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Stable vector bundles over cuspidal cubics

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Abstract: We give a complete classification of stable vector bundles over a cuspidal cubic and calculate their cohomologies. The technique of matrix problems is used, similar to [2, 3].

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1 Introduction

Stable vector bundles over projective curves have been widely investigated. In particular, it has been proved that there are coarse moduli spaces of such bundles and they have good compactifications; the dimensions of these spaces have been found, etc. (cf. [6]). Nevertheless, not so much is known about explicit structure of stable bundles, especially over singular curves. It seems that there is a unique general result in this direction, namely that of [1] describing stable vector bundles over a nodal cubic. This description has been derived from the description of all vector bundles over such a curve [3]. If we consider more complicated singularities, e.g. a *cuspidal cubic*, it follows from [3] that the description of *all* vector bundles is a *wild problem*, i.e. contains a classification of all representations of all finitely generated algebras. Thus, if we are going to study stable vector bundles over such curves, we have to find another approach. Fortunately, there is one used before in quite different situation, namely in study of representations of "mixed"

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Lie groups (i.e. neither reductive nor solvable) [2]. It combines the technique of "matrix problems" used in [3] with the concept of "general position", allowing to restrict matrix considerations by rather simple cases, especially to avoid reductions that lead to wild fragments, thus making possible a recursive construction of all stable vector bundles.

Using this method, we prove the following main result.

Theorem 1.1. Let $C \subset \mathbb{P}^2$ be a cuspidal cubic over an algebraically closed field $\mathbf{k}$.

(1) The rank r and degree d of a stable vector bundle over C are always coprime.

(2) For every pair (r,d) of coprime integers with $r > 0$ the moduli space $\mathrm{VB}(r,d)$ of stable vector bundles of rank r and degree d over C is isomorphic to the affine line $\mathbb{A}^1$.

Note that $\mathbb{A}^1 \simeq \mathbf{k}^+$ is just the Picard group $\mathrm{Pic}^\circ(C)$ [5, Example II.6.11.4].

Moreover, we explicitly construct a universal family $\mathcal{F}(r,d,\mu)$ of stable vector bundles of rank r and degree d depending on the parameter $\mu \in \mathbb{A}^1$ and calculate their cohomologies.

Note that the matrix problem used in these calculations coincides with that arising in the description of representations of groups of echelon matrices [2]. Perhaps there would be an intrinsic reason for this coincidence, but at the moment we have no idea of what nature it can be. There is also evidence that an analogous result must be valid for other degenerate cubics. We hope to present it in the near future.

2 Matrix reduction

We use the technique of [3] to reduce the description of vector bundles to some matrix calculations. Let C be a cuspidal cubic with the singular point p, $\pi : \tilde{C} \to C$ be its normalization. (C is the compactification of the curve $y^2 = x^3$ and $p = (0,0)$.) Then $\tilde{C} \simeq \mathbb{P}^1$ and $\pi^{-1}(p) = \{q\}$, one point set. We denote $\mathcal{O} = \mathcal{O}_C$ and $\tilde{\mathcal{O}} = \mathcal{O}_{\tilde{C}}$. We also denote by $\mathrm{VB}(C)$ the category of vector bundles or, the same, locally free coherent sheaves over C. For every vector bundle $\mathcal{F}$ over C set $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$. It is a vector bundle over $\tilde{C}$. We always identify $\mathcal{F}$ with $\pi^{-1}\mathcal{F} \subset \tilde{\mathcal{F}}$. Note that $\tilde{C} \setminus \{q\} \simeq C \setminus \{p\}$ and the sections of $\tilde{\mathcal{F}}$ and $\mathcal{F}$ coincide on this common part, while $\tilde{\mathcal{F}}_q \supset \mathcal{F}_p \supset t^2\tilde{\mathcal{F}}_q$, where t is the local parameter at the point $q \in \tilde{C}$. Thus $V = \mathcal{F}_p/t^2\tilde{\mathcal{F}}_q$ is a vector subspace in $W = \tilde{\mathcal{F}}_q/t^2\tilde{\mathcal{F}}_q$. The latter is a free module over the algebra $\tilde{\mathcal{O}}_q/t^2\tilde{\mathcal{O}}_q \simeq \mathbf{k}[t]/(t^2)$. Moreover, $\dim V = \mathrm{rk}\, W = \mathrm{rk}\, \tilde{\mathcal{F}}$ and V generates W as $\mathbf{k}[t]/(t^2)$-module. On the contrary, if $\mathcal{E}$ is a vector bundle over $\tilde{C}$ of rank r and V is an r-dimensional subspace in $W = \mathcal{E}_q/t^2\mathcal{E}_q$ generating W as $\mathbf{k}[t]/(t^2)$-module, its preimage in $\mathcal{E}$ is a vector bundle $\mathcal{F}$ over C of rank r. We consider the following category $\mathcal{T}$:

