ON POLYNOMIAL FUNCTORS

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This is a survey of the last results of the author on classification of polynomial functors, especially quadratic and cubic.

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Polynomial functors appeared in algebraic topology [8] and proved themselves useful in various questions of this theory, especially in studying homotopy types. So their classification is of a definite interest. Some time ago the author noticed that at least the quadratic case can be treated in more or less usual framework of the representation theory. It gave possibility to obtain their complete description [6]. Unfortunately, this is the last case when such a description can be given. The cubic case is already *wild* in the sense of the representation theory [7]. Nevertheless, some special types of cubic functors can be classified. Perhaps, the most important seems the *2-divisible* case, which is completely analogous to the quadratic one [7]. As a consequence, a conjecture appears that the situation is the same for polynomial functors of degree p (prime) if we invert all smaller primes. This survey is mainly devoted to these results. Other special types of cubic functors that have been classified are "*cubic vector spaces*," weakly alternative and torsion free functors, but we only give a brief outlook of their description, since its proper place is still unclear. The author is grateful to Professor H.-J. Baues for his enthusiastic support of this research.

1. Generalities.

We suppose all categories *pre-additive*, i.e. all morphism sets endowed with abelian group structure. On the other hand, the *functors* are not supposed additive, though we always suppose that they map zero objects to zero. If $F:\mathcal{A}\to\mathcal{B}$ is any functor, we can measure its non-additivity by its polarizations (or *cross-effects*). The latter are constructed as follows. Let $\mathcal{A}$ be additive (i.e. having finite direct sums) and $\mathcal{B}$ be *fully additive*, i.e. additive category such that every idempotent corresponds to a direct decomposition. For any objects $A_1,\dots,A_n$ from $\mathcal{A}$ consider their direct sum $A=\bigoplus_{k=1}^n A_i$ together with the embeddings $i_k:A_k\to A$ and projections $p_k:A\to A_k$.

Then $e_k=i_kp_k$ are orthogonal idempotent endomorphisms of A, hence $f(k)=F(e_k)$ are orthogonal idempotent endomorphisms of $F(A)$. Define recursively endomorphisms $f(k_1\dots k_m)$ for each $m\leqslant n,\ 1\leqslant k_1<\dots<k_m\leqslant n$ setting

$$f(k_1\dots k_m)=F(e_{k_1}+\dots+e_{k_m})-\sum_{l<m}\sum_{j_1<\dots<j_l}f(j_1\dots j_l),$$

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for instance $f(kl) = F(e_k + e_l) - F(e_k) - F(e_l)$. Set $F_n(A_1|\dots|A_n) = \mathrm{Im} f(12\dots n)$. Then

$$F(A) = \bigoplus_{m \leqslant n} \bigoplus_{k_1 < \dots < k_m} F_m(A_{k_1}|\dots|A_{k_m}).$$

The functor F is called *polynomial* if there is an integer d such that $F_n = 0$ for $n > d$. The smallest d with this property is called the *degree* of F. Certainly functors of degree 1 are just additive; those of degree 2 are called *quadratic* and of degree 3 *cubic.*

In what follows we consider the case when $\mathcal{A} = \mathbf{fab}$, the category of finitely generated free abelian groups, and $\mathcal{B} = \mathbf{R}\text{-}Mod$, the category of modules over a ring $\mathbf{R}$. As any additive functor $F : \mathbf{fab} \to \mathbf{R}\text{-}Mod$ can be identified with the $\mathbf{R}$-module $F(\mathbb{Z})$, we call polynomial functors $F : \mathbf{fab} \to \mathbf{R}\text{-}Mod$ *polynomial* $\mathbf{R}$-*modules.* Moreover, as a rule we only deal with *finitely generated* polynomial modules, i.e. polynomial functors $F : \mathbf{fab} \to \mathbf{R}\text{-}mod$, the category of finitely generated $\mathbf{R}$-modules. If $\mathbf{R} = \mathbb{Z}$, we simply say "polynomial modules" not precising the ring.

