COHERENT SHEAVES ON RATIONAL CURVES WITH SIMPLE DOUBLE POINTS AND TRANSVERSAL INTERSECTIONS

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Abstract
We study the derived categories of coherent sheaves on some singular projective curves and give a complete description of indecomposable objects using the technique of matrix problems.

Contents

1. Introduction

This paper is, in particular, motivated by the recent research in homological mirror symmetry (see [18], [24]). It is well known that any object of the derived category of coherent sheaves on a smooth elliptic curve is isomorphic to a direct sum of shifts of vector bundles and skyscraper sheaves. Indecomposable vector bundles on elliptic curves were classified in the paper of Atiyah [1] (see also [22] and [24] for a description via étale coverings). An essential feature of this description is that an indecom-

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posable vector bundle is described by two discrete parameters, rank r and degree d, and *one* continuous parameter, a point of the curve. Suppose now that a family of elliptic curves degenerates into a cycle of projective lines. A natural question is, what happens with the derived category of coherent sheaves under this degeneration?

As we shall see, the derived category of coherent sheaves on a cycle of projective lines resembles the situation in the smooth case. There are three types of indecomposable objects: shifts of skyscraper sheaves at a smooth point of the curve, and the so-called bands $\mathscr{B}(w, m, \lambda)$ and strings $\mathscr{S}(w)$. A band $\mathscr{B}(w, m, \lambda)$ depends on *one* continuous parameter $\lambda \in \mathbf{k}^*$, on a natural number $m \in \mathbb{N}$ (which can be interpreted as a "thickening" of an object), and on a quite complicated discrete parameter w. Strings $\mathscr{S}(w)$ can be viewed as degenerations of bands. For example, all vector bundles are bands. Torsion-free sheaves, which are not vector bundles, are strings. In fact, strings are exactly indecomposable complexes from $D^-(\mathrm{Coh}_X)$ of infinite homological dimension.

Suppose that a family of elliptic curves degenerates to a cuspidal curve. It was shown in [11] that the category of vector bundles on a cuspidal curve "explodes": for any natural number $n \in \mathbb{N}$, there are families of indecomposable vector bundles depending on n continuous parameters. In the language of representation theory, it means that the category of vector bundles is *wild*. The category of skyscraper sheaves at the singular point is also wild (see [9]). It agrees with the fact that the category of semistable torsion-free sheaves of degree zero is equivalent, via Fourier-Mukai transform, to the category of torsion sheaves (see [13], [27]).

In the case of a smooth curve, knowing vector bundles, one also knows all coherent sheaves; they are just direct sums of vector bundles and of skyscraper sheaves $\mathscr{O}_x/\mathfrak{m}_x^k$, where $\mathscr{O}$ is the structure sheaf, x is a (closed) point, and $\mathfrak{m}_x$ is the $\mathscr{O}_x$-ideal of functions vanishing at x. The situation is quite different in the case when a curve is no longer smooth. Then first there are indecomposable sheaves that are "mixed" (neither torsion-free nor skyscraper); second, the structure of skyscraper sheaves is much more complicated (cf. [15], [19]).

The aim of this paper is to describe all coherent sheaves in this case. Moreover, we describe all indecomposable objects in the derived categories of coherent sheaves over tame curves.

2. Vector bundles on cycles of projective lines

Recall some facts about the classification of vector bundles on projective curves. As a rule, it is a *wild problem* in the sense that it contains the description of representations of *all* finitely generated algebras. The cases when it is not so are the following:

- projective line $\mathbb{P}^1$ (see [16]) and configurations of projective lines with transversal intersection and with the intersection graph of type A (a chain of lines); in this case, indecomposable vector bundles have rank 1 and are classified by their (multi)degree (see Fig. 1);

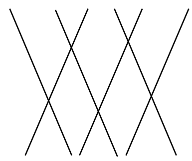

Figure 1

- elliptic curves (smooth curves of genus 1) (see [1]);
- rational curve with one node (see Fig. 2);

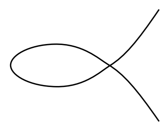

Figure 2

- configurations of projective lines of type $\tilde{\mathrm{A}}$ (a cycle) (see Fig. 3).

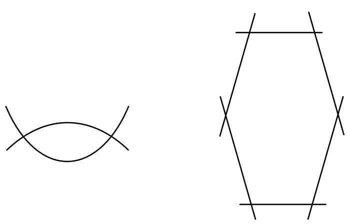

Figure 3

Let us briefly recall the description of vector bundles on cycles of projective lines (see [11]). Let $X = X_s$ be a cycle of s projective lines (in the case of $s = 1$, it is just a rational curve with one simple node), $\mathscr{O} = \mathscr{O}_X$. Then we have $\mathrm{Ext}^1_{\mathscr{O}}(\mathscr{O}, \mathscr{O}) = \mathrm{H}^1(\mathscr{O}) = \mathbf{k}$. Hence we have a unique nonsplit extension

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{O} \longrightarrow 0.$$

Repeating the arguments of [1], we obtain that the subcategory consisting of iterated extensions of the structure sheaf $\mathscr{O}$ consists of direct sums of $\mathscr{O} = \mathscr{F}_1, \mathscr{F}_2, \dots$, where each $\mathscr{F}_m$ is inductively defined by an extension

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_m \longrightarrow \mathscr{F}_{m-1} \longrightarrow 0, \quad m \geq 2.$$

The bundles $\mathscr{F}_m,\ m \geq 1$, are called unipotent.

Let us note that a line bundle on a cycle of projective lines is determined by its multidegree $\mathbf{d}$ and continuous parameter $\lambda \in \mathbf{k}^*$.

THEOREM 2.1 ([11, Th. 2.12]; see also [6])
Suppose that $\operatorname{char}(\mathbf{k}) = 0$. *Let* X_s *be a cycle of* s *lines, let*

$$p_n : X_{ns} \longrightarrow X_s$$

be an étale covering of degree n, *and let*

$$\mathbf{d} = d_1 d_2 \cdots d_s d_{s+1} d_{s+2} \cdots d_{2s} \cdots d_{(n-1)s+1} d_{(n-1)s+2} \cdots d_{ns},$$

$d_i \in \mathbb{Z},\ i = 1, \ldots, ns$, *be any nonperiodic sequence of integers. Nonperiodicity means that* $\mathbf{d}[t] \neq \mathbf{d}$ *for all* $t = 1, \ldots, n-1$, *where*

$$\mathbf{d}[1] = d_{s+1} d_{s+2} \cdots d_{2s} \cdots d_{(n-1)s+1} d_{(n-1)s+2} \cdots d_{ns} d_1 d_2 \cdots d_s,$$

$\mathbf{d}[t] = (\mathbf{d}[t-1])[1]$. *Let* $\mathscr{L} = \mathscr{L}(\mathbf{d}, \lambda)$ *be a line bundle on* X_{ns} *of multidegree* $\mathbf{d}$. *Then*

$$\mathscr{B}(\mathbf{d}, m, \lambda) = p_{n*}(\mathscr{L}) \otimes \mathscr{F}_m$$

is an indecomposable vector bundle of rank mn *on the curve* X_s. *Moreover, each indecomposable vector bundle on the curve* X_s *is isomorphic to some vector bundle* $\mathscr{B}(\mathbf{d}, m, \lambda)$. *For the sake of convenience, we denote* $\mathscr{B}(0, m, 1) = \mathscr{F}_m$.

The construction via étale coverings has the disadvantage that it does not work in the case of $\operatorname{char}(\mathbf{k}) = p > 0$. We now give the description, which does not depend on the characteristic of $\mathbf{k}$.

Let X be a cycle of projective lines, let $\pi : \tilde{X} \longrightarrow X$ be its normalization, let $\tilde{\mathscr{O}} = \pi_*(\mathscr{O}_{\tilde{X}})$, and let $\mathscr{J} = \operatorname{Ann}_{\mathscr{O}}(\tilde{\mathscr{O}}/\mathscr{O})$ be the conductor.

Let $\mathscr{E}$ be a vector bundle on X. Denote $\tilde{\mathscr{E}} = \pi_* \pi^*(\mathscr{E})$; then we have a canonical inclusion $\mathscr{E} \longrightarrow \tilde{\mathscr{E}}$, which induces an inclusion $i : \mathscr{E} \otimes_{\mathscr{O}} \mathscr{O}/\mathscr{J} \longrightarrow \tilde{\mathscr{E}} \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{O}}/\mathscr{J}$. It was shown in [11] that a vector bundle on a cycle of projective lines is completely determined by its normalization $\tilde{\mathscr{E}}$ and inclusion i.

Example 2.2
Let $\operatorname{char}(\mathbf{k}) = 0$, let $X = X_1$ be a rational curve with one node, and let $\pi_3 : X_3 \longrightarrow X_1$ be an étale covering of degree 3. Let $\mathscr{L} = \mathscr{L}((1,1,0), \lambda)$ be a line bundle on X_3; the degree of $\mathscr{L}$ is 1 on components with number 1 and 2, and 0 on the third component, and λ glues fibers of $\mathscr{L}$ at the point of intersection of the first and third components. Let

$$\mathscr{E} = \mathscr{B}\big((1,1,0), m, \lambda\big) = \pi_{3*}(\mathscr{L}) \otimes_{\mathscr{O}} \mathscr{B}(0, m, 1).$$

Then

$$\tilde{\mathscr{E}} \cong \tilde{\mathscr{O}}(1)^{2m} \oplus \tilde{\mathscr{O}}^m.$$

The map i is just a $\mathbf{k}$-linear map of a $\mathbf{k}$-module into a $(\mathbf{k} \times \mathbf{k})$-module; hence it is given by the two matrices $i(0:1)$ and $i(1:0)$. Doing basis transformation, we can reduce them to the form of Figure 4.

$$\begin{array}{ccc} \left[\begin{array}{c:c:c} I_m & & \\ \hdashline & I_m & \\ \hdashline & & I_m \end{array}\right] & \begin{array}{c}1\\1\\0\end{array} & \left[\begin{array}{c:c:c} & I_m & \\ \hdashline & & I_m \\ \hdashline J_m(\lambda) & & \end{array}\right] \\ i(0:1) & & i(1:0) \end{array}$$

Figure 4

It turns out that we can define our bundles $\mathscr{B}(\mathbf{d}, m, \lambda)$ by means of these gluing matrices i. This description is independent of the characteristic of the field $\mathbf{k}$.

Remark 2.3
Let $\mathrm{char}(\mathbf{k}) = 0$, let X_s be a cycle of s lines, and let $\mathbf{d} = d_1 d_2 \cdots d_{ns}$ be periodic; that is, let $\mathbf{d} = \mathbf{ee} \cdots \mathbf{e}$ (k times), where $k|n$ and $\mathbf{e}$ is a sequence on $X_{sn/k}$. Then we can still consider a locally free sheaf

$$\mathscr{B}(\mathbf{d}, 1, \lambda) = p_{n*}\mathscr{L}(\mathbf{d}, \lambda),$$

where $p_n : X_{ns} \longrightarrow X_s$ is an étale covering of degree n. However, it decomposes into a direct sum

$$\mathscr{B}(\mathbf{d}, 1, \lambda) = \bigoplus_{i=1}^{k} \mathscr{B}(\mathbf{e}, 1, \xi_i),$$

where $\xi_1, \xi_2, \ldots, \xi_k$ are all kth roots of λ.

3. Description of indecomposable complexes

Let $X = X_s$ be a cycle of s lines. In order to describe indecomposable complexes of the derived category $D^-(\mathrm{Coh}_X)$, we define the following combinatorial data.

Definition 3.1
(1) The *alphabet* $\mathfrak{A}$ consists of the following symbols (*letters*):
  (a) n_k, where $n, k \in \mathbb{Z}$ (we call them 0-*letters*);

(b) x_k^n, y_k^n, where $n \in \mathbb{N}$, $k \in \mathbb{Z}$ (we call them, resp., *x-letters* and *y-letters*).

We write $i(a) = k$, where a is one of the letters above.

(2) The set of *active indices* of a letter a is defined as follows:

(a) if $a = n_k$, it is $\{k\}$;

(b) if $a = x_k^n$ or y_k^n, it is $\{k, k+1\}$.

(3) A *word* is a mapping $w : \mathbb{Z} \to \mathfrak{A}$ such that

(a) if all 0-letters are omitted, x-letters and y-letters alternate;

(b) any two neighbor letters have a common active index, and every active index of a letter is also an active index of one of its neighbors.

(4) We define the functions δ_w and j_w, both $\mathbb{Z} \to \mathbb{Z}/s\mathbb{Z}$, by the following rules:

(a) if there is $m' > m$ such that $w(m')$ is a y-letter and all $w(m'')$ with $m < m'' < m'$ are 0-letters, then $\delta_w(m) = -1$; otherwise, $\delta_w(m) = 1$;

(b) $j_w(0) = 0$ and $j_w(m+1) = j_w(m) + \delta_w(m)$ for every m.

If $s = 1$, then both functions j_w and δ_w are trivial.

(5) A *string* $\mathscr{S}(w)$ is given by a word w satisfying the following conditions:

(a) there is an integer K such that $i(w(m)) \geq K$ for all $m \in \mathbb{Z}$;

(b) for every integer k, the set $\{m \in \mathbb{Z} \mid i(w(m)) = m\}$ is finite.

(6) If $l \in \mathbb{N}$, we define the *l-shift* w^l of a word l setting $w^l(m) = w(m+l), m \in \mathbb{Z}$. We call this shift *admissible* if $j_w(0) = j_w(l)$.

(7) A *cycle* is a word w such that $w^l = w$ for some admissible shift. The smallest l with this property is called the *period* of the cycle w. Evidently a cycle of period l is completely defined by its segment $w(0)w(1)\cdots w(l-1)$, and we often identify them. Note only that if $s > 1$, not every finite word defines a cycle; the condition $j_w(0) = j_w(l)$ is rather restrictive.

(8) A *band* $\mathscr{B} = \mathscr{B}(w, d, \lambda)$ is given by a triple (w, d, λ), where $d \in \mathbb{N}$, $\lambda \in \mathbf{k}^* = \mathbf{k} \setminus \{0\}$, and w is a cycle.

(9) We denote by w° the *opposite* word; that is, $w^\circ(m) = w(-m)$.

(10) We call two strings $\mathscr{S}(v)$, $\mathscr{S}(w)$ *equivalent* if there is an admissible shift v^l such that $v^l = w$ or $v^l = w^\circ$.

