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DERIVED CATEGORIES OF NODAL CURVES

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We describe derived categories of coherent sheaves over nodal noncommutative curves of string and almost string types.

Introduction

The present paper is a sequel of [1] devoted to the description of vector bundles over a certain class of noncommutative curves (nodal curves of string and almost string types). These curves are noncommutative analogs of linear configurations of the types A and $\tilde{A}$ playing an important role in the theory of vector bundles over projective curves and in the theory of Cohen–Macaulay modules [2, 3]. It is also proved that, with the exception of these curves and certain weighted projective Geigle–Lenzing straight lines [4], all other noncommutative curves are wild relative to the classification of vector bundles. In the present paper, we prove that, for nodal curves of string and almost string types, it is possible to describe not only vector bundles but also the derived categories of coherent sheaves in complete agreement with the fact that this description is possible for linear configurations of the types A and $\tilde{A}$ [5].

1. Derived Categories and Categories of Triples

We use the terminology and results from [1]. In what follows, $(X, \mathcal{A})$ is a projective *nodal* noncommutative curve over an algebraically closed field $\Bbbk$, $\operatorname{sg}\mathcal{A}_X$ is the set of its singular points, $\mathcal{H}$ is a sheaf of $\mathcal{O}_X$-algebras such that $\mathcal{H}_x = \operatorname{End}_{\mathcal{A}_x}(\operatorname{rad}\mathcal{A}_x)$ for every point $x \in X$, and $\tilde{X} = \operatorname{spec}(\operatorname{center}\mathcal{H})$. Then $(\tilde{X}, \mathcal{H})$ is a noncommutative curve all localizations of which are hereditary and the morphism of ringed spaces $\pi : (\tilde{X}, \mathcal{H}) \to (X, \mathcal{A})$ is defined. Note that $\mathcal{H}_x = \mathcal{A}_x$ whenever $x \notin \operatorname{sg}\mathcal{A}$. By $\widetilde{\operatorname{sg}}\ \mathcal{A}$ we denote a set-theoretic preimage of $\operatorname{sg}\mathcal{A}$ under this morphism. Also let $\mathcal{J}$ be a sheaf of $\mathcal{A}$-ideals such that

$$\mathcal{J}_x = \begin{cases} \mathcal{A}_x & \text{for } x \notin \operatorname{sg}\mathcal{A}, \\ \operatorname{rad}\mathcal{A}_x & \text{for } x \in \operatorname{sg}\mathcal{A}. \end{cases}$$

Since the algebra $\mathcal{A}_x$ is nodal, we have $\operatorname{rad}\mathcal{A}_x = \operatorname{rad}\mathcal{H}_x$ and, hence, $\mathcal{J}$ is a sheaf of $\mathcal{H}$-ideals. We also denote $\mathcal{S} = \mathcal{A}/\mathcal{J}$ and $\tilde{\mathcal{S}} = \mathcal{H}/\mathcal{J}$. It is possible to consider the 0-dimensional noncommutative curves $(\operatorname{sg}\mathcal{A}, \mathcal{S})$ and $(\widetilde{\operatorname{sg}}\ \mathcal{A}, \tilde{\mathcal{S}})$ and the commutative diagram of morphisms

$$\begin{array}{ccc} (\widetilde{\operatorname{sg}}\,\mathcal{A}, \tilde{\mathcal{S}}) & \xrightarrow{\bar{\pi}} & (\operatorname{sg}\mathcal{A}, \mathcal{S}) \\ {\scriptstyle\tilde{\iota}}\downarrow & & \downarrow{\scriptstyle\iota} \\ (\tilde{X}, \mathcal{H}) & \xrightarrow{\pi} & (X, \mathcal{A}). \end{array} \tag{1.1}$$

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All layers of the sheaves $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are finite-dimensional semisimple $\Bbbk$-algebras. Therefore, the coherent sheaves of modules over $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are naturally identified with finite-dimensional modules over the semisimple finite-dimensional $\Bbbk$-algebras $\mathbf{S} = \bigoplus_{x\in\mathrm{sg}\,\mathcal{A}} \mathcal{A}_x/\mathcal{J}_x$ and $\widetilde{\mathbf{S}} = \bigoplus_{x\in\widetilde{\mathrm{sg}}\,\mathcal{A}} \mathcal{H}_x/\mathcal{J}_x$, respectively.

