

ON COHEN–MACAULAY MODULES OVER NON-COMMUTATIVE SURFACE SINGULARITIES

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ABSTRACT. We generalize the results of Kahn about a correspondence between Cohen–Macaulay modules and vector bundles to non-commutative surface singularities. As an application, we give examples of non-commutative surface singularities which are not Cohen–Macaulay finite, but are Cohen–Macaulay tame.

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INTRODUCTION

Cohen–Macaulay modules over commutative Cohen–Macaulay rings have been widely studied. A good survey on this topic is the book of Yoshino [14]. In particular, for curve, surface and hypersurface singularities criteria are known for them to be *Cohen–Macaulay finite*, i.e. only having finitely many indecomposable Cohen–Macaulay modules (up to isomorphism). For curve singularities and minimally elliptic surface singularities criteria are also known for them to be *Cohen–Macaulay tame*, i.e. only having 1-parameter families of non-isomorphic indecomposable Cohen–Macaulay modules [4, 5]. Less is known if we consider non-commutative Cohen–Macaulay algebras. In [6] a criterion was given for a *primary* 1-dimensional Cohen–Macaulay algebra to be Cohen–Macaulay finite. In [1] (see also [3]) a criterion of Cohen–Macaulay finiteness is given for *normal* 2-dimensional Cohen–Macaulay algebras (maximal orders). As far as we know, there are no examples of 2-dimensional Cohen–Macaulay algebras which are not Cohen–Macaulay finite but are Cohen–Macaulay tame.

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In this paper we use the approach of Kahn [10] to study Cohen–Macaulay modules over normal non-commutative surface singularities. Just as in [10], we establish (in Section 2) a one-to-one correspondence between such modules and vector bundles over some, in general non-commutative, projective curves (Theorem 2.13). In Sections 3 and 4 we apply this result to a special case, which we call “*good elliptic*.” It is analogous to the minimally elliptic case in [10], though seems somewhat too restrictive. Unfortunately, we could not find more general conditions which ensure such analogy. As an application, we present two examples of Cohen–Macaulay tame non-commutative surface singularities (Examples 4.1 and 4.2). We hope that this approach shall be useful in more general situations too.

1. PRELIMINARIES

We fix an algebraically closed field \mathbf{k} , say *algebra* instead of \mathbf{k} -algebra, *scheme* instead of \mathbf{k} -scheme and write Hom and \otimes instead of $\mathrm{Hom}_{\mathbf{k}}$ and $\otimes_{\mathbf{k}}$. We call a scheme X a *variety* if $\mathbf{k}(x) = \mathbf{k}$ for every closed point $x \in X$.

Definition 1.1. A *non-commutative scheme* is a pair (X, A) , where X is a scheme and A is a sheaf of \mathcal{O}_X -algebras coherent as a sheaf of \mathcal{O}_X -modules. If X is a variety, (X, A) is called a *non-commutative variety*. We say that (X, A) is *affine*, *projective*, *excellent*, etc. if so is X .

A *morphism* of non-commutative schemes $(X, A) \rightarrow (Y, B)$ is their morphism as ringed spaces, i.e. a pair $(\varphi, \varphi^\sharp)$, where $\varphi : X \rightarrow Y$ is a morphism of schemes and $\varphi^\sharp : \varphi^{-1}A \rightarrow B$ is a morphism of sheaves of algebras. A morphism $(\varphi, \varphi^\sharp)$ is said to be *finite*, *projective* or *proper* if so is φ . We often omit φ^\sharp and write $\varphi : (X, A) \rightarrow (Y, B)$.

For a non-commutative scheme (X, A) we denote by $\mathrm{Coh} A$ ($\mathrm{Qcoh} A$) the category of coherent (quasi-coherent) sheaves of A -modules. Every morphism $\varphi : (X, A) \rightarrow (Y, B)$ induces functors of direct image $\varphi_* : \mathrm{Qcoh} A \rightarrow \mathrm{Qcoh} B$ and inverse image $\varphi^* : \mathrm{Qcoh} B \rightarrow \mathrm{Qcoh} A$, where $\varphi^*\mathcal{F} = A \otimes_{\varphi^{-1}B} \varphi^{-1}\mathcal{F}$. Note that this inverse image does not coincide with the inverse image of sheaves of \mathcal{O}_X -modules. The latter (when used) will be denoted by φ_X^* . Note also that φ^* maps coherent sheaves to coherent. The pair (φ^*, φ_*) is a pair of adjoint functors, i.e. there is a functorial isomorphism $\mathrm{Hom}_A(\varphi^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_B(\mathcal{F}, \varphi_*\mathcal{G})$ for any sheaf of B -modules \mathcal{F} and any sheaf of A -modules \mathcal{G} .

We call a coherent sheaf of A -modules \mathcal{F} a *vector bundle* if it is locally projective, i.e. \mathcal{F}_p is a projective A_p -module for every point $p \in X$. We denote by $\mathrm{VB}(A)$ the full subcategory of $\mathrm{Coh} A$ consisting of vector bundles.

A non-commutative scheme (X, A) is said to be *regular* if $\mathrm{gl. dim} A_p = \dim_p X$ for every point $p \in X$ (it is enough to check this property at the closed points).

We say that (X, A) is *reduced* if X is reduced and neither stalk A_p contains nilpotent ideals. Then, if $\mathcal{K} = \mathcal{K}_X$ is the sheaf of rational functions on X , $\mathcal{K}(A) = A \otimes_{\mathcal{O}_X} \mathcal{K}$ is a locally constant sheaf of semisimple \mathcal{K} -algebras. We

call it the *sheaf of rational functions* on (X, A) . In this case each stalk A_p is an *order* in the algebra $\mathcal{K}(A)_p$, i.e. an $\mathcal{O}_{X,p}$ -algebra finitely generated as $\mathcal{O}_{X,p}$ -module and such that $\mathcal{K}_p A_p = \mathcal{K}(A)_p$. We say that (X, A) is *normal* if A_p is a maximal order in $\mathcal{K}(A)_p$ for each p . Note that a regular scheme is always reduced, but not necessarily normal.

A morphism $(\varphi, \varphi^\sharp) : (X, A) \rightarrow (Y, B)$ of reduced non-commutative schemes is said to be *birational* if $\varphi : X \rightarrow Y$ is birational and the induced map $\mathcal{K}(B) \rightarrow \mathcal{K}(A)$ is an isomorphism.

A *resolution* of a non-commutative scheme (X, A) is a proper birational morphism $(\pi, \pi^\sharp) : (\tilde{X}, \tilde{A}) \rightarrow (X, A)$, where (\tilde{X}, \tilde{A}) is regular and normal.

Remark 1.2. Let (X, A) be a non-commutative scheme and $C = \text{cen}(A)$ be the center of A . (It means that $C_p = \text{cen}(A_p)$ for every point $p \in X$.) Let also $X' = \text{Spec } C$. The natural morphism $\varphi : X' \rightarrow X$ is finite and $A' = \varphi^{-1}A$ is a sheaf of $\mathcal{O}_{X'}$ -modules, so we obtain a morphism $(\varphi, \varphi^\sharp) : (X', A') \rightarrow (X, A)$, where φ^\sharp is identity. Moreover, the induced functors φ^* and φ_* define an equivalence of $\text{Qcoh } A$ and $\text{Qcoh } A'$. So, while we are interesting in study of sheaves, we can always suppose that A is a sheaf of *central \mathcal{O}_X -algebras*. Note that if (X, A) is normal and A is central, then X is also normal.

Given a non-commutative scheme (X, A) and a morphism of schemes $\varphi : Y \rightarrow X$, we can consider the non-commutative scheme $(Y, \varphi_Y^* A)$ and uniquely extend φ to the morphism $(Y, \varphi_Y^* A) \rightarrow (X, A)$ which we also denote by φ . Especially, if φ is a blow-up of a subscheme of X , we call the morphism $(Y, \varphi_Y^* A) \rightarrow (X, A)$ the blow-up of (X, A) .