(11) We call two bands $\mathscr{B}(w, d, \lambda)$, $\mathscr{B}(v, d, \mu)$ *equivalent* if there is an admissible shift v^l such that either $w = v^l$ and $\mu = \lambda$, or $w^\circ = v^l$ (not all letters in w are 0-letters) and $\mu = \lambda^{-1}$.

The functions j_w and δ_w have the following interpretation: $j_w(0)$ indicates the component of X we start with (i.e., $w(0)$ is supported on the $j_w(0)$th component); δ_w shows the direction of "jumping" from a component of X to one of its neighbors; $j_w(m)$ is the number of the component, where $w(m)$ is supported.

THEOREM 3.2
There is a one-to-one correspondence between isomorphism classes of indecomposable objects of the derived category $D^-(\mathrm{Coh}_X)$ *and equivalence classes of strings and bands.*

This theorem is proved in the following sections. Nearby we show how one can construct a complex corresponding to a string or band. For the sake of simplicity, we consider only the case of an irreducible curve. Then $s = 1$, so the function j_w plays no role; in particular, any shift is admissible. (In the general case, one only has to place each bundle over $\widetilde{X}$ on the corresponding irreducible component). To explain the rule, note that in [11] vector bundles over X were encoded by bands such that corresponding cycles contain only 0-words with $k = 0$ (so we omit this index). If $n_0, n_1, \dots, n_{l-1}$ defines such a cycle, the vector bundle corresponding to it can be described as a gluing of vector bundles over the normalization. Namely, take $\bigoplus_{i=0}^{l-1} \tilde{\mathscr{O}}(n_i)^d$, choose its local trivializations, which give bases of fibres at every point, and identify the fibres at zero with the fibres at ∞ in the following way (for details, cf. [11, Th. 2.12], especially Figs. 2 and 3 after it):

- if $0 \le i < l - 1$, the basis at ∞ of $\tilde{\mathscr{O}}(n_i)^d$ is identified with the basis at zero of $\tilde{\mathscr{O}}(n_{i+1})^d$;
- the basis at ∞ of $\tilde{\mathscr{O}}(n_{l-1})^d$ is identified with the twisted basis at zero of $\tilde{\mathscr{O}}(n_0)^d$, where the twist is given by the $d \times d$ Jordan cell $J_d(\lambda)$ with eigenvalue λ.

Quite analogous gluing can be done for the derived category. Indeed, let $w(0)w(1)\cdots w(l-1)$ define a cycle w. Replace

- every 0-letter n_k by the kth shift of the vector bundle $\tilde{\mathscr{O}}(n)^d$ over $\widetilde{X}$;
- every x-word x_k^n by the kth shift of the complex $(\tilde{\mathscr{O}}(-n) \xrightarrow{x^n} \tilde{\mathscr{O}})^d$;
- every x-word y_k^n by the kth shift of the complex $(\tilde{\mathscr{O}}(-n) \xrightarrow{y^n} \tilde{\mathscr{O}})^d$.

Here $\tilde{\mathscr{O}}(-n) \xrightarrow{x^n} \tilde{\mathscr{O}}$ and $\tilde{\mathscr{O}}(-n) \xrightarrow{y^n} \tilde{\mathscr{O}}$ are locally free resolutions, respectively, of the skyscrapers $\tilde{\mathscr{O}}/\mathfrak{m}_0^n$ and $\tilde{\mathscr{O}}/\mathfrak{m}_\infty^n$. Take the direct sum of all these complexes, fix local trivializations, and identify the fibres at zero and at ∞. Namely, let i be the common active index of $w(m)$ and $w(m+1)$ (note that $w(l) = w(0)$), and let $\delta = \delta_w(m)$. Then follow these rules.

(1) If $\delta = 1$, glue the basis at ∞ of the ith component of the mth complex defined above with the basis at zero of the ith component of the $(m+1)$st complex (twisted by the $J_d(\lambda)$ if $m = l - 1$).

(2) If $\delta = -1$, glue the basis at zero of the ith component of the mth complex with the basis at ∞ of the ith component of the $(m+1)$st complex (twisted by the $J_d(\lambda)$ if $m = l - 1$).

(3) Glue the remaining bases at zero (originating from y-letters) with the remaining bases at ∞ (originating from x-letters) in any way, keeping only the "parallelogram rule," which means that if you glue bases of two complexes corresponding to the letters x_k^n and y_k^r at the kth component, then the same must be done at the $(k+1)$st, too. The alternating condition (3.a) from Definition 3.1 guarantees that such gluing is always possible.

Remark 3.3
This construction should be modified in the case when $X = X_s$, $s \geq 3$. In order to get a complex of locally free sheaves on X, we should add some trivial complexes of $\tilde{\mathscr{O}}$-modules $\tilde{\mathscr{O}} \xrightarrow{\text{id}} \tilde{\mathscr{O}}$ to the gluing data. See Section 6 for further details.

One can easily see that as a result we obtain in each component a vector bundle over X, and the differential of the complex over $\widetilde{X}$ induces homomorphisms between these bundles; so we get a complex of vector bundles over X which defines an object of the derived category.

Here are some examples. Consider the band $\mathscr{B}(w, 1, \lambda)$, where $w = 3_0 x_0^2 y_1^1$ - $2_2 x_1^3 y_0^1$. The corresponding complex is in Figure 5. In Figure 5 (as well as in the

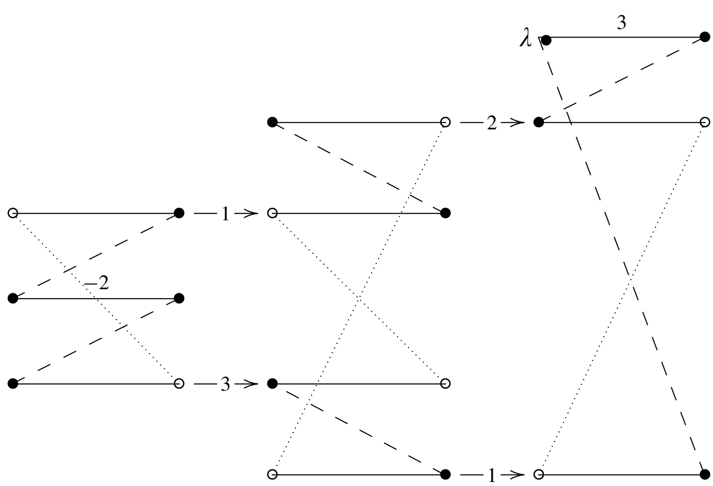

Figure 5

next ones) solid lines show the line bundles on the normalization; their left (resp., right) ends symbolize the fibres at zero (resp., at ∞). The superscripts show the degrees. Vector bundles in the same column correspond to the fixed component of the complex. (In this example the very right is the 0-component, and the very left is the 2-component.) Horizontal arrows show the differential: it is given by x^n (resp., by y^n)

if the left ends (resp., the right ends) of the corresponding lines are bullets. Note that the absolute degrees of these line bundles play no role: only their difference n (shown by the inserted number) matters. The twist by λ is shown near the corresponding end. Dashed lines show the mandatory gluing according to rules (1) and (2) above; dotted lines show the arbitrary parallel gluing of "free" ends according to rule (3). These ends are encircled.

One can easily see that this complex does not correspond to any coherent sheaf. Indeed, it has nontrivial cohomologies in degrees 0 and 2 corresponding to the lone solid lines on the first and fourth levels. We can also write down a locally free representative of this complex,

$$\mathscr{B}\big((-2,0,0),1,1\big) \longrightarrow \mathscr{B}\big((0,0,1,3),1,1\big) \longrightarrow \mathscr{B}\big((3,1,2),1,\lambda\big),$$

and differentials are those as it is shown in Figure 5.

Remark 3.4
Let $\mathscr{E}$ and $\mathscr{F}$ be two locally free $\mathscr{O}$-modules, and let $\tilde{\mathscr{E}}$ and $\tilde{\mathscr{F}}$ be corresponding $\tilde{\mathscr{O}}$-modules. We have a canonical embedding

$$\mathrm{Hom}_{\mathscr{O}}(\mathscr{E},\mathscr{F}) \longrightarrow \mathrm{Hom}_{\tilde{\mathscr{O}}}(\tilde{\mathscr{E}},\tilde{\mathscr{F}}).$$

Differentials in the figures are homomorphisms of $\tilde{\mathscr{O}}$-modules. They lie in the image of the normalization map and hence define homomorphisms of $\mathscr{O}$-modules.

The following example is the string $\mathscr{S}(w)$, where $w = \cdots y_3^1 x_2^1 y_1^1 x_0^n y_0^m x_1^1 y_2^1 x_3^1 \cdots$, which corresponds to a skyscraper sheaf (to $\mathbf{k}(p)$, where p is the singular point if $n = m = 1$). The complex defined by this string is in Figure 6. Its locally free

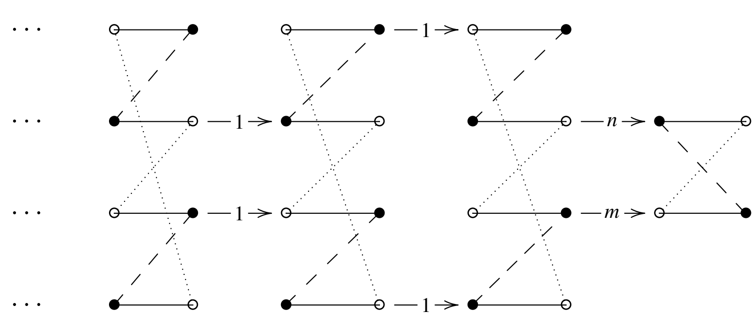

Figure 6

representative is

$$\cdots \longrightarrow \mathscr{B}\big((1,0,0,1),1,1\big) \longrightarrow \mathscr{B}\big((1,0,0,1),1,1\big) \longrightarrow \mathscr{B}\big((n,m),1,1\big).$$

The band $\mathscr{B}(w, 2, \lambda)$, where $w = \text{-}3_0 0_0 y_0^1 x_0^2 y_0^4 x_0^5 0_0$, defines a *mixed* coherent sheaf, that is, one that is neither torsion-free nor skyscraper. The corresponding complex is in Figure 7. Double horizontal lines reflect the fact that each line bundle must

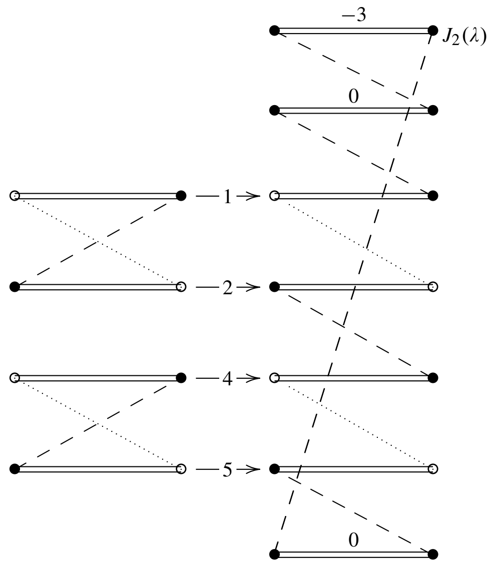

Figure 7

be taken twice. The twist by the Jordan cell is marked at the corresponding place. A locally free representative of this complex is

$$\mathscr{B}\big((0,0),2,1\big) \oplus \mathscr{B}\big((0,0),2,1\big) \longrightarrow \mathscr{B}\big((-3,0,1,2,4,5,0),2,\lambda\big).$$

Note that this time we could trace dotted lines another way, joining the first free end with the last one and the second with the third (see Fig. 8). It gives an isomorphic object in $D^-(\mathrm{Coh}_X)$,

$$\mathscr{B}\big((0,0,0,0),2,1\big) \longrightarrow \mathscr{B}\big((-3,0,1,5,0),2,\lambda\big) \oplus \mathscr{B}\big((2,4),2,1\big).$$

The last example is an object of $D^-(\mathrm{Coh}_X)$ which does not belong to $D^b(\mathrm{Coh}_X)$. It is the string $\mathscr{S}(w)$, $w = \cdots y_2^2 x_1^1 \text{-}2_1 y_0^3 0_0 x_0^2 y_1^2 1_2 \cdots$. This string has 0-letters with all indices, which immediately implies that all cohomologies are nonzero. The corresponding complex is in Figure 9. Its locally free representative is

$$\cdots \longrightarrow \mathscr{B}\big((-2,1,2,0,0),1,1\big) \longrightarrow \mathscr{B}\big((0,2,3),1,1\big).$$

4. Main construction

In what follows, we denote by the same letter $\mathbf{F}$ the derived functor $D^-(\mathscr{A}) \longrightarrow D^-(\mathscr{B})$ induced by a functor $\mathbf{F} : \mathscr{A} \longrightarrow \mathscr{B}$. In particular, $\otimes$ denotes the derived

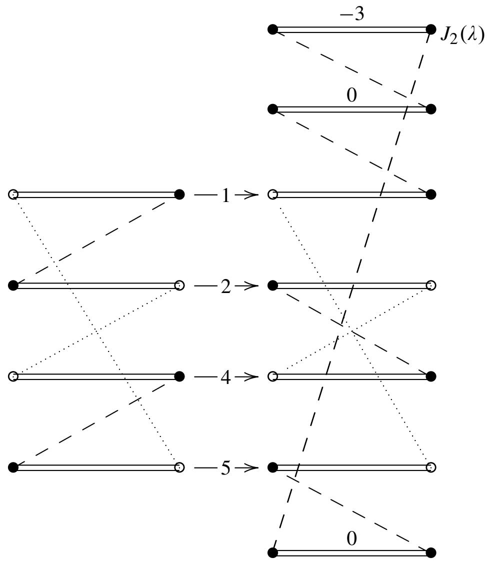

Figure 8

functor induced by the tensor product.