We write $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ instead of $\mathcal{O}_X$ and $\mathcal{O}_{\widetilde{X}}$. By $\mathcal{K}$ we denote a sheaf of rational functions on X (or, which is the same, on $\widetilde{X}$). Also let $\mathcal{K}(\mathcal{A})$ be a sheaf $\mathcal{A}\otimes_{\mathcal{O}}\mathcal{K} \simeq \mathcal{H}\otimes_{\widetilde{\mathcal{O}}}\mathcal{K}$. If $X_1, X_2, \ldots, X_s$ are irreducible components of $\widetilde{X}$, then we write $\widetilde{\mathcal{O}}_i = \widetilde{\mathcal{O}}|_{X_i}$ and $\mathcal{H}_i = \mathcal{H}|_{X_i}$.

By $\mathcal{D}^-(\mathcal{C})$ we denote a derived category of bounded (from the right) complexes over the Abelian category $\mathcal{C}$. If $\mathcal{C} = \mathrm{coh}(\mathcal{R})$ is a category of coherent sheaves on the projective noncommutative manifold $(V, \mathcal{R})$, then it follows from the Serre theorem [6] (Theorem II.5.17) that every complex from $\mathcal{D}^-(\mathcal{C})$ is isomorphic in this category to a complex of locally projective sheaves (*vector bundles* in the terminology of [1]). Diagram (1.1) induces the following diagram of derived functors:

$$\begin{array}{ccc} \mathcal{D}^-(\mathcal{A}) & \xrightarrow{L\pi^*} & \mathcal{D}^-(\mathcal{H}) \\ {\scriptstyle L\iota^*}\big\downarrow & & \big\downarrow{\scriptstyle L\tilde{\iota}^*} \\ \mathcal{D}^-(\mathcal{S}) & \xrightarrow{L\bar{\pi}^*} & \mathcal{D}^-(\widetilde{\mathcal{S}}). \end{array} \tag{1.2}$$

This diagram is commutative in a sense that there exists a natural isomorphism of functors $\gamma\colon L\bar{\pi}^* L\iota^* \xrightarrow{\sim} L\tilde{\iota}^* L\pi^*$.

By analogy with [5], we define a *category of triples* $\mathcal{T}(\mathcal{A})$ as follows:

objects of the category of triples are triples $(\mathcal{G}_\bullet, \mathcal{V}_\bullet, \theta)$, where $\mathcal{G}_\bullet$ and $\mathcal{V}_\bullet$ are complexes from $\mathcal{D}^-(\mathcal{H})$ and $\mathcal{D}^-(\mathcal{S})$, respectively, and θ is the isomorphism $L\bar{\pi}^*\mathcal{V}_\bullet \xrightarrow{\sim} L\tilde{\iota}^*\mathcal{G}_\bullet$ in the category $\mathcal{D}^-(\widetilde{\mathcal{S}})$;

the *morphism* from a triple $(\mathcal{G}_\bullet, \mathcal{V}_\bullet, \theta)$ to a triple $(\mathcal{G}'_\bullet, \mathcal{V}'_\bullet, \theta')$ is a couple of morphisms $\Phi\colon \mathcal{G}_\bullet \to \mathcal{G}'_\bullet$ and $\phi\colon \mathcal{V}_\bullet \to \mathcal{V}'_\bullet$ in the categories $\mathcal{D}^-(\mathcal{H})$ and $\mathcal{D}^-(\mathcal{S})$, respectively, such that the diagram

$$\begin{array}{ccc} L\bar{\pi}^*\mathcal{V}_\bullet & \xrightarrow{\theta} & L\tilde{\iota}^*\mathcal{G}_\bullet \\ {\scriptstyle L\bar{\pi}^*\phi}\big\downarrow & & \big\downarrow{\scriptstyle L\tilde{\iota}^*\Phi} \\ L\bar{\pi}^*\mathcal{V}'_\bullet & \xrightarrow{\theta'} & L\tilde{\iota}^*\mathcal{G}'_\bullet \end{array}$$

is commutative.