Definition 1.3. A reduced excellent non-commutative variety (X, A) is called a *non-commutative surface* if X is a surface, i.e. $\dim X = 2$. If $X = \text{Spec } R$, where R is a local complete noetherian algebra with the residue field \mathbf{k} (then it is automatically excellent), we say that (X, A) is a *germ of non-commutative surface singularity* or, for short, a *non-commutative surface singularity*. In what follows, we identify a non-commutative surface singularity (X, A) with the R -algebra $\Gamma(X, A)$ and the sheaves from $\text{Qcoh } A$ with modules over this algebra (finitely generated for the sheaves from $\text{Coh } A$).

If (X, A) is a non-commutative surface, there always is a normal non-commutative surface (X', A') and a finite birational morphism $\nu : (X', A') \rightarrow (X, A)$. We call (X', A') , as well as the morphism ν , a *normalization* of (X, A) . Note that, unlike the commutative case, such normalization is usually not unique.

Let (X, A) be a connected central non-commutative surface such that X is normal, $C \subset X$ be an irreducible curve with the general point g , $\mathcal{K}_C(A) = A_g / \text{rad } A_g$ and $\mathbf{k}_A(C) = \text{cen } \mathcal{K}_C(A)$. A is normal if and only if it is Cohen–Macaulay (or, the same, reflexive) as a sheaf of \mathcal{O}_X -modules, $\mathcal{K}_C(A)$ is a simple algebra and $\text{rad } A_g$ is a principal left (or right) A_g -ideal for every

curve C [12]. $\mathbf{k}_A(C)$ is a finite extension of the field of rational functions $\mathbf{k}(C) = \mathcal{O}_{X,g}/\text{rad } \mathcal{O}_{X,g}$ on the curve C . The integer $e_C(A) = \dim_{\mathbf{k}(C)} \mathbf{k}_A(C)$ is called the *ramification index* of A on C , and A is said to be *ramified on* C if $e_C(A) > 1$. If p is a regular closed point of C , we denote by $e_{C,p}(A)$ the ramification index of the extension $\mathbf{k}_A(C)$ over $\mathbf{k}(C)$ with respect to the discrete valuation defined by the point p . For instance, if C is smooth, $e_{C,p}(A)$ is defined for all closed points $p \in C$. We denote by $D(A)$ the *ramification divisor* $D = D(A)$ which is the union of all curves $C \subset X$ such that A is ramified on C . Note that if $p \in X \setminus D(A)$, then A_p is an Azumaya algebra over $\mathcal{O}_{X,p}$.

Suppose that (X, A) is a normal non-commutative surface and A is central. Then X is Cohen–Macaulay and A is maximal Cohen–Macaulay as a sheaf of \mathcal{O}_X -modules. We denote by $\text{CM}(A)$ the category of sheaves of maximal Cohen–Macaulay A -modules, i.e. the full subcategory of $\text{Coh } A$ consisting of sheaves \mathcal{F} which are maximal Cohen–Macaulay considered as sheaves of \mathcal{O}_X -modules. We often omit the attribute “maximal” and just say shortly “Cohen–Macaulay module.” Obviously, $\text{VB}(A) \subseteq \text{CM}(A)$ and these categories coincide if and only if A is regular. For a sheaf $\mathcal{F} \in \text{Coh } A$ we denote by \mathcal{F}^\vee the sheaf $\text{Hom}_A(\mathcal{F}, A)$. It always belongs to $\text{CM}(A)$. We also set $\mathcal{F}^\dagger = \mathcal{F}^{\vee\vee}$. There is a morphism of functors $\text{Id} \rightarrow \dagger$, which is isomorphism when restricted onto $\text{CM}(A)$. If $\varphi : (X, A) \rightarrow (Y, B)$ is a morphism of central normal non-commutative surfaces, we set $\varphi^\dagger \mathcal{F} = (\varphi^* \mathcal{F})^\dagger$.

It is known that every non-commutative surface has a regular resolution. More precisely, we can use the following procedure of Chan–Ingalls [3].¹ The non-commutative surface (X, A) is said to be *terminal* [3, Definition 2.5] if the following conditions hold:

- (1) X is smooth.
- (2) All irreducible components of $D = D(A)$ are smooth.
- (3) D only has normal crossings (i.e. nodes as singular points).
- (4) At a node $p \in D$, for one component C_1 of D containing this point, the field $\mathbf{k}_A(C_1)$ is totally ramified over $\mathbf{k}(C_1)$ of degree $e = e_{C_1}(A) = e_{C_1,p}(A)$, and for the other component C_2 also $e_{C_2,p}(A) = e$.

It is shown in [3] that every terminal non-commutative surface is regular and every non-commutative surface (X, A) has a terminal resolution $\pi : (\tilde{X}, \tilde{A}) \rightarrow (X, A)$. Moreover, such resolution can be obtained by a sequence of morphisms π_i , where each π_i is either a blow-up of a closed point or a normalization. Then π is a projective morphism. If (X, A) is a normal non-commutative surface singularity, $\check{X} = X \setminus \{o\}$, where o is the unique closed point of X , the restriction of π onto $\pi^{-1}(\check{X})$ is an isomorphism and we always identify $\pi^{-1}(\check{X})$ with \check{X} . The subscheme $E = \pi^{-1}(o)_{\text{red}}$ is a connected (though maybe reducible) projective curve called the *exceptional curve* of the resolution π .

¹Note that the term “normal” is used in [3] in more wide sense, but we only need it for our notion of normality.

Recall also that, for a normal non-commutative surface singularity A , the category $\text{CM}(A)$, as well as the ramification data of A , only depends on the algebra $\mathcal{K}(A)$ [1, (1.6)]. If A is central and *connected*, i.e. indecomposable as a ring, $\mathcal{K}(A)$ is a central simple algebra over the field \mathcal{K} , so the category $\text{CM}(A)$ is defined by the class of $\mathcal{K}(A)$ in the Brauer group $\text{Br}(\mathcal{K})$, and this class is completely characterized by its ramification data.

We also use the notion of *non-commutative formal scheme*, which is a pair $(\mathfrak{X}, \mathfrak{A})$, where \mathfrak{X} is a “usual” (commutative) formal scheme and \mathfrak{A} is a sheaf of $\mathcal{O}_{\mathfrak{X}}$ -algebras coherent as a sheaf of $\mathcal{O}_{\mathfrak{X}}$ -modules. If (X, A) is non-commutative scheme and $Y \subset X$ is a closed subscheme, the *completion* (\tilde{X}, \tilde{A}) of (X, A) along Y is well-defined and general properties of complete schemes and their completions, as in [7, 9], hold in non-commutative case too.

2. KAHN’S REDUCTION

From now on we consider a normal non-commutative surface singularity (X, A) and suppose A central. We fix a resolution $\pi : (\tilde{X}, \tilde{A}) \rightarrow (X, A)$, where \tilde{A} is also supposed central. Then $\text{CM}(\tilde{A}) = \text{VB}(\tilde{A})$ and we consider π^\dagger as a functor $\text{CM}(A) \rightarrow \text{VB}(\tilde{A})$. A vector bundle \mathcal{F} is said to be *full* if it is isomorphic to $\pi^\dagger M$ for some (maximal) Cohen–Macaulay A -module M . We denote by $\text{VB}^f(\tilde{A})$ the full subcategory of $\text{VB}(\tilde{A})$ consisting of full vector bundles. We also set $\omega_{\tilde{A}} = \mathcal{H}om_{\tilde{X}}(\tilde{A}, \omega_{\tilde{X}})$, where $\omega_{\tilde{X}}$ is a canonical sheaf over \tilde{X} , and call $\omega_{\tilde{A}}$ the *canonical sheaf* of \tilde{A} . It is locally free, i.e. belongs to $\text{VB}(\tilde{A})$.