One can show (see [1]) that a polynomial module M of degree d is completely defined by the values $M_n = M_n(\mathbb{Z}|\dots|\mathbb{Z})$ (n times) for $n \leqslant d$ and the homomorphisms $H^n_m : M_n \to M_{n+1}$, $P^n_m : M_{n+1} \to M_n$ for each $n < d, m \leqslant n$, which are defined as the following compositions:

$$H^n_m : M_n \to M(\mathbb{Z}^n) \to M(\mathbb{Z}^{n+1}) \to M_{n+1},$$

where the first mapping is just the embedding of the direct summand, the last one is the projection onto the direct summand, and the middle one equals $M(\delta_m)$, where

$$\delta_m : \mathbb{Z}^n \to \mathbb{Z}^{n+1}, \quad \delta_m(z_1, \dots, z_n) = (z_1, \dots, z_{m-1}, z_m, z_m, z_{m+1}, \dots, z_n);$$

and

$$P^n_m : M_{n+1} \to M(\mathbb{Z}^{n+1}) \to M(\mathbb{Z}^n) \to M_n,$$

where the first mapping is the projection, the last one is the embedding, and the middle one equals $M(\gamma_m)$, where

$$\gamma_m : \mathbb{Z}^{n+1} \to \mathbb{Z}^n, \quad \gamma_m(z_1, \dots, z_{n+1}) = (z_1, \dots, z_{m-1}, z_m + z_{m+1}, z_{m+2} \dots, z_n).$$

Certainly, these mappings must satisfy some relations (cf. [1]), which we shall not write in general case.

Important examples of polynomial modules are:

- tensor powers $T^n : A \mapsto A^{\otimes n}$,
- symmetric powers $\mathrm{S}^n : A \mapsto \mathrm{S}^n A$,
- exterior (skew-symmetric) powers $\Lambda^n : A \mapsto \Lambda^n A$.

In particular, tensor power T^n and its polarizations $T^{n,k} : A \to T^n_k(A|\dots|A)$ (k times) are just indecomposable projectives in the category of all polynomial modules of degree n.

2. Quadratic modules

For quadratic modules the previous construction gives two **R**-modules M_1, M_2, and two mappings $H : M_1 \to M_2$ and $P : M_2 \to M_1$, such that $PHP = 2P$ and $HPH = 2H$.

We consider the "absolute" case, when $\mathbf{R} = \mathbb{Z}$ (it is the most important for topology). Then one can easily see that a quadratic module can be considered as a module over a special ring $\mathbf{A}$, which is the subring in the direct product $\mathbb{Z} \times \mathrm{Mat}(2, \mathbb{Z}) \times \mathbb{Z}$ consisting of the triples

$$(a, b, c), \quad \text{where} \quad b = \begin{pmatrix} b_1 & 2b_2 \\ b_3 & b_4 \end{pmatrix}, \quad b_1 \equiv a, \ b_4 \equiv c \ (\mathrm{mod}\ 2).$$

Namely, if M is an $\mathbf{A}$-module, in the corresponding quadratic module $M_1 = e_1M$, $M_2 = e_2M$, H is the multiplication by h and P is the multiplication by p, where

$$e_1 = (1, e_{11}, 0), \ e_2 = (0, e_{22}, 1), \ h = (0, e_{21}, 0), \ p = (0, 2e_{12}, 0)$$

(e_{ij} are the matrix units in $\mathrm{Mat}(2, \mathbb{Z})$). Fortunately, this ring belongs to the class considered by the author in [5]. In particular, it is *tame*; moreover, its representations can be described in a rather usual language of "strings" and "bands." Indeed, this classification is a special case of the so-called *representations of bunches of chains* (cf. [2]). For details of the calculations we refer to [6]; here we only formulate the result in a bit more convenient form.

First, using the common tool of adèles groups, like in [4], we establish a sort of "Hasse principle" for quadratic modules. Remind that we always suppose our modules finitely generated.

2.1. Proposition. *Two quadratic modules M and N are isomorphic if and only if there localizations M_p and N_p are isomorphic for each prime number p.*

If $p > 2$, $\mathbf{A}_p = \mathbb{Z}_p \times \mathrm{Mat}(2, \mathbb{Z}_p) \times \mathbb{Z}_p$, so the description of $\mathbf{A}_p$-modules is quite simple: there are three indecomposable torsion free modules (direct summands of $\mathbf{A}_p$), and every other indecomposable module is isomorphic to P/p^kP for some positive integer k and one of these modules P. The description in case $p = 2$ is more interesting. First introduce some configurations of integers called *strings* and *bands*. Namely, define two symmetric relations on the set $\{1, 2, 3, 4\}$: an equivalence relation $-$ such that the only non-trivial equivalence is $2 - 3$, and $\sim$ (not an equivalence!) such that $1 \sim 2$ and $3 \sim 4$. Now a *string* is a configurations of one of the following sorts:

$$\text{(i)} \quad \begin{array}{ccccccccccc} j_1 & & & & j_2j_3 & & & & j_{2n-2}j_{2n-1} & & \\ & k_1 & & k_2 & & k_3 & & \cdots & & k_{2n-1} & \\ & & i_1i_2 & & & & i_3i_4 & & & & i_{2n-1} \end{array}$$

