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VECTOR BUNDLES OVER NONCOMMUTATIVE NODAL CURVES

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Abstract. We describe vector bundles over a class of noncommutative curves, namely, over noncommutative nodal curves of string type and of almost string type. We also prove that in other cases the classification of vector bundles over a noncommutative curve is a wild problem.

Описано векторні розшарування над деяким класом некомутативних кривих, а саме, над нодальними некомутативними кривими струнного та майже струнного типу. Встановлено також, що в інших випадках класифікація векторних розшарувань над некомутативною кривою є дикою задачею.

Classification of vector bundles over algebraic curves is a popular topic in modern mathematical literature. It is due to their importance for many branches of mathematics and mathematical physics. Vector bundles over the projective line were described by Birkhoff [2] and Grothendieck [11], vector bundles over elliptic curves were classified by Atiyah [1]. In the paper [9] Greuel and the first author described vector bundles over a class of singular curves (line configurations of types A and $\tilde{\mathrm{A}}$) and showed that in all other cases a complete classification of vector bundles is a "wild problem" in the sense of representation theory of algebras.

This paper is devoted to analogous questions for *noncommutative curves*. Perhaps, the first results in this direction were obtained by Geigle and Lenzing [10] who considered the so called *weighted projective lines*. Though the original definition of this paper was in the frames of "usual" (commutative) algebraic geometry, these curves are actually of noncommutative nature. They can be considered as such noncommutative curves that the underlying algebraic curve is a projective line and all localizations of the structure sheaf are hereditary. In some sense, it is the simplest example of noncommutative curves, though their theory is far from being simple.

We consider the "next step," namely the case when the localizations of the structure sheaf are *nodal* in the sense of [5]. In particular, this class contains all line configurations in the sense of [9]. We reduce the description of vector bundles over such curves to the study of a bimodule category in the sense of [8, 9]. Using this reduction, we describe vector bundles in two cases: *string type* and *almost string type*, see Sections 3 and 4. Note that the string type is an immediate generalization of line configurations of types A and Ã. The main tool in this description is a special sort of bimodule problems, namely, the so called *bunches of chains*. Fortunately, these problems are well elaborated and a good description of representations is given in [4]. We also show that in all other cases the classification of vector bundles is a wild problem (Section 5). Thus, in some sense, the question about the "representation type" of the category of vector bundles over noncommutative curves is completely solved.

1. Noncommutative curves, vector bundles and categories of triples

We call a *noncommutative variety* a pair $(X, \mathcal{A})$, where X is an algebraic variety over an algebraically closed field $\Bbbk$ (reduced, but maybe reducible) and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras which is coherent as a sheaf of $\mathcal{O}_X$-modules. We often speak about a "noncommutative variety $\mathcal{A}$" not mentioning explicitly the underlying variety X. We denote by $\mathcal{K}_X$ (or $\mathcal{K}$) the sheaf of total rings of fractions of $\mathcal{O}_X$ (it is locally constant) and set $\mathcal{K}(\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{K}_X$. Without loss of generality we may (and usually will) suppose that $\mathcal{A}$ is *central*, i.e. $\mathcal{O}_{X,x} = \operatorname{center}(\mathcal{A}_x)$ for each $x \in X$. Otherwise we can replace X by the variety $X' = \operatorname{spec} \mathcal{C}$, where $\mathcal{C} = \operatorname{center}(\mathcal{A})$. We define a noncommutative curve as a noncommutative variety $(X, \mathcal{A})$ such that X is a curve (that is all its components are 1-dimensional) and $\mathcal{A}$ is reduced, that is has no nilpotent ideals. A coherent sheaf of $\mathcal{A}$-modules $\mathcal{F}$ is said to be a *vector bundle* over $(X, \mathcal{A})$ if it is *locally projective*, i.e. the $\mathcal{A}_x$-module $\mathcal{F}_x$ is projective for every $x \in X$. We denote by $\mathrm{VB}(X, \mathcal{A})$ or by $\mathrm{VB}(\mathcal{A})$ the category of vector bundles over $(X, \mathcal{A})$.

We call a noncommutative curve $(X, \mathcal{A})$ *normal* if, for every point $x \in X$, the algebra $\mathcal{A}_x$ is a *maximal $\mathcal{O}_{X,x}$-order*, that is there is no $\mathcal{O}_{X,x}$-subalgebra $\mathcal{A}_x \subset \mathcal{A}' \subset \mathcal{K}_x$ which is also finitely generated as $\mathcal{O}_{X,x}$-module. Since $\mathcal{A}$ is reduced, there is a normal curve $\tilde{X} = (X, \tilde{\mathcal{A}})$ such that $\mathcal{A} \subseteq \tilde{\mathcal{A}} \subset \mathcal{K}_X$. Moreover, $\mathcal{A}_x = \tilde{\mathcal{A}}_x$ for almost all $x \in X$ (it follows from [7]). We call $(X, \tilde{\mathcal{A}})$ *a normalization* of X and denote by $\operatorname{sg} \mathcal{A}$ the set of all points $x \in X$ such that $\mathcal{A}_x \neq \tilde{\mathcal{A}}_x$. Note that such a normalization is, as a rule, not unique, though $\operatorname{sg} \mathcal{A}$ does not depend on the choice of normalization. Let $\tilde{\mathcal{C}} = \operatorname{center}(\tilde{\mathcal{A}})$, $\tilde{X} = \operatorname{spec} \tilde{\mathcal{C}}$. We can (and will) consider $\tilde{\mathcal{A}}$ as a sheaf of central $\mathcal{O}_{\tilde{X}}$-algebras, hence consider the normalization as the noncommutative curve $(\tilde{X}, \tilde{\mathcal{A}})$. The natural morphism of ringed spaces $\pi : (\tilde{X}, \tilde{\mathcal{A}}) \to (X, \mathcal{A})$ is defined. We also denote by $\widetilde{\operatorname{sg}}\,\mathcal{A}$ the set-theoretical preimage $\pi^{-1}(\operatorname{sg}\mathcal{A})$. If $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_s$ are the irreducible components of $\tilde{X}$, we set $\tilde{\mathcal{A}}_i = \tilde{\mathcal{A}}|_{\tilde{X}_i}$, so consider the noncommutative curves $(\tilde{X}_i, \tilde{\mathcal{A}}_i)$. We also set $\widetilde{\operatorname{sg}}_i\,\mathcal{A} = \widetilde{\operatorname{sg}}\,\mathcal{A} \cap \tilde{X}_i$. Let $X_i = \pi(\tilde{X}_i)$. Certainly, each X_i is an irreducible component of X, but these components need not be different. We set $\mathcal{K}_i(\mathcal{A}) = \mathcal{K}(\mathcal{A})|_{\tilde{X}_i}$. It is a constant sheaf of central $\mathcal{K}_{X_i}$-gebras. Since $\Bbbk$ is algebraically closed, the Brauer group of the field $\mathcal{K}_i = \mathcal{K}_{X_i}$ is trivial [13, Chapter II, § 3], so $\mathcal{K}_i(\mathcal{A}) \simeq \operatorname{Mat}(n_i, \mathcal{K}_i)$ for some n_i. We call a noncommutative curve $(X, \mathcal{A})$ *rational* if so is the curve X, i.e. all components of $\tilde{X}$ are isomorphic to the projective line $\mathbb{P}^1$.

