ON DERIVED CATEGORIES OF CERTAIN ASSOCIATIVE ALGEBRAS

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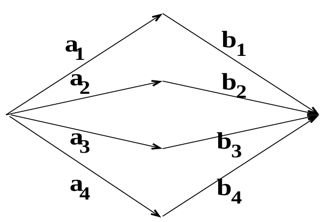
ABSTRACT. We describe indecomposable objects in derived categories of a wide class of associative algebras, including gentle, skew-gentle algebras and some degenerations of tubular algebras.

INTRODUCTION

The aim of this article is to apply the method for description of indecomposable objects in derived categories elaborated in [6] to several classes of associative algebras (both finite and infinite dimensional). We consider gentle and skew-gentle algebras, treated by other methods in [16, 18, 10, 4, 5]. An advantage of our approach is that it does not depend on finiteness of homological dimension and can be uniformly applied to new classes of algebras, for instance, to degenerations of tubular algebras.

It is well-known that canonical tubular algebras of type $(2, 2, 2, 2, \lambda)$



with relations

$$b_1a_1 + b_2a_2 + b_3a_3 = 0,$$

$$b_2a_2 + b_3a_3 + \lambda b_4a_4 = 0,$$

where $\lambda \neq 0, 1$, are derived tame of polynomial growth. They naturally arise in connection with weighted projective lines of tubular type [15]. A natural question is: what happens if a family of tubular algebras specializes to a forbidden value of parameter $\lambda = 0$? It turns out that the degenerated tubular algebra is still derived tame, but of exponential growth.

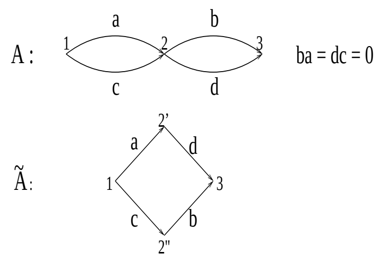
This work was supported by the CRDG Grant UM2-2094 and by the DFG Schwerpunkt "Globale Methoden in der komplexen Geometrie".

1. Category of triples

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, $\mathbf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ a right exact functor. It induces a functor between corresponding derived categories $\mathbf{D}^{-}\mathbf{F} : D^{-}(\mathcal{A}) \longrightarrow D^{-}(\mathcal{B})$. The description of the fibres of this functor is often equivalent to certain matrix problem.

In what follows we shall consider the following situation. Let A be a semi-perfect noetherian associative $\mathbf{k}$-algebra (not necessarily finite dimensional), $A \subset \tilde{A}$ be an embedding such that $rad(A) = rad(\tilde{A}) = r$. Let $I \subset A$ be a two-sided $\tilde{A}$-ideal containing r, thus A/I and $\tilde{A}/I$ are semi-simple algebras.

Example 1.1. Consider the following embedding of algebras:



Then the two-sided ideal I in A and $\tilde{A}$ generated by e_1 and e_3 satisfies the above properties.

Definition 1.2. In the above situation we define the category of triples of complexes TC_A as follows.

(1) Objects are triples $(\tilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$, where
$\tilde{\mathcal{P}}_\bullet \in D^{-}(\tilde{A}\text{-}\mathrm{mod})$,
$\mathcal{M}_\bullet \in D^{-}(A/I\text{-}\mathrm{mod})$,
$i : \mathcal{M}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \tilde{\mathcal{P}}_\bullet$ a morphism in $D^{-}(A/I\text{-}\mathrm{mod})$ such that the induced morphism $\tilde{i} : \tilde{A}/I \otimes_{A/I} \mathcal{M}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \tilde{\mathcal{P}}_\bullet$ is an isomorphism in $D^{-}(\tilde{A}/I\text{-}\mathrm{mod})$.

(2) Morphisms $(\tilde{\mathcal{P}}_{\bullet 1}, \mathcal{M}_{\bullet 1}, i_1) \longrightarrow (\tilde{\mathcal{P}}_{\bullet 2}, \mathcal{M}_{\bullet 2}, i_2)$ are pairs (Φ, φ),

$$\tilde{\mathcal{P}}_{\bullet 1} \xrightarrow{\Phi} \tilde{\mathcal{P}}_{\bullet 2}, \quad \mathcal{M}_{\bullet 1} \xrightarrow{\varphi} \mathcal{M}_{\bullet 2},$$

such that the diagram

$$\begin{tikzcd}
\tilde{A}/I \otimes_{\tilde{A}} \tilde{\mathcal{P}}_{\bullet 1} \arrow[d, "\Phi\otimes\mathrm{id}"] & \mathcal{M}_{\bullet 1} \arrow[l, "i_1"'] \arrow[d, "\varphi"] \\
\tilde{A}/I \otimes_{\tilde{A}} \tilde{\mathcal{P}}_{\bullet 2} & \mathcal{M}_{\bullet 2} \arrow[l, "i_2"]
\end{tikzcd}$$

is commutative.

Remark. If an algebra A has infinite homological dimension, we are forced to deal with the derived category of *right bounded* complexes (in order to define the left derived functor of the tensor product). In case A has finite homological dimension we can suppose that all complexes above are *bounded from both sides.*

Theorem 1.3 ([6]). *The functor* $\mathbf{F} : D^-(A\text{-}\mathrm{mod}) \longrightarrow \mathrm{TC}_A$,

$$\mathcal{P}_\bullet \mapsto (\tilde{A} \otimes_A \mathcal{P}_\bullet, A/I \otimes_A \mathcal{P}_\bullet, i : A/I \otimes_A \mathcal{P}_\bullet \longrightarrow \tilde{A}/I \otimes_A \mathcal{P}_\bullet)$$

has the following properties:

1. $\mathbf{F}$ *is* dense, *i.e. every triple* $(\tilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$ *is isomorphic to some* $\mathbf{F}(\mathcal{P}_\bullet)$.
2. $\mathbf{F}$ *is* conservative, *i.e.* $\mathbf{F}(\mathcal{P}_\bullet) \cong \mathbf{F}(\mathcal{Q}_\bullet) \Longleftrightarrow \mathcal{P}_\bullet \cong \mathcal{Q}_\bullet$.
3. $\mathbf{F}(\mathcal{P}_\bullet)$ *is indecomposable if and only if so is* $\mathcal{P}_\bullet$ *(note that this property is an easy formal consequence of the previous two).*
4. $\mathbf{F}$ *is full.*