or

$$\text{(ii)} \quad \begin{array}{ccccccccc} & & j_2j_3 & & & & j_{2n-2}j_{2n-1} & & \\ & k_2 & & k_3 & & \cdots & & k_{2n-1} & \\ i_2 & & & & i_3i_4 & & & & i_{2n-1} \end{array}$$

or

(iii) j_1 k_1 $i_1 i_2$ k_2 $j_2 j_3$ $\dots$ $i_{2n-1} i_{2n}$ k_{2n} j_{2n}

where $i_r, j_r \in \{1,2,3,4\}$, $k_r \in \mathbb{N}$ satisfy the following conditions:

- $i_{2r-1} \sim i_{2r}$ for each $r = 1,2,\dots,n$. This condition is empty for types (i) and (ii) if $r = n$ and for type (ii) if $r = 1$, but in these cases we *define* i_{2n}, respectively i_1 so that it holds.
- $j_{2r+1} \sim j_{2r}$ for each $r = 1,2,\dots,n-1$.
- $i_r - j_r$ for each $r = 1,2,\dots,2n$ (again it is empty in some cases, but here we do not define any extra values).

Consider now the following mappings acting in every quadratic module:

$$\theta(11) = 2\mathrm{id}_{M_1} - PH, \quad \theta(22) = PH, \quad \theta(23) = H,$$
$$\theta(32) = P, \quad \theta(33) = HP, \quad \theta(44) = 2\mathrm{id}_{M_2} - HP.$$

Set also $\nu\{1,2\} = 1$, $\nu\{3,4\} = 2$. Then the quadratic *string module* $M = M^{\mathrm{D}}$ corresponding to a string diagram D is generated by the elements

$$\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n \qquad \mathbf{g}_r \in M_{\nu\{i_{2r-1}, i_{2r}\}}$$

subject to the relations:

$$2^{k_{2r}}\theta(i_{2r}j_{2r})\mathbf{g}_r = 2^{k_{2r+1}}\theta(i_{2r+1}j_{2r+1})\mathbf{g}_{r+1} \quad (r = 0,1,\dots,n).$$

We set here $\mathbf{g}_0 = \mathbf{g}_{2n+1} = 0$ and omit the case $r = n$ for diagrams of types (i),(ii) and the case $r = 0$ for diagrams of type (ii).

A *band data* is a pair (D, m, ϕ), where D is a diagram of type (iii) and $\phi = \lambda_1 + \lambda_2 t + \dots + \lambda_m t^{m-1} + t^m$ is a polynomial over the residue field $\mathbb{Z}/2$ such that

- $j_{2n} \sim j_1$.
- D is non-periodic, i.e. cannot be written as a repetition $\mathrm{D}'\mathrm{D}'\dots\mathrm{D}'$ of a shorter diagram D'.
- ϕ is a power of an irreducible polynomial and $\lambda_1 \neq 0$.

The quadratic *band module* $M = M^{\mathrm{D},\phi}$ corresponding to a band data is generated by the elements

$$\mathbf{g}_{rs} \quad (r = 1,2,\dots,n,\ s = 1,2,\dots,m) \quad \mathbf{g}_{rs} \in M_{\nu\{i_{2r-1}i_{2r}\}}$$

subject to the relations:

$$2^{k_{2r}}\theta(i_{2r}j_{2r})\mathbf{g}_{rs} = 2^{k_{2r+1}}\theta(i_{2r+1}j_{2r+1})\mathbf{g}_{r+1,s} \quad (r = 0,1,\dots,n) \ \text{ if } \ 1 \leqslant r < n;$$
$$2^{k_{2n}}\theta(i_{2n}j_{2n})\mathbf{g}_{ns} = 2^{k_1}\theta(i_1j_1)\mathbf{g}_{1,s+1} \ \text{ if } \ 1 \leqslant s < m;$$
$$2^{k_{2r}}\theta(i_{2n}j_{2n})\mathbf{g}_{nm} = -2^{k_1}\theta(i_1j_1)\sum_{s=1}^{m}\lambda_s\mathbf{g}_{1s}.$$

2.2. Theorem. *(1) Every indecomposable quadratic module is isomorphic to one of the string or band modules defined above, or to a module* S^2/p^k, Λ^2/p^k, *or* Id/p^k, *where* p *is an odd prime.*

(2) The only isomorphisms between these indecomposable modules are the following:

- $M^{\mathrm{D}} \simeq M^{\mathrm{D}^*}$, *where* D *is the symmetric diagram to a diagram* D *of type (ii) or (iii).*