For calculation of vector bundles over noncommutative curves one can use the "sandwich procedure," just as it has been done in [9] in the commutative case. Let $\pi : (\tilde{X}, \tilde{\mathcal{A}}) \to (X, \mathcal{A})$ be a normalization of a noncommutative curve $(X, \mathcal{A})$. We denote by $\mathcal{J}$ the *conductor of $\tilde{\mathcal{A}}$ in $\mathcal{A}$*, that is the maximal sheaf of $\tilde{\mathcal{A}}$-ideals contained in $\mathcal{A}$. We consider the noncommutative varieties $(\operatorname{sg}\mathcal{A}, \mathcal{S})$ and $(\widetilde{\operatorname{sg}}\,\mathcal{A}, \tilde{\mathcal{S}})$, where $\mathcal{S} = \mathcal{A}/\mathcal{J}$ and $\tilde{\mathcal{S}} = \tilde{\mathcal{A}}/\mathcal{J}$. These varieties are 0-dimensional and usually not reduced. We denote by $\bar{\pi} : (\widetilde{\operatorname{sg}}\,\mathcal{A}, \tilde{\mathcal{S}}) \to (\operatorname{sg}\mathcal{A}, \mathcal{S})$ the restriction of π onto $(\widetilde{\operatorname{sg}}\,\mathcal{A}, \tilde{\mathcal{S}})$ and by ι and $\tilde{\iota}$, respectively, the closed embeddings $(\operatorname{sg}\mathcal{A}, \mathcal{S}) \to (X, \mathcal{A})$ and $(\widetilde{\operatorname{sg}}\,\mathcal{A}, \tilde{\mathcal{S}}) \to (\tilde{X}, \tilde{\mathcal{A}})$. So we have a commutative diagram of morphisms of noncommutative varieties

$$\begin{tikzcd}
(\widetilde{\operatorname{sg}}\,\mathcal{A}, \tilde{\mathcal{S}}) \arrow[r, "\tilde{\iota}"] \arrow[d, "\bar{\pi}"'] & (\tilde{X}, \tilde{\mathcal{A}}) \arrow[d, "\pi"] \\
(\operatorname{sg}\mathcal{A}, \mathcal{S}) \arrow[r, "\iota"] & (\tilde{X}, \tilde{\mathcal{A}})
\end{tikzcd}$$

Since $(\operatorname{sg}\mathcal{A}, \mathcal{S})$ and $(\widetilde{\operatorname{sg}}\,\mathcal{A}, \tilde{\mathcal{S}})$ are 0-dimensional, coherent sheaves on them can be identified with finitely generated modules over the algebras, respectively,

$$\mathbf{S} = \prod_{x \in \operatorname{sg}\mathcal{A}} \mathcal{A}_x/\mathcal{J}_x \quad \text{and} \quad \tilde{\mathbf{S}} = \prod_{y \in \widetilde{\operatorname{sg}}\,\mathcal{A}} \tilde{\mathcal{A}}_y/\mathcal{J}_y.$$

Following [5, 6], we introduce the *category of triples* $\mathcal{T}(\mathcal{A})$ as follows.

- The objects of $\mathcal{T}(\mathcal{A})$ are triples $(\mathcal{G}, P, \theta)$, where
  - $\mathcal{G}$ is a vector bundle over $\tilde{\mathcal{A}}$,
  - P is a vector bundle over $\mathcal{S}$, or, the same, a finitely generated projective $\mathbf{S}$-module,

– θ is an isomorphism $\bar{\pi}^*P \to \tilde{\iota}^*\mathcal{G}$, or, the same, an isomorphism of $\tilde{\mathbf{S}}$-modules $\tilde{\mathbf{S}} \otimes_{\mathbf{S}} P \to \prod_{y\in\widetilde{\mathrm{sg}}\,\mathcal{A}} \mathcal{G}_y/\mathcal{J}_y\mathcal{G}_y$.

- A morphism $(\mathcal{G}, P, \theta) \to (\mathcal{G}', P', \theta')$ is a pair (Φ, ϕ), where $\Phi \in \mathrm{Hom}_{\tilde{\mathcal{A}}}(\mathcal{G}, \mathcal{G}')$ and $\phi \in \mathrm{Hom}_{\mathcal{S}}(P, P')$ such that the induced diagram

$$\begin{tikzcd} \bar{\pi}^*P \arrow[r, "\bar{\pi}^*\phi"] \arrow[d, "\theta"'] & \bar{\pi}^*P' \arrow[d, "\theta'"] \\ \tilde{\iota}^*\mathcal{G} \arrow[r, "\tilde{\iota}^*\Phi"] & \tilde{\iota}^*\mathcal{G}' \end{tikzcd}$$

is commutative.

One easily sees that $\mathcal{T}(\mathcal{A})$ is indeed a full subcategory of a *bimodule category* in the sense of [8], namely, the category defined by the $\mathrm{VB}(\mathcal{S})$-$\mathrm{VB}(\tilde{\mathcal{A}})$-bimodule $\mathrm{Hom}_{\tilde{\mathcal{S}}}(\bar{\pi}^*P, \tilde{\iota}^*\mathcal{G})$. It can also be considered as the push-out of the categories $\mathrm{VB}(\tilde{\mathcal{A}})$ and $\mathrm{VB}(\mathcal{S})$ over the category $\mathrm{VB}(\tilde{\mathcal{S}})$ with respect to the functors $\tilde{\iota}^*$ and $\bar{\pi}^*$. So it is an analogue of Milnor's construction of projective modules from [12, § 2].

We define the functor $\mathbf{F} : \mathrm{VB}(\mathcal{A}) \to \mathcal{T}(\mathcal{A})$, which maps a vector bundle $\mathcal{F}$ to the triple $(\pi^*\mathcal{F}, \iota^*\mathcal{F}, \theta_{\mathcal{F}})$, where $\theta_{\mathcal{F}}$ is the natural isomorphism $\bar{\pi}^*\iota^*\mathcal{F} \to \tilde{\iota}^*\pi^*\mathcal{F}$. The same considerations as in [6, 9] give the following result.

Theorem 1.1. *The functor* $\mathbf{F}$ *induces an equivalence of the categories* $\mathrm{VB}(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}(\mathcal{A})$. *The inverse functor* $\mathbf{G} : \mathcal{T}(\mathcal{A}) \to \mathrm{VB}(\mathcal{A})$ *maps a triple* $(\mathcal{G}, P, \theta)$ *to the preimage in* $\mathcal{G}$ *of the* $\mathbf{S}$*-submodule* $\theta(1 \otimes P) \subseteq \tilde{\iota}^*\mathcal{G}$.

2. Nodal curves

Definition 2.1. (1) An algebra R over a local commutative ring O of Krull dimension 1, which is finitely generated and torsion free as O-module, is said to be *nodal* [5, 14] if the following conditions hold:
   (a) $\mathrm{End}_R(\mathrm{rad}\,R) = H$ is hereditary.
   (b) $\mathrm{rad}\,H = \mathrm{rad}\,R$ (under the natural embedding of R into H).
   (c) $\mathrm{length}_R(H \otimes_R U) \le 2$ for every simple R-module U.
   Note that a nodal algebra never has nilpotent ideals, since it holds for any hereditary O-algebra.

(2) A noncommutative curve $(X, \mathcal{A})$ is said to be *nodal* if every algebra $\mathcal{A}_x$ ($x \in X$) is a nodal $\mathcal{O}_{X,x}$-algebra.

If $\mathcal{A} = \mathcal{O}_X$, so we deal with a "usual" (commutative) curve, it means that all singular points of X are nodes (ordinary double points).

We recall the construction of nodal algebras over the ring $O = \Bbbk[[t]]$ from [14]. Up to Morita equivalence such algebra is given by a tuple $\mathbf{N} = (s; n_1, n_2, \dots, n_s; \sim)$, where s and $n_1, n_2, \dots, n_s$ are positive integers, while $\sim$ is a symmetric relation on the set of pairs $\mathbb{I} = \{\,(k,i) \mid 1 \le k \le s,\ 1 \le i \le n_k\,\}$ satisfying the following conditions:

(N1) $\#\{\,(l,j) \in \mathbb{I} \mid (l,j) \sim (k,i)\,\} \le 1$ for each pair $(k,i) \in \mathbb{I}$.

(N2) If $(k,i) \sim (k,i)$, then $i < n_k$ and $(k, i+1) \not\sim (l,j)$ for any $(l,j) \in \mathbb{I}$.