Given a coherent sheaf $\mathcal{F} \in \text{Coh } \tilde{A}$, we denote by $\text{ev}_{\mathcal{F}}$ the natural map $\Gamma(\tilde{X}, \mathcal{F}) \otimes \tilde{A} \rightarrow \mathcal{F}^2$. We say that \mathcal{F} is *globally generated* if $\text{Im } \text{ev}_{\mathcal{F}} = \mathcal{F}$ and *generically globally generated* if $\text{supp}(\mathcal{F}/\text{Im } \text{ev}_{\mathcal{F}})$ is discrete, i.e. consists of finitely many closed points.

Theorem 2.1 (Cf. [10, Proposition 1.2]). (1) *The functor π^\dagger establishes an equivalence between the categories $\text{CM}(A)$ and $\text{VB}^f(\tilde{A})$, its quasi-inverse being the functor π_* .*

(2) *A vector bundle $\mathcal{F} \in \text{VB}(\tilde{A})$ is full if and only if the following conditions hold:*

(a) *\mathcal{F} is generically globally generated.*

(b) *The restriction map $\Gamma(\tilde{X}, \mathcal{F}) \rightarrow \Gamma(\tilde{X}, \mathcal{F})$ is surjective, or equivalently, using local cohomologies,*

(b') *The map $\alpha_{\mathcal{F}} : \mathbb{H}_E^1(\tilde{X}, \mathcal{F}) \rightarrow \mathbb{H}^1(\tilde{X}, \mathcal{F})$ is injective.*

Under these conditions $\mathcal{F} \simeq \pi^\dagger \pi_ \mathcal{F}$.*

Proof. Note that there is an exact sequence

$$0 \rightarrow \text{tors}(\pi^* M) \rightarrow \pi^* M \xrightarrow{\gamma_M} \pi^\dagger M \rightarrow \overline{M} \rightarrow 0,$$

where $\text{tors}(\mathcal{M})$ denotes the periodic part of \mathcal{M} and the support of \overline{M} consists of finitely many closed points. Since $\pi^* M$ is always globally generated, so

²Recall that $\Gamma(\tilde{X}, \mathcal{F}) \simeq \text{Hom}_{\tilde{A}}(\tilde{A}, \mathcal{F})$.

is also $\text{Im } \gamma_M$. Therefore, $\pi^\dagger M$ is generically globally generated. If M is Cohen–Macaulay, the restriction map $\Gamma(X, M) \rightarrow \Gamma(\tilde{X}, M) = \Gamma(\tilde{X}, \pi^* M)$ is an isomorphism. Since M naturally embeds into $\Gamma(\tilde{X}, \pi^* M)$ and hence into $\Gamma(\tilde{X}, \pi^\dagger M)$, the restriction $\Gamma(\tilde{X}, \pi^\dagger M) \rightarrow \Gamma(\check{X}, \pi^\dagger M)$ is surjective.

Suppose now that the conditions (a) and (b) hold. Set $M = \pi_* \mathcal{F}$. Since π is projective, $M \in \text{Coh } A$. The condition (b) implies that $M \in \text{CM}(A)$. Note that $\Gamma(X, M) = \Gamma(\tilde{X}, \mathcal{F})$, so the image of the natural map $\pi^* M \rightarrow \mathcal{F}$ coincides with $\text{Im } \gamma_M$. As \mathcal{F} is generically globally generated, it implies that the natural map $\pi^\dagger M \rightarrow \mathcal{F}$ is an isomorphism. It proves (2).

Obviously, the functors $\pi^\dagger : \text{CM}(A) \rightarrow \text{VB}^f(\tilde{A})$ and $\pi_* : \text{VB}^f(\tilde{A}) \rightarrow \text{CM}(A)$ are adjoint. Moreover, if $M = \pi_* \mathcal{F}$, where $\mathcal{F} \in \text{VB}^f(\tilde{A})$, there are functorial isomorphisms

$$\begin{aligned} \text{Hom}_A(M, M) &\simeq \text{Hom}_{\tilde{A}}(\pi^* \pi_* \mathcal{F}, \mathcal{F}) \simeq \\ &\simeq \text{Hom}_{\tilde{A}}(\pi^\dagger \pi_* \mathcal{F}, \mathcal{F}) \simeq \text{Hom}_{\tilde{A}}(\mathcal{F}, \mathcal{F}). \end{aligned}$$

It proves (1). \square

Remark 2.2. A full vector bundle over \tilde{A} need not be generically globally generated as a sheaf of $\mathcal{O}_{\tilde{X}}$ -modules. Moreover, examples below show that even the sheaf $\tilde{A} = \pi^* A = \pi^\dagger A$ need not be generically globally generated as a sheaf of $\mathcal{O}_{\tilde{X}}$ -modules.

Definition 2.3. From now on we consider a sheaf of ideals \mathcal{I} in \tilde{A} such that $\text{supp}(\tilde{A}/\mathcal{I}) \subseteq E$, $\Lambda = \tilde{A}/\mathcal{I}$ and $Z = \text{Spec}(\text{cen } \Lambda)$. Then (Z, Λ) is a projective non-commutative curve, i.e. a projective non-commutative variety of dimension 1 (maybe non-reduced). We set $\omega_Z = \mathcal{E}xt_{\tilde{X}}^1(\mathcal{O}_Z, \omega_{\tilde{X}})$ and

$$\omega_\Lambda = \mathcal{E}xt_{\tilde{A}}^1(\Lambda, \omega_{\tilde{A}}) \simeq \mathcal{E}xt_{\tilde{X}}^1(\Lambda, \omega_{\tilde{X}}) \simeq \mathcal{H}om_Z(\Lambda, \omega_Z).$$

The sheaves ω_Z and ω_Λ , respectively, are canonical sheaves for Z and Λ . It means that there are Serre dualities

$$\begin{aligned} \text{Ext}_Z^i(\mathcal{F}, \omega_Z) &\simeq \text{DH}^{1-i}(E, \mathcal{F}) \quad \text{for any } \mathcal{F} \in \text{Coh } Z, \\ \text{Ext}_\Lambda^i(\mathcal{F}, \omega_\Lambda) &\simeq \text{DH}^{1-i}(E, \mathcal{F}) \quad \text{for any } \mathcal{F} \in \text{Coh } \Lambda, \end{aligned}$$

where DV denotes the vector space dual to V .

Definition 2.4. We say that an ideal I of a ring R is *bi-principal* if $I = aR = Ra$ for a non-zero-divisor $a \in R$. A sheaf of ideals $\mathcal{I} \subset \tilde{A}$ is said to be *locally bi-principal* if every point $x \in X$ has a neighbourhood U such that the ideal $\Gamma(U, \mathcal{I})$ is bi-principal in $\Gamma(U, \tilde{A})$.

Lemma 2.5. *If the sheaf of ideals \mathcal{I} is locally bi-principal, then*

$$\omega_\Lambda \simeq \mathcal{H}om_{\tilde{A}}(\mathcal{I}, \omega_{\tilde{A}}) \otimes_{\tilde{A}} \Lambda.$$

Proof. Let $\mathcal{I}' = \mathcal{H}om_{\tilde{A}}(\mathcal{I}, \omega_{\tilde{A}})$. Consider the locally free resolution $0 \rightarrow \mathcal{I} \xrightarrow{\tau} \tilde{A} \rightarrow \Lambda \rightarrow 0$ of Λ . Since $\omega_{\tilde{A}}$ is locally free over \tilde{A} , it gives an exact sequence

$$0 \rightarrow \omega_{\tilde{A}} \xrightarrow{\tau^*} \mathcal{I}' \rightarrow \mathcal{E}xt_{\tilde{A}}^1(\Lambda, \omega_{\tilde{A}}) \rightarrow 0.$$

On the other hand, tensoring the same resolution with \mathcal{I}' gives an exact sequence

$$0 \rightarrow \mathcal{I}' \otimes_{\tilde{A}} \mathcal{I} \xrightarrow{1 \otimes \tau} \mathcal{I}' \rightarrow \mathcal{I}' \otimes_{\tilde{A}} \Lambda \rightarrow 0.$$

Since \mathcal{I} is locally bi-principal, the natural map $\mathcal{I}' \otimes_{\tilde{A}} \mathcal{I} \rightarrow \omega_{\tilde{A}}$ is an isomorphism, and, if we identify $\mathcal{I}' \otimes_{\tilde{A}} \mathcal{I}$ with $\omega_{\tilde{A}}$, $1 \otimes \tau$ identifies with τ^* . It implies the claim of the Lemma. \square

Definition 2.6. Let $\mathcal{I} \subset \tilde{A}$ be a bi-principal sheaf of ideals such that $\text{supp}(\tilde{A}/\mathcal{I}) = E$, $\Lambda = \tilde{A}/\mathcal{I}$ and $I = \mathcal{I}/\mathcal{I}^2$. (Note that $I \in \text{VB}(\Lambda)$.) \mathcal{I} is said to be a *weak reduction cycle* if

- (1) I is generically globally generated as a sheaf of Λ -modules.
- (2) $H^1(E, I) = 0$.

