Збірник праць Ін-ту математики НАН України (2020) т. XX, № XX, 1–23

Matrix problems and representations of algebras

Yu. A. Drozd

Abstract. This paper is devoted to the *theory of matrix problems*, a new branch of modern algebra created and developed to a large extent by the Kyiv algebraic school. It originated from the questions of the theory of representations, but now has proved its efficiency in many areas, such as algebraic geometry, algebraic topology, linear algebra, theory of groups etc. Certainly, I could not embraced all achievements or even all directions of investigation, so their choice in the paper is rather subjective. Moreover, I only consider the "classical" results, not involving the new investigations and applications to algebraic geometry and algebraic topology (see surveys [20, 21]).

Анотація. Ця стаття присвячена *теорії матричних задач*, новій галузі сучасної алгебри, яка була створена й розвинена значною мірою Київською алгебричною школою. Вона виникла з проблем теорії зображень, але зараз вже довела свою ефективність у багатьох галузях, таких як алгебрична геометрія, алгебрична топологія, лінійна алгебра, теорія груп, тощо. Звичайно, я не міг охопити всі досягнення або хоча б усі напрямки досліджень, тож їхній вибір у статті досить суб'єктивний. Більш того, я обмежився лише «класичними» результатами, не торкаючись новітніх досліджень і застосувань до алгебричної геометрії та алгебриченої топології (дивись огляди [20, 21]).

2010 Mathematics Subject Classification: 16G20,16G60,15A21
УДК 512.64+512.552
Ключові слова: algebras,quivers,posets,representations, box, bimodule,tame,wild

1. Introduction

The new history of the representations theory of finite dimension algebras starts with the *Brauer–Thrall conjectures*. We give corresponding definitions and formulate these conjectures. In waht follows $\Bbbk$ denotes a field and we consider algebras over this field. If the opposite is not stated, all algebras and modules are supposed to be finite dimensional. By $r(d, A)$ we denote the number of isomorphism classes of indecomposable A-modules of dimension d. (possibly $r(d, A) = \infty$). Recall that for finite dimensional representations the Krull–Schmidt theorem holds true [22], that is every represemtations decomposes into a direct sum of indecomposables and this decomposition is unique up to isomorphism and permutation of summands.

Означення 1.1. *They say that an algebra A is*

- *of* finite representations type *if it only has finitely many non-isomorphic indecomposable modules;*
- *of* bounded representations type *if the dimensions of indecomposable A-modules are bounded, i.e. there is an integer d_0 such that $r(d, A) = 0$ for $d > d_0$;*
- *of* strongly unbounded representations type *if there are infinitely many d such that $r(d, A) = \infty$.*

Obviously, if the field $\Bbbk$ is finite, bounded representation type is the same as finite representation type and the strictly unbounded representation type is impossible.

Гіпотеза 1.2 (see [9],[49],[28]).

1st. *If an algebra A is of bounded representation type, it is actually of finite representation type.*

2nd. *If an algebra A over an infinite field $\Bbbk$ is of unbounded representation type, it is actually of strictly unbounded representation type.*

These conjectures we proved by Yoshii [50] if $(\operatorname{rad} A)^2 = 0$. There were more papers devoted to these conjectures, until in 1968 A. Roiter proved the 1st Brauer–Thrall conjecture completely [42]. It was, in some sense, a sensation which gave a wide publicity to the Kyiv school of the theory of representations.

The 2nd Brauer–Thrall conjecture turned out to be much more difficult. To approach it, L. Nazarova and A. Roiter started in 1970-es to develop a new branch of the representation theory, namely, the theory of *matrix problems*. The idea was completely clear and evident. Every representation of an algebra is a set of matrices. Isomorphic representations correspond to conjugate sets. One knows, from the standard course of linear algebra, how to deal with one matrix: there is the Jordan or Frobenius normal form. Why not to reduce one of the matrices to this normal form, fix it and consider only conjugations that do not disturb it.

Though the idea was so simple, its realization was far from being so. The thing is that there are relations between the matrices defining a representation and it is usually very troublesome to follow them during the described process of reduction. Actually, it seemed to be non-realistic. A countermeasure was first found in homological algebra, a powerful tool of linearization non-linear problems. Indeed, if we fix an ideal $I \subset A$ and consider its annihilator J, we can include any module M into an exact sequence $0 \to N \to M \to L \to 0$, where $N = MI$ and $L = M/MI$. Then N is a module over the quotient A/J and L is a module over the quotient A/I. These new algebras are of smaller dimensions (at least if $I \subseteq \operatorname{rad} A$), so one can suppose that the conjecture holds true for them and try to use induction.

A problem is that non-equivalent extensions (in the sense of homological algebra) can produce isomorphsic modules. On the level of the group $\operatorname{Ext}^1_A(L,N)$ it is always so if the corresponding elements can be transformed to one another by automorphisms of N and L. Moreover, usually one can choose the ideal I in such a way that this condition is also necessary in order that two extensions produce isomorphic modules.

The new problem looks much better, since $\operatorname{Ext}^1_A(L,N)$ is vector space and there are no relations between its elements. If we know a decomposition of L and N into a direct sum of indecomposable modules, the elements of $\operatorname{Ext}^1(L,N)$ are presented by sets of matrices (with no relations). The automorphisms of N and L are linear algebraic groups, so we just have to find orbits of their action. The realization of this idea gave origin to the theore of matrix problems. Roughly speaking, it is the study of some specific actions of special algebraic groups on special vector spaces. In what follows I will try to explain, what do the words ‘specific’ and ‘special’ mean in this context.

In [38] L. Nazarova and A. Roiter announced that this method was applied to prove the 2nd Brauer–Thrall conjecture, the details were contained in their preprint [40]. Unfortunately, it was never published in a journal paper. So the first such papers with a complete proof of the 2nd Brauer–Thrall conjecture only appeared in 1985 [1, 7, 25].

2. Representations of quivers and posets

The first two papers on matrix problems appeared almost simultaneously. They were the paper of P. Gabriel [26] and that of L. Nazarova and A. Roiter [39].

P. Gabriel introduced the notion of *representations of quivers*. Actually, a quiver is an oriented graph Γ, perhaps with multiple edges and loops. We denote by Γ_0 the set of its vertices, by Γ_1 the set of its arrows and suppose that both Γ_0 and Γ_1 are finite. We write $a : i \to j$ if i is the source and j is the target of the arrow a.

Означення 2.1. (1) *A* representation of the quiver Γ *over a field* $\Bbbk$ *is a map V that maps every vertex i to a vector space $V(i)$ over* $\Bbbk$ *and every arrow $a : i \to j$ to a linear map $V(a) : V(i) \to V(j)$.*

(2) *A* morphism *of a representation V to a representation W is a map φ that maps every vertex i to a linear map $\varphi(i) : V(i) \to W(i)$ so that $\varphi(j)V(a) = W(a)\varphi(i)$ for each arrow $a : i \to j$.*

In particular, the representations V and W are isomorphic if and only if there are isomorphisms of vector spaces $\varphi(i) : V(i) \to W(i)$ such that $W(a) = \varphi(j)V(a)\varphi(i)^{-1}$ for every arrow $a : i \to j$.

(3) *The* dimension $\mathbf{dim}\, V$ *of the representation V is the vector $\mathbf{d} = (d_i)_{i\in\Gamma_0}$, where $d_i = \dim V(i)$.*

In other words, a representation of quiver is a set of linear maps between vector spaces, and two representations are isomorphic if one can choose bases in these spaces in two ways so that the matrices of the first set of operators with respect to the first choice are the same as the matrices of the second set of operators with respect to the second choice. Thus it is a rather general problem of linear algebra and the classification of such representations can be considered as the problem of finding canonical forms of matrices of such sets of linear maps. For the simplest quiver $\bullet \to \bullet$ it is one linear map $A : V \to W$. For the loop $\bullet \circlearrowright$ it is one linear map $A : V \to V$, when the classification of representations is given by the Jordan or the Frobenius normal form. For the *Kronecker quiver* $\mathbf{K} :\ \bullet \rightrightarrows \bullet$ it is the problem on *matrix pencils* solved by Kronecker.

Obviously, one can define the *representations type* of a quiver in the same way as for an algebra. In [26] P. Gabriel proved both Brauer–Thrall conjectures for quivers. Moreover, he gave a criterion for a quiver to be of finite representation type and described the indecomposable representations in the finite case. Namely, he considered the quadratic form $Q_\Gamma = \sum_{i\in\Gamma_0} x_i^2 - \sum_{\substack{a\in\Gamma_1\\ a:i\to j}} x_i x_j$, known now as the *Tits form* of the quiver Γ, and proved the following theorem.