Namely, define $R(\mathbf{N})$ as the subring of $M(\mathbf{N}) = \prod_{k=1}^{s} \mathrm{Mat}(n_k, O)$ consisting of such collections of matrices $(A_1, A_2, \dots, A_s)$, where $A_k = (a_{ij}^k) \in \mathrm{Mat}(n_k, O)$, that

$$a_{ij}^k \equiv 0 \pmod t \ \text{ if } \ i > j \ \text{ or } \ i = j-1 \ \text{ and } \ (k,i) \sim (k,i), \tag{2.1}$$

$$a_{ii}^k \equiv a_{jj}^l \pmod t \ \text{ if } \ (k,i) \sim (l,j). \tag{2.2}$$

Theorem 2.2 ([14]). (1) *Every ring $R(\mathbf{N})$ is a nodal O-algebra.*

(2) *Every nodal O-algebra is Morita equivalent to one of the rings $R(\mathbf{N})$.*

(3) $\operatorname{rad} R(\mathbf{N})$ *consists of such collections $(A_1, A_2, \dots, A_k)$ that the condition* (2.1) *holds and also $a_{ii}^k \equiv 0 \pmod t$ for all k, i.*

(4) *The hereditary algebra $H(\mathbf{N}) = \mathrm{End}_{R(\mathbf{N})}(\operatorname{rad} R(\mathbf{N}))$ consists of such collections $(A_1, A_2, \dots, A_k)$ that*

$$\begin{aligned} a_{ij}^k \equiv 0 \pmod t \ \text{ if } i > j, \ \text{except the case when}\\ i = j-1 \text{ and } (k,i) \sim (k,i). \end{aligned}$$

(5) *$M(\mathbf{N})$ is a maximal order containing $R(\mathbf{N})$ such that $J(\mathbf{N}) = \operatorname{rad} M(\mathbf{N})$ is the conductor of $M(\mathbf{N})$ both in $R(\mathbf{N})$ and in $H(\mathbf{N})$, and $J(\mathbf{N}) \subseteq \operatorname{rad} R(\mathbf{N})$.*

(6) *$R(\mathbf{N})/J(\mathbf{N})$ is the subring of $M/J(\mathbf{N}) = \prod_{k=1}^{s} \mathrm{Mat}(n_k, \Bbbk)$ consisting of such collections of matrices $(A_1, A_2, \dots, A_s)$ that*

$$\begin{aligned} &a_{ij}^k = 0 \ \text{ if } \ i > j \ \text{ or } \ i = j-1 \ \text{ and } \ (k,i) \sim (k,i),\\ &a_{ii}^k = a_{jj}^l \ \text{ if } \ (k,i) \sim (l,j). \end{aligned}$$

In particular, $R(\mathbf{N})$ is hereditary if and only if the relation $\sim$ is empty. (Then we write $R = R(s; n_1, n_2, \dots, n_s)$.)

Actually, to define a ring Morita equivalent to $R(\mathbf{N})$, one only has to prescribe positive integers $m(k,i)$ for each pair $(k,i) \in \mathbb{I}$ so that $m(k,i) = m(l,j)$ if $(k,i) \sim (l,j)$, and consider a_{ij}^k in the definition of $R(\mathbf{N})$ not as elements of $\Bbbk$, but as matrices from $\mathrm{Mat}(m(k,i) \times m(k,j), \Bbbk)$, preserving all congruences modulo t. We denote such data by $(\mathbf{N},\mathbf{m})$, where $\mathbf{m}$ is the function $(k,i)\mapsto m(k,i)$, and the corresponding algebra by $R(\mathbf{N},\mathbf{m})$. Note that different data $\mathbf{N}$ or $(\mathbf{N},\mathbf{m})$ can describe isomorphic algebras, even if they do not only differ by a permutation of indices (k,i). We extend the relation $\sim$ to an equivalence relation $\approx$ setting $(k,i)\approx(l,j)$ if and only if $(k,i)=(l,j)$ or $(k,i)\sim(l,j)$.

From the well-known properties of torsion free modules over reduced rings of Krull dimension 1 (see, for instance, [7]) it follows that, given a torsion free coherent sheaf $\mathcal{F}$ over a noncommutative curve $(X,\mathcal{A})$, a finite set of closed points $x_1,x_2,\dots,x_m\in X$ and a set of coherent $\mathcal{A}_{x_i}$-submodules $\mathcal{G}_i\subset\mathcal{F}\otimes_{kO_X}\mathcal{K}$, there is a unique coherent sheaf $\mathcal{G}\subset\mathcal{F}\otimes_{kO_X}\mathcal{K}$ such that $\mathcal{G}_{x_i}=\mathcal{G}_i$ and $\mathcal{G}_y=\mathcal{F}_y$ if $y\neq x_i$ for all i. In particular, since almost all localizations $\mathcal{A}_x$ are maximal, one can construct a normalization $\tilde{\mathcal{A}}$ of $\mathcal{A}$ locally, choosing arbitrary normalizations $\tilde{\mathcal{A}}_x$ of $\mathcal{A}_x$ for $x\in\operatorname{sg}\mathcal{A}$. Therefore, given a nodal noncommutative curve $(X,\mathcal{A})$, we can (and will) suppose that the normalizations of its local components are chosen as in Theorem 2.2. Thus, if $x\in\operatorname{sg}X$, $y\in\pi^{-1}(x)=\{\,y_1,y_2,\dots,y_r\,\}$, we identify $\tilde{\mathcal{A}}_y$ with a full matrix ring $\operatorname{Mat}(n_y,\mathcal{O}_{\tilde{X},y})$ and suppose that the ring $\mathcal{A}_x$ is given by some data $(\mathbf{N},\mathbf{m})$ as above. In what follows, we write (y_k,i) instead of (k,i), so the local embeddings $\mathcal{A}_x\to\tilde{\mathcal{A}}_x=\prod_{k=1}^r\tilde{\mathcal{A}}_{y_k}$ for $x\in\operatorname{sg}\mathcal{A}$ are described by the data $\mathbf{N}(\mathcal{A})$ consisting of integers n_y and $m(y,i)$ for $y\in\widetilde{\operatorname{sg}}\,\mathcal{A}$, $1\le i\le n_y$, and an equivalence relation $\sim$ on the set of pairs (y,i) satisfying the above conditions (N1) and (N2) and such that

(N3) the sum $\mathbf{m}_y=\sum_{i=1}^{n_y}m(y,i)$ is the same for all points y belonging to the same component of $\tilde{X}$.

The last condition just expresses the fact that the sheaf $\mathcal{K}(\tilde{\mathcal{A}})$ is locally constant. One easily sees that $\pi(y)=\pi(y')$ if and only if there is at least one relation $(y,i)\sim(y',j)$. Moreover, if we suppose that X is connected and $\mathcal{A}$ is central, the set $\pi^{-1}(x)$ for each $x\in\operatorname{sg}X$ must be connected as the graph defined by the symmetric relation $y\sim y'$ which means that there is at least one pair i,j such that $(y,i)\sim(y',j)$.

From now on we fix a connected central noncommutative nodal curve $(X,\mathcal{A})$ and its normalization $\pi:(\tilde{X},\tilde{\mathcal{A}})\to(X,\mathcal{A})$ chosen as described above. We write $\mathcal{O}$ instead of $\mathcal{O}_X$ and $\tilde{\mathcal{O}}$ instead of $\mathcal{O}_{\tilde{X}}$. If $\tilde{X}_1,\tilde{X}_2,\dots,\tilde{X}_s$ are the irreducible components of $\tilde{X}$, $X_k=\pi(\tilde{X}_k)$, we write $\tilde{\mathcal{O}}_k=\tilde{\mathcal{O}}_{\tilde{X}_k}$, $\tilde{\mathcal{A}}_k=\tilde{\mathcal{A}}|_{\tilde{X}_k}$, $\mathcal{O}_k=\mathcal{O}_{X_k}$ and $\mathcal{A}_k=\mathcal{A}|_{X_k}$. Recall that the sheaves of rings $\tilde{\mathcal{O}}_k$ and $\tilde{\mathcal{A}}_k$ are Morita equivalent. Namely, there is a vector bundle $\mathcal{L}_k$ over $\tilde{\mathcal{A}}_k$ such that $\mathcal{E}nd_{\tilde{\mathcal{A}}_k}\mathcal{L}_k\simeq\tilde{\mathcal{O}}_k$, $\mathcal{E}nd_{\tilde{\mathcal{O}}_k}\mathcal{L}_k\simeq\tilde{\mathcal{A}}_k$, so the functors $\mathcal{H}om_{\tilde{\mathcal{A}}_k}(\mathcal{L}_k,\_\,)$ and $\mathcal{L}_k\otimes_{\tilde{\mathcal{O}}_k}\_$ establish an equivalence between $\mathrm{Coh}(\tilde{\mathcal{A}}_k)$ and $\mathrm{Coh}(\tilde{\mathcal{O}}_k)$. We call $\mathcal{L}_k$ a *basic vector bundle* over $\tilde{\mathcal{A}}_k$. (Note that it is not uniquely defined.)