- Its objects are pairs $(\mathcal{E}, V)$, where $\mathcal{E}$ is a vector bundle over $\tilde{C}$ and V is a subspace in $W = \mathcal{E}_q/t^2\mathcal{E}_q$ such that $\dim V = \mathrm{rk}\, \mathcal{E}$ and V generates W as $\mathbf{k}[t]/(t^2)$-module.
- A morphism $(\mathcal{E}, V) \to (\mathcal{E}', V')$ is a morphism of sheaves $\phi : \mathcal{E} \to \mathcal{E}'$ such that $\overline{\phi}(V) \subseteq V'$, where $\overline{\phi}$ is the induced morphism $\mathcal{E}_q/t^2\mathcal{E}_q \to \mathcal{E}'_q/t^2\mathcal{E}'_q$.

The preceding considerations make obvious the following result (cf. also [3]).

Proposition 2.1. The functor $T : \mathrm{VB}(C) \to \mathcal{T}$ mapping $\mathcal{F}$ to $(\tilde{\mathcal{F}}, \mathcal{F}_p/t^2\tilde{\mathcal{F}}_q)$ is an equivalence of categories.

Recall that all indecomposable vector bundles over $\tilde{C} \simeq \mathbb{P}^1$ are just twists $\tilde{\mathcal{O}}(d)$ of the structure sheaf [4]. Especially, $\tilde{\mathcal{F}} \simeq \bigoplus_d m_d\tilde{\mathcal{O}}(d)$ for some multiplicities m_d.

Note that the arithmetic genus of C is 1 and that of $\tilde{C}$ is 0. Therefore, the exact sequence $0 \to \mathcal{F} \to \tilde{\mathcal{F}} \to W/V \to 0$ and the Riemann–Roch theorem give that

$$\deg_C \mathcal{F} = \chi(\mathcal{F}) = \chi(\tilde{\mathcal{F}}) - r = \deg_{\tilde{C}} \tilde{\mathcal{F}},$$

where $\chi(\mathcal{F})$ denotes, as usual, the Euler–Poincaré characteristic: $\chi(\mathcal{F}) = \dim \mathrm{H}^0(\mathcal{F}) - \dim \mathrm{H}^1(\mathcal{F})$. Since W/V is zero outside the unique point q, $\chi(W/V) = \dim(W/V) = r$.

Denote by $\mathrm{sl}(\mathcal{F}) = \deg \mathcal{F}/\operatorname{rk} \mathcal{F}$, the *slope* of $\mathcal{F}$. Recall [6] that a vector bundle $\mathcal{F}$ is said to be *stable* if $\mathrm{sl}\,\mathcal{F}' < \mathrm{sl}\,\mathcal{F}$ for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$. As the arithmetic genus of C is 1, there is an easier criterion for $\mathcal{F}$ to be stable.

Lemma 2.2. A vector bundle $\mathcal{F}$ over C is stable if and only if $\operatorname{End} \mathcal{F} = \mathbf{k}$.