If, moreover,

- (3) $\omega_{\Lambda}^{\vee} = \mathcal{H}om_{\Lambda}(\omega_{\Lambda}, \Lambda)$ is generically globally generated over Λ ,

\mathcal{I} is called a *reduction cycle*.

For a weak reduction cycle \mathcal{I} we define the *Kahn's reduction functor* $R_{\mathcal{I}} : \text{CM}(A) \rightarrow \text{VB}(\Lambda)$ as

$$R_{\mathcal{I}}(M) = \Lambda \otimes_{\tilde{A}} \pi^{\dagger} M.$$

We fix a weak reduction cycle \mathcal{I} and keep the notation of the preceding Definition. We also set $\Lambda_n = \tilde{A}/\mathcal{I}^n$, $I_n = \mathcal{I}^n/\mathcal{I}^{n+1}$, $\mathcal{I}^{-n} = (\mathcal{I}^n)^{\vee}$ and $I_{-n} = \mathcal{I}^{-n}/\mathcal{I}^{1-n}$. In particular, $\Lambda_1 = \Lambda$ and $I_1 = I$. One easily sees that $I_n \simeq I \otimes_{\Lambda} I \otimes_{\Lambda} \dots \otimes_{\Lambda} I$ (n times) and $I_{-n} \simeq I_n^{\vee} = \mathcal{H}om_{\Lambda}(I_n, \Lambda)$.

Proposition 2.7. *If a coherent sheaf F of Λ -modules is generically globally generated, then $H^1(E, I \otimes_{\Lambda} F) = 0$. In particular, $H^1(E, I_n) = 0$*

Proof. Let $H = \Gamma(E, F)$. Consider the exact sequence

$$0 \rightarrow N \rightarrow H \otimes \Lambda \rightarrow F \rightarrow T \rightarrow 0,$$

where $N = \ker \text{ev}_F$ and $\text{supp } T$ is 0-dimensional. It gives the exact sequence

$$0 \rightarrow I \otimes_{\Lambda} N \rightarrow H \otimes I \rightarrow I \otimes_{\Lambda} F \rightarrow I \otimes_{\Lambda} T \rightarrow 0.$$

Since $H^1(E, H \otimes I) = H^1(E, I \otimes_{\Lambda} T) = 0$, we get that $H^1(E, I \otimes_{\Lambda} F) = 0$. \square

For any vector bundle \mathcal{F} over \tilde{A} set $F = \Lambda \otimes_{\tilde{A}} \mathcal{F}$ and $F_n = \Lambda_n \otimes_{\tilde{A}} \mathcal{F}$. There are exact sequences

$$(2.1) \quad \begin{aligned} 0 &\rightarrow I_n \rightarrow \Lambda_{n+1} \rightarrow \Lambda_n \rightarrow 0, \\ 0 &\rightarrow I_n \otimes_{\Lambda} F \rightarrow F_{n+1} \rightarrow F_n \rightarrow 0. \end{aligned}$$

For $n = 1$, tensoring the second one with $I^{\vee} = \mathcal{H}om_{\Lambda}(I, \Lambda)$, we get

$$0 \rightarrow F \rightarrow I^{\vee} \otimes_{\Lambda_2} F_2 \rightarrow I^{\vee} \otimes_{\Lambda} F \rightarrow 0.$$

Proposition 2.8. *Let \mathcal{I} be a weak reduction cycle and \mathcal{F} be a vector bundle over \tilde{A} such that F is generically globally generated over Λ . Then \mathcal{F} is also generically globally generated and $H^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$.*

Note that if \mathcal{F} is generically globally generated and $H^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$, then F is also generically globally generated, since the map $H^0(\tilde{X}, \mathcal{F}) \rightarrow H^0(\tilde{X}, F)$ is surjective.

Proof. We first prove the second claim. Recall that, by the Theorem on Formal Functions [7, Theorem III.11.1],

$$H^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) \simeq \varprojlim_n H^1(E, \mathcal{I}/\mathcal{I}^n \otimes_{\tilde{A}} \mathcal{F}).$$

(We need not use completion, since $H^1(\tilde{X}, \mathcal{M})$ is finite dimensional for every $\mathcal{M} \in \text{Coh } \tilde{X}$.) Since $\mathcal{I}/\mathcal{I}^n$ is filtered by I_m ($1 \leq m < n$), we have to show that $H^1(E, I_m \otimes_{\tilde{A}} \mathcal{F}) = H^1(E, I_m \otimes_{\Lambda} F) = 0$ for all m . It follows from Proposition 2.7.

Note that $\Gamma(\tilde{X}, \mathcal{F}) = \Gamma(X, \pi_* \mathcal{F})$ and $\pi_* \mathcal{F}$ is globally generated, since X is affine. Moreover, the sheaves \mathcal{F} and $\pi_* \mathcal{F}$ coincide on \tilde{X} . Hence $\Gamma(\tilde{X}, \mathcal{F})$ generate \mathcal{F}_p for all $p \in \tilde{X}$. Therefore, we only have to prove that they generate \mathcal{F}_p for almost all points $p \in E$. Since $\text{supp } \Lambda = E$, it is enough to show that the global sections of \mathcal{F} generate F_p for almost all $p \in E$. From the exact sequence $0 \rightarrow \mathcal{I} \otimes_{\tilde{A}} \mathcal{F} \rightarrow \mathcal{F} \rightarrow F \rightarrow 0$ and the equality $H^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$ we see that the restriction $\Gamma(\tilde{X}, \mathcal{F}) \rightarrow \Gamma(E, F)$ is surjective. Since F is generically globally generated, so is also \mathcal{F} . \square

Corollary 2.9. *A locally bi-principal sheaf of ideals $\mathcal{I} \subset \tilde{A}$ is a weak reduction cycle if and only if*

- (1) \mathcal{I} is generically globally generated.
- (2) $H^1(\tilde{X}, \mathcal{I}) = 0$.

It is a reduction cycle if and only if, moreover, $\omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I}$ is generically globally generated.

Proof. If \mathcal{I} is a weak reduction cycle, (1) and (2) follows from Proposition 2.8. Conversely, suppose that (1) and (2) hold. Since $H^2(\tilde{X}, -) = 0$, then $H^1(E, I) = 0$. Moreover, just as in Proposition 2.7, $H^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$ for any generically globally generated \mathcal{F} . In particular, $H^1(\tilde{X}, \mathcal{I}^2) = 0$. Hence the map $\Gamma(\tilde{X}, \mathcal{I}) \rightarrow \Gamma(E, I)$ is surjective, so I is generically globally generated.