Теорема 2.2. (1) *A quiver Γ is of finite representation type if and only if the form Q_Γ is positive definite, i.e. $Q_\Gamma(x) > 0$ for every non-zero vector $\mathbf{x}$. If not, it is of strictly unbounded representation type.*

(2) *In this case there is an indecomposable representation V of the quiver Γ such that* $\mathbf{dim}\, V(i) = \mathbf{d}$ *if and only if* $Q_\Gamma(\mathbf{d}) = 1$. *Moreover, for such dimensions there is only one, up to isomorphism, indecomposable representation.*

Actually, (2) was obtained from an explicit description of all indecomposable representations.

Note that the orientation of the arrows play no role in this theorem, since it does not imply the Tits form. So the answer only depends on the underlying non-oriented quiver $|\Gamma|$. The list of connected non-oriented graphs with positive definite Tits form had been known for years and was closely related to the theory of Lie algebras and groups. They are the so called *Dynkin graphs* (see Table 1). Moreover, the integral vectors $\mathbf{d}$ such that $Q_\Gamma(\mathbf{d}) = 1$ are just the *roots* corresponding to these graphs, which are also of great importance in Lie theory.

Таблиця 2.1. Dynkin graphs

$$A_n:\ 1 — 2 — 3 \cdots (n-1) — n$$

$$D_n:\ \begin{array}{l} 1 \diagdown \\ \qquad 3 — 4 \cdots (n-1) — n \qquad (n \geq 4) \\ 2 \diagup \end{array}$$

$$E_6:\ \begin{array}{c} \qquad\qquad 6 \\ \qquad\qquad | \\ 1 — 2 — 3 — 4 — 5 \end{array}$$

$$E_7:\ \begin{array}{c} \qquad\qquad 7 \qquad\qquad \\ \qquad\qquad | \qquad\qquad \\ 1 — 2 — 3 — 4 — 5 — 6 \end{array}$$

$$E_8:\ \begin{array}{c} \qquad\qquad 8 \qquad\qquad\qquad \\ \qquad\qquad | \qquad\qquad\qquad \\ 1 — 2 — 3 — 4 — 5 — 6 — 7 \end{array}$$

L. Nazarova and A. Roiter considered another class of matrix problems, the so called *representations of posets* (*partially ordered sets*). We give the version of their definition proposed by P. Gabriel.

Означення 2.3. *Let $\mathfrak{S}$ be a finite poset.*

(1) *A* representation of the poset $\mathfrak{S}$ *over the field* $\Bbbk$ *is a map V that maps every element $i \in \mathfrak{S}$ to a subspace $V(i)$ of a vector space $V(0)$ over* $\Bbbk$ *so that $V(i) \subseteq V(j)$ if $i \leq j$ in $\mathfrak{S}$.*

(2) *A* morphism *of a representation V to a representation W is a linear map $\varphi : V(0) \to W(0)$ such that $\varphi(V(i)) \subseteq W(i)$ for every $i \in \mathfrak{S}$.*

In particular, the representations V and W are isomorphic if and only if there is an isomorphism of vector spaces $\varphi : V(0) \to W(0)$ such that $\varphi(V(i)) = W(j)$ for every element $i \in \mathfrak{S}$.

(3) *The* dimension $\mathbf{dim}\, V$ *of the representation V is the vector $\mathbf{d} = (d_0, d_i)_{i\in\mathfrak{S}}$, where $d_i = \dim V(i)$.*

It is very easy to translate this definition to the original matrix one of [39]. Choose a basis $v_1, v_2, \dots, v_n$ of the space $V(0)$ and for every $i \in \mathfrak{S}$ choose a basis $u_1^i, u_2^i, \dots, u_{m_i}^i$ of $V(i)$ modulo $\sum_{j<i} V(j)$. Let $u_k^i = \sum_{j=1}^n a_{jk}^i v_j$. Then V is given by the set of matrices $A_1, A_2, \dots, A_s$, where $A_i = (a_{jk}^i)_{n\times m_i}$. If we change the basis of $V(0)$, the matrix A_i changes to $S(0)A_i$ for an invertible $n \times n$ matrix $S(0)$. A bit more complicated is the change of the basis $u_1^i, u_2^i, \dots, u_{m_i}^i$, since we can also add to every u_k^i the vectors from $V(j)$ for $j < i$. Therefore, under such changes A_i is transformed to $A_iS(i) + \sum_{j<i} A_jS_{ij}$. Alltogether, two such matrix representations $\{A_i\}$ and $\{B_i\}$ are isomorphic if and only if there are invertible matrices $S(0)$ (of size $n \times n$) and $S(i)$ (of size $m_i \times m_i$) and matrices S_{ij} (of size $m_i \times m_j$) for $j < i$ such that $B_i = S(0)^{-1}\big(A_iS(i) + \sum_{j<i} S(j)S_{ij}\big)$. It is just the matrix definition of representations of posets from [39].

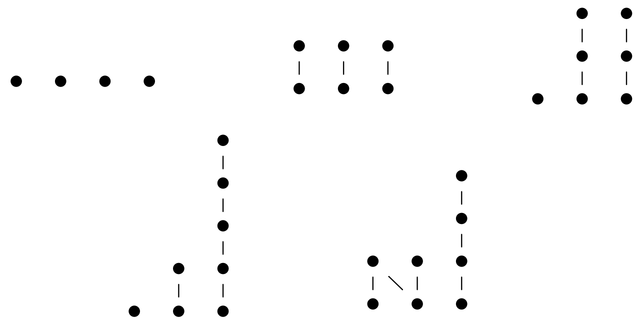
The paper [39] was devoted to the proof of the Brauer–Thrall conjecture for representations of posets.

Теорема 2.4. *Every poset is either of finite representation type or of strictly unbounded representation type.*

The proof rested upon an algorithm of reduction. Namely, choosing a maximal element $i \in \mathfrak{S}$, they constructed, for every representation V a new representation V' of a new poset $\mathfrak{S}'$ (derived poset) such that the correspondence $V \mapsto V'$ reflected isomorphisms and was one-to-one, except several "trivial" representations. In particular, representations type of $\mathfrak{S}'$ was the same as that of $\mathfrak{S}$. Afterwords, they proved that after several steps the iterated derived poset becomes either trivial or of width 4 (i.e. contaning 4 non-comparable elements).In the first case $\mathfrak{S}$ is of finite representation type, in the second case it is of strictly unbounded representation type.

This paper did not contain neither a criterion for a poset to be of finite representation type nor a description of indecomposable representatioins. These problems were solved by M. Kleiner [30],[29]. In the first paper he gave the following criterion.

Теорема 2.5. *A poset $\mathfrak{S}$ is of finite representation type if and only if it does not contain subposets of the following forms:*



These posets are called *critical* and usually named, respectively, $\{1,1,1,1\}$, $\{2,2,2\}$, $\{$ and $\{\mathsf{N},4\}$.

A simpler proof of this result was proposed by A. Roiter in [45].

Note that, contrary to the case of quivers, there is no "good" list of posets of finite representation type and their number grows rapidly with the growth of the number of elements.

In the second paper M. Kleiner gave a complete descriprion of indecomposable representations. It was possible, since, as he proved, there is only a finite number of posets of finite representation type which have *sincere* indecompoosable representations, that is such that all matrices A_i are non-empty, i.e. $\sum_{j<i} V(j) \neq V(i)$. As this list is rather big, we do not present it here.

As M. Kleiner remarked (oral communication on a seminar), these results could also be formulated in terms of some quadratic form. Namely, define the *Tits form* of the poset $\mathfrak{S}$ as $Q_{\mathfrak{S}} = x_0^2 + \sum_{i\in\mathfrak{S}} x_i^2 + \sum_{j<i} x_i x_j - \sum_{i\in\mathfrak{S}} x_0 x_i$.

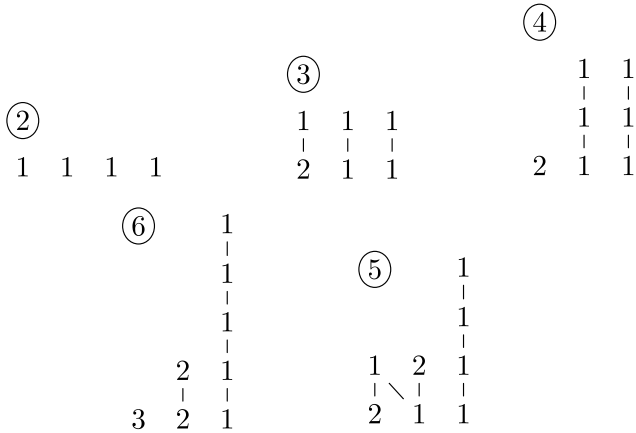
Зауваження 2.6 (Kleiner's remark)**.** (1) $\mathfrak{S}$ is of finite representation type if and only if the form $Q_{\mathfrak{S}}$ is *weakly positive* in the sense that $Q_{\mathfrak{S}}(\mathbf{x}) > 0$ for any nonzero vector with non-negative coordinates.