Let $\mathcal{J}$ be the conductor of $\tilde{\mathcal{A}}$ in $\mathcal{A}$. If $x \in \operatorname{sg}\mathcal{A}$, then $\mathcal{J}_x = \bigoplus_{\pi(y)=x} \operatorname{rad}\tilde{\mathcal{A}}_y$, $\tilde{\mathbf{S}}_y = \tilde{\mathcal{A}}_y/\mathcal{J}_y \simeq \mathrm{Mat}(\mathbf{m}_y, \mathbf{k})$ and $\mathcal{L}_y/\mathcal{J}_y\mathcal{L}_y \simeq \mathbf{m}_y U_y$, where U_y is the unique simple $\tilde{\mathbf{S}}_y$-module. For any vector bundle $\mathcal{G}$ over $\tilde{\mathcal{A}}_i$ we define its *rank*: $\operatorname{rk}\mathcal{G} = r$ if $\mathcal{G}_y/\mathcal{J}_y\mathcal{G}_y \simeq rU_y$ for some (then for any) $y \in \tilde{X}_i$.

Every pair (y, i), where $\pi(y) = x$, $1 \le i \le n_y$, defines a simple $\mathbf{S}_x$-module $V_{y,i}$, where $\mathbf{S}_x = \mathcal{A}_x/\mathcal{J}_x$, and $V_{y,i} \simeq V_{y',j}$ if and only if $(y, i) \approx (y', j)$. Moreover, $U_y \simeq \bigoplus_{i=1}^{n_y} V_{y,i}$ as $\mathbf{S}_y$-module. We denote by $P_{y,i}$ the projective $\mathbf{S}_y$-module such that $P_{y,i}/\operatorname{rad}P_{y,i} \simeq V_{y,i}$. In particular, $P_{y,i} \simeq P_{y',j}$ if and only if $(y, i) \approx (y', j)$.

To describe the category of triples $\mathcal{T}(\mathcal{A})$ it is convenient to introduce new symbols e^y_{ij}, where $1 \le i \le j \le n_y$, and the sets $E^{y,i}_{y',j}$ consisting of all $e^z_{i'j'}$ such that one of the following conditions hold:

- $z = y$, $(y, j') \sim (y', j)$ and either $i = i'$ or $(y, i) \sim (y, i')$;
- $z = y'$, $(y', i') \sim (y, i)$ and either $j = j'$ or $(y', j) \sim (y', j')$.

We also set $e^y_i = \sum_{(z,j)\approx(y,i)} e^z_{jj}$ and consider the copies Ue^y_{ii} of the simple modules U_y. Then

$$\tilde{\mathcal{S}} \otimes_{\mathbf{S}} P_{y,i} \simeq \bigoplus_{(z,j)\approx(y,i)} U^z_j e^z_{jj},$$

$$\operatorname{End}_{\mathbf{S}} P_{y,i} \simeq \begin{cases} \Bbbk e^y_i & \text{if } (y,i) \not\sim (y,j) \text{ for any } j \ne i, \\ \Bbbk e^y_i \oplus \Bbbk e^y_{ij} & \text{if } (y,i) \sim (y,j) \text{ and } i < j, \\ \Bbbk e^y_i \oplus \Bbbk e^y_{ji} & \text{if } (y,i) \sim (y,j) \text{ and } j < i. \end{cases}$$

and, for $(y, i) \not\sim (y', j)$,

$$\operatorname{Hom}_{\mathbf{S}}(P_{y,i}, P_{y',j}) \simeq \bigoplus_{E^{y,i}_{y',j}} \Bbbk e^z_{i'j'}.$$

Under such notations the maps $\tilde{\mathcal{S}} \otimes_{\mathbf{S}} P_{y,i} \to \tilde{\mathcal{S}} \otimes_{\mathbf{S}} P_{y',j}$ induced by the homomorphisms $P_{y,i} \to P_{y',j}$ as well as the multiplication of homomorphisms are given by the "matrix multiplication" on the right, i.e. by the rules:

$$e^y_{ii'} e^{y'}_{j'j} = \begin{cases} 0 & \text{if } y \ne y' \text{ or } i' \ne j', \\ e^y_{ij} & \text{if } y = y' \text{ and } i' = j'. \end{cases}$$

Let $(\mathcal{G}, P, \theta)$ be a triple from $\mathcal{T}(\mathcal{A})$. Decompose $\mathcal{G}$ and P:

- $\mathcal{G} = \bigoplus_{k,l} g_{kl}\mathcal{G}_{kl}$, where $\mathcal{G}_{kl}$ are nonisomorphic indecomposable vector bundles over $\tilde{\mathcal{A}}_k$,

- $P=\bigoplus_{y,i} p_{y,i}P_{y,i}$.

Set $r_{kl}=\operatorname{rk}\mathcal{G}_{kl}$. Then the isomorphism $\theta:\bar{\pi}^*P\to\tilde{\iota}^*\mathcal{G}$ is given by a set $\Theta=\{\,\Theta_y\mid y\in\widetilde{\operatorname{sg}}\,\mathcal{A}\,\}$ of invertible block matrices $\Theta_y=(\Theta^{y,i}_{kl})$, where $y\in\widetilde{\operatorname{sg}}_k\,\mathcal{A}$, the block $\Theta^{y,i}_{kl}$ has coefficients from $\Bbbk$ and is of size $r_{kl}g_{kl}\times p_{y,i}$. If another triple $(\mathcal{G}',P',\theta')$ is given by the matrices Θ'_y, a morphism $(\mathcal{G},P,\theta)\to(\mathcal{G}',P',\theta')$ is given by block matrices $\Phi_k=(\Phi^{kl}_{kl'})$ and $\phi_y=(\phi^{y,i}_{y,j})$ such that $\Phi_k(y)\Theta_y=\Theta'_y\phi_y$ for every $y\in\widetilde{\operatorname{sg}}_k\,\mathcal{A}$, where the elements of $\Phi^{kl}_{kl'}$ are from $\operatorname{Hom}_{\tilde{\mathcal{A}}_k}(\mathcal{G}_{kl},\mathcal{G}_{kl'})$, elements of $\phi^{y,i}_{y,j}$ are from $\Bbbk$, $\phi^{y,i}_{y,i}=\phi^{y',j}_{y',j}$ if $(y,i)\sim(y',j)$ and $\phi^{y,i}_{y,j}=0$ if $i>j$ or $i=j-1$, $(y,i)\sim(y,i)$. This morphism is an isomorphism if and only if all "diagonal" blocks Φ^{kl}_{kl} and $\phi^{y,i}_{y,i}$ are invertible.

Let $\mathcal{N}(\mathcal{A})$ be the ideal in $\mathcal{T}(\mathcal{A})$ consisting of all morphisms (Φ,ϕ) such that all values $\Phi_k(y)$, where $y\in\widetilde{\operatorname{sg}}_k\,\mathcal{A}$, are zero. In the matrix presentation it means that $\Phi^{kl}_{kl'}(y)=0$ for all possible triples (k,l,l') and all $y\in\tilde{X}_k$. Denote $\overline{\mathcal{T}}(\mathcal{A})=\mathcal{T}(\mathcal{A})/\mathcal{N}(\mathcal{A})$. These categories have the same objects and the natural functor $\mathcal{T}(\mathcal{A})\to\overline{\mathcal{T}}(\mathcal{A})$ is full (not faithful), maps nonisomorphic objects to nonisomorphic and indecomposable objects to indecomposable. Therefore, to obtain a classification of vector bundles, we actually have to study the category $\overline{\mathcal{T}}(\mathcal{A})$. Nevertheless, passing from $\mathcal{T}$ to $\overline{\mathcal{T}}$ we can lose some information. It is important, for instance, if we are looking for *stable* vector bundles (see, for instance, [3]).

3. String case

Definition 3.1. A noncommutative nodal curve $(X,\mathcal{A})$ is said to be of *string type* if it is rational and every set $\widetilde{\operatorname{sg}}_k\,\mathcal{A}$ contains at most 2 points.

If $(X,\mathcal{A})$ is of string type, we identify all components $\tilde{X}_k$ with $\mathbb{P}^1$ and fix an affine part $\mathbb{A}^1\subset\tilde{X}_k$ containing $\widetilde{\operatorname{sg}}_k\,\mathcal{A}$.

In this case the category of triples $\mathcal{T}(\mathcal{A})$ can be treated as the category of representations of a certain *bunch of chains* $\mathfrak{B}(\mathcal{A})$ in the sense of [5, Appendix B].[1] Namely, if $\mathcal{L}_k$ is a basic vector bundle over $\tilde{\mathcal{A}}_k$, then every indecomposable vector bundle over $\tilde{\mathcal{A}}_k$ is isomorphic to $\mathcal{L}_k(d)$ for some d, which is called the *degree* of $\mathcal{L}_k(d)$.[2] Moreover,

$$\operatorname{Hom}_{\tilde{\mathcal{A}}}(\mathcal{L}_k(d),\mathcal{L}_k(d'))\simeq\begin{cases}0 & \text{if } d>d',\\ \Bbbk[t]_{d'-d} & \text{if } d\le d',\end{cases}$$

[1] Or a *bundle of semi-chains* in the terms of [4].