Now let \mathcal{I} be a weak reduction cycle. Note that, by Lemma 2.5,

$$\begin{aligned} \omega_{\Lambda}^{\vee} &= \text{Hom}_{\Lambda}(\mathcal{I}^{\vee} \otimes_{\tilde{A}} \omega_{\tilde{A}} \otimes_{\tilde{A}} \Lambda, \Lambda) \simeq \\ &\simeq \text{Hom}_{\tilde{A}}(\mathcal{I}^{\vee} \otimes_{\tilde{A}} \omega_{\tilde{A}}, \Lambda) \simeq \text{Hom}_{\tilde{A}}(\omega_{\tilde{A}}, \mathcal{I} \otimes_{\tilde{A}} \Lambda) \simeq \\ &\simeq \omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I} \otimes_{\tilde{A}} \Lambda \simeq \omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I} / \omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I}^2. \end{aligned}$$

Hence, by Proposition 2.8, ω_{Λ}^{\vee} is generically globally generated if and only if so is $\omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I}$. \square

Proposition 2.10. *A reduction cycle always exists.*

Proof. Since the intersection form is negative definite on the group of divisors on \tilde{X} with support E [11], there is a divisor D with support E such that $\mathcal{O}_{\tilde{X}}(-D)$ is ample. Therefore, for some $n > 0$, $\mathcal{I} = \tilde{A}(-nD)$ as well as $\omega_{\tilde{A}}^{\vee}(-nD)$ are generically globally generated and, moreover, $H^1(\tilde{X}, \mathcal{I}) = 0$. Obviously, \mathcal{I} is bi-principal, so it is a reduction cycle. \square

Now we need the following modification of the Wahl's lemma [13, Lemma B.2].

Lemma 2.11. *If \mathcal{F} is a vector bundle over \tilde{A} , then*

$$H_E^1(\tilde{X}, \mathcal{F}) \simeq \varinjlim_n H^0(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n)$$

Moreover, the natural homomorphisms

$$H^0(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n) \rightarrow H^0(E, \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F_{n+1})$$

are injective.

Proof. Note that $H_E^1(\tilde{X}, \mathcal{F}) \simeq \varinjlim_n \text{Ext}_{\tilde{A}}^1(\Lambda_n, \mathcal{F})$. Consider the spectral sequence $H^p(\tilde{X}, \mathcal{E}xt_{\tilde{A}}^q(\Lambda_n, \mathcal{F})) \Rightarrow \text{Ext}_{\tilde{A}}^{p+q}(\Lambda_n, \mathcal{F})$. Since $\mathcal{H}om_{\tilde{A}}(\Lambda_n, \mathcal{F}) = 0$, the exact sequence of the lowest terms gives an isomorphism $\text{Ext}_{\tilde{A}}^1(\Lambda_n, \mathcal{F}) \simeq H^0(E, \mathcal{E}xt_{\tilde{A}}^1(\Lambda_n, \mathcal{F}))$. Applying $\mathcal{H}om_{\tilde{A}}(-, \mathcal{F})$ to the exact sequence $0 \rightarrow \mathcal{I}^n \rightarrow \tilde{A} \rightarrow \Lambda_n \rightarrow 0$, we get the exact sequence

$$0 \rightarrow \mathcal{F} = \tilde{A} \otimes_{\tilde{A}} \mathcal{F} \rightarrow \mathcal{H}om_{\tilde{A}}(\mathcal{I}^n, \mathcal{F}) \simeq \mathcal{I}^{-n} \otimes_{\tilde{A}} \mathcal{F} \rightarrow \mathcal{E}xt_{\tilde{A}}^1(\Lambda_n, \mathcal{F}) \rightarrow 0,$$

whence $\mathcal{E}xt_{\tilde{A}}^1(\Lambda_n, \mathcal{F}) \simeq (\mathcal{I}^{-n}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F}$. Moreover, since $\mathcal{I}^{-n}/\tilde{A} \subseteq \mathcal{I}^{-n-1}/\tilde{A}$ and \mathcal{F} is locally projective, we get an embedding $(\mathcal{I}^{-n}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F} \hookrightarrow (\mathcal{I}^{-n-1}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F}$, hence an embedding of cohomologies. It remains to note that

$$(\mathcal{I}^{-n}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F} \simeq (\mathcal{I}^{-n}/\tilde{A}) \otimes_{\Lambda_n} F_n \simeq \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n,$$

since \mathcal{I}^n annihilates $\mathcal{I}^{-n}/\tilde{A}$. \square

Since $I \otimes_{\Lambda} F \simeq \mathcal{I} \otimes_{\tilde{A}} F$, there is an exact sequence

$$0 \rightarrow \mathcal{I} \otimes_{\tilde{A}} F \rightarrow F_2 \rightarrow F \rightarrow 0.$$

Multiplying it with \mathcal{I}^{\vee} , we get an exact sequence

$$(2.2) \quad 0 \rightarrow F \rightarrow \mathcal{I}^{\vee} \otimes_{\tilde{A}} F_2 \rightarrow \mathcal{I}^{\vee} \otimes_{\tilde{A}} F \rightarrow 0,$$

which gives the coboundary map $\theta_F : H^0(E, \mathcal{I}^{\vee} \otimes_{\tilde{A}} F) \rightarrow H^1(E, F)$.

Proposition 2.12 (Cf. [10, Proposition 1.6]). *Let \mathcal{I} be a weak reduction cycle. A vector bundle $\mathcal{F} \in \text{VB}(\tilde{A})$ is full if and only if*

- (1) F is generically globally generated over Λ .
- (2) The coboundary map θ_F is injective.

Proof. Let \mathcal{F} be generically globally generated. Since $H^1(\tilde{X}, \mathcal{I}) = 0$, also $H^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$. Therefore, the map $\Gamma(\tilde{X}, \mathcal{F}) \rightarrow \Gamma(E, F)$ is surjective, so F is generically globally generated. Conversely, if F is generically globally generated, so is \mathcal{F} by Proposition 2.8. Hence, this condition (1) is equivalent to the condition (1) of Proposition 2.1. So now we suppose that both \mathcal{F} and F are generically globally generated.

Consider the commutative diagram

$$\begin{array}{ccc} H_E^1(\tilde{X}, \mathcal{F}) & \xrightarrow{\alpha_{\mathcal{F}}} & H^1(\tilde{X}, \mathcal{F}) \\ i \uparrow & & p \downarrow \\ H^0(E, \mathcal{I}^\vee \otimes_{\tilde{A}} F) & \xrightarrow{\theta_F} & H^1(E, F) \end{array}$$

Here i is an embedding from Lemma 2.11 and p is an isomorphism, since $H^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$. If \mathcal{F} is full, $\alpha_{\mathcal{F}}$ is injective, hence so is θ_F .

Conversely, suppose that θ_F is injective. We show that all embeddings

$$(2.3) \quad H^0(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n) \rightarrow H^0(E, \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F_{n+1})$$

from Lemma 2.11 are actually isomorphisms. It implies that $\alpha_{\mathcal{F}}$ is injective, so \mathcal{F} is full.

The map (2.3) comes from the exact sequence

$$(2.4) \quad 0 \rightarrow \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n \rightarrow \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F_{n+1} \rightarrow \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F \rightarrow 0$$

obtained from the exact sequence

$$0 \rightarrow \mathcal{I} \otimes_{\tilde{A}} F_n \rightarrow F_{n+1} \rightarrow F \rightarrow 0$$

by tensoring with \mathcal{I}^{-n-1} . So we have to show that the connecting homomorphism

$$\beta_n : H^0(E, \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F) \rightarrow H^1(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n)$$

is injective. We actually prove that even the map

$$\beta'_n : H^0(E, \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F) \rightarrow H^1(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F),$$

which is the composition of β_n with the natural map $H^1(E, \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F_n) \rightarrow H^1(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F)$ is injective.

Indeed, β_0 coincides with θ_F . Since all sheaves \mathcal{I}^n are generically globally generated, there is a homomorphism $m\tilde{A} \rightarrow \mathcal{I}^n$ whose cokernel has a finite support. Taking duals, we get an embedding $\mathcal{I}^{-n} \hookrightarrow m\tilde{A}$. Tensoring this embedding with the exact sequence (2.4) for $n = 0$ and taking cohomologies, we get a commutative diagram

$$\begin{array}{ccc} H^0(E, \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F) & \xrightarrow{\beta'_n} & H^1(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F) \\ \downarrow & & \downarrow \\ mH^0(E, \mathcal{I}^{-1} \otimes_{\tilde{A}} F) & \longrightarrow & mH^1(E, F) \end{array}$$

where the second horizontal and the first vertical maps are injective. Therefore, β'_n is injective too, which accomplishes the proof. \square

We call a vector bundle $F \in \text{VB}(\Lambda)$ *full* if $F \simeq \Lambda \otimes_{\tilde{A}} \mathcal{F}$, where \mathcal{F} is a full vector bundle over \tilde{A} .