Let X be a projective curve over an algebraically closed field $\mathbf{k}$. We always suppose that X is connected (although it may be reducible). We use the following notation:

- $\pi : \tilde{X} \to X$ the normalization of X;
- $\mathscr{O} = \mathscr{O}_X$ and $\tilde{\mathscr{O}} = \pi_* \mathscr{O}_{\tilde{X}}$;
- $S = \operatorname{Sing} X$ the set of singular points of X and $\tilde{S} = \pi^{-1}(S)$;
- $\tilde{\mathscr{F}}_\bullet = \mathscr{F}_\bullet \otimes_{\mathscr{O}} \tilde{\mathscr{O}}$, where $\mathscr{F}_\bullet$ is a complex of locally free $\mathscr{O}$-modules;
- $\mathscr{J} = \operatorname{Ann}_{\mathscr{O}} \tilde{\mathscr{O}}/\mathscr{O}$, the conductor of $\tilde{\mathscr{O}}$ in $\mathscr{O}$;
- $\mathscr{A} = \mathscr{O}/\mathscr{J}$ and $\tilde{\mathscr{A}} = \tilde{\mathscr{O}}/\mathscr{J}$;
- $\bar{\mathscr{F}}_\bullet = \tilde{\mathscr{F}}_\bullet \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}}$, where $\tilde{\mathscr{F}}_\bullet$ is a complex of locally free $\tilde{\mathscr{O}}$-modules;
- $\tilde{\mathscr{M}}_\bullet = \mathscr{M}_\bullet \otimes_{\mathscr{A}} \tilde{\mathscr{A}}$, where $\mathscr{M}_\bullet$ is a complex of locally free $\mathscr{A}$-modules.

Note that $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are skyscraper sheaves, and note that $\operatorname{Supp} \mathscr{A} = \operatorname{Supp} \tilde{\mathscr{A}} = \operatorname{Sing} X$.

Since the morphism π is affine, we can identify $\mathscr{O}_{\tilde{X}}$-modules and $\tilde{\mathscr{O}}$-modules. We also identify the derived category $D^-(\mathrm{Coh}_X)$ with the category of quotients $K^-(\mathrm{VB}_X)[Q^{-1}]$, where VB_X is the category of locally free sheaves of $\mathscr{O}$-modules (or, equivalently, vector bundles over X), and Q is the set of quasi-isomorphisms.

We have canonical homomorphisms $\mathscr{O} \longrightarrow \mathscr{A} \longrightarrow \tilde{\mathscr{A}}$, $\tilde{\mathscr{O}} \longrightarrow \tilde{\mathscr{A}}$. Since the forgetful functor $\mathrm{Coh}_{\mathscr{A}} \longrightarrow \mathrm{Coh}_{\mathscr{O}}$ is exact, it induces an exact functor $D^-(\mathrm{Coh}_{\mathscr{A}}) \longrightarrow$

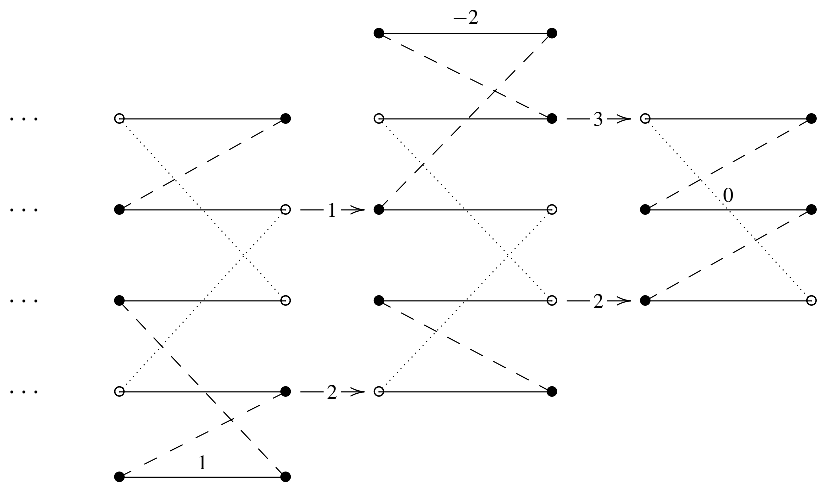

Figure 9

$D^-(\mathrm{Coh}_{\mathscr{O}})$. Its action on objects does not require any choice of special representatives, so we identify objects of $D^-(\mathrm{Coh}_{\mathscr{A}})$ with their images in $D^-(\mathrm{Coh}_{\mathscr{O}})$.

We have a commutative diagram of functors

$$\begin{tikzcd} D^-(\mathrm{Coh}_{\mathscr{O}}) \arrow[r] \arrow[d] & D^-(\mathrm{Coh}_{\mathscr{A}}) \arrow[d] \\ D^-(\mathrm{Coh}_{\tilde{\mathscr{O}}}) \arrow[r] & D^-(\mathrm{Coh}_{\tilde{\mathscr{A}}}) \end{tikzcd}$$

We want to reconstruct a complex $\mathscr{F}_\bullet \in D^-(\mathrm{Coh}_{\mathscr{O}})$ from its images in $D^-(\mathrm{Coh}_{\tilde{\mathscr{O}}})$ and $D^-(\mathrm{Coh}_{\mathscr{A}})$.

Definition 4.1

Define the category of triples of complexes TC_X in the following way.

- Its objects are triples $(\tilde{\mathscr{G}}_\bullet, \mathscr{M}_\bullet, i)$, where
  - $\tilde{\mathscr{G}}_\bullet$ is a (right bounded) complex of locally free $\tilde{\mathscr{O}}$-modules;
  - $\mathscr{M}_\bullet$ is a (right bounded) complex of locally free $\mathscr{A}$-modules;
  - i is an isomorphism $\mathscr{M}_\bullet \otimes_{\mathscr{A}} \tilde{\mathscr{A}} \to \mathscr{G}_\bullet \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}}$ in the category $D^-(\mathrm{Coh}_{\tilde{\mathscr{A}}})$.
- A morphism $(\mathscr{G}_\bullet, \mathscr{M}_\bullet, i) \to (\mathscr{G}'_\bullet, \mathscr{M}'_\bullet, i')$ is a pair (Φ, φ), where $\Phi : \mathscr{G}_\bullet \to \mathscr{G}'_\bullet$ is a morphism in $D^-(\mathrm{Coh}_{\tilde{\mathscr{O}}})$ and $\varphi : \mathscr{M}_\bullet \to \mathscr{M}'_\bullet$ is a morphism in $D^-(\mathrm{Coh}_{\mathscr{A}})$

such that the diagram

$$\begin{tikzcd}
\mathscr{M}_\bullet \otimes_{\mathscr{A}} \tilde{\mathscr{A}} \arrow[r, "i"] \arrow[d, "\varphi\otimes\mathrm{id}"'] & \mathscr{G}_\bullet \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} \arrow[d, "\Phi\otimes\mathrm{id}"] \\
\mathscr{M}'_\bullet \otimes_{\mathscr{A}} \tilde{\mathscr{A}} \arrow[r, "i'"'] & \mathscr{G}'_\bullet \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}}
\end{tikzcd}$$

is commutative in $D^-(\mathrm{Coh}_{\tilde{\mathscr{A}}})$.

- We define the functor $\mathbf{F} : D^-(\mathrm{Coh}_X) \to \mathrm{TC}_X$ which maps a complex $\mathscr{F}_\bullet$ of locally free sheaves to the triple $(\tilde{\mathscr{F}}_\bullet, \mathscr{F}_\bullet \otimes_{\mathscr{O}} \mathscr{A}, i_{\mathscr{F}_\bullet})$, where $i_{\mathscr{F}_\bullet}$ is the natural isomorphism

$$(\mathscr{F}_\bullet \otimes_{\mathscr{O}} \tilde{\mathscr{O}}) \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} \xrightarrow{\sim} \mathscr{F}_\bullet \otimes_{\mathscr{O}} \tilde{\mathscr{A}} \xrightarrow{\sim} (\mathscr{F}_\bullet \otimes_{\mathscr{O}} \mathscr{A}) \otimes_{\mathscr{A}} \tilde{\mathscr{A}}.$$

Note that any quasi-isomorphism of complexes $\mathscr{F}_\bullet \to \mathscr{F}'_\bullet$ in $K^-(\mathrm{VB}_X)$ induces quasi-isomorphisms $\tilde{\mathscr{F}}_\bullet \to \tilde{\mathscr{F}}'_\bullet$ and $\mathscr{F}_\bullet \otimes_{\mathscr{O}} \mathscr{A} \to \mathscr{F}'_\bullet \otimes_{\mathscr{O}} \mathscr{A}$. Therefore the functor $\mathbf{F}$ is well defined on the derived category.

THEOREM 4.2
The functor $\mathbf{F}$ *is dense (i.e., epimorphic on the set of isoclasses of objects) and reflects isomorphisms; that is, if* $\mathbf{F}\mathscr{F}_\bullet \simeq \mathbf{F}\mathscr{F}'_\bullet$, *then* $\mathscr{F}_\bullet \simeq \mathscr{F}'_\bullet$.

Remark 4.3
On the other hand, this functor is not faithful, though it is an equivalence on the full subcategory of $D^-(\mathrm{Coh}_X)$ consisting of the images of locally free coherent sheaves under the natural embedding $\mathrm{Coh}_X \to D^-(\mathrm{Coh}_X)$.

Proof
The main ingredient of the proof is, given a triple $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$, how can we reconstruct $\mathscr{F}_\bullet$?

The exact sequence

$$0 \longrightarrow \mathscr{J}\tilde{\mathscr{F}}_\bullet \longrightarrow \tilde{\mathscr{F}}_\bullet \longrightarrow \tilde{\mathscr{F}}_\bullet \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} \longrightarrow 0$$

in $\mathrm{Coh}(\mathrm{Coh}_X)$ gives a distinguished triangle

$$\mathscr{J}\tilde{\mathscr{F}}_\bullet \longrightarrow \tilde{\mathscr{F}}_\bullet \longrightarrow \tilde{\mathscr{F}}_\bullet \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} \longrightarrow \mathscr{J}\tilde{\mathscr{F}}_\bullet[-1]$$

in $D^-(\mathrm{Coh}_X)$. Let $\tilde{i} : \mathscr{M}_\bullet \longrightarrow \bar{\mathscr{F}}_\bullet$ be the composition of i with the canonical map $\mathscr{M}_\bullet \longrightarrow \mathscr{M}_\bullet \otimes_{\mathscr{A}} \tilde{\mathscr{A}}$.

The properties of triangulated categories imply that there is a morphism of triangles

$$\begin{tikzcd}
\mathscr{J}\tilde{\mathscr{F}}_\bullet \arrow[r] & \tilde{\mathscr{F}}_\bullet \arrow[r] & \tilde{\mathscr{F}}_\bullet \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} \arrow[r] & \mathscr{J}\tilde{\mathscr{F}}_\bullet[-1] \\
\mathscr{J}\tilde{\mathscr{F}}_\bullet \arrow[r] \arrow[u, "\mathrm{id}"] & \mathscr{F}_\bullet \arrow[r] \arrow[u, "\Phi"'] & \mathscr{M}_\bullet \arrow[r] \arrow[u, "\tilde{i}"'] & \mathscr{J}\tilde{\mathscr{F}}_\bullet[-1] \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

where $\mathscr{F}_\bullet = \mathrm{cone}(\mathscr{M}_\bullet \longrightarrow \mathscr{J}\mathscr{F}_\bullet[-1])[1]$. Taking a cone is not a functorial operation. It gives an intuitive explanation why functor $\mathbf{F}$ is not an equivalence.

The properties of triangulated categories imply immediately that the constructed map (not a functor!)

$$\mathbf{G} : \mathrm{Ob}(\mathrm{TC}_X) \longrightarrow \mathrm{Ob}\left(D^-(\mathrm{Coh}_X)\right)$$

sends isomorphic objects into isomorphic ones and satisfies $\mathbf{GF}(\mathscr{F}_\bullet) \cong \mathscr{F}_\bullet$. Now we have to show that $\mathbf{FG}(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i) \cong (\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$. □

LEMMA 4.4
For every triple $T = (\tilde{\mathscr{G}}_\bullet, \mathscr{M}_\bullet, i)$ from TC_X there is a triple $T' = (\tilde{\mathscr{G}}'_\bullet, \mathscr{M}'_\bullet, i')$ such that i' is an isomorphism of complexes and $T \simeq T'$ in TC_X.

Proof
Note that coherent sheaves of $\mathscr{A}$-modules can be identified with modules over the finite-dimensional algebra $\mathbf{A} = \prod_{x\in S} \mathscr{A}_x$, and also note that coherent sheaves of $\tilde{\mathscr{A}}$-modules can be identified with modules over $\tilde{\mathbf{A}} = \bigoplus_{x\in S} \tilde{\mathscr{A}}_x$. Moreover, considering $D^-(\mathrm{Coh}_{\mathscr{A}})$ and $D^-(\mathrm{Coh}_{\tilde{\mathscr{A}}})$, we can (and shall) always consider complexes of projective modules. Denote by $\mathbf{J}$ and $\tilde{\mathbf{J}}$, respectively, the radicals of the rings $\mathbf{A}$ and $\tilde{\mathbf{A}}$. We call such a complex $\mathscr{M}_\bullet$ of $\mathbf{A}$-modules (or the corresponding complex of $\mathscr{A}$-sheaves) *minimal* if $\mathrm{Im}\, d_n \subseteq \mathbf{J}M_{n-1}$ for each n; this is the same terminology we use for complexes of $\tilde{\mathbf{A}}$-modules and $\tilde{\mathscr{A}}$-sheaves. Since these algebras are commutative, $\mathbf{J} \subseteq \tilde{\mathbf{J}}$; hence if $\mathscr{M}_\bullet$ is minimal, so is $\tilde{\mathscr{M}}_\bullet = \mathscr{M}_\bullet \otimes_{\mathbf{A}} \tilde{\mathbf{A}}$. It follows easily from [2] that each complex $\mathscr{M}_\bullet \in D^-(\mathrm{Coh}_{\mathscr{A}})$ is a direct sum $\mathscr{M}^m_\bullet \oplus \mathscr{M}^t_\bullet$, where $\mathscr{M}^m_\bullet$ is a minimal complex and $\mathscr{M}^t_\bullet$ is a *trivial* one, that is, a direct sum of *short trivial complexes* of the form $\cdots \to 0 \to \mathscr{M} \xrightarrow{\cong} \mathscr{M} \to 0 \to \cdots$. We call $\mathscr{M}^m_\bullet$ and $\mathscr{M}^t_\bullet$, respectively, the *minimal* and the *trivial part* of the complex $\mathscr{M}_\bullet$.

One can easily see that

- the embedding $\mathscr{M}^m_\bullet \to \mathscr{M}_\bullet$ is a quasi-isomorphism;
- any quasi-isomorphism of minimal complexes is indeed an isomorphism.