This condition is always necessary. Indeed, if $\operatorname{End} \mathcal{F} \neq \mathbf{k}$, there is an endomorphism f that is neither zero nor an isomorphism. Denote by $\mathcal{F}' = \operatorname{Im} f \simeq \mathcal{F}/\operatorname{Ker} f$. If $\mathcal{F}$ is stable, $\mathrm{sl}\,\mathcal{F}' < \mathrm{sl}\,\mathcal{F}$. But it implies that $\mathrm{sl}(\operatorname{Ker} f) > \mathrm{sl}\,\mathcal{F}$, so $\mathcal{F}$ is not stable.

To prove that it is also sufficient, note the following easy result.

Lemma 2.3. Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves of $\mathcal{O}$-modules.

(1) If one of them is locally free, $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq \mathcal{F}^\vee \otimes \mathcal{G}$.

(2) If $\mathcal{F}$ is locally free, $(\mathcal{F}^\vee \otimes \mathcal{G})^\vee \simeq \mathcal{G}^\vee \otimes \mathcal{F}$.

Proof.

(1) There is a natural morphism $\phi : \mathcal{F}^\vee \otimes \mathcal{G} \to \mathcal{H}om(\mathcal{F}, \mathcal{G})$, which is isomorphism if either $\mathcal{F} = \mathcal{O}$ or $\mathcal{G} = \mathcal{O}$. Since ϕ is an isomorphism if and only if all induced morphisms of stalks are isomorphisms, it implies the claim.

(2) $\mathcal{H}om(\mathcal{F}^\vee \otimes \mathcal{G}, \mathcal{O}) \simeq \mathcal{H}om(\mathcal{G}, \mathcal{H}om(\mathcal{F}^\vee, \mathcal{O})) \simeq \mathcal{G}^\vee \otimes \mathcal{F}$.

Now the Riemann–Roch theorem implies that for any coherent sheaves $\mathcal{F}, \mathcal{G}$ over C, one of which is locally free,

$$\begin{aligned}\dim \operatorname{Hom}(\mathcal{F}, \mathcal{G}) - \dim \operatorname{Ext}(\mathcal{F}, \mathcal{G}) &= \chi(\mathcal{F}^\vee \otimes \mathcal{G}) = \deg(\mathcal{F}^\vee \otimes \mathcal{G}) \\ &= \operatorname{rk} \mathcal{F} \deg \mathcal{G} - \operatorname{rk} \mathcal{G} \deg \mathcal{F},\end{aligned}$$

especially $\dim \operatorname{Hom}(\mathcal{F}, \mathcal{G}) \neq 0$ if $\mathrm{sl}\,\mathcal{G} > \mathrm{sl}\,\mathcal{F}$. Moreover, by the Serre's duality, if $\mathcal{F}$ is locally free,

$$\begin{aligned}\dim \operatorname{Ext}(\mathcal{F}, \mathcal{G}) &= \dim \mathrm{H}^1(\mathcal{H}om(\mathcal{F}, \mathcal{G})) = \dim \mathrm{H}^1(\mathcal{F}^\vee \otimes \mathcal{G}) = \\ &= \dim \mathrm{H}^0(\mathcal{G}^\vee \otimes \mathcal{F}) = \dim \operatorname{Hom}(\mathcal{G}, \mathcal{F}).\end{aligned}$$

Thus if $\operatorname{sl}\mathcal{G} = \operatorname{sl}\mathcal{F}$ and $\operatorname{Hom}(\mathcal{G}, \mathcal{F}) \neq 0$, also $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \neq 0$.

Suppose now that $\mathcal{F}$ is not stable and $\mathcal{F}' \subset \mathcal{F}$ is such that $\operatorname{sl}\mathcal{F}' \geq \operatorname{sl}\mathcal{F}$. Then, as we have seen, $\operatorname{Hom}(\mathcal{F}, \mathcal{F}') \neq 0$, so the composition of a non-zero homomorphism $\mathcal{F} \to \mathcal{F}'$ with the embedding $\mathcal{F}' \to \mathcal{F}$ gives a nontrivial endomorphism of $\mathcal{F}$.

Corollary 2.4. If $\mathcal{F}$ is a stable vector bundle, then $\{d \,|\, m_d \neq 0\} = \{c, c+1\}$ for some integer c.