Theorem 2.13 (Cf. [10, Theorem 1.4]). *Let \mathcal{I} be a weak reduction cycle. A vector bundle $F \in \text{VB}(\Lambda)$ is full if and only if*

- (1) *F is generically globally generated.*
- (2) *There is a vector bundle $F_2 \in \text{VB}(\Lambda_2)$ such that $\Lambda \otimes_{\tilde{A}} F_2 \simeq F$ and the connecting homomorphism $\theta_F : \text{H}^0(E, \mathcal{I}^\vee \otimes_{\tilde{A}} F) \rightarrow \text{H}^1(E, F)$ coming from the exact sequence (2.2) is injective.*

If, moreover, \mathcal{I} is a reduction cycle, the full vector bundle $\mathcal{F} \in \text{VB}(\tilde{A})$ such that $\Lambda \otimes_{\tilde{A}} \mathcal{F} \simeq F$ is unique up to isomorphism. Thus the reduction functor $R_{\mathcal{I}}$ induces a one-to-one correspondence between isomorphism classes of Cohen–Macaulay A -modules and isomorphism classes of full vector bundles over Λ .

Proof. Let \mathcal{I} be a weak reduction cycle, F satisfies (1) and (2). If $U \subset E$ is an affine open subset, there is an exact sequence

$$0 \rightarrow I_n(U) \rightarrow \Lambda_{n+1}(U) \rightarrow \Lambda_n(U) \rightarrow 0,$$

where the ideal $I_n(U)$ is nilpotent (actually, $I_n(U)^2 = 0$). Therefore, given a projective $\Lambda_n(U)$ -module P_n , there is a projective $\Lambda_{n+1}(U)$ -module P_{n+1} such that $\Lambda_n(U) \otimes_{\Lambda_{n+1}(U)} P_{n+1} \simeq P_n$. Moreover, if P'_n is another projective $\Lambda_n(U)$ -module, P'_{n+1} is a projective $\Lambda_{n+1}(U)$ -module such that $\Lambda_n(U) \otimes_{\Lambda_{n+1}(U)} P'_{n+1} \simeq P'_n$ and $\varphi_n : P_n \rightarrow P'_n$ is a homomorphism, it can be lifted to a homomorphism $\varphi_{n+1} : P_{n+1} \rightarrow P'_{n+1}$, and if φ_n is an isomorphism, so is φ_{n+1} too.

Consider an affine open cover $E = U_1 \cup U_2$. Let $P_{2,i} = F_2(U_i)$. Iterating the above procedure, we get projective $\Lambda_n(U_i)$ -modules $P_{n,i}$ such that

$$\Lambda_n(U_i) \otimes_{\Lambda_{n+1}(U_i)} P_{n+1,i} \simeq P_{n,i}$$

for all $n \geq 2$. If $U = U_1 \cap U_2$, there is an isomorphism $\varphi_2 : P_{2,1}(U) \xrightarrow{\sim} P_{2,2}(U)$. It can be lifted to $\varphi_n : P_{n,1}(U) \xrightarrow{\sim} P_{n,2}(U)$ so that the restriction of φ_{n+1} to $P_{n,1}$ coincides with φ_n . Hence there are vector bundles F_n over Λ_n such that $\Lambda_n \otimes_{\tilde{A}} F_{n+1} \simeq F_n$. Taking inverse image, we get a vector bundle $\hat{\mathcal{F}} = \varprojlim_n F_n$ over the formal non-commutative scheme (\hat{X}, \hat{A}) which is the completion of (\tilde{X}, \tilde{A}) along the subscheme E . As \tilde{X} is projective, hence proper over X , $\hat{\mathcal{F}}$ uniquely arises as the completion of a vector bundle \mathcal{F} over \tilde{A} such that $\Lambda_n \otimes_{\tilde{A}} \mathcal{F} \simeq F_n$ for all n (see [9, Theorem 5.1.4]). If we choose F_2 so that the condition (2) holds, \mathcal{F} is full by Proposition 2.12. Thus F is full as well.

Let now \mathcal{I} be a reduction cycle, F be a full vector bundle over Λ and F_n be vector bundles over Λ_n such that $\Lambda_n \otimes_{\tilde{A}} F_{n+1} \simeq F_n$ for all n and F_2 satisfies the condition (2). As we have already mentioned, all choices of F_n are locally isomorphic. Therefore, if we fix one of them, their isomorphism classes are in one-to-one correspondence with the cohomology set $\text{H}^1(E, \text{Aut } F_n)$ [8].

From the exact sequence (2.1) we obtain an exact sequence of sheaves of groups

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{A}ut F_{n+1} \xrightarrow{\rho} \mathcal{A}ut F_n \rightarrow 0,$$

where $\mathcal{H} = \ker \rho \simeq \mathcal{H}om_{\Lambda_n}(F_n, I_n \otimes_{\Lambda} F) \simeq \mathcal{H}om_{\Lambda}(F, I_n \otimes_{\Lambda} F)$. It gives an exact sequence of cohomologies

$$\begin{aligned} 0 \rightarrow \mathcal{H}om_{\Lambda}(F_n, I_n \otimes_{\Lambda} F) &\rightarrow \mathcal{A}ut F_{n+1} \rightarrow \mathcal{A}ut F_n \xrightarrow{\delta} \\ &\rightarrow \text{Ext}_{\Lambda}^1(F, I_n \otimes_{\Lambda} F) \rightarrow \mathbb{H}^1(E, \mathcal{A}ut F_{n+1}) \rightarrow \mathbb{H}^1(E, \mathcal{A}ut F_n). \end{aligned}$$

The isomorphism classes of liftings F_{n+1} of a given F_n are in one-to-one correspondence with the orbits of the group $\mathcal{A}ut F_n$ naturally acting on $\text{Ext}_{\Lambda}^1(F, I_n \otimes_{\Lambda} F)$. [8, Proposition 5.3.1].

We write automorphisms of F_n in the form $1 + \varphi$ for $\varphi \in \text{End } F_n$. Then $\delta(1 + \varphi) = \delta_n(\varphi)$, where $\delta_n : \mathcal{H}om_{\Lambda_n}(F_n, F_n) \rightarrow \text{Ext}_{\Lambda}^1(F, I_n \otimes_{\Lambda} F)$ is the connecting homomorphism coming from the exact sequence (2.1). We restrict δ_n to $\mathcal{H}om_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F)$ (see the same exact sequence, with n replaced by $n - 1$). The resulting homomorphism $\delta'_n : \mathcal{H}om_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F) \rightarrow \text{Ext}_{\Lambda}^1(F, I_n \otimes_{\Lambda} F)$ coincides with the connecting homomorphism coming from the exact sequence (2.2) tensored with \mathcal{I}^{n-1} .

Claim 1. δ'_n is surjective.

Indeed, since F , I_{n-1} and ω_{Λ}^{\vee} are generically globally generated, so is their tensor product. Hence, there is a homomorphism $m\Lambda \rightarrow \omega_{\Lambda}^{\vee} \otimes_{\Lambda} I_{n-1} \otimes_{\Lambda} F$, thus also $m\omega_{\Lambda} \rightarrow I_{n-1} \otimes_{\Lambda} F$ whose cokernel has discrete support. Applying $\mathcal{H}om_{\Lambda}(F, -)$, we get a commutative diagram

$$\begin{array}{ccc} m \mathcal{H}om_{\Lambda}(F, \omega_{\Lambda}) & \longrightarrow & m \text{Ext}_{\Lambda}^1(F, I \otimes_{\Lambda} \omega_{\Lambda}) \\ \downarrow & & \eta \downarrow \\ \mathcal{H}om_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F) & \xrightarrow{\delta'_n} & \text{Ext}_{\Lambda}^1(F, I_n \otimes_{\Lambda} F), \end{array}$$

where η is surjective. Note that the first horizontal map here is the m -fold Serre dual θ_F^* to the map

$$\theta_F : \mathcal{H}om_{\Lambda}(I, F) \simeq \mathbb{H}^0(E, I^{\vee} \otimes_{\Lambda} F) \rightarrow \text{Ext}_{\Lambda}^1(\Lambda, F) \simeq \mathbb{H}^1(E, F),$$

which is injective. Therefore, θ_F^* is surjective and so is also δ'_n .