The commutativity of diagram (1.2) enables one to determine the functor $\mathbf{F}\colon \mathcal{D}^-(\mathcal{A}) \to \mathcal{T}(\mathcal{A})$ by setting $\mathbf{F}(\mathcal{F}_\bullet) = (L\pi^*\mathcal{F}_\bullet, L\iota^*\mathcal{F}_\bullet, \gamma(\mathcal{F}_\bullet))$. Repeating the reasoning from [5] (Theorem 4.2), we arrive at the following result:

Theorem 1.1. *The functor* $\mathbf{F}$ *is dense* (*i.e., each object from* $\mathcal{T}(\mathcal{A})$ *is isomorphic to an image* $\mathbf{F}(\mathcal{F}_\bullet)$) *and conservative* (*i.e., the isomorphism* $\mathbf{F}(\mathcal{F}_\bullet) \simeq \mathbf{F}(\mathcal{F}'_\bullet)$ *implies that* $\mathcal{F}_\bullet \simeq \mathcal{F}'_\bullet$). *The indicated two properties also imply that* $\mathbf{F}$ *maps indecomposable objects into indecomposable.*

As in the commutative case [5], the functor $\mathbf{F}$ is not the equivalence of categories because it is not strict (i.e., can map nonzero morphisms into zero). It turns into the equivalence relation only under a restriction imposed on the complete subcategory of vector bundles [1].

Consider an ideal $\mathcal{N}$ of the category of triples formed by morphisms of the form $(\Phi, 0)$ (in this case, $\tilde{\iota}^*\Phi = 0$). We set $\overline{\mathcal{T}}(\mathcal{A}) = \mathcal{T}(\mathcal{A})/\mathcal{N}$. It is clear that the composition $\overline{\mathbf{F}}$ of the functor $\mathbf{F}$ with the projection $\mathcal{T}(\mathcal{A}) \to$

$\bar{\mathcal{T}}(\mathcal{A})$ is also a dense and conservative functor. Hence, the classes of isomorphism of the objects from $\mathcal{D}^-(\mathcal{A})$ and $\bar{\mathcal{T}}(\mathcal{A})$ coincide.

Note that if $\mathcal{F}_\bullet = (\mathcal{F}_n, \partial_n)$ is a complex of locally projective sheaves over $\mathcal{A}$, then $L\pi^*\mathcal{F}_\bullet$ can be found term by term as $(\pi^*\mathcal{F}_n, \pi^*\partial_n)$ and the same is true for the other components of diagram (1.2). The categories $\mathrm{coh}(\mathcal{S})$ and $\mathrm{coh}(\widetilde{\mathcal{S}})$ are semisimple. Therefore, each complex in $\mathcal{D}^-(\mathcal{S})$ or $\mathcal{D}^-(\widetilde{\mathcal{S}})$ is decomposed into the direct sum of "shifted simple modules," i.e., complexes $\mathcal{U}[n]$, where $\mathcal{U}$ is a simple **S**-module and $[n]$ denotes a shift in the category of complexes. The category $\mathrm{coh}(\mathcal{H})$ is *hereditary*, i.e., in this category, we have $\mathrm{Ext}^2 = 0$. Thus, every complex $\mathcal{F}_\bullet$ from $\mathcal{D}^-(\mathcal{H})$ is isomorphic (in this derived category) to the direct sum of shifted sheaves of the homologies: $\mathcal{F}_\bullet \simeq \bigoplus_n H_n(\mathcal{F}_\bullet)[n]$ (see [7], Theorem 3.1). Moreover, every coherent sheaf over $\mathcal{H}$ is decomposed in the direct sum of vector bundles and *skyscrapers*, i.e., sheaves whose support is a single (closed) point. An indecomposable skyscraper $\mathcal{F}$ with support at a point $x \in \widetilde{\mathrm{sg}}\ \mathcal{A}$ is isomorphic to a factor P_x/P', where P_x is an indecomposable projective $\mathcal{H}_x$-module. Moreover, one can always find a vector bundle $\mathcal{P}$ over $\mathcal{H}$ such that $\mathcal{P}_x \simeq P_x$. In this case,

$$\mathrm{End}_{\mathcal{H}_x} P_x \simeq \widetilde{\mathcal{O}}_x$$

and, hence, $\mathrm{End}_{\mathcal{H}}\mathcal{P} \simeq \widetilde{\mathcal{O}}_i$, where X_i is the component of $\widetilde{X}$ containing the point x. These vector bundles are called *linear bundles*. It is known (see, e.g., [8]) that the lattice of submodules in the indecomposable projective $\mathcal{H}_x$-module is a chain. Hence, in $\mathcal{P}$, there exists a unique subsheaf $\mathcal{P}'$ such that $\mathcal{P}/\mathcal{P}' \simeq \mathcal{F}$ and, in addition, $\mathcal{F}$ is uniquely defined by its length $l = \mathrm{length}_{\mathcal{H}}\mathcal{F}$ and the factor $\mathcal{U} = \mathcal{F}/\mathcal{J}\mathcal{F}$, which is a simple $\mathcal{H}_x$-module. Thus, in the category $\mathcal{D}^-(\mathcal{H})$, the sheaf $\mathcal{F}$ can be replaced by the complex