• $M^{\mathrm{D},\phi} \simeq M^{\mathrm{D}^l,\phi}$ *, where* D^l *denotes the* l*-th cyclic shift of the diagram of type (iii), i.e. the configuration*

$$\begin{array}{cccccccccccc} j_{2l+1} & & & & & j_{2l+2}j_{2l+3} & & & & & j_{2l} \\ & k_{2l+1} & & & k_{2l+2} & & & \cdots & & k_{2l} & \\ & & & i_{2l+1}i_{2l+2} & & & & & i_{2l-1}i_{2l} & & \end{array}$$

• $M^{\mathrm{D},\phi} \simeq M^{\mathrm{D}^{*l},\phi^*}$ *, where* $\phi^*(t) = \lambda_1^{-1} t^m \phi(1/t)$ *.*

(3) Any quadratic module uniquely decomposes into a direct sum of indecomposable ones.

2.3 Corollary.

• *Every quadratic module* M *has a periodic projective resolution of period 4, namely*

$$\cdots \to P_n \xrightarrow{\alpha_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\alpha_1} P_0 \to M \to 0$$

with $P_{n+4} = P_n,\ \alpha_{n+4} = \alpha_n$ *for* $n > 2$ *.*

• *The projective dimension of a quadratic module is either 0, or 1, or* ∞ *. Hence the finitistic projective dimension of the category of quadratic modules equals 1.*

3. Cubic modules

A cubic module is given by 3 groups $M_1,\ M_2,\ M_3$ and 6 mappings

$$H : M_1 \to M_2,\ P : M_2 \to M_1,\ H_m : M_2 \to M_3,\ P_m : M_3 \to M_2 \quad (m = 1, 2)$$

subject to the conditions:

$$H_1 P_2 = H_2 P_1 = 0,\ H_1 H = H_2 H,\ P P_1 = P P_2,$$

$$H_i P_i H_i = 2H_i,\ P_i H_i P_i = 2P_i\ (i = 1, 2),$$

$$HPH = 2(H + (P_1 + P_2)\bar{P}),\ PHP = 2(P + \bar{P}(H_1 + H_2)),$$

$$\bar{H}P + H_1 + H_2 = H_1 P_1 H_2 P_2 H_1 + H_2 P_2 H_1 P_1 H_2,$$

$$H\bar{P} + P_1 + P_2 = P_1 H_2 P_2 H_2 P_1 + P_2 H_1 P_1 H_2 P_2,$$

where $\bar{H} = H_1 H = H_2 H,\ \bar{P} = P P_1 = P P_2$.

We consider the ring $\mathbf{B}$ generated by three orthogonal idempotents e_1, e_2, e_3 such that $e_1 + e_2 + e_3 = 1$ and 6 elements

$$H \in e_2 \mathbf{B} e_1,\ P \in e_1 \mathbf{B} e_2,\ H_m \in e_3 \mathbf{B} e_2,\ P_m \in e_2 \mathbf{B} e_3 \quad (m = 1, 2)$$

subject to the above relations. Then any cubic module can be considered as $\mathbf{B}$-module. Set $\mathbf{B}_1 = e_1 \mathbf{B} e_1$.

3.1. Proposition.

1. The ring $\mathbf{B}_1$ *is generated by two elements* $a = PH - \bar{P}\bar{H},\ b = PH$ *subject to the relations* $a^2 = 2a,\ b^2 = 6b,\ ab = ba = 0$ *.*

2. The ring $\mathbf{B}_1$ *(all the more* $\mathbf{B}$ *) is wild.*

Proof. The first claim is verified by straightforward calculations [7]. To prove the second, consider the free (non-commutative) algebra $\Sigma = \mathbb{Z}/4\langle x, y\rangle$ over the residue ring $\mathbb{Z}/4$ and the homomorphism $\sigma : \mathbf{B}_1 \to \Sigma$ mapping $a \mapsto 2x,\ b \mapsto 2y$. For every Σ-module L denote by ${}^\sigma L$ the $\mathbf{B}_1$-module obtained from L by the change of rings. Then one easily verifies that for any Σ-modules L, L' , which are free as $\mathbb{Z}/4$-modules,

- ${}^{\sigma}L \simeq {}^{\sigma}L'$ if and only if $L/2 \simeq L'/2$;
- ${}^{\sigma}L$ is indecomposable if and only if $L/2$ is indecomposable.

Hence the classification of $\mathbf{B}_1$-modules is at least as complicated as that of modules over $\Sigma/2 \simeq \mathbb{Z}/2\langle x, y\rangle$. It means that $\mathbf{B}_1$ is wild in the sense of the representation theory.