It is easy to see that $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ is isomorphic to $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}^m_\bullet, i^m)$, where $i^m : \tilde{\mathscr{M}}^m_\bullet \to \bar{\mathscr{F}}^m_\bullet$ is the component of $i : \tilde{\mathscr{M}}_\bullet \to \bar{\mathscr{F}}_\bullet$. By what was said above, i^m is an isomorphism of complexes. Now observe that complexes $\bar{\tilde{\mathscr{F}}}_\bullet$, $\mathscr{M}^m$ are locally free. Hence $\bar{\tilde{\mathscr{F}}}^t_\bullet$ is locally free, too. So it is a direct sum of complexes of type $\tilde{\mathscr{A}}^n_x \xrightarrow{\cong} \tilde{\mathscr{A}}^n_x$. Lift each of them to $\mathscr{A}^n_x \xrightarrow{\text{id}} \mathscr{A}^n_x$. Thus we can assume $i : \tilde{\mathscr{M}}_\bullet \to \bar{\tilde{\mathscr{F}}}_\bullet$ to be an isomorphism of complexes. □

Consider the pullback diagram in the abelian category $\mathrm{Com}(\mathrm{Coh}_X)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{J}\tilde{\mathscr{F}}_\bullet \arrow[r] \arrow[d, "\text{id}"] & \mathscr{F}_\bullet \arrow[r, "\Psi"] \arrow[d, "\Phi"] & \mathscr{M}_\bullet \arrow[r] \arrow[d, "\tilde{i}"] & 0 \\
0 \arrow[r] & \mathscr{J}\tilde{\mathscr{F}}_\bullet \arrow[r] & \tilde{\mathscr{F}}_\bullet \arrow[r, "\pi"] & \bar{\tilde{\mathscr{F}}}_\bullet \arrow[r] & 0
\end{tikzcd}$$

Just as in [11], establish that

(1) $\mathscr{F}_\bullet$ is a complex of locally free $\mathscr{O}$-modules;

(2) $(\tilde{\Phi}, \tilde{\Psi}) : (\mathscr{F}_\bullet \otimes_{\mathscr{O}} \tilde{\mathscr{O}}, \mathscr{F}_\bullet \otimes_{\mathscr{O}} \mathscr{A}, i_{\mathscr{F}_\bullet}) \longrightarrow (\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ is an isomorphism in the category of triples.

The following example shows that **F** is not faithful. Let

$$\mathscr{E}_\bullet[-1] = \cdots \longrightarrow 0 \longrightarrow \underbrace{\mathscr{E}}_{1} \longrightarrow 0 \longrightarrow \cdots$$

and

$$\mathscr{F}_\bullet = \cdots \longrightarrow 0 \longrightarrow \underbrace{\mathscr{F}}_{0} \longrightarrow 0 \cdots$$

be two complexes ($\mathscr{F}$ and $\mathscr{E}$ are vector bundles). We have $\mathrm{Hom}(\mathscr{F}_\bullet, \mathscr{E}_\bullet[1]) = \mathrm{Ext}^1(\mathscr{F}, \mathscr{E})$. The map

$$\begin{aligned}\mathrm{Hom}_{D^-(\mathrm{Coh}_X)}(\mathscr{F}_\bullet, \mathscr{E}_\bullet[1]) = \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}) &\longrightarrow \mathrm{Ext}^1_{\tilde{\mathscr{O}}_X}(\tilde{\mathscr{F}}, \tilde{\mathscr{E}}) \\ &= \mathrm{Hom}_{TC_X}\big(\mathbf{F}(\mathscr{F}), \mathbf{F}(\mathscr{G})\big)\end{aligned}$$

is not always a monomorphism: for instance, $\mathrm{Ext}^1(\mathscr{O}_{\mathbf{P}^1}, \mathscr{O}_{\mathbf{P}^1}) = H^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}) = \mathbf{0}$, but $H^1(X, \mathscr{O}_X) = 1$ for curves of arithmetic genus 1. So our functor is not faithful.

5. Coherent sheaves on a rational curve with one node

Let us consider first the case of the rational curve with one simple node. Suppose that its equation is $zy^2 - x^3 - x^2z = 0$. Then its normalization is $\tilde{X} = \mathbf{P}^1$ and we may suppose that the preimages of the singular point are $(0 : 1) = 0$ and $(1 : 0) = \infty$ (see Fig. 10).

What does the result of Section 4 mean? As a data structure, a complex $\mathscr{F}_\bullet$ from the derived category $D^-(\mathrm{Coh}_X)$ is uniquely defined by some triple $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$.

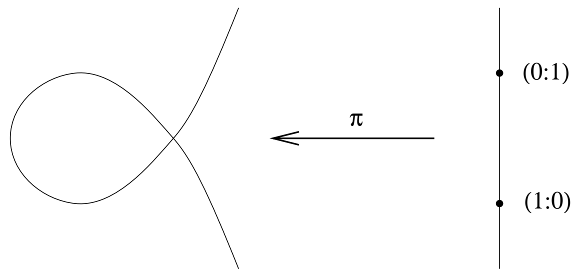

Figure 10

What is $\tilde{\mathscr{F}}_\bullet$? The category $\mathrm{Coh}_{\mathbf{P1}}$ has global dimension 1. It means (see [7]) that indecomposable objects of $D^-(\mathrm{Coh}_{\mathbf{P1}})$ are

$$\mathscr{E}_n[r] : \cdots \longrightarrow 0 \longrightarrow \underbrace{\mathscr{O}_{\mathbf{P1}}(n)}_{r} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

and

$$\mathscr{T}_{kx}[s] : \cdots \longrightarrow 0 \longrightarrow \mathscr{O}_{\mathbf{P1}}(-kx) \hookrightarrow \underbrace{\mathscr{O}_{\mathbf{P1}}}_{s} \longrightarrow 0 \longrightarrow \cdots .$$

A complex $\tilde{\mathscr{F}}_\bullet$ is just a direct sum

$$\tilde{\mathscr{F}}_\bullet \cong \bigoplus \big((\mathscr{E}_n[r]^{N_{n,r}}) \oplus (\mathscr{T}_{kx}[s]^{M_{x,k,s}})\big).$$

Now let us explain what $\mathscr{M}_\bullet$ and i are. $\mathscr{A}$ is a skyscraper sheaf $\mathbf{k}_p$ (with the stalk $\mathbf{k}$ at the singular point p), and $\tilde{\mathscr{A}} = (\mathbf{k} \times \mathbf{k})_p$. This means that the categories $\mathrm{Coh}_{\tilde{\mathscr{A}}}$ and $\mathrm{Coh}_{\mathscr{A}}$ are semisimple. So $\mathscr{M}_\bullet \otimes_{\mathscr{A}} \tilde{\mathscr{A}} \cong (H_\bullet(\tilde{\mathscr{M}}_\bullet), 0)$ and we get, moreover, a commutative diagram

$$\begin{tikzcd} \tilde{\mathscr{M}}_\bullet \arrow[r, "i"] \arrow[d] & \bar{\mathscr{F}}_\bullet \arrow[d] \\ H_\bullet(\tilde{\mathscr{M}}_\bullet) \arrow[r, "H_\bullet(i)"] & H_\bullet(\bar{\mathscr{F}}_\bullet) \end{tikzcd}$$

The map $H_k(i) : H_k(\tilde{\mathscr{M}}_\bullet) \longrightarrow H_k(\bar{\mathscr{F}}_\bullet)$ is simply a map of two $(\mathbf{k} \times \mathbf{k})$-modules. This implies that $H_k(i)$ is given by two matrices $H_k(i|0)$ and $H_k(i|\infty)$ (intuitively, one corresponds to the point zero, the other to ∞). Moreover, both of these matrices have the same size and are nondegenerate.

Consider the images of the complexes $\mathscr{E}[n]$ and $\mathscr{T}_{kx}[n]$ after applying $\bigotimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}}$:

$$\tilde{\mathscr{O}}(n) \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} = \mathbf{k}_0 \oplus \mathbf{k}_\infty.$$

Let $x = 0 = (0:1)$. Then

$$\mathscr{T}_{kx} \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} = \big(\mathscr{O}_{\mathbf{P}^1}(-kx) \hookrightarrow \mathscr{O}_{\mathbf{P}^1}\big) \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} = \Big(\mathbf{k}_0 \oplus \mathbf{k}_\infty \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathbf{k}_0 \oplus \mathbf{k}_\infty\Big).$$

This complex is quasi-isomorphic to

$$\mathbf{k}_0 \xrightarrow{0} \mathbf{k}_0.$$

Let $x = \infty = (1:0)$. In the same way, we obtain

$$\mathscr{T}_{kx} \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} \cong \mathbf{k}_\infty \xrightarrow{0} \mathbf{k}_\infty.$$

All other skyscraper sheaves vanish after tensoring with $\bigotimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}}$:

$$\big(\mathscr{O}_{\mathbf{P}^1}(-kx) \hookrightarrow \mathscr{O}_{\mathbf{P}^1}\big) \otimes_{\tilde{\mathscr{O}}} \tilde{\mathscr{A}} = \Big(\mathbf{k}_0 \oplus \mathbf{k}_\infty \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathbf{k}_0 \oplus \mathbf{k}_\infty\Big) \cong 0.$$

Let us consider morphisms in the derived category $D^b(\mathrm{Coh}_{\tilde{\mathscr{O}}}) = D^b(\mathrm{Coh}_{\mathbb{P}^1})$.

(1) Let $n < m$. Then we have a morphism $\tilde{\mathscr{O}}(n) \xrightarrow{p} \tilde{\mathscr{O}}(m)$, given by a homogeneous form $p = p(x_0, x_1)$ of degree $m - n$. The induced map modulo the conductor is

$$\mathbf{k}_0 \oplus \mathbf{k}_\infty \xrightarrow{\mathrm{diag}(a,b)} \mathbf{k}_0 \oplus \mathbf{k}_\infty,$$

where $a = p(0:1), b = p(1:0)$.

(2) Let $n < m$. Then we have a morphism $\mathscr{T}_{m0} \longrightarrow \mathscr{T}_{n0}$:

$$\begin{tikzcd} \tilde{\mathscr{O}}(-m) \arrow[r, "x_0^m"] \arrow[d, "x_0^{(m-n)}p"'] & \tilde{\mathscr{O}} \arrow[d, "p"] \\ \tilde{\mathscr{O}}(-n) \arrow[r, "x_0^n"] & \tilde{\mathscr{O}} \end{tikzcd}$$

where p is a homogeneous form of degree $m - n$, which induces

$$\begin{tikzcd} \mathbf{k}_0 \arrow[r, "0"] \arrow[d, "0"] & \mathbf{k}_0 \arrow[d, "\lambda"] \\ \mathbf{k}_0 \arrow[r, "0"] & \mathbf{k}_0 \end{tikzcd}$$

where $\lambda = p(0:1)$.

(3) Let $n < m$. We also have a morphism $\mathscr{T}_{n0} \longrightarrow \mathscr{T}_{m0}$:

$$\begin{tikzcd} \tilde{\mathscr{O}}(-m) \arrow[r, "x_0^m"] & \tilde{\mathscr{O}} \\ \tilde{\mathscr{O}}(-m) \arrow[u, "p"] \arrow[r, "x_0^n"] \arrow[d] & \tilde{\mathscr{O}}(n-m) \arrow[u, "x_0^{(m-n)}p"'] \arrow[d] \\ \tilde{\mathscr{O}}(-n) \arrow[r, "x_0^n"] & \tilde{\mathscr{O}} \end{tikzcd}$$

where this map is a composition of an inverse to a quasi-isomorphism and a map of complexes. It induces

$$\begin{tikzcd} \mathbf{k}_0 \arrow[r, "0"] & \mathbf{k}_0 \\ \mathbf{k}_0 \arrow[u, "\lambda"] \arrow[r, "0"] & \mathbf{k}_0 \arrow[u, "0"'] \end{tikzcd}$$

where $\lambda = p(0:1)$.

(4) In the same way, we can consider all other cases. Let $n_2 < n_1$, $m_1 > m_2$ be natural numbers, and let $k_2 < k_1$ be integers. Then we have a chain of morphisms, which induce nonzero maps modulo the conductor:

$$\mathscr{T}_{n_1 0}[1] \longrightarrow \mathscr{T}_{n_2 0}[1] \longrightarrow \tilde{\mathscr{O}}(k_2) \longrightarrow \tilde{\mathscr{O}}(k_1) \longrightarrow \mathscr{T}_{m_1 0} \longrightarrow \mathscr{T}_{m_2 0} :$$

$$\begin{tikzcd} & \tilde{\mathscr{O}}(k_2) \arrow[r, "x_0^{n_2}"] \arrow[d] & \tilde{\mathscr{O}}(n_2+k_2) \arrow[d] \\ & \tilde{\mathscr{O}}(k_2) \arrow[r, "x_0^{n_1}"] \arrow[d] & \tilde{\mathscr{O}}(n_1+k_2) \\ & \tilde{\mathscr{O}}(k_2) \arrow[d] & \\ & \tilde{\mathscr{O}}(k_1) \arrow[d] & \\ \tilde{\mathscr{O}}(-m_1) \arrow[r, "x_0^{m_1}"] \arrow[d] & \tilde{\mathscr{O}} \arrow[d] & \\ \tilde{\mathscr{O}}(-m_2) \arrow[r, "x_0^{m_2}"] & \tilde{\mathscr{O}} & \end{tikzcd}$$

(5) The same holds, of course, at the point $(1:0)=\infty$.

(6) Finally, let us consider the case of endomorphisms of indecomposable objects of $D^b(\mathrm{Coh}_{\tilde{\mathscr{O}}})$. An endomorphism of $\tilde{\mathscr{O}}(n)$ is scalar and hence induces

$$\mathbf{k}_0 \oplus \mathbf{k}_\infty \xrightarrow{\mathrm{diag}(a,a)} \mathbf{k}_0 \oplus \mathbf{k}_\infty, \quad a \in \mathbf{k}.$$

An endomorphism of $(\mathscr{O}_{\mathbf{P^1}}(-k0) \hookrightarrow \mathscr{O}_{\mathbf{P^1}})$ always induces a map of the type

$$\begin{tikzcd}
\mathbf{k}_0 \arrow[r, "0"] \arrow[d, "\lambda"] & \mathbf{k}_0 \arrow[d, "\lambda"] \\
\mathbf{k}_0 \arrow[r, "0"] & \mathbf{k}_0
\end{tikzcd}$$

Remark 5.1
Let

$$\tilde{\mathscr{F}}_\bullet \cong \bigoplus \big((\mathscr{E}_n[r]^{N_{n,r}}) \oplus (\mathscr{T}_{kx}[s]^{M_{x,k,s}}) \big)$$

be a direct sum decomposition of an object $\tilde{\mathscr{F}}_\bullet \in \mathrm{Ob}(D^b(\mathrm{Coh}_{\tilde{\mathscr{O}}}))$. An endomorphism $\Phi : \tilde{\mathscr{F}}_\bullet \longrightarrow \tilde{\mathscr{F}}_\bullet$ is an isomorphism if and only if the induced endomorphism of each component $\mathscr{E}_n[r]^{N_{n,r}}$ and $\mathscr{T}_{kx}[s]^{M_{x,k,s}}$ is an isomorphism.