$$\mathcal{P}(l, x, \mathcal{U})\colon\ 0 \to \mathcal{P}' \to \mathcal{P} \to 0$$

with $\mathcal{P}$ at the zero position. The shifts of this complex are denoted by $\mathcal{P}(x, l, \mathcal{U})[n]$ (in this complex, $\mathcal{P}$ is at the nth position). Thus, each object from $\mathcal{D}^-(\mathcal{H})$ can be regarded as the direct sum of shifted vector bundles $\mathcal{P}[n]$ and shifted complexes $\mathcal{P}(x, l, \mathcal{U})[n]$.

2. Curves of the String Type

Recall that a nodal noncommutative curve is called a curve of *string type* if all components X_k are rational, i.e., isomorphic to $\mathbb{P}^1$ and every intersection $\widetilde{\mathrm{sg}}_k\, \mathcal{A} = \widetilde{\mathrm{sg}}\ \mathcal{A} \cap X_k$ contains at most two points. In this case, every indecomposable vector bundle over $\mathcal{H}$ is linear [1, 4] and defined (to within twisting) by its localizations at singular points. Thus, it is convenient to number these bundles in the indicated way.

Case 1. Let x be a unique point from $\widetilde{\mathrm{sg}}_k\, \mathcal{A}$. Moreover, we assume that $\mathcal{H}_x$ has n simple modules, i.e., $\mathcal{H}_x$ is Morita-equivalent to the algebra $R(1; n)$ in the notation of [1] (Theorem 2.1). The indecomposable projective $\mathcal{H}_x$-modules $P_1, P_2, \ldots, P_n$ can be chosen so that $P_{i+1} = \mathcal{J}_x P_i$ for $1 \le i < n$ and $\mathcal{J}_x P_n = tP_1$, where t is a uniformizing element from $\widetilde{\mathcal{O}}_x$. We fix linear bundles $\mathcal{P}(x, i)$, $1 \le i \le n$, such that $\mathcal{P}(x, i)_x = P_i$. Then every linear bundle over $\mathcal{H}_i$ is isomorphic to $\mathcal{P}(x, i)(d)$ for some d. Denote $U(x, i) = P_i/\mathcal{J}_x P_i$ (this is a simple $\mathcal{H}_x$-module).

Case 2. Now let $\widetilde{\mathrm{sg}}_k\, \mathcal{A} = \{x, y\}$. Moreover, we assume that $\mathcal{H}_x$ has n simple modules and $\mathcal{H}_y$ has m simple modules. We choose indecomposable projective $\mathcal{H}_x$-modules $P_1, P_2, \ldots, P_n$ such that $P_{i+1} = \mathcal{J}_x P_i$ for $1 \le i < n$ and $\mathcal{J}_x P_n = tP_1$, where t is a uniformizing element from $\widetilde{\mathcal{O}}_x$. Further, we choose indecomposable projective $\mathcal{H}_y$-modules $P'_1, P'_2, \ldots, P'_m$ such that $P'_{i+1} = \mathcal{J}_y P'_i$ for $1 \le i < m$ and $\mathcal{J}_y P'_m = t' P'_1$, where t' is a uniformizing element from $\widetilde{\mathcal{O}}_y$, and fix linear bundles $\mathcal{P}(x, i, j)$, $1 \le i \le n$, $1 \le j \le m$, such that

$\mathcal{P}(x,i,j)_x = P_i$ and $\mathcal{P}(x,i,j)_y = P'_j$. Then every linear bundle over $\mathcal{H}_i$ is isomorphic to $\mathcal{P}(x,i,j)(d)$ for some d. Note that, in this case, we can exchange the roles played by the points x and y. Thus, the sheaf $\mathcal{P}(x,i,j)$ is renamed into $\mathcal{P}(y,j,i)$ and we denote $U(x,i) = P_i/\mathcal{J}_x P_x$ and $U(y,j) = P'_j/\mathcal{J}_y P'_j$.

In what follows, we fix the proposed enumeration. Hence, the indecomposable skyscrapers with support x are complexes of the form

$$\mathcal{P}(l,x,i)\colon 0 \to \mathcal{P}(x,i')(d) \to \mathcal{P}(x,i) \to 0$$

or

$$\mathcal{P}(l,x,i)\colon 0 \to \mathcal{P}(x,i',j)(d) \to \mathcal{P}(x,i,j) \to 0,$$

where $l = i' - i + dn$ and, in addition, in the second case, different indices j give complexes isomorphic in $\mathcal{D}^-(\mathcal{H})$. Moreover, these complexes are isomorphic to any twisting.