It gives no hope to obtain a good classification of cubic modules. Nevertheless, the situation becomes much better if we "invert 2," that is consider cubic modules over the ring $\mathbb{Z}' = \mathbb{Z}[1/2]$. We call them *2-divisible cubic modules.* Then straightforward, though rather cumbersome, calculations give the following result.

3.2. Proposition. *The ring* $\mathbf{B}[1/2]$ *is Morita equivalent to the direct product* $\mathbb{Z}' \times \mathbb{Z}' \times \mathbf{B}'$, *where* $\mathbf{B}'$ *is the subring of* $\mathbb{Z}' \times \mathrm{Mat}(2, \mathbb{Z}') \times \mathrm{Mat}(2, \mathbb{Z}') \times \mathbb{Z}'$ *consisting of quadruples*

$$(a, b, c, d), \quad \textit{where}\ \ b = \begin{pmatrix} b_1 & 3b_2 \\ b_3 & b_4 \end{pmatrix},\ c = \begin{pmatrix} c_1 & 3c_2 \\ c_3 & c_4 \end{pmatrix},$$

such that $a \equiv b_1,\ b_4 \equiv c_1,\ c_4 \equiv d \pmod 3$.

The cubic modules corresponding to the first two factor $\mathbb{Z}'$ are just S^2/p^k and Λ^2/p^k for odd primes p (they are indeed quadratic modules). The description of $\mathbf{B}'$-modules can be given in the same frames as that of quadratic modules. The corresponding string and bands only differs from those of the preceding section by the features that now the indices i_r, j_r are taken from the set $\{1, 2, 3, 4, 5, 6\}$ with the relations $2-3,\ 4-5,\ 1 \sim 2,\ 3 \sim 4,\ 5 \sim 6$, polynomials ϕ are taken from $\mathbb{Z}/3[t]$, and the mappings $\theta(ij)$ are defined as follows:

$$\theta(11) = 3\mathrm{Id}_{M_1} - \beta_1\alpha_1,\ \theta(22) = \beta_1\alpha_1,\ \theta(23) = \alpha_1,\ \theta(32) = \beta_1,\ \theta(33) = \alpha_1\beta_1,$$

$$\theta(44) = \beta_2\alpha_2,\ \theta(45) = \alpha_2,\ th(54) = \beta_2,\ \theta(55) = \alpha_2\beta_2,\ \theta(66) = 3\mathrm{Id}_{M_3} - \alpha_2\beta_2,$$

where $\alpha_1 : M_1 \to M_2$ corresponds to the quadruple $(0, e_{21}, 0, 0)$, $\beta_1 : M_2 \to M_1$ to the quadruple $(0, 3e_{12}, 0, 0)$, $\alpha_2 : M_2 \to M_3$ to the quadruple $(0, 0, e_{21}, 0)$, and $\beta_2 : M_3 \to M_2$ to the quadruple $(0, 0, 3e_{12}, 0)$.

So we get the following results.

3.3. Theorem. *(1) Two cubic 2-divisible modules* M, N *are isomorphic if and only if* $M_p \simeq N_p$ *for each odd prime* p.

(2) Every indecomposable 2-divisible cubic module is isomorphic to one of the following:

- *string or band module;*
- S^3/p^k, S^{3*}/p^k, Λ^3/p^k, Id/p^k, *where* $\mathrm{S}^{3*}(A) = \mathrm{S}^2_3(A|A)$ *and* $p > 3$ *is a prime;*
- S^2/p^k *or* Λ^2/p^k, *where* p *is an odd prime.*

(3) The only isomorphisms between these indecomposable cubic modules are:

- $M^{\mathrm{D}} \simeq M^{\mathrm{D}^*}$, *where* D *is the symmetric diagram to a diagram* D *of type (ii) or (iii).*
- $M^{\mathrm{D},\phi} \simeq M^{\mathrm{D}^l,\phi}$, *where* D^l *denotes the* l*-th shift of the diagram of type (iii), i.e. the configuration*

$$\begin{matrix} j_{2l+1} & & & & j_{2l+2}j_{2l+3} & & & & j_{2l} \\ & k_{2l+1} & & k_{2l+2} & & \cdots & & k_{2l} & \\ & & i_{2l+1}i_{2l+2} & & & & i_{2l-1}i_{2l} & & \end{matrix}$$

• $M^{\mathrm{D},\phi} \simeq M^{\mathrm{D}^{*l},\phi^*}$, *where* $\phi^*(t) = \lambda_1^{-1} t^m \phi(1/t)$.