(2) If $\mathfrak{S}$ is of finite representation type, the dimensions $\mathbf{d}$ of its indecomnposable representations are just the *roots* of this form in the sense that $Q_{\mathfrak{S}}(\mathbf{d}) = 1$.

Note that if the Tits form of a quiver is weakly positive, it is positive definite, but it is not so for the Tits forms of posets.

Theorem 2.5 was further generalized by Yu. Drozd and E. Kubichka [23]. Namely, they considered representations over infinite fields and *dimensions of finite type*, i.e. such that there is finitely many non-isomorphic representations of this dimension, and proved the following result. We call a dimension $\mathbf{d}$ *critical*, if its *support* $\{i \in \mathfrak{S} \mid d_i \neq 0\}$ is a critical set and the restriction of $\mathbf{d}$ onto its support is of one of the dimensions from Table 2.2.

Таблиця 2.2. Critical dimensions



Here we write dimensions d_i instead of the elements i of $\mathfrak{S}$. The encircled number denotes the dimension of $V(0)$.

We write $\mathbf{d}' \leq \mathbf{d}$ if $d'_i \leq d_i$ for all i. Using the approach analogous to [45], they proved the following result.

Теорема 2.7. *(1) The following conditions for a dimension* $\mathbf{d}$ *are equivalent:*

(a) $\mathbf{d}$ *is of finite type.*

(b) $Q_{\mathfrak{S}}(\mathbf{d}') > 0$ *for any* $\mathbf{d}' \leq \mathbf{d}$.

(c) There is no critical dimension $\mathbf{d}' \leq \mathbf{d}$.

(2) If $\mathbf{d}$ *is of finite type, the following conditions are equivalent:*

(a) There is an indecomposable representation of dimension $\mathbf{d}$.

(b) $Q_{\mathfrak{S}}(\mathbf{d}) = 1$.

Moreover, in this case there is a unique (up to isomorphism) indecomposable representation V *of dimension* $\mathbf{d}$ *and* $\operatorname{End} V \simeq \Bbbk$.

Obviously, this theorem implies the Kleiner's criterion (Theorem 2.5) as well as his remark (Remark 2.6). An analogous result for quivers was obtained by P. Magyar, J. Weyman and A. Zelevinsky [34].

Зауваження 2.8. Note that in all theorems 2.2–2.7 the ground field $\Bbbk$ plays no role. The answer does not depend on this field.

3. Reflections and Coxeter functors

The proofs of Theorems 2.2–2.5 rested on some matrix calculations. Actually, as J. Tits remarked, rather simple geometrical considerations show that if a quiver Q_Γ is of finite type, then Q_Γ is positive definite, as well as if a poset $\mathfrak{S}$ is of finite type, then $Q_{\mathfrak{S}}$ is weakly positive. One only has to compare the dimension of the space of representations of a prescribed dimension and the dimenion of the group of transformations that define isomorphisms of representations. Since there was also a one-to-one correspondence between dimensions of representations and positive roots of the quiver, one could presume that there must be an *a priory* way to establish this correspondence as well as to prove that the positive definiteness of the Tits form implies finite representation type. This presumption was realized by I. Benstein, I. Gelfand and A. Ponomarev [4]. Their idea was the *categorification* of reflections, which played an important role in the study of Lie algebras and Coxeter groups.

Namely, let Q be a quadratic form on $\mathbb{R}^n$, B be the corresponding symmetric bilinear form $\mathbf{e} \in \mathbb{R}^n$ be a vector with $Q(\mathbf{e}) \neq 0$. The *reflection with respect to* $\mathbf{e}$ is defined as the linear map $s_{\mathbf{e}} : \mathbf{x} \mapsto \mathbf{x} - 2\frac{B(\mathbf{x},\mathbf{e})}{Q(\mathbf{e})}$. This map is *isometric*: $Q(s_{\mathbf{e}}\mathbf{x}) = Q(\mathbf{x})$, and *involutive*: $s_{\mathbf{e}}^2 = \mathrm{id}$. If $Q = Q_\Gamma$ is the Tits form of a quiver Γ and $\mathbf{e}_i$ $(i \in \Gamma_0)$ are the basic vectors (the i-th coordinate equals 1, all other coordinates are 0), set $s_i = s_{\mathbf{e}_i}$. The subgroup $W \subseteq \mathrm{GL}(n, \mathbb{R})$ generated by these reflections is called the *Weyl group* of the quiver. It is known (and rather easy to prove) that if Q_Γ is positive defiite, every root is of the form $w\mathbf{e}_i$ for some i and some $w \in W$. Moreover, in this case W is finite, so there are finitely many roots.

I. Benstein, I. Gelfand and A. Ponomarev categorified these reflections. Namely, if the vertex i is either a source (or a sink) in the quiver Γ, they defined the action of s_i on representations. Actually, if V is a representation of Γ, s_iV is a representation of the quiver $s_i\Gamma$ obtained by reversing all arrows that starts (or ends) at i. Moreover, except trivial cases, $\mathbf{dim}\, s_iV = s_i\, \mathbf{dim}\, V$. Using this categorification, they proved the following theorem. They used a numeration of the vertices $i_1, i_2, \dots, i_n$ such that i_1 is a source, i_2 is a source in $s_{i_1}\Gamma$, i_3 is a source in $s_{i_2}s_{i_1}\Gamma$ etc. The crucial role in their proof played the *Coxeter transformation* $C = s_{i_n} \dots s_{i_2}s_{i_1}$.

Теорема 3.1. *Let Q_Γ be positive definite. For every indecomposable representation V there is an integer m such that $V = s_{i_1}s_{i_2}\dots s_{i_m}\mathbf{e}_{i_{m+1}}$ (here we set $i_{k+nq} = i_k$). The integer m is uniquely defined by* $\mathbf{dim}\, V$. *On the contrary, all such representations are indecomposable.*

Obviously, it implies that Γ is of finite representation type and there is a one-to-one correspondence between indecomposable representations and positive roots.

The Kleiner's remark 2.6 gave a hint that something similar could be done for representations of posets. Indeed, in [14] Yu. Drozd constructed an analogue C of the Coxeter transformation for representations of posets and proved the following result.

Лема 3.2. *Let $Q_{\mathfrak{S}}$ is weakly positive. Then every indecomposbale representation is of the form C^kU, where either U is not sincere or $U = U_0$, where $U_0(0) = \Bbbk = U_0(i)$ for all $i \in \mathfrak{S}$.*

It implies the main theorem.

Теорема 3.3. *Let $Q_{\mathfrak{S}}$ is weakly positive.*

(1) $\mathfrak{S}$ is of finite representation type.

(2) There is an indecomposable representation of dimension $\mathbf{d}$ if and only if $Q_{\mathfrak{S}}(\mathbf{d}) = 1$. Moreover, in this case there is a unique (up to isomorphism) indecomposable representation of dimension $\mathbf{d}$.

The weakness of this paper was that the author could not construct all reflections. Actually, only two operations, σ and ρ, were constructed, where ρ could be considered as the reflection s_0, while σ replaced the product of all reflections s_i $(i \in \mathfrak{S})$. To improve this defect, in the paper [17] a generalization of representations of posets was considered.

Означення 3.4. *(1) A* bisected poset *is a poset $\mathfrak{S}$ together with a partition $\mathfrak{S} = \mathfrak{S}^+ \sqcup \mathfrak{S}^-$ such that if $i \in \mathfrak{S}^-$ and $j < i$, then also $j \in \mathfrak{S}^-$.*

We set $\hat{\mathfrak{S}} = \mathfrak{S} \cup \{0\}$, where $0 \notin \mathfrak{S}$ is a new symbol.

