Springer–Verlag, 1994.)

[7] C. Geiß and J. A. de la Peña. Auslander–Reiten components for clans. Bol. Soc. Mat. Mex., III. Ser. 5, No. 2 (1999), 307–326.

[8] M. M. Kleiner. Partially ordered sets of finite type. Zapiski Nauch. Semin. LOMI, 28 (1972), 32–41.

[9] M. M. Kleiner. Pairs of partially ordered sets of tame representation type. Linear Algebra Appl. 104 (1988), 103–115.

[10] L. A. Nazarova. Partially ordered sets of infinite type. Izv. Akad. Nauk SSSR, Ser. Mat. 39 (1975), 963–991.

[11] L. A. Nazarova and A. V. Roiter. Representations of partially ordered sets. Zapiski Nauch. Semin. LOMI, 28 (1972), 5–31.

[12] A. Skowronski and J. Waschbusch, Representation-finite biserial algebras. J. Reine Angew. Math. 345 (1983), 172–181.

[13] D. E. Voloshyn. Structure of nodal algebras. Ukr. Mat. Zh. 63, No. 7 (2011), 880–888.

Institute of Mathematics, National Academy of Sciences of Ukraine, 01601 Kyiv, Ukraine

E-mail address: `y.a.drozd@gmail.com, drozd@imath.kiev.ua`

URL: `www.imath.kiev.ua/~drozd`

E-mail address: `vaszem@rambler.ru`