[2] Note that it is not the degree of $\mathcal{L}_k(d)$ as of $\tilde{\mathcal{O}}_k$-sheaf; the latter equals dn_k.

where $\Bbbk[t]_m$ denotes the set of polynomials $f(t)$ such that $\deg f(t) \le m$. Therefore, in the decomposition of a vector bundle $\mathcal{G}_k$ over $\tilde{\mathcal{A}}_k$ we can suppose that $\mathcal{G}_{kl} = \mathcal{L}_k(l)$. Then the elements of the matrices $\Phi^{kl}_{kl'}$ can be considered as the polynomials of degree $l'-l$ if $l' \ge l$; they are zero if $l' < l$. If $y \ne y'$ are two points from $\widetilde{\mathrm{sg}}_k\,\mathcal{A}$ and $l' > l$, we can always choose a polynomial $f(t) \in \Bbbk[t]_{l'-l}$ such that $f(y) = a,\ f(y') = b$ for any prescribed values $a, b \in \Bbbk$. It means that the values of the matrices $\Phi^{kl}_{kl'}$ at the points y and y' can be prescribed arbitrary. Therefore, the rule $\Phi_k(y)\Theta_y = \Theta'_y\phi_y$ from the matrix description of morphisms in $\mathcal{T}(\mathcal{A})$ can be rewritten as $F(y)\Theta_y = \Theta'_y\phi_y$, where $F(y)$ is an arbitrary lower block triangular matrix $F(y) = (F(y)^{kl}_{kl'})$ ($F(y)^{kl}_{kl'} = 0$ if $l < l'$) over the field $\Bbbk$ and the only restrictions for these blocks is that $F(y)^{kl}_{kl} = F(y')^{kl}_{kl}$ if y and y' are in the same component $\tilde{X}_k$.

Thus we define the bunch of chains $\mathfrak{B}(\mathcal{A})$ as follows. We consider $\widetilde{\mathrm{sg}}\,\mathcal{A}$ as the index set of this bunch and for every $y \in \widetilde{\mathrm{sg}}\,\mathcal{A}$ set

$$\begin{aligned}
&\mathfrak{E}_y = \{\,(y,i) \mid 1 \le i \le n_y\,\} \setminus \{\,(y,i) \mid (y,i-1) \sim (y,i-1)\,\},\\
&\mathfrak{F}_y = \{\,(d,y) \mid d \in \mathbb{Z}\,\},\\
&(y,i) < (y,j) \ \text{ if } i < j,\\
&(d,y) < (d',y) \ \text{ if } d < d',\\
&(y,i) \sim (y',j) \text{ if and only if they are so in the nodal data } \mathbf{N}(\mathcal{A}),\\
&(d,y) \sim (d',z) \text{ if and only if } d = d',\ y \ne z \text{ but } y \text{ and } z \text{ belong}\\
&\qquad\qquad\qquad \text{to the same component } \tilde{X}_k.
\end{aligned}$$

Recall [4, 5] that a representation M of this bunch of chains is given by a set of block matrices $M_y = (M^{yi}_{dy})$, where $y \in \widetilde{\mathrm{sg}}\,\mathcal{A},\ 1 \le i \le n_y,\ M^{yi}_{dy} \in \mathrm{Mat}(m_{dy} \times n_{yi}, \Bbbk)$ for some integers m_{dy}, n_{yi} such that $m_{dy} = m_{dy'}$ if $(d,y) \sim (d,y')$ and $n_{yi} = n_{y'j}$ if $(y,i) \sim (y',j)$. Here we identify the symbols $(y,i)'$ and $(y,i)''$ from [5, Definition B.1], where $(y,i) \sim (y,i)$, with the pairs (y,i) and $(y,i+1)$. A morphism $\alpha : M \to M'$ given by a set of block matrices α'_y, α''_y, where $y \in \widetilde{\mathrm{sg}}\,\mathcal{A}$, $\alpha'_y = (\alpha^{dy}_{d'y})$, $\alpha''_y = (\alpha^{yi}_{yi'})$, such that

$$\begin{aligned}
\alpha_{d'y}^{dy} &\in \operatorname{Mat}(m_{d'y} \times m_{dy}, \Bbbk),\\
\alpha_{yi'}^{yi} &\in \operatorname{Mat}(m_{yi'} \times m_{yi}, \Bbbk),\\
\alpha_{d'y}^{dy} &= 0 \quad \text{if } d > d',\\
\alpha_{yi'}^{yi} &= 0 \quad \text{if } i > i' \text{ or } i' = i+1 \text{ and } (y,i) \sim (y,i),\\
\alpha_{dy}^{dy} &= \alpha_{dy'}^{dy'} \quad \text{if } (d,y) \sim (d,y'),\\
\alpha_{yi}^{yi} &= \alpha_{y'j}^{y'j} \quad \text{if } (y,i) \sim (y',j),
\end{aligned}$$

and

$$\alpha'_y M_y = M'_y \alpha''_y \quad \text{for all } y \in \widetilde{\operatorname{sg}}\,\mathcal{A}.$$

The matrix presentations described above imply the following fact.

Proposition 3.2. *Let the noncommutative nodal curve $(X, \mathcal{A})$ is of string type, $\mathfrak{B} = \mathfrak{B}(\mathcal{A})$. Then the category $\overline{\mathcal{T}}(\mathcal{A})$ is equivalent to the full subcategory* $\operatorname{rep}_0(\mathfrak{B})$ *of the category of representations of the bunch of chains $\mathfrak{B}$ consisting of such representations M that all matrices M_y are invertible.*

In particular, the category $\overline{\mathcal{T}}(\mathcal{A})$ and hence the category $\operatorname{VB}(\mathcal{A})$ are *tame* in the sense that they have at most 1-parameter families of indecomposable objects. Moreover, from the description of representations of a bunch of chains given in [4] one can deduce a description of vector bundles over a noncommutative nodal curve of string type. For the corresponding combinatorics we use the terminology from [5] adopted to our situation.

Definition 3.3. (1) Let $\mathfrak{E} = \bigcup_y \mathfrak{E}_y$, $\mathfrak{F} = \bigcup_y \mathfrak{F}_y$, $\mathfrak{X} = \mathfrak{E} \cup \mathfrak{F}$. We define the symmetric relation $-$ on $\mathfrak{X}$ setting $(d,y) - (y,i)$ for all possible d, i, y. We also write $\xi \parallel \xi'$ if either both ξ and ξ' belong to $\mathfrak{E}$ or both of them belong to $\mathfrak{F}$, and $\xi \perp \xi'$ if one of them belongs to $\mathfrak{E}$ while the other belongs to $\mathfrak{F}$.

(2) We define a *word* (more precisely, an $\mathfrak{X}$-*word*) as a sequence $\xi_1 r_1 \xi_2 r_2 \dots \xi_{l-1} r_{l-1} \xi_l$ such that
  - (a) $\xi_i \in \mathfrak{X}$, $r_i \in \{\sim, -\}$;
  - (b) $\xi_i r_i \xi_{i+1}$ for each $1 \le i < l$ accordingly to the definition of the relations $\sim$ and $-$;
  - (c) $r_i \ne r_{i+1}$ for all $1 \le i < l-1$.

  We call $l = l(w)$ the *length* of the word w and ξ_1, ξ_l the *ends* of this word.

(3) We call the word w *full* if the following conditions hold:

(a) either $r_1 =\sim$ or $\xi_1 \not\sim \xi'$ for any $\xi' \neq \xi_1$;
(b) either $r_{l-1} =\sim$ or $\xi_l \not\sim \xi'$ for any $\xi' \neq \xi_l$.

(4) We call the word w *terminating* if it is full and $r_1 = r_{l-1} = -$.
(5) The end ξ_1 (ξ_l) is said to be *special* if $r_1 = -$ and $\xi_1 \sim \xi_1$ (respectively, $\xi_l \sim \xi_l$ and $r_{l-1} = -$). Otherwise it is said to be *usual*.
(6) The terminating word w is said to be
- *usual* if both its ends are usual;
- *special* if one of its ends, but not both, is special;
- *bispecial* if both its ends are special.