(2) A representation *V of such a bisected poset consists of vector spaces $V(i)$ $(i \in \hat{\mathfrak{S}})$ and linear maps $v(i)$ $(i \in \mathfrak{S})$, where $v(i) : V(i) \to V(0)$ if $i \in \mathfrak{S}^-$ and $v(i) : V(0) \to V(i)$ if $i \in \mathfrak{S}^+$, such that $V(i)V(j) = 0$ if $j < i$, $j \in \mathfrak{S}^-$, $i \in \mathfrak{S}^+$.*

(3) A morphism *φ of a representation V to a representation W is a set of linear maps $\varphi(i) : V(i) \to W(i)$ $(i \in \hat{\mathfrak{S}})$ and $\varphi(ij) : V(i) \to W(j)$, where $i < j$ and either $i, j \in \mathfrak{S}^+$ or $i, j \in \mathfrak{S}^-$, such that*

$$\varphi(0)v(i = w(i)\varphi(i) + \sum_{j<i} w(j)\varphi(ji) \quad \text{if } i \in \mathfrak{S}^-,$$

$$w(i)\varphi(0) = \varphi(i)v(i) + \sum_{i<j} \varphi(ij)v(j) \quad \text{if } i \in \mathfrak{S}^+.$$

One easily checks that φ is an isomorphism if and only if all $\varphi(i)$ are so.

Зауваження 3.5. (1) If $\mathfrak{S}^+ = \emptyset$, this definition becomes equivalent to the usual definition of representations of a poset.

(2) This definition can be naturally formulated in the language of *boxes* (see Section 4), as it was done in [17].

The Tits form for a bisected poset does not depend on the partition, it is the same form $Q_{\mathfrak{S}} = x_0^2 + \sum_{i\in\mathfrak{S}} x_i^2 + \sum_{j<i} x_i x_j - \sum_{i\in\mathfrak{S}} x_0 x_i$. An element $i \in \hat{\mathfrak{S}}$ is called *positive* if either i is minimal in $\mathfrak{S}^+$ or $i = 0$ and $\mathfrak{S}^+ = \emptyset$. It is called *negative* if either i is maximal in $\mathfrak{S}^-$ or $i = 0$ and $\mathfrak{S}^- = \emptyset$. In [17] all reflections s_i are defined for positive or negative $i \in \hat{\mathfrak{S}}$. Just as before, when doing reflections, the bisposet also changes. Namely, the positive (respectively, negative) vertex i is displaced from $\mathfrak{S}^+$ to $\mathfrak{S}^-$ (respectively, from $\mathfrak{S}^-$ to $\mathfrak{S}^+$), becoming negative (respectively, positive). It is proved that, if $Q_{\mathfrak{S}}$ is weakly positive and V is sincere (that is all $V(i) \neq 0$), then $\mathbf{dim}\, s_i V = s_i\, \mathbf{dim}\, V$ and $s_i^2 V \simeq V$. It gives the proof of the following theorem.

Теорема 3.6. $\mathfrak{S}$ *is of finite representation type if and only if* $Q_{\mathfrak{S}}$ *is weakly positive. In this case*

(1) Every indecomposable representation is of the form $s_{i_1} s_{i_2} \dots s_{i_m} W$ *for some* m *and some non-sincere indecomposable representation* W.
(2) There is an indecomposable representation of dimension $\mathbf{d}$ *if and only if* $Q_{\mathfrak{S}}(\mathbf{d}) = 1$. *Moreover, such representation is unique up to isomorphism.*

4. Boxes, bimodules and reduction

In [31] A. Roiter and M. Kleiner gave a very general definition that had to cover most cases of matrix problems known at that time. It was formulated in terms of *differential graded categories*. An equivalent, but more easy-to-use formulation was proposed by A. Roiter in [44] in terms of *boxes* (or *bocses*, ukrainian , from *bimodule over a category with coalgebra structure*).

Означення 4.1. *(1) A* box *is quadruple* $\mathfrak{A} = (\mathcal{A}, \mathcal{V}, \mu, \varepsilon)$, *where* $\mathcal{A}$ *is a category,* $\mathcal{V}$ *is an* $\mathcal{A}$*-bimodule, i.e. an additive bifunctor* $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to$ **Vec** *(the category of vector spaces),* $\mu : \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \to \mathcal{V}$ *(*comultiplication*) and* $\varepsilon : \mathcal{V} \to \mathcal{A}$ *(*counit*) are morphisms of* $\mathcal{A}$*-bimodules furnishing* $\mathcal{V}$ *with the structure of* $\mathcal{A}$-coalgebra. *Recall that it means that the following diagrams are commutative:*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \arrow[d, "\mu\otimes\mathrm{id}"] \\ \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \arrow[r, "\mathrm{id}\otimes\mu"'] & \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \end{tikzcd} \qquad \textit{(coassociativity)}$$

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\sim"'] \arrow[d, "\mu"] \arrow[dr, "\sim"] & \\ \mathcal{V} \otimes_{\mathcal{A}} \mathcal{A} & \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \arrow[l, "\mathrm{id}\otimes\varepsilon"] \arrow[r, "\varepsilon\otimes\mathrm{id}"'] & \mathcal{A} \otimes_{\mathcal{A}} \mathcal{V} \end{tikzcd}$$ *(counit axiom)*

($\sim$ marks the natural isomorphisms).

(2) *A* representation *of the box $\mathfrak{A}$ is an $\mathcal{A}$-module, i.e. a functor $M : \mathcal{A} \to \mathbf{Vec}$.*

(3) *A* morphism *of a representation M to a representation N is a morphism of $\mathcal{A}$-modules $\varphi : \mathcal{V} \otimes_{\mathcal{A}} M \to N$.*

(4) *The product $\varphi\psi$ of morphisms $\varphi : M \to N$ and $\psi : L \to N$ is the composition*

$$\mathcal{V} \otimes \mathcal{A}L \xrightarrow{\mu\otimes\mathrm{id}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} L \xrightarrow{\mathrm{id}\otimes\psi} \mathcal{V} \otimes_{\mathcal{A}} M \xrightarrow{\varphi} N.$$

We denote by $\operatorname{Rep}\mathfrak{A}$ *the category of representations of the* $\mathfrak{A}$.

In most applications they use special classes of boxes.

Означення 4.2. *Let $\mathfrak{A} = (\mathcal{A}, \mathcal{V}, \mu, \varepsilon)$ be a box. Denote by $\bar{\mathcal{V}}$ the kernel* $\ker\varepsilon$. *A* section *is a morphism of (left) $\mathcal{A}$-modules (usually not of bimodules) $\omega : \mathcal{A} \to \mathcal{V}$. Then $\mathcal{V} = \operatorname{Im}\omega \oplus \bar{\mathcal{V}}$ as $\mathcal{A}$-module. The box $\mathfrak{A}$ is called*

- free *if $\mathcal{A}$ is a* free category *(i.e. isomorpic to the category of parths of a quiver Γ) and $\bar{\mathcal{V}}$ is a* free $\mathcal{A}$-bimodule, *i.e. a direct sum of bimodules of the form $\mathcal{A}e_i\mathcal{A}$, where e_i is the trivial path at a vertex i. In other words it means that there are sets of generators $\mathbf{A}_0$ of morphisms of the category $\mathcal{A}$ and $\mathbf{A}_1$ of the $\mathcal{A}$-bimodule $\bar{\mathcal{V}}$ such that there are no nontrivial relations between them. They say that $\mathbf{A} = \mathbf{A}_0 \cup \mathbf{A}_1$ is a* set of free generators *of the box $\mathfrak{A}$.*
- normal *if there is a* section *ω such that $\mu(\omega_i) = \omega_i \otimes \omega_i$ for every vertex $i \in \Gamma$, where $\omega_i = \omega(e_i)$.*
  In this case we set $\partial a = a\omega_i - \omega_j a$ for a morphism $a : i \to j$ and $\partial v = \mu(v) - v \otimes \omega_i - \omega_j \otimes v$, where $v \in \bar{\mathcal{V}}(i,j)$. Note that $\partial a \in \bar{\mathcal{V}}$ and $\partial v \in \bar{\mathcal{V}} \otimes_{\mathcal{A}} \bar{\mathcal{V}}$.
- triangular *or* Roiter box *if it is free, normal and the set $\mathbf{A}$ of free generators of $\mathcal{A}$ can be linear ordered so that, for every $a \in \mathbf{A}$, ∂a is contained in the subbox generated by the set $\{b \in \mathbf{A} \mid b < a\}$. Such set of generators is called* triangular.

Зауваження 4.3. If $\mathcal{V} = \mathcal{A}$, the box $\mathfrak{A}$ is called the *principal box* over the category $\mathcal{A}$. It is quite obvious that in this case $\operatorname{Rep}\mathfrak{A}$ coincides with the category $\mathcal{A}$-mod of $\mathcal{A}$-modules.