Proof. Otherwise there are integers a, b such that $a \leq b-2$ and $m_a \neq 0$, $m_b \neq 0$. There is a nonzero homomorphism $f : \tilde{\mathcal{O}}(a) \to \tilde{\mathcal{O}}(b)$ such that $f(\tilde{\mathcal{O}}(a)_q) \subseteq t^2\tilde{\mathcal{O}}(b)_q$. It gives rise to an endomorphism ϕ of $\tilde{\mathcal{F}}$ such that $\overline{\phi} = 0$, so $(\phi, 0) \in \operatorname{End}(\tilde{\mathcal{F}}, \mathcal{F}_p/t^2\tilde{\mathcal{F}}_q)$ is a nontrivial endomorphism. By Proposition 2.1 it corresponds to a nontrivial endomorphism of $\mathcal{F}$. Hence $\mathcal{F}$ is not stable.

From now on we consider the full subcategories $\mathrm{VB}_c(C) \subset \mathrm{VB}(C)$ consisting of all vector bundles $\mathcal{F}$ such that $\tilde{\mathcal{F}} \simeq a\tilde{\mathcal{O}}(c) \oplus b\tilde{\mathcal{O}}(c+1)$ for some integers a, b. Note that in this case $\operatorname{rk}\mathcal{F} = a+b$ and $\deg\mathcal{F} = (a+b)c+b$. Under the equivalence T it corresponds to the full subcategory $\mathcal{T}_c \subset \mathcal{T}$ consisting of all pairs $(\mathcal{E}, V)$ with $\mathcal{E} = a\tilde{\mathcal{O}}(c) + b\tilde{\mathcal{O}}(c+1)$. The shift $\mathcal{F} \mapsto \mathcal{F}(c)$ of the category of coherent sheaves induces equivalences $\mathrm{VB}_0(C) \to \mathrm{VB}_c(C)$ and $\mathcal{T}_0 \to \mathcal{T}_c$, so we only have to consider $\mathcal{T}_0$. If $\mathcal{E} = a\tilde{\mathcal{O}} \oplus b\tilde{\mathcal{O}}(1)$, then $\mathcal{E}_q/t^2\mathcal{E}_q \simeq aW_0 \oplus bW_1$, where $W_0 = W_1 = \mathbf{k}[t]/(t^2)$. Homomorphisms $\tilde{\mathcal{O}} \to \tilde{\mathcal{O}}(1)$ induce homomorphisms of $\mathbf{k}[t]/(t^2)$-modules $W_0 \to W_1$ mapping $1 \mapsto \lambda + \mu t$ and $t \mapsto \lambda t$. Note that there are no non-zero homomorphisms $\tilde{\mathcal{O}}(1) \to \tilde{\mathcal{O}}$ and all endomorphisms of both $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}(1)$ are just multiplications by a scalar.

Given a pair $(\mathcal{E}, V)$ from $\mathcal{T}_0$, choose bases

$$\begin{array}{rl} w_1^0, w_2^0, \dots, w_a^0 & \text{of } aW_0, \\ w_1^1, w_2^1, \dots, w_b^1 & \text{of } bW_1, \\ v_1, v_2, \dots, v_r & \text{of } V \end{array}$$

and write

$$v_j = \sum_{i=1}^{a} ((\alpha_{ij}^0 + \beta_{ij}^0 t)w_i^0 + (\alpha_{ij}^1 + \beta_{ij}^1 t)w_i^1).$$

Since $v_1, v_2, \dots, v_r$ must generate W, the matrix

$$L = \begin{pmatrix} \alpha_{11}^0 & \dots & \alpha_{1r}^0 \\ \dots & \ddots & \dots \\ \alpha_{a1}^0 & \dots & \alpha_{ar}^0 \\ \alpha_{11}^1 & \dots & \alpha_{1r}^1 \\ \dots & \ddots & \dots \\ \alpha_{b1}^1 & \dots & \alpha_{br}^1 \end{pmatrix}$$