Choose trivializations of each component of $\tilde{\mathscr{F}}_\bullet$ in neighborhoods of the points zero and ∞. They induce some basis in $H_\bullet(\bar{\mathscr{F}}_\bullet)$. Choose some basis in $H_k(\mathscr{M}_\bullet)$. With respect to such a choice, the map $H_\bullet(i)$ is given by a collection of matrices in Figure 11.

There are two types of blocks: those which came from vector bundles and those which came from skyscraper sheaves or, equivalently, from complexes $\mathscr{T}_{k0}[s]$ and $\mathscr{T}_{k\infty}[s]$. The blocks are numbered by integers and natural numbers, respectively. This numbering defines some "weights" of the blocks of vertical matrices. Blocks corresponding to the same skyscraper or vector bundle are called conjugate. Conjugate blocks have the same number of rows. Indeed, all but finitely many blocks of $H_k(i|0)$ and $H_k(i|\infty)$ have zero size. But if one of the conjugate blocks is nonempty, then the other one is nonempty, too.

Now we should answer the following question: which triples $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, H_\bullet(i))$ correspond to isomorphic complexes $\mathscr{F}_\bullet$? Surely, we have to consider the automorphisms of $\tilde{\mathscr{F}}_\bullet$ and $\mathscr{M}_\bullet$ and look at what they induce in homologies. As a result, we get the following matrix problem:

(1) we can do any simultaneous elementary transformations of columns of the matrices $H_k(i|0)$ and $H_k(i|\infty)$;

(2) we can do any simultaneous transformations of rows inside conjugate blocks;

(3) we can add a scalar multiple of any row from a block with lower weight to any row of a block of a higher weight (inside the big matrix, of course); these

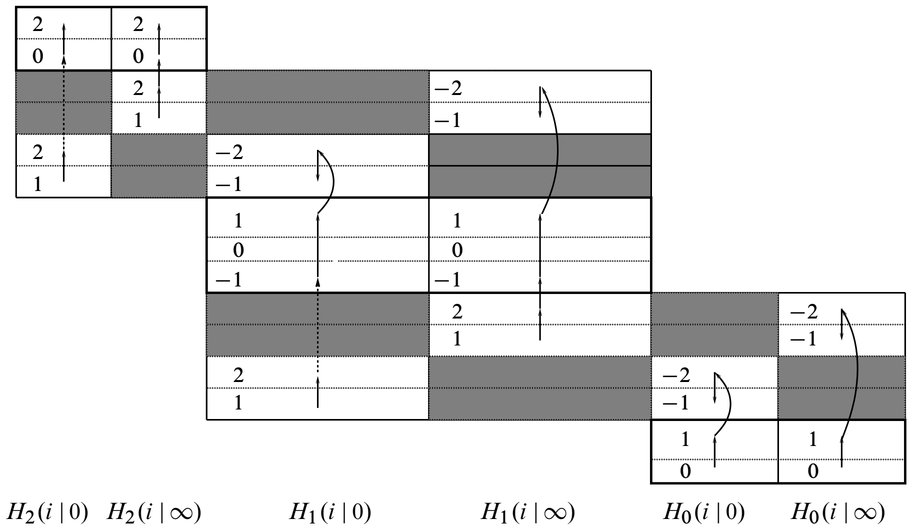

Figure 11

transformations can proceed independently inside $H_k(i|0)$ and $H_k(i|\infty)$ (see Sec. 6 for more details).

These types of problems are well known in representation theory. They first appeared in the work of Nazarova and Roiter [19] about the classification of $\big(\mathbf{k}[[x, y]]/(xy)\big)$-modules. They are sometimes called Gelfand problems in honor of I. M. Gelfand, who formulated a conjecture (at the International Congress of Mathematics in Nice, 1970) about the structure of Harish-Chandra modules at the singular point of $\mathrm{SL}_2(\mathbb{R})$ (see [14]). This problem was reduced to a matrix problem of this type (see [20]).

The strict categorical formulation and then a solution of this type of problem was done by Nazarova and Roiter [20] and by Bondarenko [4] (see also [21, App. A]). It means that these matrices correspond to the objects of some category and that these objects are isomorphic if and only if one matrix can be transformed into another one by the above set of transformations. Certainly, it is enough to describe the indecomposable objects.

Let us recall a combinatoric of the answer in this case. There are two types of indecomposable objects: bands and strings. We give the definitions in Section 6. Here we want to stress that a band object depends on one continuous parameter and several discrete parameters. A string object depends only on discrete parameters. Let us consider some examples (the same examples we considered in Sec. 3).

Example 5.2
The data $\mathscr{B}(w, 1, \lambda)$ (band), where $w = 3_0 x_0^2 y_1^1\text{-}2_2 x_1^3 y_0^1$, define an object of the bounded category $D^b(\mathrm{Coh}_X)$ which is not a coherent sheaf. The corresponding triple $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ is

$$\tilde{\mathscr{F}}_\bullet = \mathscr{E}_{-2}[-2] \oplus \mathscr{T}_{30}[-1] \oplus \mathscr{T}_{1\infty}[-1] \oplus \mathscr{T}_{20} \oplus \mathscr{T}_{1\infty} \oplus \mathscr{E}_3,$$
$$\mathscr{M}_\bullet = \mathbf{k}^2 \xrightarrow{0} \mathbf{k}^2 \xrightarrow{0} \mathbf{k}^2,$$

and i is given by matrices in Figure 12.

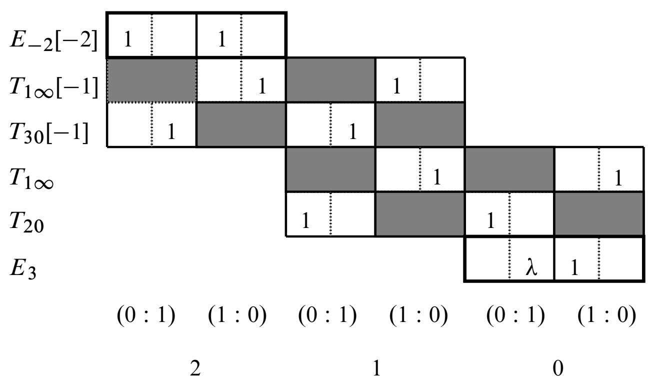

Figure 12

Example 5.3
The data $\mathscr{S}(w)$ (string), where $w = \cdots y_3^1 x_2^1 y_1^1 x_0^n y_0^m x_1^1 y_2^1 x_3^1 \cdots$, define the skyscraper sheaf at a singular point. The corresponding triple $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ is

$$\tilde{\mathscr{F}}_\bullet = \bigoplus_{i=1}^{\infty} (\mathscr{T}_{10}[-i] \oplus \mathscr{T}_{1\infty}[-i]) \oplus \mathscr{T}_{n0} \oplus \mathscr{T}_{m\infty},$$
$$\mathscr{M}_\bullet = \cdots \longrightarrow \mathbf{k}^2 \xrightarrow{0} \mathbf{k}^2 \xrightarrow{0} \mathbf{k},$$

and matrices defining i are in Figure 13.

Example 5.4
The data $\mathscr{B}(w, 2, \lambda)$ (band), where $w = \text{-}3_0 0_0 y_0^1 x_0^2 y_0^4 x_0^5 0_0$, define a mixed sheaf

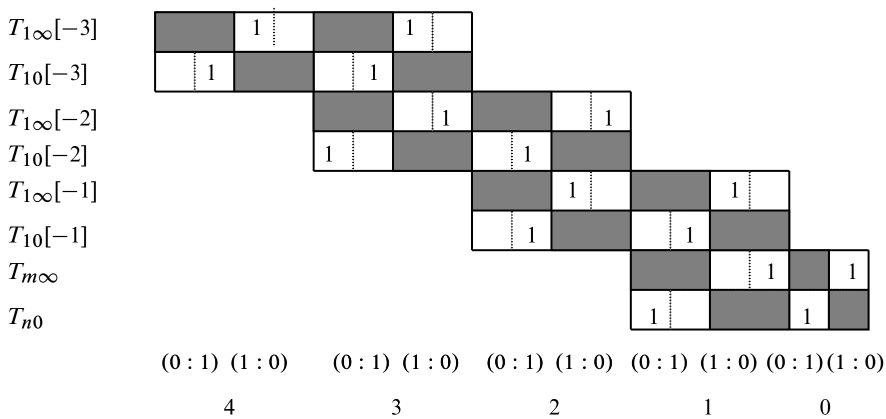

Figure 13

(sheaf that is neither torsion nor torsion-free):

$$\tilde{\mathscr{F}}_\bullet = \mathscr{T}_{20} \oplus \mathscr{T}_{50} \oplus \mathscr{T}_{1\infty} \oplus \mathscr{T}_{4\infty} \oplus \mathscr{E}_{-3} \oplus \mathscr{E}_0^2,$$
$$\mathscr{M}_\bullet = \mathbf{k}^2 \xrightarrow{0} \mathbf{k}^5,$$

and matrices are in Figure 14.

Example 5.5

The data $\mathscr{S}(w)$ (string), where $w = \cdots y_2^2 x_1^1\text{-}2_1 y_0^3 0_0 x_0^2 y_1^2 1_2 \cdots$, define an object of the category $D^-(\mathrm{Coh}_X)$ which is not an object of the bounded derived category $D^b(\mathrm{Coh}_X)$:

$$\tilde{\mathscr{F}}_\bullet = \cdots \longrightarrow \mathscr{E}_1[-2] \oplus \mathscr{T}_{10}[-1] \oplus \mathscr{T}_{2\infty}[-1] \oplus \mathscr{E}_{-2}[-1] \oplus \mathscr{T}_{20} \oplus \mathscr{T}_{3\infty} \oplus \mathscr{E}_0,$$
$$\mathscr{M}_\bullet = \cdots \longrightarrow \mathbf{k}^3 \xrightarrow{0} \mathbf{k}^3 \xrightarrow{0} \mathbf{k}^2,$$

and matrices are in Figure 15 on page 212.

Let us now consider a general case.

6. Reduction to the matrix problem

Consider the case when X is a configuration of projective lines of type $\tilde{\mathrm{A}}_n$ ($n \geq 1$); if $n = 1$, then X is just an irreducible rational curve with one simple node. At the end of this section we explain the difference that occurs in the case of A_n (which is a bit simpler).

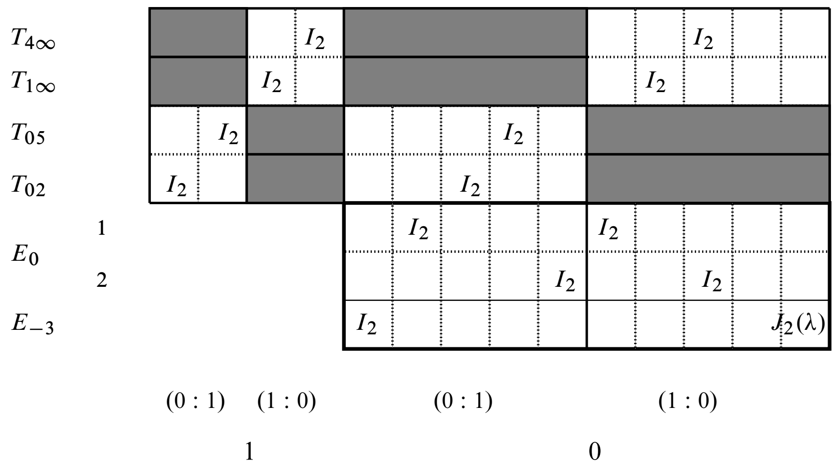

Figure 14

We keep the notation of Section 3; in particular, $\pi : \widetilde{X} \to X$ is the normalization, and S is the set of singular points of X. $\widetilde{X}$ consists of n irreducible components $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$. We identify them with irreducible components $X_1, X_2, \dots, X_n$ of X. The set S consists of n points $x_1, x_2, \dots, x_n$. These points and components can be arranged so that, for any $i < j$,

$$X_i \cap X_j = \begin{cases} x_i & \text{if } j = i+1, \\ x_n & \text{if } i = 1,\ j = n, \\ \emptyset & \text{in all other cases.} \end{cases}$$

For any integer k, we set $x_{n+k} = x_k,\ X_{n+k} = X_k,\ \widetilde{X}_{n+k} = \widetilde{X}_k$. Set also $\pi^{-1}(x_i) = \{x_i', x_i''\}$ so that $x_i' \in \widetilde{X}_i,\ x_i'' \in \widetilde{X}_{i+1}$.

The category $\mathrm{Coh}_{\widetilde{X}}$ is equivalent to $\prod_{i=1}^{n} \mathrm{Coh}_{\widetilde{X}_i}$, and the same is true of their derived categories.