The main point of the proof to be clarified is: given a triple $\mathcal{T} = (\tilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$, how can we reconstruct $\mathcal{P}_\bullet$? Here is the procedure. The exact sequence

$$0 \longrightarrow I\tilde{\mathcal{P}}_\bullet \longrightarrow \tilde{\mathcal{P}}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \tilde{\mathcal{P}}_\bullet \longrightarrow 0$$

of complexes in A-mod gives a distinguished triangle

$$I\tilde{\mathcal{P}}_\bullet \longrightarrow \tilde{\mathcal{P}}_\bullet \longrightarrow \tilde{A}/I \otimes_{\tilde{A}} \tilde{\mathcal{P}}_\bullet \longrightarrow I\tilde{\mathcal{P}}_\bullet[-1]$$

in $D^-(A\text{-}\mathrm{mod})$. The properties of triangulated categories imply that there is a morphism of triangles

$$\begin{tikzcd}
I\tilde{\mathcal{P}}_\bullet \arrow[r] & \tilde{\mathcal{P}}_\bullet \arrow[r] & \tilde{A}/I \otimes_{\tilde{A}} \tilde{\mathcal{P}}_\bullet \arrow[r] & I\tilde{\mathcal{P}}_\bullet[-1] \\
I\tilde{\mathcal{P}}_\bullet \arrow[r] \arrow[u, "id"] & \mathcal{P}_\bullet \arrow[r] \arrow[u, "\Phi"'] & \mathcal{M}_\bullet \arrow[r] \arrow[u, "i"'] & I\tilde{\mathcal{P}}_\bullet[-1], \arrow[u, "id"]
\end{tikzcd}$$

where $\mathcal{P}_\bullet = cone(\mathcal{M}_\bullet \longrightarrow I\tilde{\mathcal{P}}_\bullet[-1])[1]$. Set $\mathbf{G}(\mathcal{T}) = \mathcal{P}_\bullet$. The properties of triangulated categories immediately imply that the constructed map (not a functor!)

$$\mathbf{G} : Ob(\mathrm{TC}_A) \longrightarrow Ob\big(D^-(A\text{-}\mathrm{mod})\big)$$

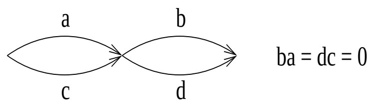
sends isomorphic objects into isomorphic ones and $\mathbf{GF}(\mathcal{P}_\bullet) \cong \mathcal{P}_\bullet$. For more details see [6].

Remark. Taking a cone is not a functorial operation. It gives an intuitive explanation why the functor $\mathbf{F}$ is not an equivalence.

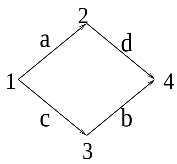
2. Derived categories of gentle algebras

It was observed that the representation theory of gentle (or, more general, string) algebras is closely related to representations of quivers of type $\tilde{A}_n$. We shall sketch an explicit calculation of indecomposable objects in derived categories of gentle algebras based on a new way of reduction to a matrix problem (see also [16, 4]). It implies, in particular, the known result that these algebras are derived tame.

Let A be the path algebra of the quiver



We can embed it into the path algebra $\tilde{A}$ of the quiver



In this case set $I = \langle e_1, e_4\rangle$. So $A/I = \mathbf{k}$ and $\tilde{A}/I = \mathbf{k}\times\mathbf{k}$, $A/I \longrightarrow \tilde{A}/I$ is a diagonal embedding.

As we have seen in the previous section, a complex $\mathcal{P}_\bullet$ of the derived category $D^-(A\text{-}\operatorname{mod})$ is defined by some triple $(\tilde{\mathcal{P}}_\bullet, \mathcal{M}_\bullet, i)$. Since $A/I\text{-}\operatorname{mod}$ can be identified with the category of $\mathbf{k}$-vector spaces, the map $i : \mathcal{M}_\bullet \longrightarrow \tilde{\mathcal{P}}_\bullet/I\tilde{\mathcal{P}}_\bullet$ is given by a collection of linear maps $H_k(i) : H_k(\mathcal{M}_\bullet) \longrightarrow H_k(\tilde{\mathcal{P}}_\bullet/I\tilde{\mathcal{P}}_\bullet)$. The map $H_k(i)$ is a $\mathbf{k}$-linear map of a $\mathbf{k}$-module into a $\mathbf{k}\times\mathbf{k}$-module. Hence it is given by two matrices $H_k(i)(1)$ and $H_k(i)(2)$. The "non-degeneration condition" imposed on $\tilde{i}$ in the definition of the category of triples implies that both of these matrices are square and non-degenerate.

The algebra $\tilde{A}$ has homological dimension 1. By a theorem of Dold (see [8]), any indecomposable complex from $D^-(\tilde{A}\text{-}\operatorname{mod})$ is isomorphic to a shift

$$\dots \longrightarrow 0 \longrightarrow \underbrace{M}_{i} \longrightarrow 0 \longrightarrow \dots,$$

where M is an indecomposable $\tilde{A}$-module, or, equivalently, to its projective resolution

$$\dots \longrightarrow 0 \longrightarrow \underbrace{P'}_{i+1} \xrightarrow{\phi} \underbrace{P}_{i} \longrightarrow 0 \longrightarrow \dots,$$

where $M \simeq \operatorname{Coker}\phi$.

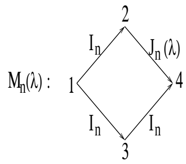
The next question is: which transformations can we perform with the matrices $H_k(i)(1)$ and $H_k(i)(2)$? First, change of bases in the spaces $H_k(\mathcal{M}_\bullet)$ induces simultaneous elementary transformations of columns of matrices $H_k(i)(1)$ and $H_k(i)(2)$. The row transformations are induced by morphisms in $D^-(\tilde{A}\text{-}\operatorname{mod})$.