The category of representations $\operatorname{Rep}\mathfrak{A}$ of a Roiter box $\mathfrak{A}$ has “usual” properties of the categories of modules, as was proved in [31]. Namely, for a morphism $\varphi : M \to N$ and an element $\xi \in \mathcal{V}(i,j)$ we denote by $\varphi(\xi) : M(j) \to N(i)$ the map sending $x \in M(j)$ to $\varphi(\xi \otimes x) \in N(i)$.

Теорема 4.4. *Let $\mathfrak{A}$ is a Roiter box, $M, N \in \operatorname{Rep}\mathfrak{A}$.*

(1) A morphism $\varphi : M \to N$ is an isomorphism if and only if $\varphi(\omega_i)$ is an isomorphism for every i.

(2) $\operatorname{Rep}\mathfrak{A}$ *is* fully additive *(or* Karubian*, or* idenpotent complete*), that is, for every idempotent endomorphism e of M there are morphisms*

$$M \underset{\iota}{\overset{\pi}{\rightleftarrows}} M' \quad \text{such that } e = \iota\pi \text{ and } \pi\iota = \mathrm{id}_{M'}.$$

(Then $M \simeq M' \oplus M''$, where M'' is the object constructed in the same way from the idempotent $\mathrm{id}_M - e$.)

A lot of applications of boxes appear through another categorical constructi-on, namely, *bimodule categories*, first considered in [12].

Означення 4.5. *Let $\mathcal{B}$ be a bimodule over a category $\mathcal{A}$. The* bimodule category $\mathrm{El}(\mathcal{B})$ *or the* category of elements *of the bimodule $\mathcal{B}$ is defined as follows.*

- *Its set of objects is $\bigcup_{i \in \operatorname{Ob}\mathcal{A}} \mathcal{B}(i,i)$.*
- *If $A \in \mathcal{B}(i,i), B \in \mathcal{B}(j,j)$, a morphism $A \to B$ is a morphism $\varphi : i \to j$ such that $\varphi A = B\varphi$ (note that both paths are in $\mathcal{B}(i,j)$).*
- *The bimodule $\mathcal{B}$ is called* locally finite dimentional *if all spaces $\mathcal{A}(i,j)$ and $\mathcal{B}(i,j)$ are finite dimensional.*

For instance, such a bimodule category appeared in the Introduction, where $\mathcal{A}$ was the product of the categories of A/I-modules and A/J-modules, while $\mathcal{B}(X,Y) = \operatorname{Ext}^1_A(X,Y)$.

Another application of the bimodule categories was proposed by Y. Drozd in [16]. Let A be a finite dimensional algebra, A-pro be the category of projective A-modules, $\mathcal{A} = A\text{-pro} \times A\text{-pro}$, $\mathcal{B}$ be the $\mathcal{A}$-bimodule such that $\mathcal{B}(Q,P) = \operatorname{Hom}_A(Q, \operatorname{rad} P)$, $I \subset \mathrm{El}(\mathcal{B})$ be the ideal generated by the objects which are homomorphisms of the form $Q \to 0$ and $C : \mathrm{El}(\mathcal{B})/I \to A\text{-mod}$ be the functor that maps a homomorphism α to its cokernel, .

Твердження 4.6. *The functor C is an equivalence of the categories* $\mathrm{El}(\mathcal{B})/I \xrightarrow{\sim} A\text{-mod}$.

Note that the indecomposable objects of $\mathrm{El}(\mathcal{B})/I$ are the same as in $\mathcal{B}$, except the “trivial” objects $Q \to 0$, where Q is an indecomposable projective A-module.

Приклад 4.7. Let $\Bbbk$ be algebraically closed and $(\operatorname{rad} A)^2 = 0$. If $P_1, P_2, \dots, P_n$ are all pairwise non-isomorphic indecomposable modules, the indecomposable objects of $\mathcal{A}$ are the pairs (P_i, P_j) and if $a \in \operatorname{End}_A P$ is in radical, it acts trivially on $\mathcal{B}$. Therefore, we can factor out such morphisms. Reduced in this way, the bimodule category $\operatorname{El}(\mathcal{B})$ can be considered as that of representations of the quiver Γ whose vertices are $\{1, 2, \dots, n, 1', 2', \dots, n'\}$ and there are c_{ij} arrows $i \to j'$, where $c_{ij} = \dim \operatorname{Hom}_A(P_i, P_j)$.

This approach was used by P. Gabriel in [26], who deduced from his result on representations of quivers a criterion for an algebra A with $(\operatorname{rad} A)^2 = 0$ to be of finite representation type. The same approach was used by S. Kruglyak [33]. In fact, he also proved the Gabriel's Theorem 2.2 for this specific sort of quivers and obtained the same finiteness criterion for algebras with $(\operatorname{rad} A)^2 = 0$.

There is a close relation of bimodule categories to representations of boxes. Namely, in [16] the following result was proved.

Теорема 4.8. *Suppose that $\Bbbk$ is algebraically closed and the bimodule $\mathcal{B}$ is locally finite dimensional. There is a Roiter box $\mathfrak{A}$ such that the categories* $\operatorname{Rep}\mathfrak{A}$ *and* $\operatorname{El}(\mathcal{B})$ *are equivalent.*

The box $\mathfrak{A}$ and this equiuvalence were explicitly constructed.

Together with Proposition 4.6, it gives a tool to replace the study of modules over an algebra A by representations of boxes.

The language of boxes has a powerful advantage. Namely, it allows to make *change of rings* easily. Note that if $\gamma : A \to B$ is a homomorphism of rings, every B-module can be considered as A-module, which defines a functor $\gamma^* : B\text{-mod} \to A\text{-mod}$. Unfortunately, this functor is far from being full, or faithful, or dense (essentially surjective). On the contrary, on the level of boxes an analogous change of rings can be done perfectly, as shown in [16] (see also [18]).

Теорема 4.9. *Let $\mathfrak{A} = (\mathcal{A}, \mathcal{V}, \mu, \varepsilon)$ be a box and $F : \mathcal{A} \to \mathcal{B}$ be a functor. Set $\mathcal{W} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B}$.*

(1) There are natural morphisms $\nu : \mathcal{W} \otimes_{\mathcal{B}} \mathcal{W} \to \mathcal{W}$ and $\theta : \mathcal{W} \to \mathcal{B}$ such that $\mathfrak{A}^F = (\mathcal{B}, \mathcal{W}, \nu, \theta)$ is a box.

Actually, ν arises from the map $\mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{V}$ that maps $a \otimes b$ to $a \otimes 1 \otimes b$ and θ is the composition

$$\mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{1\otimes\mu\otimes 1} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\mathsf{mult}} \mathcal{B},$$

where mult is the multiplication.

(2) *The natural map* $F^* : \operatorname{Rep}\mathfrak{A}^F \to \operatorname{Rep}\mathfrak{A}$ *gives a fully faithful functor which establishes an equivalence of the category* $\operatorname{Rep}\mathfrak{A}^F$ *and the full subcategory of* $\operatorname{Rep}\mathfrak{A}$ *consisting of representations of the form* MF*, where* M *is a* $\mathcal{B}$*-module.*

Often one can choose F such that every representation of $\mathfrak{A}$ is isomorphic to such composition. Then F^* is an equivalence $\operatorname{Rep}\mathfrak{A}^F \xrightarrow{\sim} \operatorname{Rep}\mathfrak{A}$.

Usually, the functor F is constructed as follows [18]. We choose a "simple" subcategory $\mathcal{A}' \subset \mathcal{A}$ such that the $\mathcal{A}'$-modules can be easily described. It gives rise to a fucntor $F' : \mathcal{A}' \to \mathcal{B}'$ such that every $\mathcal{A}'$-module of some class $\mathcal{C}$ is isomorphic to $M'F'$ for some $\mathcal{B}'$-module. Then we take for $\mathcal{B}$ the *pullback* $\mathcal{A} \coprod_{\mathcal{A}'} \mathcal{B}'$ of $\mathcal{A}$ and $\mathcal{B}'$ over $\mathcal{A}'$ and for F the extension of F' onto $\mathcal{A}$. In this case the image of F^* contains all representations such that their restrictions onto $\mathcal{A}'$ are form the class $\mathcal{C}$. Note that there is a rather simple algorithm that calculates the box $\mathfrak{A}^F$. One can see [18, Sec. 6] for examples of explicit calculations.