Thus, in the category $\mathcal{D}^-(\mathcal{H})$, every complex is isomorphic to the direct sum of shifted linear bundles $\mathcal{P}(x,i)(d)[r]$ or $\mathcal{P}(x,i,j)(d)[r]$ and shifted complexes $\mathcal{P}(l,x,i)[r]$. In the category $\mathcal{D}^-(\widetilde{\mathcal{S}})$,

the image of $\mathcal{P}(x,i)[r]$ is a shifted simple module $U(x,i)[r]$ over $\mathcal{H}_x$;

the image of $\mathcal{P}(x,i,j)(d)[r]$ is the direct sum of shifted simple modules $U(x,i)(d)[r] \oplus U(y,j)(d)[r]$ over $\mathcal{H}_x$ and $\mathcal{H}_y$, respectively;

the image of the complex $\mathcal{P}(l,x,i)[r]$ is the direct sum of shifted simple modules $U(x,i)[r]$ and $U(x,i')[r+1]$.

It is easy to see that, by analogy with [5], the morphisms of complexes from $\mathcal{D}^-(\mathcal{H})$ induce nonzero morphisms of their images in $\mathcal{D}^-(\widetilde{\mathcal{S}})$ only in the following cases (to within a shift):

(1) the morphisms $\mathcal{P}(x,i)(d) \to \mathcal{P}(x,i)(d')$ for $d \le d'$;

(2) the morphisms $\mathcal{P}(x,i,j)(d) \to \mathcal{P}(x,i,j')(d')$ for $d < d'$ or $d = d'$, $j' < j$, which induce a nonzero map on $U(x,i)$ and a zero map on $U(y,j)$;

(3) the morphisms $\mathcal{P}(x,i,j)(d) \to \mathcal{P}(x,i',j)(d')$ for $d < d'$ or $d = d'$, $i' < i$, which induce a nonzero map on $U(y,j)$ and a zero map on $U(x,i)$;

(4) the morphisms $\mathcal{P}(x,i,j)(d) \to \mathcal{P}(x,i,j)(d)$, which induce identical maps on $U(x,i)$ and $U(y,j)$;

(5) the morphisms $\mathcal{P}(x,i)(d) \to \mathcal{P}(l,x,i)$ or $\mathcal{P}(x,i,j) \to \mathcal{P}(l,x,i)(d)$ for any d and j;

(6) the morphisms $\mathcal{P}(l,x,i) \to \mathcal{P}(x,i')(d)[1]$ or $\mathcal{P}(l,x,i) \to \mathcal{P}(x,i',j)(d)$ [1] for any d and j;

(7) the morphisms $\mathcal{P}(l,x,i) \to \mathcal{P}(l_1,x,i)$ for $l_1 < l$, which induce a nonzero map on the component $U(x,i)$ and a zero map on the component $U(x,i')$ [1];

(8) the morphisms $\mathcal{P}(l,x,i) \to \mathcal{P}(l_1,x,i_1)$ for $l < l_1$, $l + i \equiv l_1 + i_1 \pmod n$, which induce a nonzero map on the component $U(x,i')$ [1] and a zero map on the component $U(x,i)$;

(9) the morphisms $\mathcal{P}(l,x,i) \to \mathcal{P}(l,x,i)$, which induce identical maps on both components $U(x,i)$ and $U(x,i')$.

This enables us to identify the reduced category of triples $\overline{\mathcal{T}}(\mathcal{A})$ with a certain category of images of the bundle of chains in a sense of [9, 10].

We define a bundle of chains $\mathfrak{B} = \mathfrak{B}(\mathcal{A})$ as follows:

The *set of indices* of the bundle $\mathfrak{B}$ is defined as a set of triples $\mathbf{I} = \{(x,i)[r]\}$, where $x \in \widetilde{\operatorname{sg}}\ \mathcal{A}$, $r \in \mathbb{Z}$, and $1 \le i \le n$, where, in turn, n is the number of simple $\mathcal{H}_x$-modules.

$\mathfrak{F}_{(x,i)[r]} = \{(x,i)[r]\}$.

$\mathfrak{E}_{(x,i,r)}$ is formed by the following symbols:

quadruples $(x,i,d)[r]$ $(d \in \mathbb{Z})$ if x is a unique singular point in its component;

fives $(x,i,j,d)[r]$ $(d \in \mathbb{Z},\ 1 \le j \le m)$ if this component contains not only x but also another singular point y and, in addition, $\mathcal{H}_y$ has m simple modules;

quadruples $(l,x,i)[r]$, where $l \in \mathbb{Z} \setminus \{0\}$.