If $\tilde{P}_i$ and $\mathbf{k}(i)$ denote respectively the indecomposable projective and the simple $\tilde{A}$-module corresponding to the vertex i, then

$$\tilde{A}/I \otimes_{\tilde{A}} \tilde{P}_1 = \tilde{A}/I \otimes_{\tilde{A}} \tilde{P}_4 = 0,$$

$$\tilde{A}/I \otimes_{\tilde{A}} \tilde{P}_2 = \mathbf{k}(2), \quad \tilde{A}/I \otimes_{\tilde{A}} \tilde{P}_3 = \mathbf{k}(3).$$

Consider the continuous series of representations of the quiver $\tilde{A} = \tilde{A}_4$:

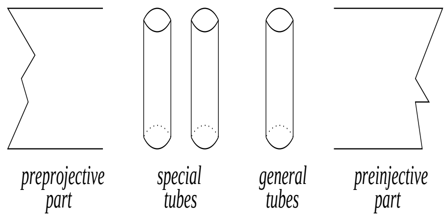


$M_n(\lambda)$ has a projective resolution:

$$0 \longrightarrow \tilde{P}_4^n \longrightarrow \tilde{P}_1^n \longrightarrow M_n(\lambda) \longrightarrow 0.$$

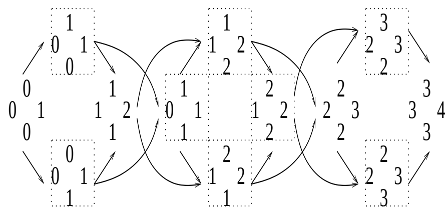
Hence $\tilde{A}/I \otimes_{\tilde{A}} M_n(\lambda) = 0$ in the derived category $D^-(\tilde{A}/I)$. So $\tilde{A}/I \otimes_{\tilde{A}}$ *kills* the continuous series of representations of $\tilde{A}$. We only have to consider the discrete series of representations.

Recall some basic facts about representations of tame hereditary algebras (see [17] for more details). The Auslander-Reiten quiver has the following structure:



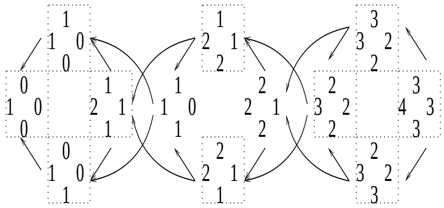
In this concrete case:

the preprojective series is

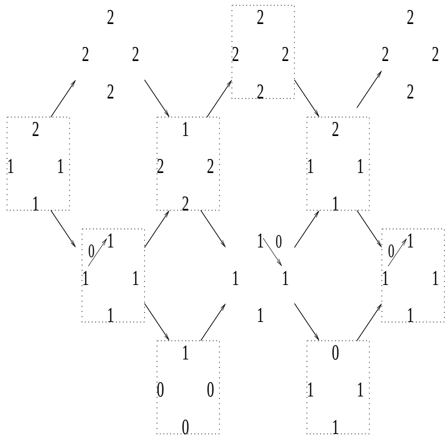


(We mark with dotted boxes the objects that remain nonzero after tensoring by $\tilde{A}/I$.)

the preinjective series is



two special tubes are



and the symmetric one.

We see that the preprojective series, as well as the preinjective one and two special tubes are 2-periodic. Let M be a preprojective module with the dimension vector (d_1, d_2, d_3, d_4). Then $\tau^{-1} \circ \tau^{-1}(M)$ has the dimension vector $(d_1+2, d_2+2, d_3+2, d_4+2)$. The same holds for a preinjective module N and $\tau \circ \tau(N)$, as well as it holds for special tubes if one goes two floors upstairs.

Consider the module M from the preprojective series

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 2 \arrow[ur] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd}$$

It has a projective resolution

$$0 \longrightarrow P_4 \longrightarrow P_1^2 \longrightarrow M \longrightarrow 0.$$

Therefore $\tilde{A}/I \otimes_{\tilde{A}} M = 0$ in $D^-(\tilde{A}/I)$. In other words, the module M gives no input to our matrix problem.

The list of modules, which are relevant for this problem is the following:

(1) Preprojective modules:

$$\begin{tikzcd} & \mathbf{k}^{n+1} \arrow[dr] & \\ \mathbf{k}^{n} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n+1} \\ & \mathbf{k}^{n} \arrow[ur] & \end{tikzcd}, \qquad \begin{tikzcd} & \mathbf{k}^{n} \arrow[dr] & \\ \mathbf{k}^{n} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n+1} \\ & \mathbf{k}^{n+1} \arrow[ur] & \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathbf{k}^{n+1} \arrow[dr] & \\ \mathbf{k}^{n} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n+1} \\ & \mathbf{k}^{n+1} \arrow[ur] & \end{tikzcd}$$

(2) Preinjective modules

$$\begin{tikzcd} & \mathbf{k}^{n+1} \arrow[dr] & \\ \mathbf{k}^{n+1} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n} \\ & \mathbf{k}^{n} \arrow[ur] & \end{tikzcd}, \qquad \begin{tikzcd} & \mathbf{k}^{n} \arrow[dr] & \\ \mathbf{k}^{n+1} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n} \\ & \mathbf{k}^{n+1} \arrow[ur] & \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathbf{k}^{n} \arrow[dr] & \\ \mathbf{k}^{n+1} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n} \\ & \mathbf{k}^{n} \arrow[ur] & \end{tikzcd}$$

(3) Modules from special tubes:

$$\begin{tikzcd}[column sep=small] & \mathbf{k}^{n+1} \arrow[dr] & \\ \mathbf{k}^n \arrow[ur] \arrow[dr] & & \mathbf{k}^n \\ & \mathbf{k}^n \arrow[ur] & \end{tikzcd}, \qquad \begin{tikzcd}[column sep=small] & \mathbf{k}^{n} \arrow[dr] & \\ \mathbf{k}^n \arrow[ur, "J_n(0)"] \arrow[dr] & & \mathbf{k}^n \\ & \mathbf{k}^n \arrow[ur] & \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small] & \mathbf{k}^{n} \arrow[dr] & \\ \mathbf{k}^{n+1} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n+1} \\ & \mathbf{k}^{n+1} \arrow[ur] & \end{tikzcd},$$

and symmetric ones.

We have to compute their images under the left derived functor $\tilde{A}/I \otimes_{\tilde{A}}$. In order to do it we have to consider the minimal projective resolutions of all these modules and then to apply the tensor product $\tilde{A}/I \otimes_{\tilde{A}}$.