(7) The word $w^* = \xi_l r_{l-1} \dots \xi_2 r_1 \xi_1$ is called *inverse* to the word w.
(8) We call w *symmetric* if $w = w^*$ and *quasisymmetric* if it can be presented as $v \sim v^* \sim v \sim \dots \sim v^* \sim v$ for a shorter word v. Note that a quasisymmetric word is always bispecial.
(9) The word w is said to be *cyclic* if $r_1 = r_{l-1} =\sim$ and $\xi_l - \xi_1$ in $\mathfrak{B}$. Then we set $r_0 = -$ and $\xi_{i+kl} = \xi_i$, $r_{i+kl} = r_i$ for any $k \in \mathbb{Z}$.
(10) A *shift* of the cyclic word w is the cyclic word

$$w^{[k]} = \xi_{k+1} r_{k+1} \xi_{k+2} \dots r_0 \xi_1 r_1 \dots \xi_k,$$

where k is even. In this case we set $\varepsilon(w,k) = (-1)^{k/2}$.
(11) The cyclic word w is said to be *aperiodic* if $w^{[k]} \neq w$ for $0 < k < l$. It is said to be *cyclic-symmetric* if $w^* = w^{[k]}$ for some k.

Note that the length of a terminating or cyclic word is always divisible by 4.

Definition 3.4. (1) A *usual string* is a usual nonsymmetric terminating word.
(2) A *special string* is a pair (w, δ), where w is a special terminating word and $\delta \in \{0, 1\}$.
(3) A *bispecial string* is a quadruple $(w, m, \delta_0, \delta_1)$, where w is a bispecial terminating word that is neither symmetric nor quasisymetric, $m \in \mathbb{N}$ and $\delta_i \in \{0, 1\}$ $(i = 0, 1)$.
(4) A *band* is a triple (w, m, λ), where w is a cyclic word, $m \in \mathbb{N}$, $\lambda \in \Bbbk^\times$ and, if w is cyclic-symmetric, also $\lambda \neq 1$.
(5) The following strings are said to be *equivalent*:
- w and w^*;
- (w, δ) and (w^*, δ);
- $(w, m, \delta_0, \delta_1)$ and $(w^*, m, \delta_1, \delta_0)$.

(6) Two bands are said to be *equivalent* if they can be obtained from one another by a sequence of the following transformations:
- replacing (w, m, λ) by $(w^{[k]}, m, \lambda^{\varepsilon(w,k)})$;
- replacing (w, m, λ) by (w^*, m, λ^{-1}).

Note that if $w^* = w^{[k]}$, then $k \equiv 2 \pmod 4$, so $\varepsilon(w,k) = -1$.

Now the results of [4] imply the following theorem.

Theorem 3.5. *The isomorphism classes of indecomposable vector bundles over a noncommutative nodal curve curve of string type $(X, \mathcal{A})$ are in one-to-one correspondence with the equivalence classes of strings and bands for the bunch of chains $\mathfrak{B}(\mathcal{A})$. The rank of the vector bundle corresponding to a string or a band equals $l/4$, where l is the length of the word w entering into this string or band.*

We refer to [4] for an explicit construction of representations corresponding to strings and bands, hence of vector bundles over noncommutative nodal curves of string type.

Note that it can so happen that there are no strings or no bands. For instance, if all localizations $\mathcal{A}_x$ are hereditary, there are no bands as well as no special and bispecial strings. Then there are only finitely many isomorphism classes of indecomposable vector bundles *up to twist*, i.e. up to change of degrees d in the pairs (d, y) occurring in a word. On the other hand, if each $\widetilde{\mathrm{sg}}_k\,\mathcal{A}$ consists of 2 points and for every pair (y, i) there is another pair $(z, j) \neq (y, i)$ such that $(z, j) \sim (y, i)$, then there are no terminating strings.

Actually, one can easily deduce the following criterion of finiteness.

Corollary 3.6. *The following conditions for a noncommutative nodal curve of string type $(X, \mathcal{A})$ are equivalent:*

1. *There are only finitely many isomorphism classes of indecomposable vector bundles over $\mathcal{A}$ up to twist.*
2. *There are no cycles for the bunch of chains $\mathfrak{B}(\mathcal{A})$.*
3. *There are no sequences of points $y_1, y_2, \dots, y_n, y_{n+1} = y_1$ from $\widetilde{\mathrm{sg}}\,\mathcal{A}$ such that, for $1 \le k \le n$,*
   - *if k is odd, then the points y_k and y_{k+1} are different and belong to the same component of $\tilde{X}$;*
   - *if k is even, there are indices i, j such that $(y_k, i) \sim (y_{k+1}, j)$ (possibly $y_k = y_{k+1}$).*

4. Almost string case

We consider one more case when there is a good description of vector bundles.

Definition 4.1. A noncommutative nodal curve $(X, \mathcal{A})$ is said to be of *almost string type* if every set $\widetilde{\mathrm{sg}}_k\,\mathcal{A}$ contains at most 3 point, and if it contains three points then for 2 of them the algebra $\mathcal{A}_{\pi(y)}$ is hereditary and Morita equivalent to the algebra $R(1;2)$ from Theorem 2.2 (with the empty relation $\sim$).

Note that if $\mathcal{A}_{\pi(y)}$ is hereditary, y is the unique point of $\widetilde{\mathrm{sg}}\,\mathcal{A}$ with the image $\pi(y)$. Hence, if X is connected, either $\tilde{X}$ consists of a unique component or there must be another point z on the same component of $\tilde{X}$ such that $\mathcal{A}_{\pi(z)}$ is not hereditary.

Let $\widetilde{\mathrm{sg}}_k\,\mathcal{A} = \{\, y_0, y_1, y_2 \,\}$ so that $\mathcal{A}_{\pi(y_1)}$ and $\mathcal{A}_{\pi(y_2)}$ are Morita equivalent to $R(1;2)$. In this case we call y_1, y_2 *extra points* and y_0 a *marked point*. Then the horizontal stripes of the matrices $\Theta_{y_1}, \Theta_{y_2}$ corresponding to the vector bundle $\mathcal{L}_k(d)$ can be reduced to the form

$$\Theta^{y_1,1}_{kd} = \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}, \quad \Theta^{y_1,2}_{kd} = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad \Theta^{y_2,1}_{kd} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix}, \quad \Theta^{y_2,2}_{kd} = \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

where I denote identity matrices of some sizes (equal if they are in the same row). From now on we only consider the objects from $\mathcal{T}(\mathcal{A})$ such that these matrices have the form (4.1), calling them *precanonical.* If (Φ, ϕ) is a morphism between precanonical objects, then the matrix Φ^{kd}_{kd} must be of the 4×4 block form

$$\Phi^{kd}_{kd} = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & * & * & * \end{pmatrix},$$

where stars denote arbitrary matrices of appropriate sizes. Moreover, if we consider $\Phi^{k,d-1}_{kd}$ also as a 4×4 block matrix (f_{ab}) $(a, b \in \{\, 1, 2, 3, 4 \,\})$, where the blocks f_{ab} consist of linear polynomials, then $f_{14}(y_1) = f_{14}(y_2) = 0$, so $f_{14} = 0$. Note that the values $f_{ab}(y_0)$ can be chosen arbitrary for $(ab) \neq (14)$, as well as the values of $\Phi^{kd'}_{kd}$ for $d' < d - 1$. Therefore, the full subcategory of $\overline{\mathcal{T}}(\mathcal{A})$ consisting of precanonical objects can again be treated as the category of representations of a bunch of chains $\mathfrak{B}' = \mathfrak{B}'(\mathcal{A})$. Namely, let $\mathrm{ex}\,\mathcal{A}$ be the set of all extra points. The index set for the bunch $\mathfrak{B}'$ is $\widetilde{\mathrm{sg}}\,\mathcal{A} \setminus \mathrm{ex}\,\mathcal{A}$. If a point y is not marked, the sets $\mathfrak{E}_y$ and $\mathfrak{F}_y$ are defined just as in Section 3 (page 9). If y is marked, the set $\mathfrak{F}_y$ is also defined as in Section 3, but the set $\mathfrak{E}_y$ consists of the triples (d, y, α), where $\alpha \in \{\, 0, 1 \,\}$, such that

- $(d', y, \alpha') < (d, y, \alpha)$ if and only if either $d' < d$ or $d' = d$ and $\alpha' < \alpha$;
- $(d, y, \alpha) \sim (d, y, \alpha)$ for all d, α.