(3) Any 2-divisible cubic module uniquely decomposes into a direct sum of indecomposable ones.

3.4. Corollary.

• *Every 2-divisible cubic module M has a periodic projective resolution of period 6, namely*

$$\cdots \to P_n \xrightarrow{\alpha_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\alpha_1} P_0 \to M \to 0$$

with $P_{n+6} = P_n,\ \alpha_{n+6} = \alpha_n$ *for* $n > 2$.

• *The projective dimension of a 2-divisible cubic module is either 0, or 1, or* ∞. *Hence the finitistic projective dimension of the category of 2-divisible cubic modules equals 1.*

3.5. Conjecture. *Let* p *be a prime,* $\mathbb{Z}^{(p)} = \mathbb{Z}[1/(p-1)!]$. *Then the category of polynomial* $\mathbb{Z}^{(p)}$*-modules of degree* p *is equivalent to the category* $\mathbf{A}^{(p)}$*-modules, where* $\mathbf{A}^{(p)}$ *is a direct product of several copies of* $\mathbb{Z}^{(p)}$ *and of the subring of* $\mathbb{Z}^{(p)} \times \mathrm{Mat}(2, \mathbb{Z}^{(p)})^{p-1} \times \mathbb{Z}^{(p)}$ *consisting of* $(p+1)$*-tuples* $(a, b^1, \dots, b^{p-1}, c)$, *where*

$$b^m = \begin{pmatrix} b_{11}^m & pb_{12}^m \\ b_{21}^m & b_{22}^m \end{pmatrix} \quad \text{with} \quad b_{22}^m \equiv b_{11}^{m+1} \pmod p \quad \text{for} \quad m = 1, \dots, p-2,$$

$a \equiv b_{11}^1,\ c \equiv b_{22}^{p-1} \pmod p$.

If this conjecture is true, the description of $\mathbb{Z}^{(p)}$-modules of degree p (we call them ($<p$)*-divisible* p*-modules*) becomes quite analogous to that of quadratic or 2-divisible cubic modules. Namely:

• Two ($<p$)-divisible p-modules M, N are isomorphic if and only if $M_q \simeq N_q$ for all prime $q \geqslant p$.

• Indecomposable ($<p$)-divisible p-modules, except some "trivial" ones, are string and band modules defined as above. Now i_r, j_r are taken from the set $\{1, 2, \dots, 2p\}$ with corresponding changes of $-$, $\sim$ and $\theta(ij)$. The isomorphisms between these modules are the same as in Theorems 2.2 and 3.3.

• Every ($<p$)-divisible p-module uniquely decomposes into a direct sum of indecomposable ones.

• Every ($<p$)-divisible p-module has a periodic projective resolution of period $2p$ starting from α_2. Therefore a projective dimension of such a module is $0, 1$ or ∞. In particular, the finitistic projective dimension of the category of ($<p$)-divisible p-modules equals 1.

4. Other classes of cubic modules

We shortly outline three other classes of cubic modules that allow an acceptable description referring for details to [7].

A. Cubic vector spaces

They are functors $\mathbf{fab} \to \mathrm{vect}_{\mathbf{k}}$, the category of vector spaces over a field $\mathbf{k}$. The interesting case is $\mathrm{char}\mathbf{k} = 2$, because otherwise such functors are special cases of 2-divisible ones. Rewriting the relations for the mappings H, P, H_i, P_i for this special case gives the following result.

4.1. Proposition. *The category of cubic vector spaces is equivalent to the direct product of a trivial* $\mathbf{k}$*-linear category with one object (it corresponds to the functor* $\mathrm{Id}\otimes$

Λ^2) and the category of modules over the $\mathbf{k}$-algebra $\mathbf{A}$ generated by three orthogonal idempotents e_1, e_2, e_3 such that $e_1+e_2+e_3=1$ and four elements

$$h \in e_2\mathbf{A}e_1,\ p \in e_1\mathbf{A}e_2,\ h_1 \in e_3\mathbf{A}e_2,\ p_1 \in e_3\mathbf{A}e_2$$

subject to the relations

$$hph = php = h_1p_1h_1 = p_1h_1p_1 = 0, \quad h_1p_1 = h_1hpp_1.$$

We consider $\mathbf{A}$-modules as diagrams of vector spaces

$$M_1 \rightleftarrows M_2 \rightleftarrows M_3,$$

where $M_i = e_iM$ and the arrows correspond to the action of h, p, h_1, p_1. As $hph = php = 0$, the fragment $M_1 \rightleftarrows M_2$ decomposes into blocks of dimension at most 3 (the dimensions of M_1, M_2 at most 2, and only one of them can be 2-dimensional).