The quadruple $(l,x,i)[r]$ corresponds to the rth component of the complex $\mathcal{P}(l,x,i)[r]$ for $l > 0$ and the rth component of the complex $\mathcal{P}(-l,x,i')[r-1]$, where $i' \equiv i + l \pmod{n}$ for $l < 0$.

The *ordering* on $\mathfrak{E}_{(x,i)[r]}$ is specified as follows:

$(x,i,d)[r] < (x,i,d')[r]$ if and only if $d < d'$;

$(x,i,j,d)[r] < (x,i,j',d')[r]$ if and only if $d < d'$ or $d = d',\ j > j'$;

$(l,x,i)[r] < (x,i,d)[r] < (l',x,i)[r]$ for any $l < 0,\ l' > 0$, and d;

$(l,x,i)[r] < (x,i,j,d)[r] < (l',x,i)[r]$ for any $l < 0,\ l' > 0$, j, and d;

$(l,x,i)[r] < (l',x,i)[r]$ if and only if $l < l'$.

The *relation* $\sim$ is defined as follows:

$(x,i,j,d)[r] \sim (y,j,i,d)[r]$ if x and y belong to the same irreducible component;

$(l,x,i)[r] \sim (-l,x,i')[r+1]$ if $l > 0$ and $i' \equiv l + i \pmod{n}$;

$(x,i)[r] \sim (x,i)[r]$ if there exist two different simple $\mathcal{S}$-modules V and V' for which $\bar{\pi}^*V \simeq \bar{\pi}^*V' \simeq U(x,i)$ (recall that the number of these modules never exceeds two [11]);

$(x,i) \sim (x',i')$ if there exists a simple $\mathcal{A}$-module V such that $\bar{\pi}^*V \simeq U(x,i) \oplus U(x',i')$.

The following main theorem is a direct consequence of the previous results:

Theorem 2.1. *If $(X,\mathcal{A})$ is a nodal noncommutative curve of the string type, then the reduced category of triples $\overline{\mathcal{T}}(\mathcal{A})$ is equivalent to a complete subcategory of images of the bundle of chains $\mathfrak{B}(\mathcal{A})$ formed by the images M for which all matrices $M_{(x,i)[r]}$ are invertible.*

Since we consider the category $\mathcal{D}^-(\mathcal{A})$ in which the complexes are bounded solely from the right, in this theorem (as well as in Theorem 3.1 from the next section), it is necessary to take into account the *infinite* images of the bundle $\mathfrak{B}(\mathcal{A})$ studied in [10] (Appendix C). The finite images describe objects of the derived category $\mathcal{D}^{\mathrm{per}}(\mathcal{A})$ of *perfect complexes*, i.e., complexes isomorphic (in the derived category) to finite complexes of the vector bundles.

Since the indecomposable images of a bundle of chains are strings and lines (see [9, 10]) and, in addition the number of strings is finite for a fixed dimension and the lines are parametrized by elements of the field $\Bbbk$, we get the following corollary:

Corollary 2.1. *Every nodal noncommutative curve of the string type is derived tame in a sense of [12].*

Remark 2.1. Recall that a perfect derived category $\mathcal{D}^{\mathrm{per}}(\mathcal{A})$ is *dense* in the derived category of bounded complexes $\mathcal{D}^b(\mathcal{A})$ [13]. Hence, the defined factor category

$$\mathcal{D}^{\mathrm{sg}}(\mathcal{A}) = \mathcal{D}^b(\mathcal{A})/\mathcal{D}^{\mathrm{per}}(\mathcal{A})$$

measures the "irregularity" of the curve $\mathcal{A}$, i.e., the degree of its deviation from the curve for which the category of coherent sheaves has finite homologic dimension. Note that, in the description of objects from $\mathcal{D}^-(\mathcal{A})$, the parameter appears only in the lines which definitely correspond to the complexes from $\mathcal{D}^{\mathrm{per}}(\mathcal{A})$. Thus, for curves of the string type, the category $\mathcal{D}^{\mathrm{sg}}(\mathcal{A})$ is *discrete*, i.e., does not contain nontrivial families of nonisomorphic indecomposable complexes.

The same is true for curves of almost string type studied in Sec. 3.