5. Tame and wild algebras

When studying the algebras of infinite representation type, they found out that there are two completely different sorts of them. The first paper where it was remarked was, pehaps, that of S. Krugluak [32]. Namely, he considered representations of the group P of type (p,p): $G = \langle a, b \mid a^p = b^p = 1, ab = ba\rangle$, for prime $p > 2$. He noticed that for any n-tuple of square matrices $X_1, X_2, \dots, X_n$ one can construct a representation $M(X_1, X_2, \dots, X_n)$ of the group P depending on this tuple so that $M(X_1, X_2, \dots, X_n) \simeq M(Y_1, Y_2, \dots, Y_n)$ if and only if there is an invertible matrix S such that $Y_i = S^{-1}X_iS$ for all i. Thus, to classify all representations of G one has to classify all tuples of matrices up to conjugation. Quite another story is if $p = 2$, when, for each dimension, there is either finitely many indecomposable representations or they form a 1-parameter family [2].

An analogous effect was considered by I. Gelfand and V. Ponimarev [27] who considered pairs of commuting operators, or, the same, representations of the polynomial algebra $\Bbbk[x, y]$. Again, they constructed representations $M(X_1, X_2, \dots, X_n)$ with the same property as above. On the other hand, they have classified all pairs (A, B) of mutually annihilating matrices, i.e. such that $AB = BA = 0$. In 1972 Yu. Drozd divided all commutative finitely generated algebras over an algebraically closed field into two types [13], whose representations behaved just as representations of the (p, p) group, respectively, for $p = 2$ and for $p > 2$.

In their paper [11] P. Donovan and M. R. Freislich conjectured that all finite dimensional algebras of infinite representation type can be split into two classes, which they called *tame* and *wild*. Formal definitions of these two classes were proposed by Yu. Drozd in [15]. We present here a bit changed but equivalent version of this definition.

Означення 5.1. *An algebra A over an algebraically closed field $\Bbbk$ is called*

- tame *if, for every dimention d its indecomposable representations form a finite set of algebraic families parametrized by an open subset of the projective line;*
  (Note that it is is allowed that some of these families are trivial*, i.e. consist of isomorphic representations. Thus representation finite algebras are also tame according to this definition.)*
- wild *if for every finitely generated algebra B there is an exact functor $F : B\text{-mod} \to A\text{-mod}$ such that*
  - *$FM \simeq FN$ if and only if $M \simeq N$;*
  - *FM is indecomposable if and only if so is M.*

In the same paper it was proved that neither algebra can be both tame and wild.

By this time a lot of examples were already known when the "*tame-wild dichotomy*" took place. For representations of quivers P. Donovan and M. R. Freislich [10] and, independently, L. Nazarova [36] proved that a quiver Γ is tame if and only if its underlying graph is a disjoint union of *Euclidean* (or *extended Dynkin*) graphs presented in Table 5.1. Otherwise Γ is wild. Moreover, they gave an explicit description of representations of Euclidiean quivers. We only present the qualitative part of this description.

Теорема 5.2. *Let Γ be a Euclidean quiver, Q_Γ be its Tits form and $\mathbf{d}$ be a dimension for its representations.*

(1) There is an indecomposable representation of dimension $\mathbf{d}$ if and only if $Q_\Gamma(\mathbf{d}) \leq 1$.
(2) If $Q_\Gamma(\mathbf{d}) = 1$, there is exactly one indecomposable representation of dimension $\mathbf{d}$ (up to isomorphism).
(3) If $Q_\Gamma(\mathbf{d}) = 0$, there are infinitely many non-isomorphic indecomposable representations of dimension $\mathbf{d}$ that form an algebraic family of such representations parametrized by the projective line.

Note that every dimension $\mathbf{d}$ with $Q_\Gamma(\mathbf{d}) = 0$ is an integral multiple of the smallest one. The coordinates of such smallest dimensions are also given in Table 5.1.

Таблиця 5.1. Euclidean graphs

$$\tilde{A}_n: \ 1 - 1 - 1 \cdots 1 - 1$$

$$\tilde{D}_n: \ \begin{array}{ccccc} 1 & & & & 1 \\ & \diagdown & & \diagup & \\ & & 2 - 2 \cdots 2 & & \\ & \diagup & & \diagdown & \\ 1 & & & & 1 \end{array} \qquad (n \geq 4)$$

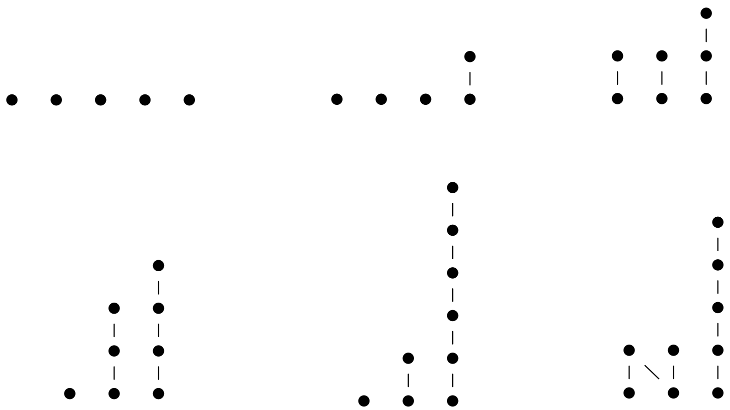
$$\tilde{E}_6: \ \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 1 - 2 - 3 - 2 - 1 \end{array}$$

$$\tilde{E}_7: \ 1 - 2 - 3 - \overset{\overset{2}{|}}{4} - 3 - 2 - 1$$

$$\tilde{E}_8: \ 2 - 4 - \overset{\overset{3}{|}}{6} - 5 - 4 - 3 - 2 - 1$$

In [37] L. Nazarova proved that a poset $\mathfrak{S}$ is tame if and only if it does not contain *supercritical subsets* listed in Table 5.2. Otherwise $\mathfrak{S}$ is wild.

Таблиця 5.2. Supercritical posets



These posets are called, respectively,
$(1,1,1,1,1)$, $(1,1,1,2)$, $(2,2,3)$, $(1,3,4)$, $(1,2,6)$ and $(\mathsf{N},5)$.

Finally, in [16] Yu. Drozd proved the result known now as "tame-wild dichotomy".

Теорема 5.3. *Every Roiter box, as well as every finite dimensional algebra over an algenraically closed field is either tame or wild.*

The proof for boxes was based on the reduction procedure described in Theorem 4.9. Using this procedure it turned necessary to consider a bit bigger class of boxes obtained from Roiter boxes by a localization with respect to a polynomial $f(a)$, where $a \in \mathbf{A}_0$ was a loop such that $\partial a = 0$. Then this result was extended to finite dimensional algebras using Proposition 4.6 and Theorem 4.8.

The same approach was used by V. Bekkert and Yu. Drozd [3, 19] to prove that tame-wild dichotomy also holds true for *derived categories* of modules over finite dimensional algebras. Another important result was obtained by Yu. Drozd and S. Ovsienko [24] who proved that tameness is preserved in *Galois coverings* with torsion free Galois groups. Note that coverings introduced by K. Bongartz and P. Gabriel [8] are now a powerful tool in the representation theory of algebras and in the study of matrix problems. Again, in these papers they had to widen the considered class of boxes.

Perhaps, the first paper, where the "tame-wild dichotomy" was effectively used, was that of V. Bondarenko and Yu. Drozd [6] devoted to the representation type of finite groups. Let G a finite group and $\Bbbk$ be a field of characteristic $p > 0$. By that time it had been known that the group algebra $\Bbbk G$ is representation finite if and only if the Sylov p-subgroup of G is cyclic. The result of Kruglyak [32] cited above had shown that, if $p > 2$, all other group algebras are wild. If $p = 2$, V. Bondarenko [5] and C. Ringel [41] described the representations of *dihedral* groups and actually showed that they are tame. It was also known that there were at most two more classes of 2-groups that are not wild. They were *quasi-dihedral* and *generalized quaternion* groups. In [6] the representations of quasi-dihedral groups were classified. From the tame-wild dichotomy it followed that any subgroup of a tame group is tame and if the Sylov p-subgroup is tame, so is the group G. Since generalized quatrenion groups are subgroups of quasi-dihedral ones, the following result was proved.

Теорема 5.4. *The group algebra $\Bbbk G$ is tame if and only if either the Sylov subgroup G_p of G is cyclic or $p = 2$ and G_p is dihedral, quasi-dihedral or generalized quaternion. In all other cases $\Bbbk G$ is wild.*

6. Involution

Papers [43] and [48] initiated the study of matrix problems with involution. Namely, in [43] A. Roiter introduced the general notion of *boxes with involution* and proved an important result, which was a far-reaching generalization of [35, § 23.2, Thm. 3].