If $n > 1$, every homomorphism $1 + \varphi$ with $\varphi \in \mathcal{H}om_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F)$ is invertible. Hence δ is surjective and F_{n+1} is unique up to isomorphism. If $n = 1$, the set $\{\varphi \in \mathcal{H}om_{\Lambda}(F, F) \mid 1 + \varphi \text{ is invertible}\}$ is open in $\mathcal{A}ut F$. Therefore, its image in $\text{Ext}_{\Lambda}^1(F, I \otimes_{\Lambda} F)$ is an open orbit of $\mathcal{A}ut F$. If we choose another lifting F'_2 of \tilde{F} so that the condition (2) holds, it also gives an open orbit. Since there can be at most one open orbit, they coincide, hence $F'_2 \simeq F_2$. Now, if \mathcal{F} and \mathcal{F}' are two full vector bundles over \tilde{A} such that $\Lambda \otimes_{\tilde{A}} \mathcal{F} \simeq \Lambda \otimes_{\tilde{A}} \mathcal{F}' \simeq F$, we can glue isomorphisms $\Lambda_n \otimes_{\tilde{A}} \mathcal{F} \xrightarrow{\sim} \Lambda_n \otimes_{\tilde{A}} \mathcal{F}'$ into an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}'$. \square

Claim 1 also implies the following result.

Corollary 2.14 (Cf. [10, Corollary 1.10]). *If $\mathcal{F} \in \text{VB}^f(\tilde{A})$, then $\text{Ext}_{\tilde{A}}^1(\mathcal{F}, \mathcal{F}) \simeq \text{Ext}_{\Lambda}^1(F, F)$.*

We omit the proof since it just copies that from [10].

3. GOOD ELLIPTIC CASE

There is one special case when the conditions of Theorem 2.13 can be made much simpler. It is analogous to the case of *minimally elliptic* surface singularities considered in [10, Section 2]. We are not aware of the full generality when it can be done, so we only confine ourselves to a rather restricted situation. Thus the following definition shall be considered as very preliminary. It will be used in the examples studied in the next section.

Definition 3.1. Let $\pi : (\tilde{X}, \tilde{A}) \rightarrow (X, A)$ be a resolution of a non-commutative surface singularity, \mathcal{I} be a weak reduction cycle and $\Lambda = \tilde{A}/\mathcal{I}$. We say that the weak reduction cycle \mathcal{I} is *good elliptic* if $\Lambda \simeq \mathcal{O}_Z$ where Z is a reduced curve of arithmetic genus 1 (hence $\omega_Z \simeq \mathcal{O}_Z$). Obviously, then \mathcal{I} is a reduction cycle. If a non-commutative surface singularity (X, A) has a resolution (\tilde{X}, \tilde{A}) such that there is a good elliptic reduction cycle $\mathcal{I} \subset \tilde{A}$, we say that (X, A) is a *good elliptic non-commutative surface singularity*.

Remark 3.2. One easily sees that being good elliptic is equivalent to fulfillments of the following conditions for some resolution:

- (1) $h^1(\tilde{X}, \tilde{A}) = 1$.
- (2) There is a weak reduction cycle \mathcal{I} such that Λ is commutative and reduced.

Then \mathcal{I} is also a reduction cycle.

For good elliptic non-commutative surface singularity we can state a complete analogue of [10, Theorem 2.1]. Moreover, the proof is just a copy of the Kahn's proof, so we omit it.

Theorem 3.3. *Suppose that \mathcal{I} is a good elliptic reduction cycle for a resolution (\tilde{X}, \tilde{A}) of a non-commutative surface singularity (X, A) , $\Lambda = \tilde{A}/\mathcal{I}$ and $I = \mathcal{I}/\mathcal{I}^2$. A vector bundle F over Λ is full if and only if $F \simeq G \oplus m\Lambda$, where the following conditions hold:*

- (1) G is generically globally generated.
- (2) $H^1(E, G) = 0$.
- (3) $m \geq h^0(E, I^\vee \otimes_\Lambda G)$.³

If these conditions hold and M is the Cohen–Macaulay A -module such that $F \simeq R_{\mathcal{I}}M$, then M is indecomposable if and only if either $m = h^0(E, I^\vee \otimes_\Lambda G)$ or $F = \Lambda$ (then $M = A$).

Now, just as in [5] (and with the same proof), we obtain the following result.

³If we identify Λ with \mathcal{O}_Z , then $I^\vee \otimes_\Lambda G$ is identified with $G(Z)$.

Corollary 3.4. *Suppose that \mathcal{I} is a good elliptic reduction cycle for a resolution (\tilde{X}, \tilde{A}) of a non-commutative surface singularity (X, A) and $\Lambda = \tilde{A}/\mathcal{I} \simeq \mathcal{O}_Z$. The non-commutative surface singularity (X, A) is Cohen–Macaulay tame if and only if Z is either a smooth elliptic curve or a Kodaira cycle (a cyclic configuration in the sense of [5]). Otherwise it is Cohen–Macaulay wild.*

For the definitions of Cohen–Macaulay tame and wild singularities see [5, Section 4]. Though in this paper only the commutative case is considered, the definitions are completely the same in the non-commutative one.

4. EXAMPLES

In what follows we consider non-commutative surface singularities (X, A) , where $X = \text{Spec } R$ and $R = \mathbf{k}[[u, v]]$. We define A by generators and relations. The ramification divisor $D = D(A)$ is then given by one relation $F = 0$ for some $F \in R$, so it is a *plane curve singularity*.

When blowing up the closed point o , we get the subset $\tilde{X} \subseteq \text{Proj } R[\alpha, \beta]$ given by the equation $u\beta = v\alpha$. We cover it by the affine charts $U_1 : \beta \neq 0$ and $U_2 : \alpha \neq 0$, so their coordinate rings are, respectively, $R_1 = R[\xi]/(u - \xi v)$ and $R_2 = R[\eta]/(v - \eta u)$, where $\xi = \alpha/\beta$ and $\eta = \beta/\alpha$.

Example 4.1.

$$A = R\langle x, y \mid x^2 = v, y^2 = u(u^2 + \lambda v^2), xy + yx = 2\varepsilon uv \rangle,$$

where $\lambda \notin \{0, 1\}$ and $\varepsilon^2 = 1 + \lambda$. Then $F = uv(u - v)(u - \lambda v)$, so D is of type T_{44} . We set $z = xy$, so $\{1, x, y, z\}$ is an R -basis of A and $z^2 = 2\varepsilon uvz - uv(u^2 + \lambda v^2)$. One can check that $\mathbf{k}_C(A)$ is a field, namely a quadratic extension of $\mathbf{k}(C)$, for every component of D . For instance, if this component is $u = v$, and g is its general point, then, modulo the ideal $(u - v)A_g$, $(z - \varepsilon uv)^2 = 0$, so $z - \varepsilon uv \in \text{rad } A_g$. Moreover,

$$\begin{aligned} (z - \varepsilon uv)^2 &= z^2 - 2\varepsilon uvz + (1 + \lambda)u^2v^2 = \\ &= -uv(u^2 + \lambda v^2) + (1 + \lambda)u^2v^2 = \\ &= uv(u - v)(\lambda v - u). \end{aligned}$$

Since $uv(\lambda v - u)$ is invertible in A_g , $u - v \in (z - \varepsilon uv)A_g$. One easily sees that $(z - \varepsilon uv)A_g$ is a two-sided ideal and $A_g/(z - \varepsilon uv)A_g \simeq \mathbf{k}[[u]][x]/(x^2 - u)$ is a field. (Note that in this factor $\varepsilon uvx = zx = vy$, so $y = \varepsilon ux$.) Therefore, (X, A) is normal and its ramification index equals 2 on every component of D .