3. Curves of Almost String Type

Recall that a nodal noncommutative curve is called a curve of *almost string type* if all its components X_k are rational, i.e., isomorphic to $\mathbb{P}^1$, each intersection $\widetilde{\mathrm{sg}}_k\,\mathcal{A} = \widetilde{\mathrm{sg}}\,\mathcal{A} \cap X_k$ contains at most three points, and moreover, if there are three points of this kind, then, for two of them, the algebra $\mathcal{A}_{\pi(x)}$ is hereditary and has two simple modules, i.e., is Morita-equivalent to the algebra $R(1;2)$ in the notation of [1] (Theorem 2.2). These points are called "*excess points*" and the set $\widetilde{\mathrm{sg}}_k\,\mathcal{A}$ in which the excess points $\widetilde{\mathrm{sg}}\,'\mathcal{A} = \cup_k \widetilde{\mathrm{sg}}\,'\mathcal{A}$ are excluded is denoted by $\widetilde{\mathrm{sg}}\,'_k\,\mathcal{A}$. A point $x \in \widetilde{\mathrm{sg}}\,'_k\,\mathcal{A}$ is called *special* if the component X_k contains excess points. A component X_k is called *special* if it contains special points.

The vector bundles on nonspecial components $\mathcal{H}_k$ remain the same as in the case of string type. Let X_k be a special component, let $x \in X_k$ be a special point, and let x_1 and x_2 be excess points from the component X_k. Assume that $\mathcal{H}_x$ has n simple modules, i.e., is Morita-equivalent to $R(1;n)$. Then it follows from [1, 4] that the indecomposable vector bundles over $\mathcal{H}_k$ can be described as follows:

1. The linear bundles $\mathcal{P}(x, i \mid c_1, c_2)(d)$, where $1 \le i \le n$, $d \in \mathbb{Z}$, and $c_1, c_2 \in \{1,2\}$. These are linear bundles $\mathcal{P}$ of degree d such that

$$\mathcal{P}/\mathcal{J}\mathcal{P} \simeq U(x,i) \oplus U(x_1,c_1) \oplus U(x_2,c_2).$$

2. For any couple $1 \le i < j \le n$ and any $d \in \mathbb{Z}$, there are two additional indecomposable vector bundles $\mathcal{P}(i,j \mid c)(d)$, where $c \in \{1,2\}$, such that $\deg \mathcal{P}(i,j \mid c)(d) = 2d - c + 1$,

$$\mathcal{P}(i,j \mid c)(d)/\mathcal{J}\mathcal{P}(i,j \mid c)(d) \simeq U(x,i) \oplus U(x,j) \oplus U(x_1,1) \oplus U(x_1,2) \oplus U(x_2,1) \oplus U(x_2,2).$$

Moreover, there exist the exact sequences

$$0 \to \mathcal{P}(j \mid 1, 2)(d) \to \mathcal{P}(i, j \mid 1)(d) \to \mathcal{P}(i \mid 2, 1)(d) \to 0, \tag{3.1}$$

$$0 \to \mathcal{P}(j \mid 2, 1)(d) \to \mathcal{P}(i, j \mid 1)(d) \to \mathcal{P}(i \mid 1, 2)(d) \to 0, \tag{3.2}$$

$$0 \to \mathcal{P}(j \mid 1, 1)(d-1) \to \mathcal{P}(i, j \mid 2)(d) \to \mathcal{P}(i \mid 2, 2)(d) \to 0, \tag{3.3}$$

$$0 \to \mathcal{P}(j \mid 2, 2)(d) \to \mathcal{P}(i, j \mid 2)(d) \to \mathcal{P}(i \mid 1, 1)(d-1) \to 0, \tag{3.4}$$

and, in addition,

$$0 \to \mathcal{P}(i \mid 1, 1)(d-1) \to \mathcal{P}(i, j \mid 1)(d) \to \mathcal{P}(j \mid 2, 2)(d+1) \to 0, \tag{3.5}$$

$$0 \to \mathcal{P}(i \mid 2, 2)(d) \to \mathcal{P}(i, j \mid 1)(d) \to \mathcal{P}(j \mid 1, 1)(d) \to 0, \tag{3.6}$$

$$0 \to \mathcal{P}(i \mid 1, 2)(d-1) \to \mathcal{P}(i, j \mid 2)(d) \to \mathcal{P}(j \mid 2, 1)(d) \to 0, \tag{3.7}$$

$$0 \to \mathcal{P}(i \mid 2, 1)(d-1) \to \mathcal{P}(i, j \mid 2)(d) \to \mathcal{P}(j \mid 1, 2)(d \to 0). \tag{3.8}$$