must be invertible. Therefore, changing the basis $v_1, v_2, \dots, v_r$, we may suppose that it is the unit $r \times r$ matrix. Then, using automorphisms of $\mathcal{E}$ of the form $\mathrm{id} + f$, where f arises from a homomorphism $\tilde{\mathcal{O}} \to \tilde{\mathcal{O}}(1)$, we can make $\beta^1_{ij} = 0$ for $j \le a$. Thus the remaining part of coefficients β^k_{ij} form a matrix

$$M = \begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix} \tag{1}$$

where M_1 is of size $a \times a$, M_2 is of size $a \times b$ and M_3 is of size $b \times b$. We shall always suppose that the bases $v_1, v_2, \dots, v_r$, $w^0_1, w^0_2, \dots, w^0_a$ and $w^1_1, w^1_2, \dots, w^1_b$ are chosen this way and call M a *defining matrix* of the pair $(\mathcal{E}, V)$. If M' is a defining matrix for another pair $(\mathcal{E}', V')$ and $\phi : (\mathcal{E}, V) \to (\mathcal{E}', V')$ is a morphism of pairs, let $\Phi = \Phi_0 + \Phi_1 t$ be the matrix of the homomorphism $\overline{\phi}$ with respect to the bases $w^0_1, w^0_2, \dots, w^0_a, w^1_1, w^1_2, \dots, w^1_b$ of W and $w'^0_1, w'^0_2, \dots, w'^0_{a'}, w'^1_1, w'^1_2, \dots, w'^1_{b'}$ of W'. Note that

$$\Phi_0 = \begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix} \quad \text{and} \quad \Phi_1 = \begin{pmatrix} 0 & 0 \\ S_4 & 0 \end{pmatrix} \tag{2}$$

where S_1 is of size $a' \times a$, S_2 and S_4 of size $b' \times a$, and S_3 of size $b' \times b$, and the matrices Φ_0, Φ_1 uniquely define the morphism ϕ. The condition $\overline{\phi}(V) \subseteq V'$ means that

$$\begin{aligned} S_1 M_1 &= M'_1 S_1 + M'_2 S_2, \\ S_1 M_2 &= M'_2 S_3, \\ S_3 M_3 + S_2 M_2 &= M'_3 S_3. \end{aligned} \tag{3}$$

On the contrary, if S_1, S_2, S_3 satisfy equations 3, there exists a unique matrix S_4 such that the pair Φ_0, Φ_1 given by equations 2 arises from a uniquely defined morphism $\phi : (\mathcal{E}, V) \to (\mathcal{E}', V')$.

We consider the category of *matrix triples* $\mathbf{M}$. Its objects are triples (M_1, M_2, M_3) as above and morphisms $(M_1, M_2, M_3) \to (M'_1, M'_2, M'_3)$ are triples (S_1, S_2, S_3) satisfying conditions 3. A triple from $\mathbf{M}$ is called *stable* if it only has scalar endomorphisms. We denote by $\mathbf{M}^s$ the full subcategory of stable triples. Then we get the following result.

Theorem 2.5. The category $\mathbf{M}$ is equivalent to the category $\mathrm{VB}_c(C)$. Especially, the category $\mathbf{M}^s$ is equivalent to the full subcategory $\mathrm{VB}^s_c(C) \subset \mathrm{VB}_c(C)$ of stable vector bundles.

We call the pair (a, b) the *size* of the triple $M = (M_1, M_2, M_3)$, the sum $r = a + b$ the *rank* of this triple and b the *degree* of this triple. Note that r coincides with the rank of the corresponding vector bundle from VB_c, while the actual degree of this vector bundle is $d = rc + b$. Especially r, d are coprime if and only if so are a, b. We also call the matrix

$$\begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$$

the *defining matrix* corresponding to the triple M. It is indeed a defining triple of a pair $(\mathcal{E}, V)$ from the category $\mathcal{T}_c$, and thus defines a vector bundle $\mathcal{F} \in \mathrm{VB}_c(C)$. Especially $\mathcal{F}$ is stable if and only if so is the triple (M_1, M_2, M_3).

3 Proof of the main theorem

Theorem 2.5 shows that the main theorem 1.1 is equivalent to the following.