(1) Preprojective series.
From the resolution

$$0 \longrightarrow P_4^n \longrightarrow P_1^n \oplus P_2 \longrightarrow 0$$

we get

$$0 \longrightarrow 0 \longrightarrow \mathbf{k}(2) \longrightarrow 0;$$

from the resolution

$$0 \longrightarrow P_4^n \longrightarrow P_1^n \oplus P_3 \longrightarrow 0$$

will be

$$0 \longrightarrow 0 \longrightarrow \mathbf{k}(3) \longrightarrow 0;$$

from the resolution

$$0 \longrightarrow P_4^{n+1} \longrightarrow P_1^n \oplus P_2 \oplus P_3 \longrightarrow 0$$

we get

$$0 \longrightarrow 0 \longrightarrow \mathbf{k}(2) \oplus \mathbf{k}(3) \longrightarrow 0.$$

(2) Preinjective series.
From the resolution

$$0 \longrightarrow P_2 \oplus P_3 \oplus P_4^n \longrightarrow P_1^{n+1} \longrightarrow 0$$

we get

$$0 \longrightarrow \mathbf{k}(2) \oplus \mathbf{k}(3) \longrightarrow 0 \longrightarrow 0;$$

from the resolution

$$0 \longrightarrow P_2 \oplus \ P_4^n \longrightarrow P_1^{n+1} \longrightarrow 0$$

will be

$$0 \longrightarrow \mathbf{k}(2) \longrightarrow 0 \longrightarrow 0;$$

from the resolution

$$0 \longrightarrow P_3 \oplus P_4^n \longrightarrow P_1^{n+1} \longrightarrow 0$$

will be

$$0 \longrightarrow \mathbf{k}(3) \longrightarrow 0 \longrightarrow 0;$$

(3) Special tubes.
From the resolution

$$0 \longrightarrow P_4^{n+1} \longrightarrow P_1^n \oplus P_2 \longrightarrow 0$$

we get

$$0 \longrightarrow 0 \longrightarrow \mathbf{k}(2) \longrightarrow 0;$$

from the resolution

$$0 \longrightarrow P_4^n \oplus P_2 \longrightarrow P_1^n \oplus P_2 \longrightarrow 0$$

will be

$$0 \longrightarrow \mathbf{k}(2) \xrightarrow{0} \mathbf{k}(2) \longrightarrow 0;$$

from the resolution

$$0 \longrightarrow P_4^n \oplus P_2 \longrightarrow P_1^{n+1} \longrightarrow 0$$

we get

$$0 \longrightarrow \mathbf{k}(2) \longrightarrow 0 \longrightarrow 0.$$

The case of the second special tube is completely symmetric.

What are induced morphisms between all these modules after applying the functor $\tilde{A}/I \otimes_{\tilde{A}}$? The image of the preprojective series is:

$$\begin{tikzcd}
\mathbf{k}(2) \arrow[dr] & & \mathbf{k}(3) \arrow[dr] & & \mathbf{k}(2) & \cdots \\
& \mathbf{k}(2) \oplus \mathbf{k}(3) \arrow[ur] \arrow[dr] & & \mathbf{k}(2) \oplus \mathbf{k}(3) \arrow[ur] \arrow[dr] & & \cdots \\
\mathbf{k}(3) \arrow[ur] & & \mathbf{k}(2) \arrow[ur] & & \mathbf{k}(3) & \cdots
\end{tikzcd}$$

We want to prove that all induced morphisms in this diagram are non-zero. It is well-known (see for instance [17]) that all morphisms between preprojective modules are determined by the Auslander-Reiten quiver. So every morphism is a linear combination of finite paths of irreducible morphisms. Consider the morphism

$$\begin{tikzcd}
 & \mathbf{k} \arrow[dr] & & & & \mathbf{k}^{n+1} \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & \mathbf{k} & \longrightarrow & \mathbf{k}^n \arrow[ur] \arrow[dr] & & \mathbf{k}^{n+1} \\
 & 0 \arrow[ur] & & & & \mathbf{k}^n \arrow[ur] &
\end{tikzcd}$$

It is clear that this map remains non-zero after applying $\tilde{A}/I \otimes_{\tilde{A}}$. It immediately implies that all morphisms we are interested in are non-zero after applying $\tilde{A}/I \otimes_{\tilde{A}}$. The same argument can be applied to preinjective modules.

Consider finally the case of special tubes. The module

$$\begin{tikzcd}
 & \mathbf{k}^n \arrow[dr] & \\
\mathbf{k}^n \arrow[ur, "J_n(0)"] \arrow[dr] & & \mathbf{k}^n \\
 & \mathbf{k}^n \arrow[ur] &
\end{tikzcd}$$

has a resolution

$$0 \longrightarrow P_4^n \oplus P_2 \longrightarrow P_1^n \oplus P_2 \longrightarrow 0.$$

It is indecomposable and its endomorphism algebra is local. Any endomorphism of this module is therefore either invertible or nilpotent. An isomorphism induces a map of the form:

$$\begin{tikzcd}
\mathbf{k}(2) \arrow[r, "0"] \arrow[d, "\lambda"] & \mathbf{k}(2) \arrow[d, "\lambda"] \\
\mathbf{k}(2) \arrow[r, "0"] & \mathbf{k}(2)
\end{tikzcd}$$

Let f be a nilpotent endomorphism. Consider an induced map of its projective resolution $f_\bullet$:

$$\begin{tikzcd}
0 \arrow[r] & P_4^n \oplus P_2 \arrow[r] \arrow[d, "f_1"] & P_1^n \oplus P_2 \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & P_4^n \oplus P_2 \arrow[r] & P_1^n \oplus P_2 \arrow[r] & 0.
\end{tikzcd}$$