Hence the mappings h and p can be chosen in the form

$$h = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(I denotes the identity matrix). Now, if we reduce the matrix of h_1 to the simplest possible form, the matrix of p_1 splits into 8 horizontal and 10 vertical stripes, which we denote respectively by R_i $(i = 1, \dots, 8)$ and S_j $(j = 1, \dots, 10)$. Moreover, one can check that the admissible transformations of these stripes can be described as representations of a bunch of semi-chains in the sense of [2], namely, we have two semi-chains

$$\mathcal{E} = \{R_1 > R_2 > R_3 > R_4 > R_6 > R_7 > R_8,\ R_3 > R_5 > R_6\},$$
$$\mathcal{F} = \{S_1 < S_2 < S_3 < S_4 < S_5 < S_7 < S_8 < S_9 < S_{10},\ S_4 < S_6 < S_7\}$$

with the involution σ such that $\sigma(x) = x$ except for the cases:

$$\sigma(R_1) = R_8,\ \sigma(R_2) = S_8,\ \sigma(R_6) = S_4,\ \sigma(S_2) = S_9.$$

Hence, the description of cubic vector spaces fits again the frames of strings and bands, though this time they are more complicated than before. We shall not precise their shape (rather complicated) here, referring to [7].

B. Weakly alternative cubic modules

We call a cubic module M *weakly alternative* if $M(\mathbb{Z}) = 0$. Examples of such modules are Λ^3 and $\Lambda^2 \otimes \mathrm{Id}$. For the corresponding diagram it means that $M_1 = 0$. Then, reducing the relations with respect to the conditions $h = p = 0$, one obtains the following result.

4.2. Proposition. *The category of weakly alternative cubic modules is equivalent to the category of $\mathbf{C}$-modules, where $\mathbf{C}$ is a semi-direct product $\mathbf{C} = (\mathbb{Z} \times \mathbf{C}_0) \ltimes D$, where $\mathbf{C}_0$ is the subring of $\mathbb{Z} \times \mathrm{Mat}(2, \mathbb{Z})$ consisting of pairs (a, b) such that $a \equiv b_{11}, b_{12} \equiv 0 \pmod 2$, D is an elementary abelian 2-group with three generators ξ, η, θ, with the multiplication $\xi\eta = \theta, \eta\xi = 0$ and the $\mathbf{C}$-action:*

$$e\xi = \xi,\ \eta e = \eta,\ \xi(0, a, b) = a\xi,\ (0, a, b)\eta = a\eta, \quad \text{where} \quad e = (1, 0, 0),\ (a, b) \in \mathbf{C}_0.$$

The same observations as for quadratic and 2-divisible cubic modules imply

4.3. Proposition. *Two weakly alternative cubic modules M, N are isomorphic if and only if $M_p \simeq N_p$ for all p.*

The only non-trivial cases are, of course, $p = 2$ and $p = 3$. In the former case $(\mathbf{C}_0)_2 \simeq \mathbb{Z}_2 \times \mathrm{Mat}(2, \mathbb{Z}_2)$ and the second factor acts trivially on $D = D_2$. So the problem reduces to the classification of diagrams of $\mathbb{Z}_2$-modules

$$W_1 \underset{\eta}{\overset{\xi}{\rightleftarrows}} W_2$$

such that $2\xi = 2\eta = \eta\xi = 0$. Splitting each of W_i into direct sum of free modules $C_\infty = \mathbb{Z}_2$ and finite cyclic groups $C_k = \mathbb{Z}/2^k$, one can reduce ξ and η to a normal form. Namely, consider (finite) words ω of the shape

$$\dots \xi^{i_r} \eta_{j_r} \xi^{i_{r+1}} \eta_{j_{r+1}} \dots \qquad (i_r, j_r \in \mathbb{Z} \cup \{\infty\}$$

not containing subwords ${}_\infty\xi$, ${}^\infty\eta$, $\eta_1\xi$. Such a diagram gives rise to a weakly alternative module $W = W(\omega)$. Namely,

$$W_1 = \bigoplus_r C_{i_r},\ W_2 = \bigoplus_r C_{j_r},\ \xi(C_{i_r}) \subset C_{j_{r-1}},\ \eta(C_{j_r}) \subset C_{i_r},$$

and the induced mappings are non-zero of period 2. Note that such mappings are unique; we denote them by γ (not precising indices). The modules $W(\omega)$ are called *string* $\mathbf{C}_2$*-modules.* A *band* $\mathbf{C}_2$*-module* depends on a pair (ω, ϕ), where