Choose the coordinates on each line X_i in such a way that the preimages of the singular points are either $(0:1)$ or $(1:0)$. We can interpret coherent $\tilde{\mathscr{O}}_i$-modules just as $\mathrm{Coh}_{\mathbf{P}^1}$. By $\mathscr{J}_i$ we denote the restriction of the conductor $\mathscr{J}$ onto X_i. Then $\mathscr{J}_i$ is just the ideal sheaf of points $(0:1)$ and $(1:0)$. Both $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are skyscraper sheaves with support in the singular points of X. The canonical morphism $\mathscr{A} \longrightarrow \tilde{\mathscr{A}}$ is then the diagonal morphism

$$\mathbf{k} \times \mathbf{k} \times \cdots \times \mathbf{k} \longrightarrow (\mathbf{k} \times \mathbf{k}) \times (\mathbf{k} \times \mathbf{k}) \times \cdots \times (\mathbf{k} \times \mathbf{k}).$$

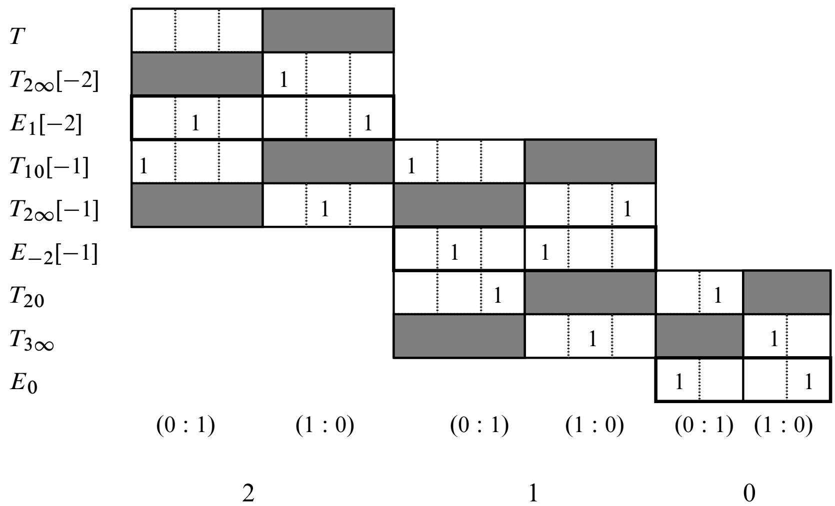

Figure 15

Let $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ be some triple. Then $\tilde{\mathscr{F}}_\bullet \cong \tilde{\mathscr{F}}_{1\bullet} \oplus \tilde{\mathscr{F}}_{2\bullet} \oplus \cdots \oplus \tilde{\mathscr{F}}_{n\bullet}$, where $\tilde{\mathscr{F}}_{i\bullet} \in D^-(\mathrm{Coh}_{X_i}) = D^-(\mathrm{Coh}_{\mathbf{P}^1})$.

How do we get a matrix problem in this case? $\mathrm{Coh}_{\mathbf{P}^1}$ has homological dimension 1, which implies that every indecomposable object of $D^-(\mathrm{Coh}_{\mathbf{P}^1})$ is isomorphic to some object of the type

$$\cdots \longrightarrow 0 \longrightarrow \underbrace{\mathscr{F}}_{i} \longrightarrow 0 \longrightarrow \cdots,$$

where $\mathscr{F} \in \mathrm{Ob}(\mathrm{Coh}_{\mathbf{P}^1})$ is indecomposable. Indecomposable objects of $\mathrm{Coh}_{\mathbf{P}^1}$ are known: line bundles $\mathscr{O}_{\mathbf{P}^1}(n)$ and skyscraper sheaves $\mathscr{O}_{\mathbf{P}^1,x}/\mathfrak{m}_x^n$.

The skyscraper sheaf $\mathscr{O}_{\mathbf{P}^1,x}/\mathfrak{m}_x^n$ has a locally free resolution

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}^1}(-n) \longrightarrow \mathscr{O}_{\mathbf{P}^1} \longrightarrow \mathscr{O}_{\mathbf{P}^1,x}/\mathfrak{m}_x^n \longrightarrow 0,$$

which means that, in the derived category, $(\mathscr{O}_{\mathbf{P}^1}(-n) \longrightarrow \mathscr{O}_{\mathbf{P}^1}) \cong \mathscr{O}_x/\mathfrak{m}_x^n$ holds.

Choose trivializations in neighborhoods of singular points of each component of the complex $\tilde{\mathscr{F}}$. The map $H_k(i) : H_k(\tilde{\mathscr{M}}_\bullet) \longrightarrow H_k(\bar{\mathscr{F}}_\bullet)$ is given by n matrices, corresponding to singular points of X. Each of these matrices itself consists of two nondegenerate components of the same size.

The question is, which transformations can we do with the matrices defining the homology. From the definition of the category of triples, it follows that we have to consider automorphisms $\Phi : \tilde{\bar{\mathscr{F}}}_\bullet \longrightarrow \tilde{\bar{\mathscr{F}}}_\bullet$ and $\varphi : \mathscr{M}_\bullet \longrightarrow \mathscr{M}_\bullet$ which make the diagram

$$\begin{tikzcd} H_\bullet(\tilde{\mathscr{M}}_\bullet) \arrow[r, "H_\bullet(i)"] \arrow[d, "H_\bullet(\tilde{\varphi})"] & H_\bullet(\bar{\mathscr{F}}_\bullet) \arrow[d, "H_\bullet(\bar{\Phi})"] \\ H_\bullet(\tilde{\mathscr{M}}_\bullet) \arrow[r, "H_\bullet(i)"] & H_\bullet(\bar{\mathscr{F}}_\bullet) \end{tikzcd}$$

commutative.

Since the description of indecomposable objects can be done by ignoring the shifts, we restrict ourselves to the complexes whose highest nonzero component is a zero component.

To each configuration of projective lines we can associate a partially ordered set. Let $\omega_{-1} < \omega_0 < \omega_1$ be three cardinal numbers. (This means that $n\omega_{-1} < m\omega_0 < k\omega_1,\ \forall n, m, k \in \mathbb{Z}$.) The algorithm is now the following: consider the set of pairs (L, a), where L is a component of $\tilde{X}$ and $a \in L$ is some preimage of the singular point.

(1) To an object

$$\mathscr{T}_{na}[s] : \cdots \longrightarrow 0 \longrightarrow \mathscr{O}_{\mathbf{P}^1}(-na) \hookrightarrow \underbrace{\mathscr{O}_{\mathbf{P}^1}}_{s} \longrightarrow 0 \longrightarrow \cdots$$

we set in correspondence two symbols $E_{(L,a)}(s, n\omega_{-1})$ and $E_{(L,a)}(s + 1, -n\omega_1)$.

(2) An object

$$\mathscr{E}_n[r] : \cdots \longrightarrow 0 \longrightarrow \underbrace{\mathscr{O}_{\mathbf{P}^1}(n)}_{r} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

corresponds to $E_{(L,a)}(r, n\omega_0)$.

Note that skyscraper sheaves $\mathscr{T}_{kx}[s]$, where x is not a preimage of a singular point, do not take part in the "gluing" of complexes. (They become equal to zero after tensoring with the conductor and taking a homology.)

On the sets $E_{(L,a)}(i) = \bigcup E_{(L,a)}(i, *)$, we get a total order in the following way: we say that $E_1 < E_2$ if there is a morphism of corresponding objects of the derived category $D^-(\mathrm{Coh}_{\mathbf{P}^1})$ inducing a nonzero evaluation map in a. To each triple (L, a, i) corresponds also a set $F_{(L,a)}(i)$ consisting of one element.

Consider the union of all points

$$\mathbf{E} \cup \mathbf{F} = \Big(\bigcup_{(L,a),i} E_{(L,a)}(i) \Big) \cup \Big(\bigcup_{(L,a),i} F_{(L,a)}(i) \Big).$$

In this set, let us introduce the following equivalence relations:

(1) $E_{(L,a)}(i, n\omega_{-1}) \sim E_{(L,a)}(i+1, -n\omega_1)$, $i \geq 0$, $n \geq 1$ (which means that the points are coming from the same skyscraper sheaf);

(2) $E_{(L,a)}(i, m\omega_0) \sim E_{(L',a)}(i, m\omega_0)$, $i \geq 0$, $m \in \mathbb{Z}$ (which means that the points are coming from the same vector bundle);

(3) $F_{(L,a)}(i) \sim F_{(L,a')}(i)$, $i \geq 0$.

Example 6.1
We have a rational curve with one node (see Fig. 16).

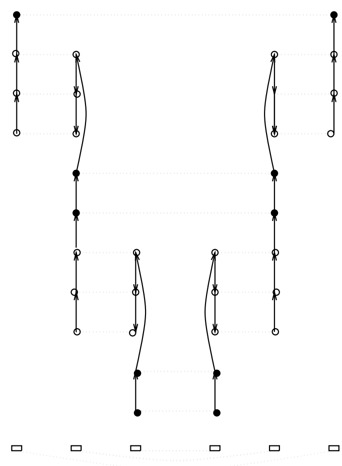

Figure 16

Example 6.2
We have a transversal intersection of two lines at one point (the A_1-case) (see Fig. 17).

Example 6.3
We have a transversal intersection of two lines at two points (the $\tilde{A}_2$-case) (see Fig. 18 on page 216).

What we have is called a bunch of chains (see [5]; see also [11]), which encodes a matrix problem. The number of matrices can be infinite, but this does not disturb the general theory.

Specifically, the matrix problem is the following.

(1) Each triple (L, a, i) (L is a component of $\tilde{X}$, $a \in L$, $i \geq 0$ the integer number) corresponds to some matrix $M(L, a, i)$. These matrices are divided into

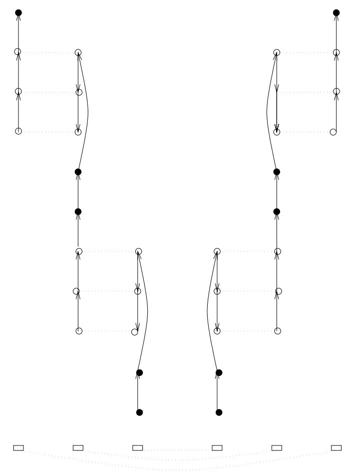

Figure 17

horizontal blocks, numbered by the points of $E_{(L,a)}(i)$. Since $F_{(L,a)}$ consists of only one element, we do not have a vertical division in this case. Indeed, some blocks may have zero size, that is, be empty.

(2) Blocks corresponding to conjugate points from **E** have an equal number of rows; blocks corresponding to conjugate points from **F** have an equal number of columns.

(3) We have a partial order on the set of points. Let us say that the horizontal blocks are supplied with some "weights," and the weight of one block is bigger than the weight of another block if the point corresponding to the first block is bigger than the point corresponding to the second one.

(4) We can do the following transformations with our matrices:
   (a) *simultaneous:* elementary transformations with the columns of matrices $M(L, a, i)$ and $M(L', a, i)$;
   (b) *simultaneous:* any elementary transformations inside conjugate blocks;
   (c) *independent:* adding a scalar multiple of any row from a block with lower weight to any row of a block of the higher weight.

In our case, there are some additional restrictions on our matrices.

(1) All big matrices are square and nondegenerate.

(2) If one of the conjugate blocks is nonempty, then the other one is nonempty, too.

There are two types of indecomposable objects: bands and strings.

(1) Band data $\mathscr{B}(w, m, \lambda)$ are given by two discrete parameters, a word w and a natural number m, and one continuous parameter $\lambda \in \mathbf{k}^*$. A word w is just

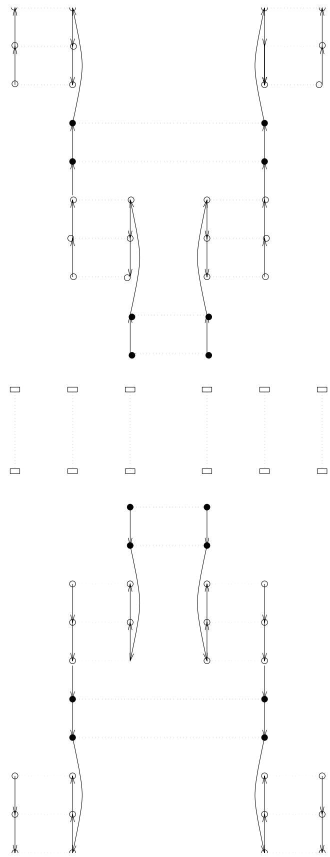

Figure 18

a sequence of points of $\mathbf{E} \cup \mathbf{F}$, $x_1 - x_2 \sim x_3 - x_4 \sim \cdots - x_N$, connected by the symbols of two types, $-$ and $\sim$. The symbol $\sim$ should stand between conjugate points, and $-$ only between a point of the type $E_{(L,a)}(*, i)$ and a point $F_{(L,a)}(i)$. If one link was $-$, then the next one should be $\sim$, and vice versa. In band data a word w should be closed: $x_N \sim x_1$. This means that it can be written as a cycle. We require that w not be a power of some other word.

(2) String data $\mathscr{S}(w)$ depend only on some *full* word w. Full means that w contains each point x_i the same number of times as its conjugate. In the case of a cycle of projective lines, a word w must be infinite. We require, however, that each point x_i appear only a finite number of times and that there be $k \in \mathbb{Z}$ such that w does not contain any elements from $F_{(L,a)}(i)$ for $i \geq k$.

Let us briefly recall the algorithm, giving a concrete description of matrices, corresponding to band and string data.

(1) Let the band data be $\mathscr{B}(w, m, \lambda)$. We count the entrance of each class of conjugate points. Let point x_i occur k_i times. The block corresponding to x_i should be divided into k_i strips. We have a division of a big matrix $M(L, a, i)$ into smaller blocks. Let us now look at the subwords $x_i - x_{i+1}$. Suppose that we have the kth appearance of the class $[x_i]$ and the lth appearance of the class $[x_{i+1}]$. One of the points x_i, x_{i+1} belongs to $\mathbf{E}$, the other to $\mathbf{F}$. If $x_{i+1} \neq x_N$, then we put the identity matrix I_m (here our second discrete parameter appears) in the entry with the coordinates (k, l) (with respect to the subpartition of the $(x_i \times x_{i+1})$-submatrix). If $x_{i+1} = x_N$, then we put, on the corresponding place, the Jordan block $J_m(\lambda)$. All other entries are zero.

(2) Let the string data be $\mathscr{S}(w)$. The algorithm is basically the same as in the case of bands. The only difference is that we have to put in the (1×1)-matrix 1 instead of I_m or $J_m(\lambda)$.

Example 6.4

Let $X = C_2$ be a cycle of two lines. Let L_1, L_2 be its irreducible components, and let a_1, a_2 be its singular points. Consider the following band $B(w, 1, \lambda)$:

$$\begin{aligned} w = {} & E_{(L_1,a_1)}(0, 1\omega_0) - F_{(L_1,a_1)}(0) \sim F_{(L_2,a_1)}(0) \\ & - E_{(L_2,a_1)}(0, -2\omega_1) \sim E_{(L_2,a_1)}(1, 2\omega_{-1}) \\ & - F_{(L_2,a_1)}(1) \sim F_{(L_1,a_1)}(1) - E_{(L_1,a_1)}(1, 1\omega_{-1}) \sim E_{(L_1,a_1)}(0, -1\omega_1) \\ & - F_{(L_1,a_1)}(0) \sim F_{(L_2,a_1)}(0) - E_{(L_2,a_1)}(0, 0\omega_0) \sim E_{(L_2,a_2)}(0, 0\omega_0) \\ & - F_{(L_2,a_2)}(0) \sim F_{(L_1,a_2)}(0) - E_{(L_1,a_2)}(0, 1\omega_0). \end{aligned}$$

The corresponding triple $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ is

$$\begin{aligned} \tilde{\mathscr{F}}_\bullet &= \big(\tilde{\mathscr{O}}_1(1)\big) \oplus \big(\tilde{\mathscr{O}}_1(-1) \xrightarrow{x} \tilde{\mathscr{O}}_1(0)\big) \\ &\qquad \oplus (\tilde{\mathscr{O}}_2) \oplus \big(\tilde{\mathscr{O}}_2(-2) \xrightarrow{y^2} \tilde{\mathscr{O}}_2(0)\big), \\ \mathscr{M}_\bullet &= \mathbf{k}_{a_1} \xrightarrow{0} \mathbf{k}_{a_1}^2 \oplus \mathbf{k}_{a_2}, \end{aligned}$$

and i is given by matrices in Figure 19.