Theorem 3.1.
(1) There is a stable triple of size (a, b) if and only if a, b are coprime.
(2) For every pair (a, b) of coprime integers the moduli space $\mathcal{T}(a, b)$ of stable triples of size (a, b) is isomorphic to the affine line $\mathbb{A}^1$.

Proof. Suppose that a triple $M = (M_1, M_2, M_3)$ is stable. If $b = 0$ (the same for $a = 0$), we only have one matrix M_1. If $a > 1$, there is a nonscalar matrix S_1 commuting with M_1, hence defining a non-trivial endomorphism of the triple. If $a = 1$, $M_1 = \mu \in \mathbf{k}$. Hence, if either a or b is zero, then $r = 1$ and the moduli space is $\mathbb{A}^1$ (it coincides with the Picard group $\mathrm{Pic}^\circ(C)$). We denote the corresponding vector bundle (actually line bundle) from $\mathrm{VB}_c(C)$ by $\mathcal{F}(1, c, \mu)$ (it is of degree c).

From now on we suppose that both a and b are non-zero. First of all we show that $\mathrm{rk}\, M_2 = \min(a, b)$. Indeed, if $\mathrm{rk}\, M_2 < \min(a, b)$, there are invertible matrices S_1, S_3 such that both the first row and the last column of the matrix $M_2' = S_1 M_2 S_3^{-1}$ are zero. Replacing M by an isomorphic triple, we may suppose that $M_2 = M_2'$. Then the triple (I_a, S_2, I_b), where S_2 has only one non–zero element in the lower left corner, defines a nontrivial endomorphism, so M is not stable.

If $a = b = \mathrm{rk}\, M_2$, we can make M_2 a unit matrix and $M_3 = 0$. Then conditions 3 for $M = M'$ become $S_1 = S_3$, $S_2 = 0$ and $S_1 M_1 = M_1 S_1$. If $a > 1$, one can easily find a non-scalar matrix S_1 such that these conditions hold, so M is not stable. If $a = 1$, S_1 is just an element from $\mathbf{k}$, so M is stable. Moreover, one cannot change M_3 (which is also an element $\mu \in \mathbf{k}$) without changing M_2 and M_1, so the stable triples of this shape are $(0, 1, \mu)$. they are of size $(1, 1)$ and their moduli space is $\mathbb{A}^1$.

Suppose that $a < b$; then M_2 can be chosen in the form $(I_a\ 0)$. Using transformations 3, we can make M_1 zero and transform M_3 to the form

$$\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}, \tag{4}$$

where N_1, N_2, N_3 are of sizes, respectively, $a \times a$, $a \times (b - a)$, $(b - a) \times (b - a)$. Moreover, if a triple (S_1, S_2, S_3) is a homomorphism of M to another triple M' of the same form, with N_i replaced by N_i', one can check that S_3 is of the form

$$S_3 = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}$$

such that

$$\begin{aligned} T_1 N_1 &= N_1' T_1 + N_2' T_2, \\ T_1 N_2 &= N_2' T_3, \\ T_3 N_3 + T_2 N_2 &= N_3' T_3, \end{aligned}$$

$S_1 = T_1$ and the part S_2 is also uniquely defined by S_3. Thus mapping a triple (N_1, N_2, N_3) of size $(a, b-a)$ to the triple

$$\left(0,\ (I_a\ 0),\ \begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}\right)$$

we get a full embedding $\mathbf{M} \to \mathbf{M}$, which induces a one-to-one correspondence between stable triples of size $(a, b-a)$ and those of size (a, b). The same result can be obtained if $a > b$: then we map a triple (N_1, N_2, N_3) of size $(a-b, b)$ to the triple

$$\left(\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix},\ \begin{pmatrix} 0 \\ I_b \end{pmatrix},\ 0\right).$$

Therefore we are able to use induction on $\max(a, b)$, which immediately implies the claim of the theorem.

Remark 3.2. One can easily check that the matrix problem given by the category $\mathbf{M}$ is actually *wild*, hence so is also a description of *all* vector bundles from $\mathrm{VB}_c(X)$.