Теорема 6.1. *Let $\mathfrak{A}$ be a box with involution over an algebraically closed field $\Bbbk$ of characteristic $\neq 2$, M, N be its self-adjoint representations in the involutive category of vector spaces with the standard involution $V \mapsto V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. If $M \simeq N$ in the category* Rep $\mathfrak{A}$, *they are* conguent, *i.e. there is an isomorphism $\phi : M \xrightarrow{\sim} N$ such that $\phi^{-1} = \phi^*$.*

In [48] this approach was applied to *simple involutive quivers*, i.e. such quivers with involution * that $a \neq a^*$ for any vertex a. The simplest example of such a quiver is

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}; \quad 1^* = 2,\ \alpha^* = \beta. \tag{6.1}$$

A self-adjont representation of this quiver is actually given by a linear map $V \to V^*$ or, the same, by a bilinear form in the space V. The Roiter's theorem 6.1 shows that to classify such forms one has to classify self-adjoint representations of the Kronecker quiver (6.1). Just in the same way, self-adjoint representations of a simple involutive quiver can be identified with systems of linear maps and bilinear forms. In [48] V. Sergeichuk defined the *scheme* of a simple involutive quiver Γ as the (non-oriented) graph whose vertices are in one-to-one correspondence with the pairs a, a^* of vertices of Γ and the edges between a, a^* and b, b^* are in one-to-one correspondence with the arrows $a \to b$, $a \to b^*$, $a^* \to b$ and $a^* \to b^*$[1]. For instance, the scheme of the involutive quiver (6.1) is just a loop $\begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$

In [48] V. Sergeichuk proved the following criterion.

Теорема 6.2. *With respect to the classification of self-adjoint representations a simple involutive quiver with the scheme* $\mathbf{B}$ *is*

- *representation finite if and only if* $\mathbf{B}$ *is Dynkin;*
- *tame if and only if* $\mathbf{B}$ *is Euclidean;*
- *wild otherwise.*

In the paper [47] V. Sergeichuk developed a general theory of representations of categories with involution. Namely, let $\Bbbk$ be a field with involution $\lambda \mapsto \bar{\lambda}$ (maybe, trivial). An *involution* * on a $\Bbbk$-linear category $\mathcal{C}$ maps

[1] Actually, the definition in [48] was a bit different but equivalent to this one.

objects to objects, morphisms to morphisms so that $a^{**} = a$, $(\alpha\beta)^* = \beta^*\alpha^*$, $(\alpha+\beta)^* = \alpha^* + \beta^*$ and $(\lambda\alpha)^* = \bar{\lambda}\alpha^*$ for $\lambda \in \Bbbk$. For a vector space V over $\Bbbk$ denote by V^* the space of semilinear maps $V \to \Bbbk$ and by $\varphi^* : W^* \to V^*$ the adjoint map for the linear map $\varphi : V \to W$. If $F : \mathcal{C} \to \mathbf{Vec}_{\Bbbk}$ is a representation of $\mathcal{C}$ over $\Bbbk$, the *adjoint representation* F^* is such that $F^*(a) = F(a^*)^*$ for any object or morphism a.

Suppose that the involution * is *simple*, i.e. $a \not\simeq a^*$ for every object a. Then there is a bisection $\operatorname{Ob}\mathcal{C} = \operatorname{Ob}_0 \sqcup \operatorname{Ob}_1$ such that if $b \simeq a^*$ the objects a and b are in different parts of this bisection. The following procedure was proposed in [47] for the description of self-adjoint representations of $\mathcal{C}$. Let $\operatorname{ind}_0\mathcal{C}$ be a full set of representatives of isomorphism classes of indecomposable *self-adjoint* representations (i.e. such that $F^* = F$) and $\operatorname{ind}_1\mathcal{C}$ be a full set of representatives of isomorphism classes of indecomposable representations which are not isomorphic to any self-adjoint one. For any representation $F \in \operatorname{ind}_1\mathcal{C}$ define the self-adoint representation F^+ as follows.

- If $a \in \operatorname{Ob}_0$, then $F^+(a) = F(a) \oplus F^*(a)$; if $a \in \operatorname{Ob}_1$, then $F^+(a) = F^*(a) \oplus F(a)$.
- If $\alpha : a \to b$, then the matrix presentation of $F^+(a)$ with respect to the preceding decompositions of $F^+(a)$ and $F^+(a)$ is
  - $\begin{pmatrix} F(\alpha) & 0 \\ 0 & F^*(\alpha) \end{pmatrix}$ if $a, b \in \operatorname{Ob}_0$;
  - $\begin{pmatrix} F^*(\alpha) & 0 \\ 0 & F(\alpha) \end{pmatrix}$ if $a, b \in \operatorname{Ob}_1$;
  - $\begin{pmatrix} 0 & F^*(\alpha) \\ F(\alpha) & 0 \end{pmatrix}$ if $a \in \operatorname{Ob}_0$, $b \in \operatorname{Ob}_1$;
  - $\begin{pmatrix} 0 & F(\alpha) \\ F^*(\alpha) & 0 \end{pmatrix}$ if $a \in \operatorname{Ob}_1$, $b \in \operatorname{Ob}_0$.

Obviously, $F^+ \simeq F \oplus F^*$, but the latter representation is not self-adjoint.

Let now $F \in \operatorname{ind}_0\mathcal{C}$, $\Lambda(F) = \operatorname{End} F$ and $\Delta(F) = \Lambda(F)/\operatorname{rad}\Lambda(F)$. Note that $\Lambda(F)$ is an algebra with involution * and $\Delta(F)$ is a skewfield with the induced involution, which we denote by $^-$. If $\phi \in \Lambda(F)$ is invertible and $\phi = \phi^*$, set $\tilde{\phi}(a) = 1_{F(a)}$ for $a \in \operatorname{Ob}_0$, $\tilde{\phi}(a) = \phi(a)$ for $a \in \operatorname{Ob}_1$ and define the representation F^ϕ as follows.

- $F^\phi(a) = F(a)$ for any object a.
- $F^\phi(\alpha) = \tilde{\phi}(b)^{-1}F(a)\tilde{\phi}(a)$ for $\alpha : a \to b$.

One easily sees that $\tilde{\phi}$ is an isomorphism $A^\phi \xrightarrow{\sim} A$ (though not a congruence). For a self-adjoint element $\xi \neq 0$ of the skewfield $\Delta(F)$ choose a self-adjoint preimage $\phi \in \Lambda(F)$ and set $F^\xi = F^\phi$. For a vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_m) \in \Delta(F)^m$ with invertible components set $F^{\boldsymbol{\xi}} = \bigoplus_{i=1}^m F^{\xi_i}$ and $Q_{\boldsymbol{\xi}} = \sum_{i=1}^m \bar{x}_i\xi_i x_i$ (it is a Hermitian form over the skewfield $\Delta(F)$).

The following theorem (see [47, Thm 1]) gives a complete description of self-adjoint representations.

Теорема 6.3. *Let $\Bbbk$ be a field of characteristic $\neq 2$, $\mathcal{C}$ be a $\Bbbk$-category with a simple involution. Every self-adjoint representation of $\mathcal{C}$ is congruent to a direct sum*

$$F_1^+ \oplus F_2^+ \oplus \ldots \oplus F_k^+ \oplus F_{k+1}^{\boldsymbol{\xi}_1} \oplus F_{k+2}^{\boldsymbol{\xi}_2} \oplus \ldots \oplus F_n^{\boldsymbol{\xi}_{n-k}},$$

where $F_i \in \operatorname{ind}_1 \mathcal{C}$ for $1 \leq i \leq k$ and $F_i \in \operatorname{ind}_0 \mathcal{C}$ for $k < i \leq n$. This decomposition is unique up to permutation of summands and replacing $F_{k+l}^{\boldsymbol{\xi}_l}$ by $F_{k+l}^{\boldsymbol{\xi}'_l}$ such that the Hermitian forms $Q_{\boldsymbol{\xi}_l}$ and $Q_{\boldsymbol{\xi}'_l}$ are equivalent over the skewfield $\Delta(F_{k+l})$.

This description becomes simpler if one knows the classification of hermitian forms, for instance, if $\Bbbk = \mathbb{C}$ (either with trivial or with non-trivial involution), or $\mathbb{R}$, or a finite field (see [47, Thm. 2]).

Applying this theorem to the representations of simple involutive quivers, one obtains classification results for a lot of problems of linear algebra such as classification of bilinear forms, some sorts of operators in spaces with bilinear metric etc. See, for instance, the book [46] for some results of this sort.