Let us mention that we can code our word w in a more economical way. First note that expression $E' - F_{(L',a)}(i) \sim F_{(L'',a)}(i) - E''$ just means that local parameters corresponding to E' and E'' have to be glued at the point a. Here E' (E'') belongs to the ith component of a complex of $\tilde{\mathscr{O}}_{L'}$-modules ($\tilde{\mathscr{O}}_{L''}$-modules). So the symbols $F_{(L,a)}$ can be skipped. We represent a complex $\tilde{\mathscr{O}}_L(i)[k]$ just as i_k, $(\tilde{\mathscr{O}}_L(-j) \xrightarrow{x^j}$

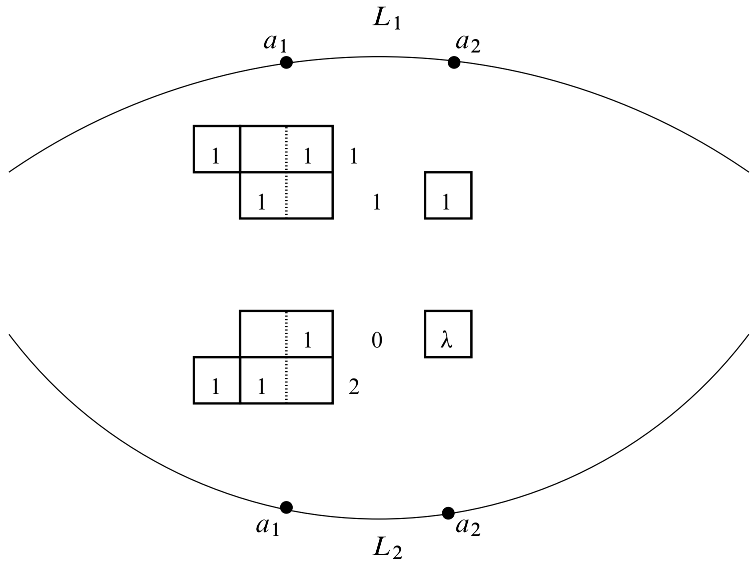

Figure 19

$\tilde{\mathscr{O}}_L)[k]$ by x_k^j, and $(\tilde{\mathscr{O}}_L(-j) \xrightarrow{y^j} \tilde{\mathscr{O}}_L)[k]$ by y_k^j. We can also skip the index L. The word w from Example 6.4 is transformed to the form

$$w = 1_0 y_0^2 x_0^1 0_0,$$

which is the *segment of a cycle* in the notation of Section 3. We just have to determine precisely that 1_0 lives on the first component. Then we can uniquely reconstruct our initial word w. In this representation of w, we just get the description of indecomposable objects given in Section 3.

Remark 6.5
Let $X = C_n, n \geq 3$, let $\mathscr{B}(w, m, \lambda)$ be a band, and let $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ be the corresponding triple. It is possible that different components of

$$\tilde{\mathscr{F}}_\bullet \in \mathrm{Ob}\left(D^-(\mathrm{Coh}_{\tilde{\mathscr{O}}})\right) = \mathrm{Ob}\left(D^-\left(\prod_i \mathrm{Coh}_{\tilde{\mathscr{O}}_i}\right)\right)$$

have different rank. In order to glue them into a complex of locally free $\mathscr{O}$-modules, we have to add some trivial complexes $\tilde{\mathscr{O}}_i \xrightarrow{\mathrm{id}} \tilde{\mathscr{O}}_i$ and then glue everything using the "parallelogram rule" from Section 3. The same concerns strings, of course.

Example 6.6
Let $X = C_3$ be a cycle of projective lines, and let $\mathscr{F}_\bullet = \mathscr{B}(w, 1, \lambda)$, where $w = x_0^2 y_0^1$ $(j(x_0^2) = 1)$, be a skyscraper sheaf at a singular point. Then the normalization of $\mathscr{F}_\bullet$ is

$$\big(\tilde{\mathscr{O}}_1(-2) \xrightarrow{x^2} \tilde{\mathscr{O}}_1\big) \oplus \big(\tilde{\mathscr{O}}_2(-1) \xrightarrow{y} \tilde{\mathscr{O}}_2\big).$$

We add the trivial complex $(\tilde{\mathscr{O}}_3 \xrightarrow{\mathrm{id}} \tilde{\mathscr{O}}_3)$ and glue everything using the "parallelogram rule." We get a resolution

$$\mathscr{B}\big((0,0,0), 1, 1\big) \longrightarrow \mathscr{B}\big((1,2,0), 1, \lambda\big).$$

Remark 6.7
Let X be a chain of projective lines. The only difference with the case of cycles of projective lines is that we have finite strings in this case (for instance, vector bundles are finite strings). The combinatoric of the answer remains the same.

THEOREM 6.8 (see [5])

(1) *All representations $\mathscr{B}(w, m, \lambda)$, $\mathscr{S}(w)$ are indecomposable. Each indecomposable representation is isomorphic either to some band representation $\mathscr{B}(w, m, \lambda)$ or to some string representation $\mathscr{S}(w)$.*

(2) *The only isomorphisms between these objects are*
   - (a) *$\mathscr{S}(w) \cong \mathscr{S}(w^\circ)$, where w° is the opposite word;*
   - (b) *$\mathscr{B}(w, m, \lambda) = \mathscr{B}(w', m, \lambda')$, where $w = a_0 - a_1 \sim a_2 - \cdots - a_m$, $w' = a_{2k} - a_{2k+2} \sim a_{2k+3} - \cdots - a_{2k-1}$ is a cyclic permutation of w, and $\lambda' = \lambda$ for k even and $\lambda' = \lambda^{-1}$ for k odd;*
   - (c) *$\mathscr{B}(w^\circ, m, \lambda) = \mathscr{B}(w, m, \lambda^{-1})$, where w° is the inverse word.*

So the main result of this paper can be formulated as the following.

THEOREM 6.9
Let X be a cycle of projective lines. Then there are the following types of indecomposable objects in $D^-(\mathrm{Coh}_X)$:

- *shifts of skyscraper sheaves $\mathscr{O}_x/\mathfrak{m}_x^n$, where x is a regular point of X,*
- *bands $\mathscr{B}(w, m, \lambda)$, where w is a closed noncyclic word, m is a natural number, and $\lambda \in \mathbf{k}^*$,*
- *strings $\mathscr{S}(w)$, where w is a full word, with properties described above.*

We now want to illustrate the convenience of our description of the complexes in the derived category $D^-(\mathrm{Coh}_X)$. Let $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$ be two objects, given by triples

$(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ and $(\tilde{\mathscr{G}}_\bullet, \mathscr{N}_\bullet, j)$. We can ask which triple corresponds to the tensor product of complexes $\mathscr{F}_\bullet \otimes \mathscr{G}_\bullet$. As one can easily see, it should be $(\tilde{\mathscr{F}}_\bullet \otimes \tilde{\mathscr{G}}_\bullet, \mathscr{M}_\bullet \otimes \mathscr{N}_\bullet, i \otimes j)$.

By the Künneth formula, we have a functorial isomorphism (since the homological dimension is zero)

$$\bigoplus_{k+l=n} \big(H_k(\mathscr{M}_\bullet) \otimes H_l(\mathscr{N}_\bullet)\big) \xrightarrow{\bigoplus(H_k(i)\otimes H_l(j))} H_n(\mathscr{M}_\bullet \otimes \mathscr{N}_\bullet).$$

This means that we can compute the matrices corresponding to the tensor product of complexes.

7. Description of coherent sheaves, vector bundles, torsion-free sheaves, mixed sheaves, and skyscraper sheaves

Now we want to show what corresponds to coherent sheaves. Let complex $\mathscr{F}_\bullet$ be given by a triple $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$. We have to write the conditions $H_i(\mathscr{F}_\bullet) = 0$ $(i \geq 1)$ in the language of matrices. Recall that we have the following diagram:

$$\begin{tikzcd}
\mathscr{J}\tilde{\mathscr{F}}_\bullet \arrow[r] & \tilde{\mathscr{F}}_\bullet \arrow[r] & \bar{\mathscr{F}}_\bullet \arrow[r] & \mathscr{J}\tilde{\mathscr{F}}_\bullet[-1] \\
\mathscr{J}\tilde{\mathscr{F}}_\bullet \arrow[r] \arrow[u, "\mathrm{id}"] & \mathscr{F}_\bullet \arrow[r] \arrow[u, "\Phi"'] & \mathscr{M}_\bullet \arrow[r] \arrow[u, "\tilde{i}"'] & \mathscr{J}\tilde{\mathscr{F}}_\bullet[-1] \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

Write the long exact sequence of homologies associated with this morphism of triangles:

$$\begin{tikzcd}
0 & H_0(\bar{\mathscr{F}}_\bullet) \arrow[l] & H_0(\tilde{\mathscr{F}}_\bullet) \arrow[l] & H_0(\mathscr{J}\tilde{\mathscr{F}}_\bullet) \arrow[l] & H_1(\bar{\mathscr{F}}_\bullet) \arrow[l] & \cdots \arrow[l] \\
0 & H_0(\mathscr{M}_\bullet) \arrow[l] \arrow[u, "H_0(\tilde{i})"] & H_0(\mathscr{F}_\bullet) \arrow[l] \arrow[u] & H_0(\mathscr{J}\tilde{\mathscr{F}}_\bullet) \arrow[l] \arrow[u, "\mathrm{id}"] & H_1(\mathscr{M}_\bullet) \arrow[l] \arrow[u, "H_1(\tilde{i})"] & \cdots \arrow[l]
\end{tikzcd}$$

We get that $H_i(\mathscr{F}_\bullet) = 0$ $(i \geq 1)$ is equivalent to the following two conditions:

(1) composition $H_1(\mathscr{M}_\bullet) \longrightarrow H_1(\bar{\mathscr{F}}_\bullet) \longrightarrow H_0(\mathscr{J}\tilde{\mathscr{F}}_\bullet)$ is a monomorphism;

(2) composition $H_{k+1}(\mathscr{M}_\bullet) \longrightarrow H_{k+1}(\bar{\mathscr{F}}_\bullet) \longrightarrow H_k(\mathscr{J}\tilde{\mathscr{F}}_\bullet)$ is an isomorphism for $k > 0$.

Let us give a combinatorical interpretation of these conditions (for both bands and strings). Let w be a parameter either of $\mathscr{B}(w, m, \lambda)$ or of $\mathscr{S}(w)$. These conditions imply the following:

(1) a word w does not contain any n_k, $k \geq 1$, $n \in \mathbb{Z}$;

(2) w does not contain any x_k^n, y_k^n, $k \geq 2$, $n \in \mathbb{N}$;

(3) a word w does not contain any subword of type $x_k^1 y_k^1$, $y_k^1 x_k^1$, $k \geq 2$.

This gives us the following description of coherent sheaves:

(1) all the bands $\mathscr{B}(w, m, \lambda)$ such that the word w does not contain letters n_k, x_k^i, y_k^j with $k \geq 1$;

(2) strings $\mathscr{S}(w)$ with the following properties:

  (a) there are no letters x_k^n, y_k^n, $k \geq 2$, $n \in \mathbb{N}$;

  (b) w does not contain any subword of type $x_k^1 y_k^1$, $y_k^1 x_k^1$, $k \geq 2$; note that the condition of fullness of w together with the restrictions above imply that w is infinite and looks like $\cdots x_3^1 y_2^1 x_1^1 u y_1^1 x_2^1 y_3^1 \cdots$, where the subword u contains only letters a with $i(a) = 0$.

In a similar way, we can describe the *bounded* derived category $D^b(\mathrm{Coh}_X)$; the conditions $H_i(\mathscr{F}_\bullet)$, $i \gg 0$, can also be described in a similar way.

In particular, we get a description of

(1) *vector bundles* (we get just the matrix problem from the work of Drozd and Greuel [11]): bands $\mathscr{B}(w, m, \lambda)$ with w not containing x_i^l, y_i^m $(i \geq 1)$, l, m arbitrary;

(2) *skyscraper sheaves:* bands and strings defining a coherent sheaf and not containing $n_i, i \geq 0, n \in \mathbb{Z}$. (This follows from the observation that $\mathscr{F}$ and $\mathscr{F} \otimes_{\mathscr{O}} \tilde{\mathscr{O}}$ have the same support.)

We see that for a coherent sheaf $\mathscr{F}$ either we have $\mathscr{T}or^i_{\mathscr{O}}(\mathscr{F}, \mathscr{A}) = 0$ for $i > 1$ or it is nonzero for all $i \geq 2$. As a corollary, we see that the homological dimension of an object of Coh_X is either 0 or 1 or ∞ (which coincides with the result of the Auslander-Buchsbaum formula).

We are going to do the last step in our classification: we describe, among all coherent sheaves, torsion-free sheaves. Those that are not vector bundles have infinite homological dimension, and hence we should look for them among strings. Let $\mathscr{F}$ be a coherent sheaf on X. It is torsion-free if and only if all its localizations $\mathscr{F}_x$ are torsion-free $\mathscr{O}_{X,x}$-modules. At regular points, this condition is obvious. But we go further; $\mathscr{F}_x$ is a torsion-free $\mathscr{O}_{X,x}$-module if and only if its completion $\hat{\mathscr{F}}_x$ is a torsion-free $\hat{\mathscr{O}}_{X,x}$-module. But in our case, if x is singular, then $\hat{\mathscr{O}}_{X,x} = \mathbf{k}[[x, y]]/(xy)$. The indecomposable torsion-free modules are known in this case; they are either $\mathbf{k}[[x]]$ or $\mathbf{k}[[y]]$ or the regular module $\mathbf{k}[[x, y]]/(xy)$ (see [3]).