The map $f_\bullet^n$ is homotopic to the zero map. Consider the component $f_1^n|_{P_2} : P_2 \longrightarrow P_2$ of the map f_1^n. It is zero modulo the radical and hence is equal to zero itself. But then $f_1|_{P_2} : P_2 \longrightarrow P_2$ is also zero. The same holds of course for $f_0|_{P_2} : P_2 \longrightarrow P_2$. We have shown that nilpotent morphisms induce the zero map modulo $\tilde{A}/I \otimes_{\tilde{A}}$. Finally observe that the chain of morphisms

$$\begin{tikzcd}
 & \mathbf{k}^{n+1} \arrow[dr] & \\
\mathbf{k}^n \arrow[ur] \arrow[dr] & & \mathbf{k}^n \\
 & \mathbf{k}^n \arrow[ur] \arrow[d] & \\
 & \mathbf{k}^n \arrow[dr] & \\
\mathbf{k}^n \arrow[ur, "J_n(0)"] \arrow[dr] & & \mathbf{k}^n \\
 & \mathbf{k}^n \arrow[ur] \arrow[d] & \\
 & \mathbf{k}^n \arrow[dr] & \\
\mathbf{k}^{n-1} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n-1} \\
 & \mathbf{k}^{n-1} \arrow[ur] &
\end{tikzcd}$$

induces modulo $\tilde{A}/I \otimes_{\tilde{A}}$ the following maps:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbf{k}(2) \arrow[d, "1"] \\
\mathbf{k}(2) \arrow[r, "0"] \arrow[d] & \mathbf{k}(2) \arrow[d, "1"] \\
0 \arrow[r] & \mathbf{k}(2).
\end{tikzcd}$$

In the same way the chain of morphisms

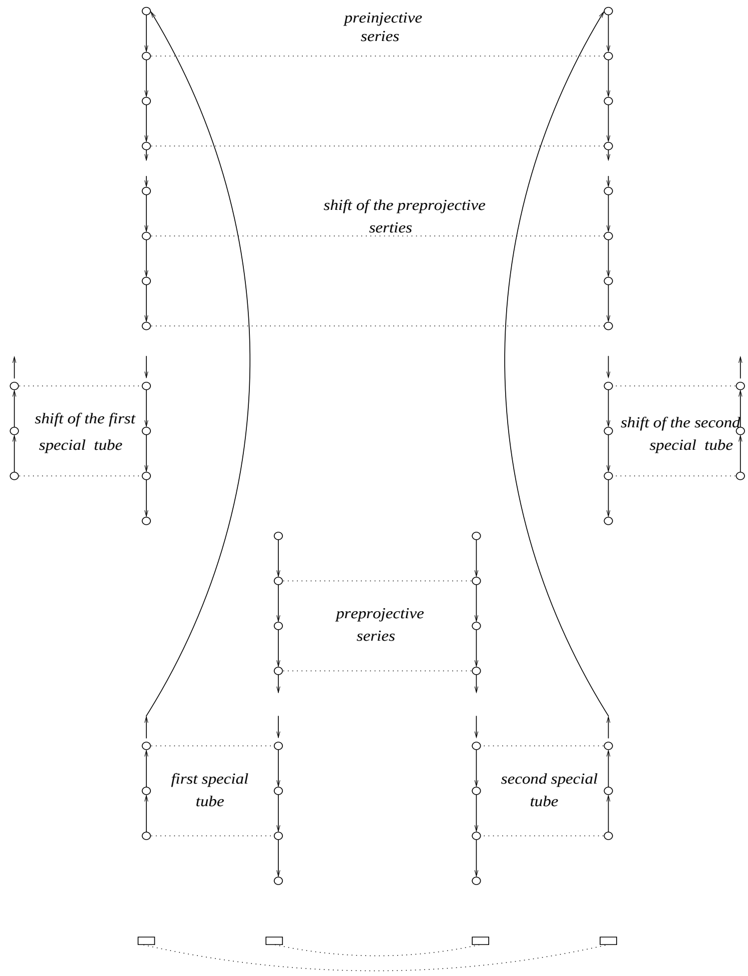
$$
\begin{tikzcd}
 & \mathbf{k}^{n-1} \arrow[dr] & \\
\mathbf{k}^{n} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n} \\
 & \mathbf{k}^{n} \arrow[ur] \arrow[d] & \\
 & \mathbf{k}^{n} \arrow[dr] & \\
\mathbf{k}^{n} \arrow[ur, "J_n(0)"] \arrow[dr] & & \mathbf{k}^{n} \\
 & \mathbf{k}^{n} \arrow[ur] \arrow[d] & \\
 & \mathbf{k}^{n} \arrow[dr] & \\
\mathbf{k}^{n+1} \arrow[ur] \arrow[dr] & & \mathbf{k}^{n+1} \\
 & \mathbf{k}^{n+1} \arrow[ur] &
\end{tikzcd}
$$

induces the maps

$$
\begin{tikzcd}
\mathbf{k}(2) \arrow[r] \arrow[d, "1"] & 0 \arrow[d] \\
\mathbf{k}(2) \arrow[r, "0"] \arrow[d, "1"] & \mathbf{k}(2) \arrow[d] \\
\mathbf{k}(2) \arrow[r] & 0
\end{tikzcd}
$$

We have the same picture for the second symmetric tube.

From these pictures we get that the matrix problem describing the derived category $D^{-}(A-\mathrm{mod})$ is given by the following bunch of chains (see [1], [7]), where small circles correspond to horizontal stripes, small rectangles correspond to vertical stripes, dotted lines show the related stripes and arrows describe the ordering in the chains or, the same, possible transformations between different horizontal stripes:



It means the following.

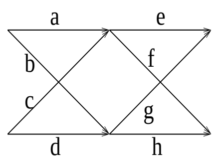
(1) We can do any simultaneous elementary transformations of columns of the matrices $H_k(i)(0)$ and $H_k(i)(\infty)$.
(2) We can do any simultaneous transformations of rows inside related blocks.
(3) We can add a scalar multiple of any row from a block with lower weight to any row of a block of a higher weight (inside the big matrix, of course). These transformations can be done independently inside $H_k(i)(0)$ and $H_k(i)(\infty)$.

This type of matrix problems is well known in the representation theory. First they appeared in the work of Nazarova–Roiter ([19]) about the classification of $\mathbf{k}[[x, y]]/(xy)$-modules. They are called, sometimes, Gelfand problems in honour of I. M. Gelfand, since they originated in a problem that first appeared in Gelfand's investigation of Harish-Chandra modules over $SL_2(\mathbb{R})$ [12] (see also [1], [7] and [6]). The explicit description of indecomposable objects given in [7, 1, 2] implies an explicit description of indecomposable objects in $D^-(A\text{-}\mathrm{mod})$, just as in [6]. The same considerations can be applied to any gentle algebra.