Note that the proof above is *effective*, i.e. enables to get an explicit description of stable vector bundles of any prescribed rank r and degree d (which must be coprime). To do it, we have first to find a, b, c such that $r = a + b$, $d = rc + b$. It means that b is the residue of d modulo r, $c = [d/r]$ and $a = r - b$. Having (a, b), suppose that $a < b$. If $a = 1$, the canonical defining matrix $\mathcal{M}(1, b, \mu)$ is

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & \mu \end{pmatrix}.$$

Let $a > 1$, $b = qa + b'$ with $0 < b' < b$. Subdivide the defining matrix

$$\begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$$

into $a \times a$ blocks starting from the left upper corner; the last horizontal and vertical stripes will be of width b'. Set all blocks zero, except those immediately over diagonal and the last two diagonal blocks, and set all square blocks immediately over the diagonal, except the last one, equal I_a, obtaining

$$\begin{pmatrix} 0 & I_a & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I_a & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & I_a & 0 \\ 0 & 0 & 0 & \dots & 0 & N_1 & N_2 \\ 0 & 0 & 0 & \dots & 0 & 0 & N_3 \end{pmatrix}$$

(there are $q+2$ horizontal and vertical stripes). If $a > b = 1$, the canonical defining matrix $\mathcal{M}(a,1,\mu)$ is

$$\begin{pmatrix} \mu & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

If $a > b > 1$, $a = qb + a'$ with $0 < a' < b$, start from the lower right corner and make $b \times b$ blocks, obtaining

$$\begin{pmatrix} N_1 & N_2 & 0 & \dots & 0 & 0 \\ 0 & N_3 & I_b & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & I_b \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Here the first horizontal and vertical stripes are of width a'. Now pass to the triple (N_1, N_2, N_3) applying the same procedure, etc. The resulting triple is called the *canonical form* of a stable triple of size (a,b), and its defining matrix $\mathcal{M} = \mathcal{M}(a,b,\mu)$ is called the *canonical defining matrix*. To obtain the corresponding vector bundle, consider the vector bundle $\mathcal{E} = a\tilde{\mathcal{O}}(c) \oplus b\tilde{\mathcal{O}}(c+1)$ over $\tilde{C}$ and take the $\mathcal{O}$-subsheaf $\mathcal{F} = \mathcal{F}(r,d,\mu)$ which coincides with $\mathcal{E}$ outside p and is generated by the preimages of columns of the matrix $I_r + t\mathcal{M}$ at the point p.

Example 3.3. Let $a = 3, b = 11$. The first step of reduction gives the matrix

$$\begin{pmatrix} 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & N_1 & N_2 \\ 0 & 0 & 0 & 0 & N_3 \end{pmatrix},$$

the triple (N_1, N_2, N_3) being of size $(3,2)$. The second step replaces

$$\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$$

by

$$\begin{pmatrix} L_1 & L_2 & 0 \\ 0 & L_3 & I_2 \\ 0 & 0 & 0 \end{pmatrix},$$

where the triple (L_1, L_2, L_3) is of size $(1,2)$. So we have to set

$$\begin{pmatrix} L_1 & L_2 \\ 0 & L_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \mu \end{pmatrix}.$$

Thus the canonical defining matrix $\mathcal{M}(3, 11, \mu)$ is

$$
\begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix} = \left[\begin{array}{ccc||ccc|ccc|ccc|cc}
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
\hline\hline
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&\mu&0&1\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].
$$

Here we keep lines showing the subdivision of the first step; double lines denotes the original division of this matrix into M_1, M_2, M_3. In the same way, the canonical defining matrix $\mathcal{M}(7, 4, \mu)$ is

$$
\left[\begin{array}{ccc|cccc||cccc}
0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
\hline
0&0&0&\mu&1&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1\\
\hline\hline
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].
$$

We note also the following easy consequence of our reduction procedure, which is useful, for instance, in calculating cohomologies.