$$\omega = {}_j\xi^{i_1}\eta_{j_1}\xi^{i_2}\eta_{j_2}\dots{}^{i_n}\eta_j$$

and $\phi \neq t^m$ is a power of an irreducible polynomial over $\mathbb{Z}/2$. The corresponding band module $W = W(\omega, \phi)$ is defined as follows:

$$W_1 = \bigoplus_r mC_{i_r}, \quad W_2 = mC_j \oplus (\bigoplus_r mC_{j_r}),$$

$$\xi(mC_{i_r}) \subset mC_{j_{r-1}},\ \eta(mC_{j_r}) \subset mC_{i_r},\ \xi(mC_{i_1}) \subset mC_j,\ \eta(nC_j) \subset mC_{i_n},$$

where $m = \deg\phi$, and the induced mappings coincide with $\gamma\mathrm{Id}$, except for $mC_j \to mC_{i_n}$ that is given by the matrix $\gamma\Phi$, where Φ is the Frobenius cell with the characteristic polynomial ϕ. In the case $p = 3$, $D_3 = 0$ and we are in the situation analogous to that of quadratic or 2-divisbile cubic modules. This time the values i_r, j_r are taken from the set $\{1, 2, 3, 4\}$ with $3 - 4$ and $2 \sim 3$. Gluing $\mathbf{C}_2$- and $\mathbf{C}_3$-modules gives the following

4.4. Theorem. *Indecomposable weakly alternative cubic modules correspond to the* $\mathbf{C}$*-modules of the following types:*

(1) Torsion modules: *(a)* 2*-torsion:* $W(\omega)$ *and* $W(\omega, \phi)$ *such that* ξ^∞ *does not occur in* ω*;* *(b)* 3*-torsion: all band* $\mathbf{C}_3$*-modules and string modules of type* (iii)*;* *(c) p-torsion for* $p > 3$*, which are* P/p^kP*, where* P *is an irreducible torsion free* $\mathbf{C}_p$*-module.*

(2) Torsion free modules, which are just irreducible modules and the projective module $\mathbf{C}(0, 1, e_{11})$.

(3) "Mixed" modules M*, which are also of three possible shapes given by their localization at* $p = 2$ *and* $p = 3$*:* *(a)* $M_2 = W(\omega)$*, where* ω *contains* ξ^∞*,* $M_3 = M^{\mathrm{D}}$*, where* D *is a string of type* (i) *or* (ii) *with* $i_{2n-1} = 2$ *or* $i_2 = 2$*; if both*

occur, it gives two non-isomorphic modules; (b) $M_2 = W(\omega)\oplus W(\omega')$ *, where both* ω *and* ω' *contains* ξ^∞ *,* $M_3 = M^{\mathrm{D}}$ *, where* D *is of type* (ii) *with* $j_{2n-1} = j_2 = 2$ *; (c)* $M_3 = M^{\mathrm{D}}$ *, where* D *is of type* (i) *or* (ii)*,* M_2 *is torsion free (hence uniquely determined).*

C. Torsion free cubic modules

They are such modules that all groups M_i $(i = 1, 2, 3)$ are torsion free. As usually, we study them locally. The only non-trivial case is $p = 2$. Then the calculations of subsection 4A imply that the corresponding (localized) ring is isomorphic to the subring in $\mathbb{Z}_2^3 \times \mathrm{Mat}(2, \mathbb{Z}_2) \times \mathrm{Mat}(4, \mathbb{Z}_2)^2$ consisting of all sextuples satisfying the following congruences modulo 2:

$$(a_1, a_2, a_3, b, c, d) \quad \text{with} \quad a_1 \equiv b_{11} \equiv c_{11}\,,\ a_2 \equiv b_{22} \equiv c_{22} \equiv c_{33}\,,$$

$$a_3 \equiv c_{44}\,,\ b_{12} \equiv 0 \ \text{ and } \ c_{ij} \equiv 0 \ \text{ if } \ i < j\,.$$

It is a *Backström order*, i.e. its radical coincides with the radical of a hereditary order. Therefore we can apply the method of [9] that reduces the description of torsion free modules to some diagrams of vector spaces. The precise shape of our ring implies that in this case the corresponding diagram is a disjoint union of 4 diagrams of types A_2, A_3, D_4 and $\tilde{D}_4$. Hence the classification of such modules is again a tame (and rather easy) problem (cf. [3]). Moreover, the specific form of this order implies the following important corollary for *all* cubic modules, extending the claim (1) of Theorem 3.3.

4.5. Corollary. *Two cubic modules are isomorphic if and only if all their localizations are isomorphic.*

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