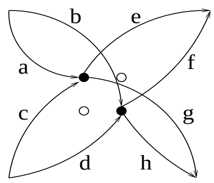
3. Derived categories of skew-gentle algebras

In the same way as the representation theory of gentle algebras is based on the representation theory of hereditary algebras $\tilde{A}_n$, that of skew-gentle algebras rests on representations of $\tilde{D}_n$.

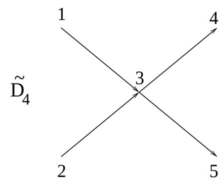
Consider the following example:



where all squares are commutative. We can embed this algebra into



where $M_2(\mathbf{k})$ stands in the middle. The last algebra is Morita-equivalent to the quiver algebra



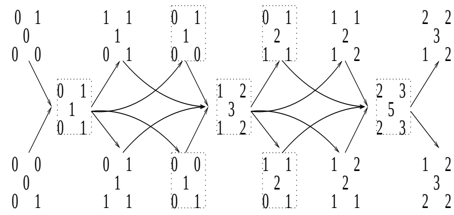
In this case the embedding $A/I \longrightarrow \tilde{A}/I$ is $\mathbf{k}\times\mathbf{k} \longrightarrow M_2(\mathbf{k})$. It implies that this time we obtain a matrix problem, which is called "representations of a bunch of semi-chains" (see [1], [7] and [6] for more details).

We have $\tilde{A} \otimes_{\tilde{A}/I} P_i = 0$, $i = 1, 2, 4, 5$. The continuous series of representations of $\tilde{D}_4$ has a projective resolution

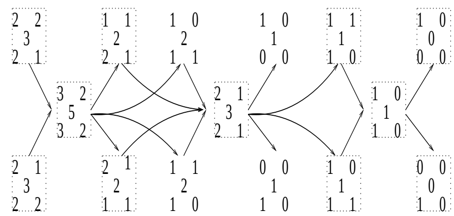
$$0 \longrightarrow P_4^n \oplus P_5^n \longrightarrow P_1^n \oplus P_2^n \longrightarrow M_n(\lambda) \longrightarrow 0,$$

hence $\tilde{A}/I \otimes_{\tilde{A}} M_n(\lambda) = 0$. We again have to take care only of discrete series of $\tilde{D}_4$. It consists of preprojective series, preinjective series, and three special tubes.

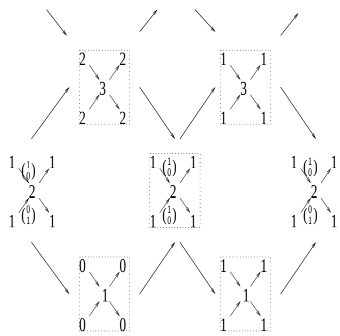
(1) The preprojective series is:



(2) The preinjective series is:

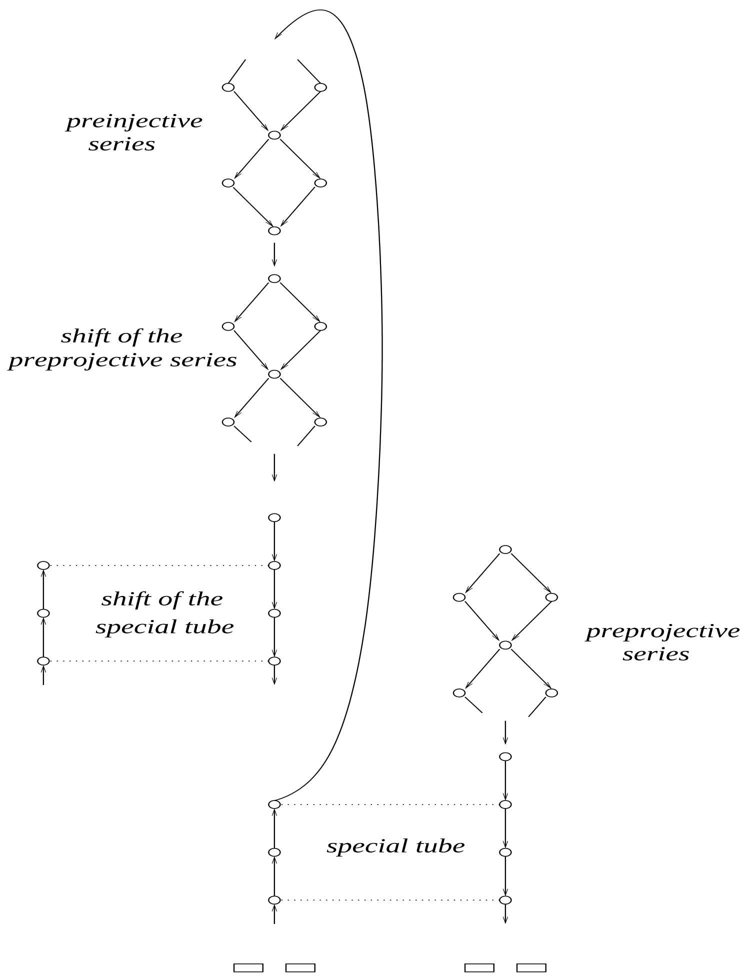


(3) The only special tube that gives an input into the matrix problem is:



The representations from two other special tubes vanish under tensoring by $\tilde{A}/I$.

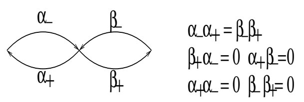
In the same way as for gentle algebras, one can see that our matrix problem is given by the following partially ordered set:



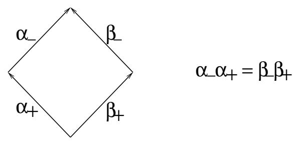
The generalization of this approach to other skew-gentle algebras of finite homological dimension gives an explicit description of indecomposables, which implies in particular a new proof of the result of Ch. Geiss and J. A. de la Pena [10] that these algebras are derived tame.