Corollary 3.4. For every stable trip (M_1, M_2, M_3) of size (a, b)

$$
\operatorname{rk} \begin{pmatrix} M_1 & M_2 \end{pmatrix} = a \quad \text{and} \quad \operatorname{rk} \begin{pmatrix} M_2 \\ M_3 \end{pmatrix} = b.
$$

4 Cohomologies

Having an explicit description, we can calculate cohomologies of stable vector bundles $\mathcal{F}(r, d, \mu)$.

Theorem 4.1. The sheaves $\mathcal{F} = \mathcal{F}(r, d, \mu)$ have the following dimensions of cohomologies:

$$\dim \mathrm{H}^0(C, \mathcal{F}) = \begin{cases} d & \text{if } d > 0, \\ 1 & \text{if } d = 0 \text{ and } \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

$$\dim \mathrm{H}^1(C, \mathcal{F}) = \begin{cases} -d & \text{if } d < 0, \\ 1 & \text{if } d = 0 \text{ and } \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Proof. Note first that if $\mathcal{F}, \mathcal{G}$ are line bundles, then any non-zero homomorphism $\mathcal{G} \to \mathcal{F}$ is a monomorphism. Therefore, if $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \neq 0$, necessarily, $\deg \mathcal{G} \leq \deg \mathcal{F}$ and if these degrees coincide, then $\mathcal{G} \simeq \mathcal{F}$. Recall also that $\dim \mathrm{Ext}(\mathcal{G}, \mathcal{F}) = \dim \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Especially, if $\mathcal{G} = \mathcal{O}$, it gives

$$\begin{aligned} &\mathrm{H}^0(\mathcal{F}) \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{F}) = 0 \quad \text{if } \deg \mathcal{F} < 0 \text{ or } \deg \mathcal{F} = 0 \text{ and } \mathcal{F} \not\simeq \mathcal{O}; \\ &\mathrm{H}^1(\mathcal{F}) \simeq \mathrm{Ext}(\mathcal{O}, \mathcal{F}) = 0 \quad \text{if } \deg \mathcal{F} > 0 \text{ or } \deg \mathcal{F} = 0 \text{ and } \mathcal{F} \not\simeq \mathcal{O} \end{aligned}$$

for any line bundles $\mathcal{F}$. Together with the Riemann–Roch theorem it implies the formulas 5 and 6 for line bundles, i.e. for the case when $b = 0$ or $a = 0$.

Suppose now that $r > 1$, hence $a > 0, b > 0$ and $d \neq 0$. The exact sequence $0 \to \mathcal{F} \to \tilde{\mathcal{F}} \to W/V \to 0$ gives rise to the exact sequence of cohomologies

$$0 \to \mathrm{H}^0(\mathcal{F}) \to \mathrm{H}^0(\tilde{\mathcal{F}}) \xrightarrow{h} W/V \to \mathrm{H}^1(\mathcal{F}) \to \mathrm{H}^1(\tilde{\mathcal{F}}) \to 0,$$

and $\dim W/V = r$. Let $\tilde{\mathcal{F}} = a\tilde{\mathcal{O}}(c) \oplus b\tilde{\mathcal{O}}(c+1)$, then $r = a + b$, $d = cr + b$ and

$$\dim \mathrm{H}^0(\tilde{\mathcal{F}}) = \begin{cases} (c+1)r + b & \text{if } c \geq -1, \\ 0 & \text{if } c < -1. \end{cases}$$

Note that $d > 0$ if and only if $c = [d/r] \geq 0$. Let H be the image of $\mathrm{H}^0(\tilde{\mathcal{F}})$ in $W = \tilde{\mathcal{F}}_q / t^2 \tilde{\mathcal{F}}_q$. If $c \geq 0$, global sections generate $\tilde{\mathcal{F}}$, so $H = W$ and $\mathrm{Im}\, h = W/V$, wherefrom $\dim \mathrm{H}^0(\mathcal{F}) = cr + b = d$. If $c = -1$, Corollary 3.4 easily implies that $\dim(H + V) = a + 2b$, so $\dim(\mathrm{Im}\, h) = b$ and $\mathrm{H}^0(\mathcal{F}) = 0$. We have proven formula 5. Now the Riemann–Roch theorem implies formula 6.

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