Actually the element $(d, y, 0)$ represents in this bunch the first horizontal row of the stripe (d, y) and the fourth horizontal row of the stripe $(d-1, y)$ in the precanonical form (4.1), while the element $(d, y, 1)$ represents the second and the third horizontal rows of the stripe (d, y).

The preceding observations imply

Theorem 4.2. *Let $(X, \mathcal{A})$ be a noncommutative nodal curve of almost string type. The category $\overline{\mathcal{T}}(\mathcal{A})$ is equivalent to the full subcategory of the category of representations of the bunch of chains $\mathfrak{B}'(\mathcal{A})$ consisting of such representations M that all matrices M_y are invertible.*

Just as in Section 3, these representations (hence, vector bundles over $\mathcal{A}$) correspond to terminating strings and bands. In particular, the category of vector bundles over a noncommutative nodal curve of almost string type is also tame.

Corollary 4.3. *The following conditions for a noncommutative nodal curve of almost string type $(X, \mathcal{A})$ are equivalent:*

(1) *There are only finitely many isomorphism classes of indecomposable vector bundles over $\mathcal{A}$ up to twist.*
(2) *There are no cycles for the bunch of chains $\mathfrak{B}'(\mathcal{A})$.*
(3) *There are no sequences of points $y_1, y_2, \dots, y_n, y_{n+1} = y_1$ from $\widetilde{\mathrm{sg}}\,\mathcal{A} \setminus \mathrm{ex}\,\mathcal{A}$ such that, for $1 \le k \le n$,*
   - *if k is odd, then either the points y_k and y_{k+1} are different and belong to the same component of $\tilde{X}$ or $y_k = y_{k+1}$ is a marked point;*
   - *if k is even, there are indices i, j such that $(y_k, i) \sim (y_{k+1}, j)$ (possibly $y_k = y_{k+1}$).*

5. Wild cases

If a noncommutative curve $(X, \mathcal{A})$ is rational and connected and all localizations $\mathcal{A}_x$ are hereditary, then $X \simeq \mathbb{P}^1$ and the category $\mathrm{Coh}(\mathcal{A})$ is equivalent to the category of coherent sheaves over a *weighted projective line* $C(\mathbf{p}, \boldsymbol{\lambda})$ in the sense of [10]. Here $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \dots, \lambda_s\} = \mathrm{sg}\,\mathcal{A}$ and $\mathbf{p} = (p_1, p_2, \dots, p_s)$ are the integers such that $\mathcal{A}_{\lambda_k}$ is Morita equivalent to the hereditary algebra $R(1; p_k)$. Then it is known that $\mathrm{VB}(\mathcal{A})$ is of finite type if and only if $\sum_{k=1}^{s} 1/p_k > 1$ and is tame if $\sum_{k=1}^{s} 1/p_k = 1$. If $\sum_{k=1}^{s} 1/p_k < 1$, it is *wild*. It means that the classification of vector bundle over such noncommutative curve contains the classification of representations of every finitely generated $\Bbbk$-algebra (see [9] for formal definitions). Note also that if $(X, \mathcal{A})$ is normal, then, just as X itself, it is of finite type if $X \simeq \mathbb{P}^1$, tame if X is an elliptic curve and wild otherwise [9]. So the next theorem completes the answer to the question about the representation type of $\mathrm{VB}(\mathcal{A})$.

Theorem 5.1. *In the following cases the category* $\mathrm{VB}(\mathcal{A})$ *is wild:*

(1) $(X,\mathcal{A})$ *is neither rational nor normal.*
(2) *At least one of the localizations* $\mathcal{A}_x$ *is not nodal.*
(3) $(X,\mathcal{A})$ *is nodal, at least one of the localizations* $\mathcal{A}_x$ *is not hereditary and* $(X,\mathcal{A})$ *is neither of string nor of almost string type.*

Proof. The cases (1) and (2) are considered quite analogously to the commutative case [9, Proposition 2.5], so we omit their proofs. The proof of (3) we shall give in two cases:

(3a) $X=\mathbb{P}^1$, $\operatorname{sg}\mathcal{A}=\{\,x,x_2,x_3\,\}$, $\mathcal{A}_{x_k}$ is Morita equivalent to $R(1;k)$ for $k=2,3$, while, $\mathcal{A}_x$ is Morita equivalent to $R(1;2;\sim)$, where either $(1,1)\sim(1,2)$ or $(1,1)\sim(1,1)$.
(3b) $X=X_1\cup X_2$ so that $X_1\simeq X_2\simeq\mathbb{P}^1$, $X_1\cap X_2=\{x\}$ and this intersection is transversal (i.e. $\mathcal{O}_x$ is nodal), there are two more singular points $x_2,x_3\in X_1$ and $\mathcal{A}_{x_k}$ is Morita equivalent to $R(1;k)$ for $k=2,3$, while $\mathcal{A}_x$ is Morita equivalent to $R(2;1,1;\sim)$, where $(1,1)\sim(2,1)$.

All other cases easily reduce to these ones.

In both cases $\pi^{-1}(x_k)=\{y_k\}$ for $k=2,3$ and the d-th horizontal stripe of the matrices Θ_{y_k} can be reduced to the form:

$$\Theta_{2d}=\left(\begin{array}{cc|cc|cc} 0&0&0&0&I&0\\ 0&0&I&0&0&0\\ I&0&0&0&0&0\\ 0&0&0&0&0&I\\ 0&0&0&I&0&0\\ 0&I&0&0&0&0 \end{array}\right),\quad \Theta_{3d}=\left(\begin{array}{ccc|ccc} 0&0&0&I&0&0\\ 0&0&0&0&I&0\\ 0&0&0&0&0&I\\ I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&I&0&0&0 \end{array}\right),$$

where the vertical lines divide these matrices into the stripes corresponding to the projective modules P_{ki}. In the case (3a) we only consider such triples that the 1st, 5th and 6th horizontal rows of these matrices are empty. Then the matrix Θ_y, where $y\in\widetilde{\operatorname{sg}}_1\mathcal{A}$ and $\pi(y)=x$, is divided into 3 horizontal stripes and if (Φ,ϕ) is a morphism of such representations, then

$$\Phi^{1d}_{1d}=\begin{pmatrix} * & 0 & 0\\ * & * & 0\\ 0 & 0 & * \end{pmatrix}.$$

The classification of such triples can be considered as a bimodule problem (see [8, 9] for definitions and details) so that the corresponding

Tits form is either

$$Q_1 = 2t_1^2 + z_1^2 + z_2^2 + z_1 z_2 + z_3^2 - 2t_1(z_1 + z_2 + z_3)$$

or

$$Q_2 = t_1^2 + t_2^2 + z_1^2 + z_2^2 + z_1 z_2 + z_3^2 - (t_1 + t_2)(z_1 + z_2 + z_3),$$

where t_i are the sizes of vertical stripes and z_i are the sizes of horizontal stripes (if $(1,1) \sim (1,2)$, then $t_1 = t_2$). Since $Q_1(2,1,1,1) = Q_2(2,2,1,1,1) = -1$, this bimodule is wild, hence so is the category $\mathrm{VB}(\mathcal{A})$. Note that we need to check that $t_1 + t_2 = z_1 + z_2 + z_3$, since the matrix Θ_y must be invertible.

In the case (3b) we only omit the 1st and the 6th row of the matrices Θ_{y_k}. Then the matrix Φ_{1d}^{1d} will be of the form

$$\Phi_{1d}^{1d} = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & * & * \end{pmatrix}$$

We have one more matrix Θ_z, where $z \in \widetilde{\mathrm{sg}}_2\,\mathcal{A}$ and $\pi(z) = x$. We consider the triples such that $\mathcal{G}|_{Y_2} = \bigoplus_{d=1}^{8} r_d \mathcal{G}_{2d}$. The matrix Θ_z reduces to the form

$$\Theta_z = \left(\begin{array}{c|c|c|c|c|c|c|c} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right)$$

Then the matrix $\phi_{y,1}^{y,1} = \phi_{z,1}^{z,1}$ from a morphism (Φ, ϕ) of such triples must be triangular and we obtain a matrix problem with the Tits form

$$Q = t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_1 t_2 + t_1 t_4 + t_3 t_4 + \sum_{i \le j} r_i r_j - \sum_{i,j} t_i r_j.$$

Now $Q(1,3,3,1,1,1,1,1,1,1,1,1) = -1$, so we again obtain a wild problem. $\square$

6. Example

We consider a simple but typical example. Let $(X, \mathcal{A})$ be defined as follows.