4. Skew-gentle algebras of infinite homological dimension

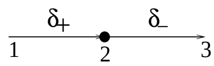
Consider now our next example:



This algebra is skew-gentle of infinite homological dimension. We can embed it into



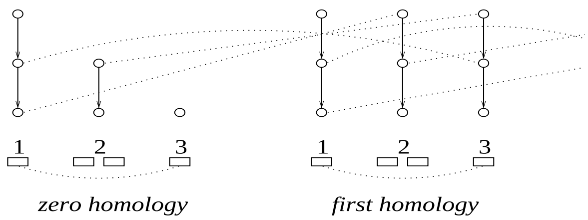
It is well known (see, for example, [14]) that this algebra is derived equivalent to the algebra $\tilde{D}_4$. In such a way we can obtain our matrix problem. But we can also embed it into



where the fat point in the middle means $M_2(\mathbf{k})$. The last algebra is Morita-equivalent to the algebra A_2. Note that there are only the following indecomposable complexes in $D^-(A_2\text{-}\operatorname{mod})$ (up to shifts):

$$\begin{gathered}
\ldots 0 \longrightarrow P_1 \longrightarrow 0 \longrightarrow \ldots, \\
\ldots 0 \longrightarrow P_2 \longrightarrow 0 \longrightarrow \ldots, \\
\ldots 0 \longrightarrow P_3 \longrightarrow 0 \longrightarrow \ldots; \\
\ldots 0 \longrightarrow P_3 \longrightarrow P_1 \longrightarrow 0 \longrightarrow \ldots, \\
\ldots 0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow 0 \longrightarrow \ldots, \\
\ldots 0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0 \longrightarrow \ldots.
\end{gathered}$$

It is easy to obtain a matrix problem now. It is again a bunch of semichains:



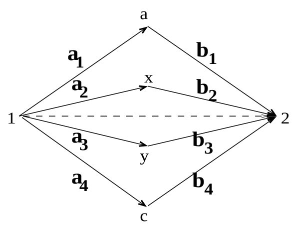
which gives an explicit description of indecomposables and, as a corollary, derived tameness.

Derived categories of skew-gentle algebras of infinite homological dimension were independently considered in [5].

5. Derived category of degenerated tubular algebra $(2,2,2,2;0)$

In all previous examples we embedded our path algebra A into a hereditary algebra $\tilde{A}$. As we shall see in the following example, it is also possible to consider embeddings into tame concealed algebras.

If we consider a canonical algebra of tubular type $(2,2,2,2;\lambda)$ and set the forbidden value of parameter $\lambda = 0$, then we get the following quiver



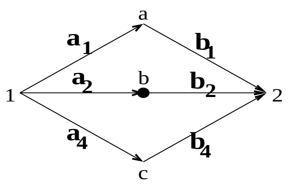
with relations

$$b_2 a_2 = b_3 a_3$$

and

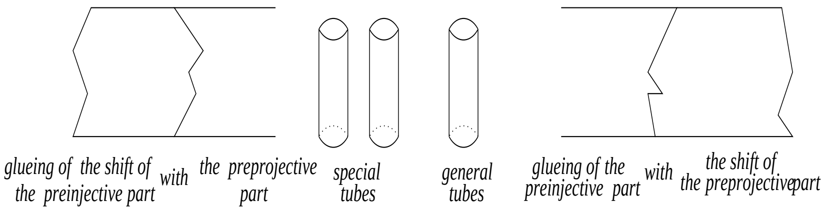
$$b_1 a_1 + b_2 a_2 + b_4 a_4 = 0.$$

We can do our trick with gluing idempotents x and y and embed this algebra into



where the fat point as usually means $M_2(\mathbf{k})$. The corresponding basic algebra is a tame-concealed algebra of type $(2,2,2)$. It is well known [17] that it is derived-equivalent to the algebra $\tilde{D}_4$. Take for I the ideal generated by all idempotents e_1, e_2, e_a, e_c. Then we have: $A/I = \mathbf{k} \times \mathbf{k}$, $\tilde{A}/I = M_2(\mathbf{k})$ and the map $A/I \longrightarrow \tilde{A}/I$ is the diagonal embedding.

It is no longer true that any complex of $D^b(\tilde{A}\text{-}\mathrm{mod})$ is isomorphic to its homology. Nevertheless, the structure of the Auslander-Reiten quiver is the same as for derived categories of tame hereditary algebras.

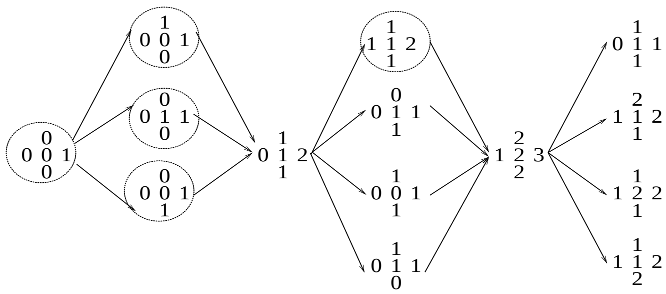


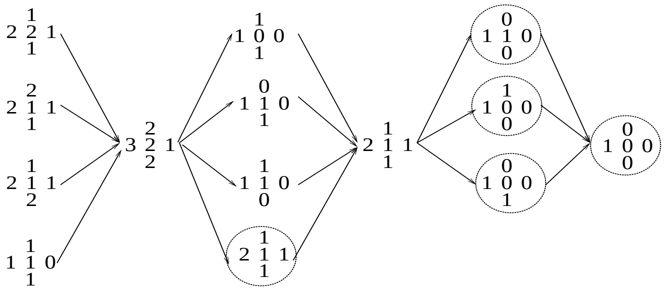
In particular, continuous series of complexes are just shifts of modules of tubular type. But they have the following form:

$$\begin{tikzcd}
 & \mathbf{k}^n \arrow[dr, "I_n"] & \\
\mathbf{k}^n \arrow[ur, "I_n"] \arrow[r, "I_n"] \arrow[dr, "I_n"'] & \mathbf{k}^n \arrow[r, "-I_n-J_n(\lambda)"] & \mathbf{k}^n \\
 & \mathbf{k}^n \arrow[ur, "J_n(\lambda)"'] &
\end{tikzcd}$$

It is easy to see that their minimal projective resolutions have the form $P_2^n \longrightarrow P_2^n$, and since $\tilde{A}/I \otimes_{\tilde{A}} (P_2^n \longrightarrow P_2^n) = 0$, they do not affect the resulting matrix problem.

Let us first consider the structure of preprojective and preinjective components of the Auslander-Reiten quiver of $\tilde{A}$- mod.





We can use the following lemma (see [14]).