Now, let us mention that in the same way we have dealt with curves, we can deal with the local ring $\mathbf{k}[[x, y]]/(xy)$. Namely, we consider its normalization $\mathbf{k}[[x]] \times \mathbf{k}[[y]]$ and the conductor $J = (x, y)$, and we just repeat the construction of the category of triples. As a result, we get the matrix problem in Figure 20 (see also the appendix).

Let $x \in X$ be singular. Consider the functor $\mathrm{Coh}_X \longrightarrow (\mathscr{O}_{X,x} - \mathrm{mod}) \longrightarrow (\mathbf{k}[[x, y]]/(xy) - \mathrm{mod})$ (composition of the localization and completion). This functor is exact and so induces the functor between the derived categories $D^-(\mathrm{Coh}_X) \longrightarrow$

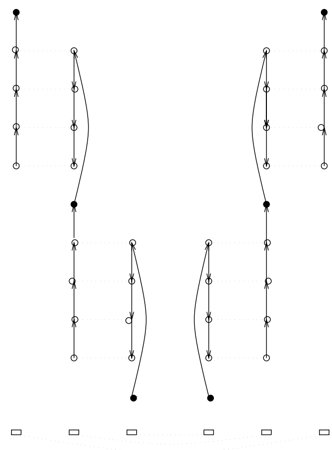

Figure 20

$D^-(\mathbf{k}[[x,y]]/(xy)-\mathrm{mod})$. What does it look like on triples? Obviously, $(\tilde{\mathscr{F}}_\bullet, \mathscr{M}_\bullet, i)$ is mapped to $(\widehat{(\tilde{\mathscr{F}}_\bullet)}_x, \widehat{(\mathscr{M}_\bullet)}_x, i_x)$. So the image of the triple is described by the same matrices! But surely there is one important difference: blocks corresponding to vector bundles are united, and there are no relations between them anymore. But we know how the modules $\mathbf{k}[[x]]$, $\mathbf{k}[[y]]$, and $\mathbf{k}[[x,y]]/(xy)$ are given in the language of triples. Let us denote $T_{nx} : \mathbf{k}[[x]] \xrightarrow{x^n} \mathbf{k}[[x]]$, $T_{ny} : \mathbf{k}[[y]] \xrightarrow{y^n} \mathbf{k}[[y]]$. Then $\mathbf{k}[[x]]$ is given by the normalization

$$\mathbf{k}[[x]] \oplus \Big(\bigoplus_{i=0}^{\infty} T_{1y}[-i] \oplus T_{1x}[-i-1] \Big)$$

and matrices are in Figure 21.

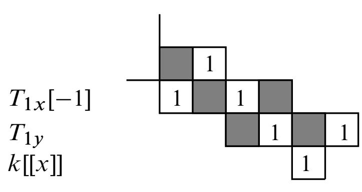

Figure 21

Hence we can deduce the answer: torsion-free sheaves, which are not vector bundles, are strings $\mathscr{S}(w)$, where w does not contain any x_i^k, y_i^k with $i \geq 1$, $k \geq 2$.

Moreover, each x_i^1, y_i^1 can occur in word w at most one time. Hence w looks like

$$\cdots y_3^1 x_2^1 y_1^1 u x_1^1 y_2^2 x_3^3 \cdots,$$

where u contains only letters n_0^i, $i \in \mathbb{Z}$.

8. Concluding remarks

In a recent paper A. Polishchuk [23] showed a connection between the structure of the derived category of the coherent sheaves $D^b(\mathrm{Coh}_X)$, where X is a projective curve of arithmetical genus 1 with nodal singularities, and the trigonometric solutions of the classical Yang-Baxter equation. The main role in this construction is played by the so-called spherical objects.

Definition 8.1 ([25, Def. 2.9])
Let $\mathscr{D}$ be a triangulated category over a field $\mathbf{k}$ such that all spaces $\mathrm{Hom}(X, Y)$ are finite-dimensional. We have $\mathrm{Hom}^i(M, N) := \mathrm{Hom}(M, N[i])$. An object M is called n-spherical if

(1) $\mathrm{Hom}^i(M, M) = 0,\ \forall i \neq 0, n$, and $\mathrm{Hom}^0(M, M) \cong \mathrm{Hom}^n(M, M) \cong \mathbf{k}$;
(2) for all $F \in \mathrm{Ob}(\mathscr{D})$, the composition map $\mathrm{Hom}^i(M, F) \times \mathrm{Hom}^{n-i}(F, M) \longrightarrow \mathrm{Hom}^n(M, M) \cong \mathbf{k}$ is nondegenerate.

One can easily show that the skyscraper sheaves $\mathscr{O}_x/\mathfrak{m}_x$ (x is regular) and simple vector bundles are 1-spherical. Simple vector bundles on tame curves were explicitly described in [6]. An interesting problem is to describe all spherical objects in the derived category $D^b(\mathrm{Coh}_X)$ (see [23]).

Another interesting problem is to describe the group of exact autoequivalences of the derived category $D^b(\mathrm{Coh}_X)$.

A. Appendix: Finite-dimensional $\big(\mathbf{k}[[x, y]]/(xy)\big)$-modules

Let $R = \mathbf{k}[[x, y]]/(xy)$. We can apply our construction to describe the derived category of finitely generated R-modules $D^-(R - \mathrm{mod})$. Again, we consider the embedding $R \longrightarrow \bar{R}$, where $\bar{R} = \mathbf{k}[[x] \times \mathbf{k}[[y]]$ is the normalization of R, $J = (x, y)$. Then $R/J = \mathbf{k}$, $\bar{R}/J = \mathbf{k} \times \mathbf{k}$, and the canonical map $R/J \longrightarrow \bar{R}/J$ is the diagonal embedding. In the same way we dealt with coherent sheaves, we can work with R-modules. Moreover, the computation of the matrix problem is almost the same, so we skip it.

An indecomposable complex form $D^-(R - \mathrm{mod})$ is a gluing of the complexes

$$0 \longrightarrow \mathbf{k}[[x]] \longrightarrow 0, \qquad 0 \longrightarrow \mathbf{k}[[y]] \longrightarrow 0,$$

$$0 \longrightarrow \mathbf{k}[[x]] \xrightarrow{x^n} \mathbf{k}[[x]] \longrightarrow 0, \qquad 0 \longrightarrow \mathbf{k}[[y]] \xrightarrow{y^m} \mathbf{k}[[y]] \longrightarrow 0.$$

It is convenient to describe a gluing of the complexes using the notation from Figure 22.

$$\begin{aligned}
\mathbf{k}[[x]] \xrightarrow{x^n} \mathbf{k}[[x]] &= \bullet \text{———} \circ \xrightarrow{n} \bullet \text{———} \circ\,,\\
\mathbf{k}[[y]] \xrightarrow{y^m} \mathbf{k}[[x]] &= \circ \text{———} \bullet \xrightarrow{m} \circ \text{———} \bullet,\\
\mathbf{k}[[x]] &= \bullet \text{———} \circ\,,\\
\mathbf{k}[[y]] &= \circ \text{———} \bullet
\end{aligned}$$

Figure 22

Example A.1
Consider the gluing diagram in Figure 23. The dashed line shows the "gluing" of

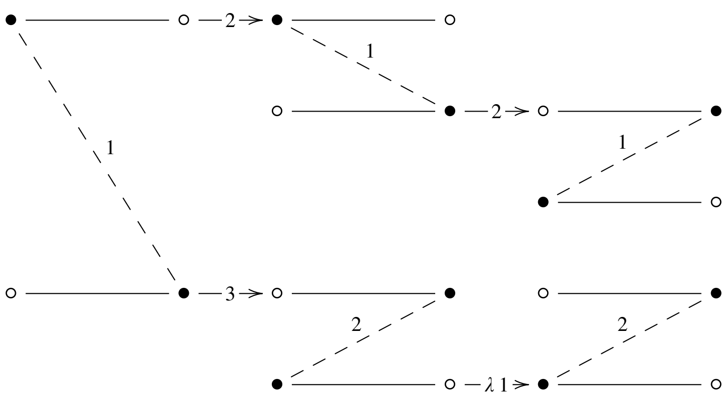

Figure 23

$\mathbf{k}[[x]]$ and $\mathbf{k}[[y]]$ into $\mathbf{k}[[x, y]]/(xy)$. We can easily write down the corresponding complex of R-modules:

$$R \xrightarrow{\begin{pmatrix} x^2 \\ y^3 \end{pmatrix}} R \oplus R \xrightarrow{\begin{pmatrix} y^2 & 0 \\ 0 & \lambda x \end{pmatrix}} R \oplus R.$$

As a consequence of our construction, we get the description of indecomposable finitely generated $\big(\mathbf{k}[[x, y]]/(xy)\big)$-modules.

(1) We have continuous series $\mathscr{M}(\mathbf{d}, s, \lambda)$:

$$0 \longrightarrow R^{sN} \xrightarrow{M(\mathbf{d}, s\lambda)} R^{sN} \longrightarrow \mathscr{M}(\mathbf{d}, s, \lambda) \longrightarrow 0,$$

where

$$M(\mathbf{d}, s, \lambda) = \begin{pmatrix} x^{n_1} I_s & 0 & 0 & \cdots & y^{m_N} J_s(\lambda) \\ y^{m_1} I_s & x^{n_2} I_s & 0 & \cdots & 0 \\ 0 & y^{m_2} I_s & x^{n_3} I_s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y^{m_{N-1}} I_s & x^{n_N} I_s \end{pmatrix}$$

and $\mathbf{d} = (n_1 m_1)(n_2 m_2) \cdots (n_N m_N)$ is a nonperiodic sequence. In the case of $N = 1$, it should be rewritten in the form

$$0 \longrightarrow R^s \xrightarrow{(x^n I_s - y^m J_s(\lambda))} R^{sN} \longrightarrow \mathscr{M}\big((n, m), s, \lambda\big) \longrightarrow 0.$$

(2) We have the first discrete series $\mathscr{S}(\mathbf{d})$:

$$\cdots R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & x \end{smallmatrix}\right)} R^{N+1} \xrightarrow{S(\mathbf{d})} R^N \longrightarrow \mathscr{S}(\mathbf{d}) \longrightarrow 0,$$

where $\mathbf{d} = (n_1 m_1)(n_2 m_2) \cdots (n_N m_N)$,

$$S(\mathbf{d}) = \begin{pmatrix} x^{n_1} & y^{m_1} & 0 & 0 & \cdots & 0 \\ 0 & x^{n_2} & y^{m_2} & 0 & \cdots & 0 \\ 0 & 0 & x^{n_3} & y^{m_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x^{n_N} & y^{m_N} \end{pmatrix}.$$

The continuous and first discrete series are finite-dimensional modules. There are also discrete series of finitely generated R-modules, which are not finite-dimensional.

(3) We have the second discrete series $\mathscr{D}(\mathbf{d})$:

$$0 \longrightarrow R^N \xrightarrow{D(\mathbf{d})} R^{N+1} \longrightarrow \mathscr{D}(\mathbf{d}) \longrightarrow 0,$$

where $\mathbf{d} = (n_1 m_1)(n_2 m_2) \cdots (n_N m_N)$,

$$D(\mathbf{d}) = \begin{pmatrix} x^{n_1} & y^{m_1} & 0 & 0 & \cdots & 0 \\ 0 & x^{n_2} & y^{m_2} & 0 & \cdots & 0 \\ 0 & 0 & x^{n_3} & y^{m_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x^{n_N} & y^{m_N} \end{pmatrix}.$$

(4) There are also discrete series of finitely generated and infinite-dimensional modules of infinite homological dimension, $\mathscr{E}_x(\mathbf{d})$ and $\mathscr{E}_y(\mathbf{d})$ (third discrete series):

$$\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x \end{pmatrix}} R^N \xrightarrow{E_x(\mathbf{d})} R^N \longrightarrow \mathscr{E}_x(\mathbf{d}) \longrightarrow 0,$$

where $\mathbf{d} = (n_1 m_1)(n_2 m_2) \cdots (n_{N-1} m_{N-1}) m_N$,

$$E_x(\mathbf{d}) = \begin{pmatrix} y^{m_1} & 0 & 0 & \cdots & 0 \\ x^{n_1} & y^{m_2} & 0 & \cdots & 0 \\ 0 & x^{n_2} & y^{m_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{n_{N-1}} & y^{m_N} \end{pmatrix},$$

and

$$\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y \end{pmatrix}} R^N \xrightarrow{E_y(\mathbf{d})} R^N \longrightarrow \mathscr{E}_y(\mathbf{d}) \longrightarrow 0,$$

where $\mathbf{d} = (n_1 m_1)(n_2 m_2) \cdots (n_{N-1} m_{N-1}) n_N$,

$$E_y(\mathbf{d}) = \begin{pmatrix} x^{n_1} & 0 & 0 & \cdots & 0 \\ y^{m_1} & x^{n_2} & 0 & \cdots & 0 \\ 0 & y^{m_2} & x^{n_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y^{m_{N-1}} & x^{n_N} \end{pmatrix}.$$

Remark A.2

Let M be a finite-dimensional $\big(\mathbf{k}[[x, y]]/(xy)\big)$-module. Then it is given by a pair of mutually annihilating nilpotent matrices X and Y. It is not difficult to get these matrices from the minimal free resolution of a module. For example, let M be given by its presentation

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} x^2 & \lambda y \\ y^3 & x^2 \end{pmatrix}} R^2 \longrightarrow M \longrightarrow 0.$$

Then M as a $\mathbf{k}$-vectorspace is given by a basis

$$\begin{aligned}&\begin{pmatrix}\bar{1}\\ \bar{0}\end{pmatrix},\begin{pmatrix}\bar{x}\\ \bar{0}\end{pmatrix},\begin{pmatrix}\bar{x}^2\\ \bar{0}\end{pmatrix}\\ &\qquad=\begin{pmatrix}\bar{0}\\ -\bar{y}^3\end{pmatrix},\begin{pmatrix}\bar{0}\\ \bar{1}\end{pmatrix},\begin{pmatrix}\bar{0}\\ \bar{y}\end{pmatrix},\begin{pmatrix}\bar{0}\\ \bar{y}^2\end{pmatrix},\begin{pmatrix}\bar{0}\\ \bar{x}\end{pmatrix},\begin{pmatrix}\bar{0}\\ \bar{x}^2\end{pmatrix}=\begin{pmatrix}-\bar{y}\\ \bar{0}\end{pmatrix}.\end{aligned}$$

The actions of X and Y are just given by multiplication with x and y. Writing down the matrices, we get an answer in the form obtained in the classical paper of Gelfand and Ponomarev [15].

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