It is possible to show that nonzero maps in the category $\mathcal{D}^-(\widetilde{\mathcal{S}})$ are induced solely by the morphisms from the previous section (1)–(9), the morphisms contained in sequences (3.1)–(3.8), and their compositions. Hence, the category $\overline{\mathcal{T}}(\mathcal{A})$ can also be identified with the category of images of a bundle of chains $\mathfrak{B} = \mathfrak{B}(\mathcal{A})$ constructed by analogy with the string case:

A *set of indices* of the bundle $\mathfrak{B}$ is a set of triples $\mathbf{I} = \{(x, i)[r]\}$, where $x \in \widetilde{\mathrm{sg}}\,'\mathcal{A}$, $r \in \mathbb{Z}$, $1 \le i \le n_x$, and n_x is the number of simple $\mathcal{H}_x$-modules.

$\mathfrak{F}_{(x,i)[r]} = \{(x, i)[r]\}$.

If a point x is not special, then the set $\mathfrak{E}_{(x,i)[r]}$ and the order on this set are defined as in the string case.

If a point x is special, then the set $\mathfrak{E}_{(x,i)[r]}$ is formed by the following symbols:

fives $(x, i, d \mid c)[r]$ ($d \in \mathbb{Z}$, $c \in \{1, 2\}$);

sixes $(x, i, j, d \mid c)[r]$ ($d \in \mathbb{Z}$, $c \in \{1, 2\}$, $j \ne i$, $1 \le j \le n_x$);

quadruples $(l, x, i)[r]$ ($l \in \mathbb{Z} \setminus \{0\}$).

In this case, the order on $\mathfrak{E}_{(x,i)[r]}$ is defined as follows:

$$(x,i,j,d \mid 2)[r] < (x,i,j',d \mid 2)[r] < (x,i,d \mid 2)[r] < (x,i,j,d \mid 1)[r] < (x,i,j',d \mid 1)[r]$$

$$< (x,i \mid 1)[r] < (x,i,j,d' \mid 2)[r] \quad \text{for any} \quad d, j, d' > d, j' < j < i,$$

$$(x,i,d \mid 2)[r] < (x,i,j,d \mid 2)[r] < (x,i,j',d \mid 2)[r] < (x,i,d \mid 1)[r] < (x,i,j,d \mid 1)[r]$$

$$< (x,i,j',d \mid 1)[r] < (x,i,d' \mid 1)[r] \quad \text{for any} \quad d, j, d' > d, j > j' > i,$$

$$(l, x, i)[r] < (x, i, d \mid c)[r] < (l', x, i)[r] \quad \text{for any} \quad c, d, l < 0, l' > 0,$$

$$(l, x, i)[r] < (x, i, j, d \mid c)[r] < (l', x, i)[r] \quad \text{for any} \quad c, d, j, l < 0, l' > 0,$$

$$(l, x, i)[r] < (l', x, i)[r] \quad \text{if and only if} \quad l < l'.$$

The relationship $\sim$ is defined as in the string case with the following additional requirements:

$$(x, i, d \mid c)[r] \sim (x, i, d \mid c)[r] \qquad \text{and} \qquad (x, i, j, d \mid c)[r] \simeq (x, j, i, d \mid c)[r].$$

In this encoding, the symbol $(x, i, d \mid 1)$ corresponds to the bundles $\mathcal{P}(x, i \mid 1, 2)(d)$ and $\mathcal{P}(x, i \mid 2, 1)(d)$, the symbol $(x, i, d \mid 2)$ corresponds to the bundles $\mathcal{P}(x, i \mid 2, 2)(d)$ and $\mathcal{P}(x, i \mid 1, 1)(d - 1)$, and the symbol $(x, i, j \mid c)(d)$ corresponds to the bundle $\mathcal{P}(x, i, j \mid c)(d)$ for $i < j$ and the bundle $\mathcal{P}(x, j, i \mid c)(d)$ for $i > j$.

The reasoning presented above yields the following theorem:

Theorem 3.1. *If $(X, \mathcal{A})$ is a nodal noncommutative curve of almost string type, then the reduced category of triples $\overline{\mathcal{T}}(\mathcal{A})$ is equivalent to the complete subcategory of the category of images of the bundle of chains $\mathfrak{B}(\mathcal{A})$ formed by the images M all matrices of which $M_{(x,i)[r]}$ are invertible.*

Corollary 3.1. *Every nodal noncommutative curve of almost string type is derived tame.*

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