Lemma 5.1. *Let A be an associative* $\mathbf{k}$*-algebra,* $0 \to M \xrightarrow{u} N \xrightarrow{v} K \to 0$ *be an Auslander-Reiten sequence in* $A\text{-}\mathrm{mod}$*,* $w \in \mathrm{Ext}^1(K,M) = \mathrm{Hom}_{D^b(A\text{-}\mathrm{mod})}(K, T(M))$ *be the corresponding element. The following conditions are equivalent:*

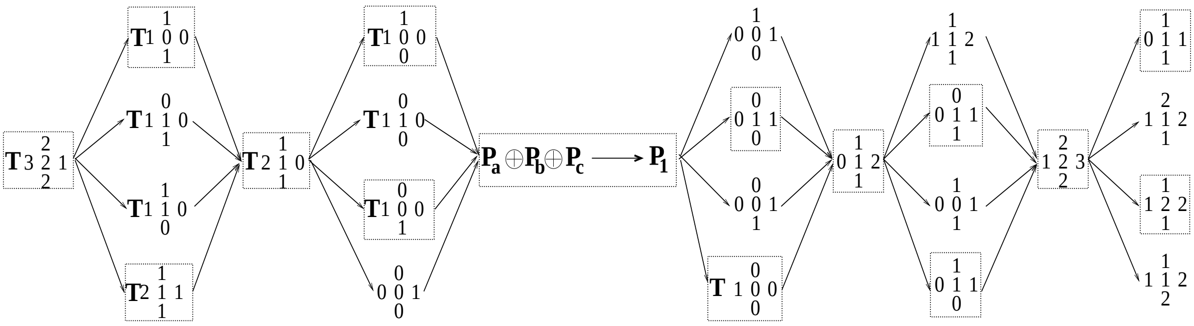
(1) $M \xrightarrow{u} N \xrightarrow{v} K \xrightarrow{w} T(M)$ *is an Auslander-Reiten triangle in* $D^b(A\text{-}\mathrm{mod})$.
(2) $\mathrm{inj.dim}(M) \le 1$ *and* $\mathrm{proj.dim}(K) \le 1$.
(3) $\mathrm{Hom}_A(I,M) = 0$ *for any injective A-module I and* $\mathrm{Hom}_A(K,P) = 0$ *for any projective A-module P.*

This lemma means that the structure of the Auslander-Reiten quiver of the category $D^b(A\text{-}\mathrm{mod})$ is basically the same as for $A\text{-}\mathrm{mod}$. For instance, all morphisms from tubes are still almost split in the derived category.

There is exactly one indecomposable complex in $D^b(A\text{-}\mathrm{mod})$, which is not isomorphic to a shift of some module: it is

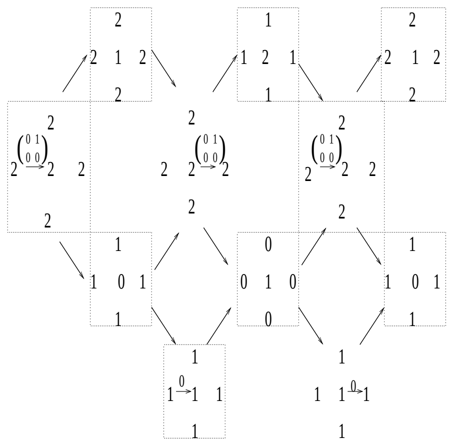
$$P_a \oplus P_b \oplus P_c \longrightarrow P_1.$$[1]

It is easy to see that this complex has two non-trivial homologies and is indecomposable. Now from the lemma above and the fact that there is only one non-trivial indecomposable complex we can deduce the exact form of the "gluing" of the preprojective component with the shift of the preinjective one in the Auslander-Reiten quiver:



Finally there are 3 special tubes of length 2. They are completely symmetric with respect to permutation of vertices a, b and c and only one of them is relevant for the matrix problem:

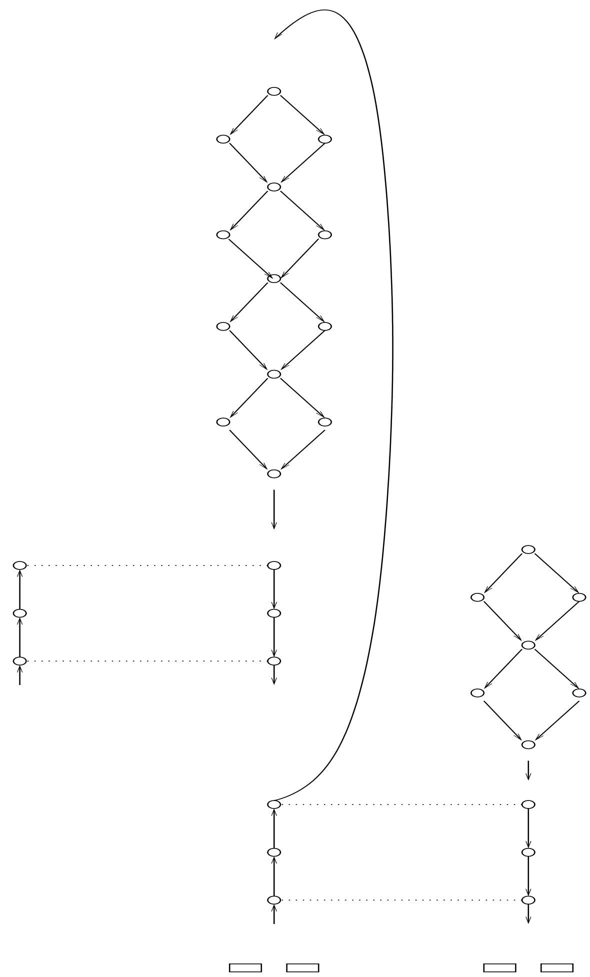
[1]The first author is grateful to C.-M.Ringel for explaining him this fact.



The resulting matrix problem is given by the partially ordered set designed at the next page (which is very similar to that for a skew-gentle algebra considered above)

Therefore the degenerated tubular algebra $(2, 2, 2, 2; 0)$ is derived-tame of exponential growth.

Is seems to be very plausible that this algebra is closely related to "weighted projective lines with a singularity of virtual genus one" and the map $A \longrightarrow \tilde{A}$ plays the role of non-commutative normalization (and note that $\tilde{A}$ correspond to a weighted projective line of virtual genus zero). We are planning to come back to this problem in the future.



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