After blowing up the closed point $o \in X$, we get

$$\pi^*A(U_1) \simeq R_1\langle x, y \mid x^2 = v, y^2 = \xi(\xi^2 + \lambda)v^3, xy + yx = 2\varepsilon\xi v^2 \rangle$$

and $z^2 = 2\varepsilon\xi v^2z - \xi v^4(\xi^2 + \lambda)$. So we can consider the R_1 -subalgebra $A_1 = \pi^*A(U_1)\langle z_1 \rangle$ of $\mathcal{K}(A)$, where $z_1 = v^{-2}z - \varepsilon\xi$. Note that $y_1 = v^{-1}y =$

$xz_1 + \varepsilon\xi x \in A_1$.

$$\pi^* A(U_2) \simeq R_2 \langle x, y \mid x^2 = \eta u, y^2 = u^3(1 + \lambda\eta^2), xy + yx = 2\varepsilon\eta u^2 \rangle$$

and $z^2 = 2\varepsilon\eta u^2 z - \eta u^4(1 + \eta^2\lambda)$. So we can consider the R_2 -subalgebra $A_2 = \pi^* A(U_2) \langle y_2, z_2 \rangle$ of $\mathcal{K}(A)$, where $y_2 = u^{-1}y$, $z_2 = u^{-2}z - \varepsilon\eta$.

Since $y_2 = \eta y_1$ and $z_2 = \eta^2 z_1$, $A_1(U_1 \cap U_2) = A_2(U_1 \cap U_2)$, so we can consider the non-commutative surface (\tilde{X}, \tilde{A}) , where $\tilde{A}(U_1) = A_1$, $\tilde{A}(U_2) = A_2$. One can check, just as above, that it is normal. Its ramification divisor \tilde{D} is given on U_1 by the equation $\xi v(\xi - 1)(\xi - \lambda) = 0$ and on U_2 by $u\eta(1 - \eta)(1 - \lambda\eta) = 0$, so its components are projective lines and have normal crossings. Moreover, $e_C(A) = 2$ for every component C of \tilde{D} , and if $x \in C$ is a node of \tilde{D} , then $e_{C,x}(A) = 2$. Hence (\tilde{X}, \tilde{A}) is a terminal resolution of (X, A) .

Consider the ideal $\mathcal{I} \subset \tilde{A}$ such that $\mathcal{I}(U_1) = (x)$ and $\mathcal{I}(U_2) = (x, y_2)$. Note that $\eta y_2 = xz_2 - \varepsilon\eta x$, $z_2 y_2 = (1 + \lambda\eta^2)x - \varepsilon\eta y_2$ and $y_2^2 = (1 + \lambda\eta^2)u$. Therefore, if $p \in U_2$ and $\eta(p) \neq 0$, then $y_2, u \in \tilde{A}_p x = x\tilde{A}_p$, while if $p \in U_2$ and $\eta(p) = 0$, then $x, u \in \tilde{A}_p y_2 = y_2 \tilde{A}_p$. Hence \mathcal{I} is bi-principal.

$$A_1/\mathcal{I}(U_1) \simeq \mathbf{k}[\xi, z_1]/(z_1^2 - \xi(\xi - 1)(\lambda - \xi)),$$

and

$$A_2/\mathcal{I}(U_2) \simeq \mathbf{k}[\eta, z_2]/(z_2^2 - \eta(1 - \eta)(\lambda\eta - 1)),$$

hence $\Lambda = \tilde{A}/\mathcal{I} \simeq \mathcal{O}_Z$, where Z is an elliptic curve. Moreover, x is a global section of \mathcal{I} , hence of $I = \mathcal{I}/\mathcal{I}^2$, and it generates I_p for every point $p \in Z$ except the point ∞ on the chart U_2 , where $\eta = 0$. So \mathcal{I} is a good elliptic reduction cycle and $I \simeq \mathcal{O}_Z(\infty)$.

Now, by Theorem 3.3, Cohen–Macaulay modules over A can be obtained as follows. We identify Z with $\text{Pic}^0(Z)$ taking ∞ as the zero point. Denote by $G(r, d; p)$ the indecomposable vector bundle over Z of rank r , degree d and the Chern class $p \in Z = \text{Pic}^0(Z)$ (see [2]). It is generically globally generated if and only if either $d > 0$ or $d = 0$, $r = 1$ and $p = \infty$. In the latter case $G(1, 0; \infty) \simeq \mathcal{O}_Z$. Then $I^\vee \otimes_\Lambda G(r, d; p) \simeq G(r, d - r; p)$. Moreover,

$$h^0(Z, G(r, d; p)) = \begin{cases} 0 & \text{if } 1 \leq d < 0 \text{ or } d = 0 \text{ and } p \neq \infty, \\ 1 & \text{if } d = 0 \text{ and } p = \infty, \\ d & \text{if } d > 0. \end{cases}$$

So if M is an indecomposable Cohen–Macaulay A -module and $M \not\simeq A$, then it is uniquely determined by its Kahn reduction $R_{\mathcal{I}}M$ which is one of the following vector bundles:

$$\begin{aligned} &G(r, d; p), \text{ where } d < r \text{ or } d = r, p \neq \infty; \text{ then } \text{rk } M = r. \\ &G(r, r; \infty) \oplus \mathcal{O}_Z, \text{ where } r > 1; \text{ then } \text{rk } M = r + 1. \\ &G(r, d; p) \oplus (d - r)\mathcal{O}_Z, \text{ where } d > r; \text{ then } \text{rk } M = d. \end{aligned}$$

In particular, A is *Cohen–Macaulay tame* in the sense of [5]. Namely, for a fixed rank r , Cohen–Macaulay A -modules of rank r , except one of them, form

$2(r-1)$ families parametrized by Z and one family parametrized by $Z \setminus \{\infty\}$, arising, respectively, from $G(d, r, p)$ ($1 \leq d < r$), $G(r', r, p)$ ($1 \leq r' < r$) and $G(r, r, p)$ ($p \neq \infty$).

Example 4.2.

$$A = R\langle x, y \mid x^3 = v, y^3 = u(u-v), xy = \zeta yx \rangle, \text{ where } \zeta^3 = 1, \zeta \neq 1.$$

Then $F = uv(u-v)$ (the singularity of type D_4). Just as above, one can check that A is normal and $e_c(A) = 3$ for every component C of D . After blowing up, on the chart U_1 we can consider the algebra $A_1 = \pi^*A(U_1)\langle w_1, z_1 \rangle$, where $w_1 = v^{-1}y^2$, $z_1 = v^{-1}xy$, and on the chart U_2 we can consider the algebra $A_2 = \pi^*A(U_2)\langle w_2, z_2 \rangle$, where $w_2 = u^{-1}y^2$, $z_2 = u^{-1}xy$. Again $A_1(U_1 \cap U_2) = A_2(U_1 \cap U_2)$, so we can glue them into a non-commutative surface (\tilde{X}, \tilde{A}) . One can verify that it is terminal. Let \mathcal{I} be the locally bi-principal ideal in \tilde{A} such that $\mathcal{I}(U_1) = (x)$ and $\mathcal{I}(U_2) = (x, w_2)$. Then $\tilde{A}/\mathcal{I} \simeq \mathcal{O}_Z$, where Z is the elliptic curve given by the equation $z_1^3 = \xi(\xi-1)$ on U_1 and by $z_2^3 = \eta(1-\eta)$ on U_2 . Again x defines a global section of \mathcal{I} , hence of I , and $I \simeq \mathcal{O}_Z(\infty)$, where ∞ is the point on U_2 with $\eta = 0$. Therefore, \mathcal{I} is a good elliptic reduction cycle and Cohen–Macaulay modules over A are described in the same way as in Example 4.1. In particular, A is also Cohen–Macaulay tame.

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