Література

[1] R. Bautista. On algebras of strongly unbounded representation type. *Comment. Math. Helv.*, 60(3):392–399, 1985, doi: 10.1007/BF02567422.

[2] V. A. Bašev. Representations of the group $Z_2 \times Z_2$ in a field of characteristic 2. *Dokl. Akad. Nauk SSSR*, 141:1015–1018, 1961.

[3] V. I. Bekkert, Yu. A. Drozd. Tame-wild dichotomy for derived categories. arXiv:math.RT/0310352, 2003.

[4] I. N. Bernšteĭn, I. M. Gelfand, V. A. Ponomarev. Coxeter functors, and Gabriel's theorem. *Uspehi Mat. Nauk*, 28(2(170)):19–33, 1973.

[5] V. M. Bondarenko. Representations of dihedral groups over a field of characteristic 2. *Mat. Sb. (N.S.)*, 96(138):63–74, 167, 1975.

[6] V. M. Bondarenko, Yu. A. Drozd. The representation type of finite groups. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 71:24–41, 282, 1977. Modules and representations.

[7] K. Bongartz. Indecomposables are standard. *Comment. Math. Helv.*, 60(3):400–410, 1985, doi: 10.1007/BF02567423.

[8] K. Bongartz, P. Gabriel. Covering spaces in representation-theory. *Invent. Math.*, 65(3):331–378, 1981/82, doi: 10.1007/BF01396624.

[9] R. Brauer. On the indecomposable representations of algebras. *Bull. Amer. Math. Soc.*, 47:684, 1941.

[10] P. Donovan, M. R. Freislich. *The representation theory of finite graphs and associated algebras.* Carleton University, Ottawa, Ont., 1973. Carleton Mathematical Lecture Notes, No. 5.

[11] P. Donovan, M. R. Freislich. Some evidence for an extension of the Brauer–Thrall conjectures. *Sonderforschungsbereich Theor. Math., Bonn*, 40, 1973.

[12] Yu. A. Drozd. Matrix problems, and categories of matrices. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 28:144–153, 1972. Investigations on the theory of representations.

[13] Yu. A. Drozd. Representations of commutative algebras. *Funkcional. Anal. i Priložen.*, 6(4):41–43, 1972.

[14] Yu. A. Drozd. Coxeter transformations and representations of partially ordered sets. *Funkcional. Anal. i Priložen.*, 8(3):34–42, 1974.

[15] Yu. A. Drozd. On tame and wild matrix problems. In *Matrix problems (Russian)*, pages 104–114. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1977.

[16] Yu. A. Drozd. Tame and wild matrix problems. In *Representations and quadratic forms (Russian)*, pages 39–74, 154. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979.

[17] Yu. A. Drozd. Representations of bisected posets and reflection functors. In *Algebras and modules, II (Geiranger, 1996)*, volume 24 of *CMS Conf. Proc.*, pages 153–165. Amer. Math. Soc., Providence, RI, 1998.

[18] Yu. A. Drozd. Reduction algorithm and representations of boxes and algebras. *C. R. Math. Acad. Sci. Soc. R. Can.*, 23(4):97–125, 2001.

[19] Yu. A. Drozd. Derived tame and derived wild algebras. *Algebra Discrete Math.*, 3(1):57–74, 2004.

[20] Yu. A. Drozd. Vector bundles and Cohen-Macaulay modules. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 189–222. Amer. Math. Soc., Providence, RI, 2004.

[21] Yu. A. Drozd. Matrix problems, triangulated categories and stable homotopy types. *São Paulo J. Math. Sci.*, 4(2):209–249, 2010, doi: 10.11606/issn.2316-9028.v4i2p209-249.

[22] Yu. A. Drozd, V. V. Kirichenko. *Finite-dimensional algebras*. Springer-Verlag, Berlin, 1994, doi: 10.1007/978-3-642-76244-4. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab.

[23] Yu. A. Drozd, E. A. Kubichka. Dimensions of finite type for representations of partially ordered sets. *Algebra Discrete Math.*, (3):21–37, 2004.

[24] Yu. A. Drozd, S. A. Ovsienko. Coverings of tame boxes and algebras. Preprint MI 00-26, 2000. Max-Plank-Institut für Mathematik.

[25] U. Fischbacher. Une nouvelle preuve d'un théorème de Nazarova et Roiter. *C. R. Acad. Sci. Paris Sér. I Math.*, 300(9):259–262, 1985.

[26] P. Gabriel. Unzerlegbare Darstellungen. I. *Manuscripta Math.*, 6:71–103; correction, ibid. 6 (1972), 309, 1972, doi: 10.1007/BF01298413.

[27] I. M. Gelfand, V. A. Ponomarev. Remarks on the classification of a pair of commuting linear transformations in a finite-dimensional space. *Funkcional. Anal. i Priložen.*, 3(4):81–82, 1969.

[28] J. P. Jans. On the indecomposable representations of algebras. *Ann. of Math. (2)*, 66:418–429, 1957, doi: 10.2307/1969899.

[29] M. M. Kleĭner. Faithful representations of partially ordered sets of finite type. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 28:42–59, 1972. Investigations on the theory of representations.

[30] M. M. Kleĭner. Partially ordered sets of finite type. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 28:32–41, 1972. Investigations on the theory of representations.

[31] M. M. Kleĭner, A. V. Roĭter. Representations of differential graded categories. In *Matrix problems (Russian)*, pages 5–70. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1977.

[32] S. A. Krugljak. Representations of the group (p, p) over a field of characteristic p. *Dokl. Akad. Nauk SSSR*, 153:1253–1256, 1963.

[33] S. A. Krugljak. Representations of algebras for which the square of the radical equals zero. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 28:60–68, 1972. Investigations on the theory of representations.

[34] Peter Magyar, Jerzy Weyman, Andrei Zelevinsky. Multiple flag varieties of finite type. *Adv. Math.*, 141(1):97–118, 1999, doi: 10.1006/aima.1998.1776.

[35] A. I. Malcev. *Foundations of Linear Algebra (Russian)*. Izdat. “Nauka”, Moscow, 1970.

[36] L. A. Nazarova. Representations of quivers of infinite type. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:752–791, 1973.

[37] L. A. Nazarova. Partially ordered sets of infinite type. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(5):963–991, 1219, 1975.

[38] L. A. Nazarova, A. V. Roĭter. Matrix questions and the Brauer-Thrall conjectures on algebras with an infinite number of indecomposable representations. In *Representation theory of finite groups and related topics (Proc. Sympos. Pure Math., Vol. XXI, Univ. Wisconsin, Madison, Wis., 1970)*, pages 111–115, 1971.

[39] L. A. Nazarova, A. V. Roĭter. Representations of partially ordered sets. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 28:5–31, 1972. Investigations on the theory of representations.

[40] L. A. Nazarova, A. V. Roĭter. *Categorical matrix problems and the Brauer-Thrall conjecture (Russian)*. Izdat. “Naukova Dumka”, Kiev, 1973.

[41] Claus Michael Ringel. The indecomposable representations of the dihedral 2-groups. *Math. Ann.*, 214:19–34, 1975, doi: 10.1007/BF01428252.

[42] A. V. Roĭter. Unboundedness of the dimensions of the indecomposable representations of an algebra which has infinitely many indecomposable representations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 32:1275–1282, 1968.

[43] A. V. Roĭter. BOCSs with involution. In *Representations and quadratic forms (Russian)*, pages 124–126, 155. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979.

[44] A. V. Roĭter. Matrix problems and representations of BOCSs. In *Representations and quadratic forms (Russian)*, pages 3–38, 154. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979.

[45] A. V. Roĭter. Representations of posets and tame matrix problems. In *Representations of algebras (Durham, 1985)*, volume 116 of *London Math. Soc. Lecture Note Ser.*, pages 91–107. Cambridge Univ. Press, Cambridge, 1986.

[46] V. V. Sergeichuk. *Canonical matrices and related questions*. Kyïv: Instytut Matematyky NAN Ukraïny, 2006 (in English).

[47] V. V. Sergeĭchuk. Classification problems for systems of forms and linear mappings. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(6):1170–1190, 1358, 1987, doi: 10.1070/IM1988v031n03ABEH001086.

[48] V. V. Sergeĭčuk. Representations of simple involutive quivers. In *Representations and quadratic forms (Russian)*, pages 127–148, 156. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979.

[49] R. M. Thrall. On ahdir algebras. *Bull. Amer. Math. Soc.*, 53:49, 1947.

[50] Tensho Yoshii. On algebras of bounded representation type. *Osaka Math. J.*, 8:51–105, 1956.