- $X = X_1 \cup X_2$, where $X_1 \simeq X_2 \simeq \mathbb{P}^1$, $X_1 \cap X_2 = \{x\}$ and the intersection is transversal;
- $\operatorname{sg} \mathcal{A} = \{ x, x_1, x_2 \}$, where $x_1 \in X_1$, $x_2 \in X_2$;
- $\mathcal{K}(\mathcal{A}) = \operatorname{Mat}(2, \mathcal{K}_1) \times \operatorname{Mat}(2, \mathcal{K}_2)$;
- The singular localizations are:

$$\begin{aligned} \mathcal{A}_x &= R(2; 2, 2; \sim), \text{ where } (1,1) \sim (2,1); \\ \mathcal{A}_{x_1} &= R(1; 2; \sim), \text{ where } (1,1) \sim (1,1); \\ \mathcal{A}_{x_2} &= R(1; 2; \sim), \text{ where } (1,1) \sim (1,2). \end{aligned}$$

Then

- $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$, where $X_1 \simeq X_2 \simeq \mathbb{P}^1$, $X_1 \cap X_2 = \emptyset$;
- $\widetilde{\operatorname{sg}} \mathcal{A} = \{ y_1, y_2, y_3, y_4 \}$, where $y_1, y_3 \in Y_1$, $y_2, y_4 \in Y_2$, $\pi(y_3) = \pi(y_4) = x$, $\pi(y_1) = x_1$, $\pi(y_2) = x_2$.

Therefore the corresponding bunch of chains is

$$\begin{aligned} &\mathfrak{E}_1 = \{ (d1) \mid d \in \mathbb{Z} \}, \ \mathfrak{F}_1 = \{ (1,1) \}, \\ &\mathfrak{E}_2 = \{ (d2) \mid d \in \mathbb{Z} \}, \ \mathfrak{F}_2 = \{ (2,1) < (2,2) \}, \\ &\mathfrak{E}_3 = \{ (d3) \mid d \in \mathbb{Z} \}, \ \mathfrak{F}_3 = \{ (3,1) < (3,2) \}, \\ &\mathfrak{E}_4 = \{ (d4) \mid d \in \mathbb{Z} \}, \ \mathfrak{F}_4 = \{ (4,1) < (4,2) \}, \end{aligned}$$
$$(1,1) \sim (1,1), \ (2,2) \sim (2,1), \ (3,1) \sim (4,1), \ (d1) \sim (d3), \ (d2) \sim (d4).$$

(We write (dk) and (k,i) instead of (d, y_k) and (y_k, i).) We fix a basic vector bundle $\mathcal{L}_k$ over $\tilde{\mathcal{A}}_k$ $(k = 1, 2)$. Then $\mathcal{L}_1(d)/\mathcal{J}\mathcal{L}_1(d)$ has a $\Bbbk$-basis $e^1_i(d), e^3_j(d)$, $(1 \le i, j \le 2)$ and $\mathcal{L}_2(d)/\mathcal{J}\mathcal{L}_2(d)$ has a $\Bbbk$-basis $e^2_i(d), e^4_j(d)$, $(1 \le i, j \le 2)$, the upper index showing the point y_k where the corresponding element is supported.

An example of a usual string is given by the word

$$(4,2) - (d_1 4) \sim (d_1 2) - (2,2) \sim (2,1) - (d_2 2) \sim (d_2 4) - (4,2)$$

with $d_1 \neq d_2$ in order that the word be not symmetric. The corresponding vector bundle $\mathcal{F}$ is the $\mathcal{A}$-submodule in $\mathcal{G} = \mathcal{L}_2(d_1) \oplus \mathcal{L}_2(d_2)$ such that $\mathcal{F}_x = \mathcal{G}_x$ for $x \notin \operatorname{sg} \mathcal{A}$, $\mathcal{F}_{x_2}$ is generated by the preimages of $e^2_2(d_1)$ and $e^2_1(d_2)$, and $\mathcal{F}_x$ is generated by the preimages of $e^4_2(d_1)$ and $e^4_2(d_2)$. Since $\operatorname{supp} \mathcal{G} = X_2$, $\mathcal{F}_{x_1} = 0$.

An example of a special string is $(w, 1)$, where.

$$w = (1,1) - (d1) \sim (d3) - (3,2).$$

Here $\mathcal{G} = \mathcal{L}_1(d)$, $\mathcal{F}_{x_1}$ is generated by the preimage of e^1_2 and $\mathcal{F}_x$ is generated by the preimage of e^3_2.

An example of a bispecial string is $(w, m, 1, 0)$, where

$$w = (1,1) - (d_1 1) \sim (d_1 3) - (3,1) \sim (4,1) - (d_2 4) \sim (d_2 2) - (2,1) \sim$$
$$\sim (2,2) - (d_3 2) \sim (d_3 4) - (4,1) \sim (3,1) - (d_4 3) \sim (d_4 1) - (1,1).$$

The degrees d_i can be arbitrary with the only restriction that $d_2 \neq d_3$ or $d_1 \neq d_4$.

- $\mathcal{G} = m(\mathcal{L}_1(d_1) \oplus \mathcal{L}_2(d_2) \oplus \mathcal{L}_2(d_3) \oplus \mathcal{L}_1(d_4))$;
- $\mathcal{F}_x$ is generated by the preimages of the columns of the matrices $I_m e_1^3(d_1)$, $I_m e_1^3(d_4)$, $I_m e_1^4(d_2)$ and $I_m e_1^4(d_3)$, where I_m denotes the identity $m \times m$ matrix;
- $\mathcal{F}_{x_2}$ is generated by the preimages of the columns of the matrices $I_m e_1^2(d_2)$ and $I_m e_2^2(d_3)$;
- $\mathcal{F}_{x_1}$ is generated by the preimages of the columns of the matrices

$$\begin{pmatrix} I_q \\ 0 \end{pmatrix} e_2^1(d_1), \quad \begin{pmatrix} 0 \\ I_{m-q} \end{pmatrix} e_1^1(d_1), \quad \begin{pmatrix} I_q \\ A_q \end{pmatrix} e_1^1(d_4) \quad \text{and} \quad \begin{pmatrix} B_q \\ I_{m-q} \end{pmatrix} e_2^1(d_4),$$

  where $q = [(m+1)/2]$ and
  - if $m = 2q$, then $A_q = I_q$, $B_q = J_q(0)$, the Jordan $q \times q$ matrix with eigenvalue 0;
  - if $m = 2q - 1$, then A_q is of size $(q-1) \times q$ and B_q is of size $q \times (q-1)$, namely,

$$A_q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad B_q = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Finally, an example of a band is (w, m, λ), where

$$w = (2,2) \sim (2,1) - (d_1 2) \sim (d_1 4) - (4,1) \sim (3,1) - (d_2 3) \sim (d_2 1) -$$
$$- (1,1) \sim (1,1) - (d_3 1) \sim (d_3 3) - (3,1) \sim (4,1) - (d_4 4) \sim (d_4 2).$$

We suppose that $d_3 < d_2$ or $d_3 = d_2$, $d_4 \leq d_1$. Then

- $\mathcal{G} = m(\mathcal{L}_1(d_1) \oplus \mathcal{L}_2(d_2) \oplus \mathcal{L}_2(d_3) \oplus \mathcal{L}_1(d_4))$;
- $\mathcal{F}_{x_1}$ is generated by the preimages of the columns of the matrices

$$\begin{pmatrix} I_m e_1^1(d_2) \\ I_m e_1^1(d_3) \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ I_m e_2^1(d_3) \end{pmatrix};$$

- $\mathcal{F}_x$ is generated by the preimages of the columns of the matrices $I_m e_1^4(d_1)$, $I_m e_1^3(d_2)$, $I_m e_1^3(d_3)$ and $I_m e_1^4(d_4)$;
- $\mathcal{F}_{x_2}$ is generated by the preimages of the columns of the matrices $I_m e_1^2(d_1)$ and $J_m(\lambda) e_2^2(d_4)$ (the Jordan $m \times m$ matrix with eigenvalue λ).

If $d_2 < d_3$ or $d_2 = d_3$, $d_1 < d_4$, one has to permute d_2 and d_3 in the generators of $\mathcal{F}_{x_1}$, also permuting the rows.

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