REPRESENTATIONS OF BISECTED POSETS AND REFLECTION FUNCTORS

YURI A. DROZD

ABSTRACT. We construct a complete set of reflection functors for the representations of posets and prove that they really have the usual properties. In particular, when the poset is of finite representation type, all of its indecomposable representations can be obtained from some "trivial" ones via reflections. To define such reflection functors, a wider class of matrix problem is introduced, called "representations of bisected posets".

Representations of partially ordered sets ("posets") introduced by Nazarova and Roiter [8] were, maybe, the first example of matrix problems, whose theory was circumstantially elaborated. Later on they played an eminent role in the whole theory of matrix problems as well as of representations of algebras. So it is not surprising that it was also the first case, when one tried to introduce something like Bernstein–Gelfand–Ponomarev's reflection functors. Namely, one could not define all of them though in [3] the author managed to introduce some substitutes for one of them and the product of all others. That was enough to consider the "Coxeter transformation" and to obtain almost all necessary results for the posets of finite representation type. Nevertheless, some unsatisfaction remained, as one would like to get all reflections, not only their composition. Some hint was contained in Ovsienko's paper [10], where all reflections were introduced for another class of matrix problems. The thing was that, though starting with usual representations of algebras, Ovsienko had to extend the considered class of problems in such a way that the sought construction became possible. The resulting class was far from being too evident as it contained both algebras and "boxes".

The aim of this paper is to present an analogous construction for the representations of posets. To do it, we also need to widen the frames. The resulting class of matrix problems, which we call "representations of bisected posets" ("bisposets"), is rather unusual as it

This research was partially supported by the Foundation for Fundamental Research of Ukraine Grant 1.4/76 and INTAS Grant 93-0183.

contains non-free (even non-semi-free) boxes. As a corollary, they cannot be considered as "bimodule matrix problems" as well. Remark that the boxes arising in [10] could also be non-free, but this "non-freeness" was more or less imaginary. Namely, no generator ("arrow") of those boxes could be involved both into the relations and into the differential of the box, i.e. "additional transformations" of representations. On the contrary, in our case the interaction between relations and differential becomes nearly indispensable. Nevertheless, we are able to establish all "usual" properties for the new reflection functors as well as to exhibit their applicability in the simplest case of posets of finite representation type.

Remind that another approach to the reflection functors (for quivers) led to the tilting theory. One may hope that a more detailed study of the new reflection functors would lead to some extended tilting theory applicable both to algebras and to boxes and showing their interrelations at least in the "simply connected case" (whatever the last term could mean).

Of course, we use freely the language and technique of boxes (=bocses=BOCS's) referring to [5] or [12] as recommended sources. For the readers of the English translation of [3] we should also remark that its translator called indecomposable representations "irreducible". Fortunately, irreducible representations themselves do not occur there, so no misunderstanding is possible.

1. Bisposets and their representations

Throughout this paper we fix some field $\mathbf{k}$ and all algebras, categories etc., which we are considering, are algebras, categories etc. over this field.

A *bisected poset* (or *bisposet*) is, by definition, a poset $(\mathbf{S},<)$ together with a bisection: $\mathbf{S}=\mathbf{S}^-\sqcup\mathbf{S}^+$, such that $j\in\mathbf{S}^+$ and $i<j$ implies that $i\in\mathbf{S}^+$ too. Choose a new symbol 0 not lying in $\mathbf{S}$ and put $\hat{\mathbf{S}}=\mathbf{S}\sqcup\{\,0\,\}$. For each bisposet $\mathbf{S}$ define the corresponding box $\mathcal{A}(\mathbf{S})=(\mathsf{A},\mathsf{V})$ in the following way:

1. $\mathsf{Ob}\,\mathsf{A}=\hat{\mathbf{S}}$.
2. The morphisms of the category A are generated by the set $\{\,a_i\mid i\in\mathbf{S}\,\}$, where $a_i:i\to 0$ if $i\in\mathbf{S}^-$ and $a_i:0\to i$ if $i\in\mathbf{S}^+$. These generators are subject to the relations:

$$a_ia_j=0 \text{ if } j\in\mathbf{S}^-,\ i\in\mathbf{S}^+ \text{ and } j<i\,.$$

3. The box $\mathcal{A}$ is *normal* in the sense of [5] or [12] and its kernel $\overline{\mathsf{V}}$ is freely generated by the set
$$\{\, \varphi_{ij} \mid i < j \text{ and either both } i,j \in \mathbf{S}^- \text{ or both } i,j \in \mathbf{S}^+ \,\} ,$$
where $\varphi_{ij} \in \overline{\mathsf{V}}(j,i)$.
4. The differential ∂ of the box $\mathcal{A}$ is given by the formulae:
$$\begin{aligned} \partial a_i &= \sum_{j<i} a_j \varphi_{ji} \ \text{ if } i \in \mathbf{S}^- ; \\ \partial a_i &= \sum_{j>i} \varphi_{ij} a_j \ \text{ if } i \in \mathbf{S}^+ ; \\ \partial \varphi_{ij} &= \pm \sum_{j<k<i} \varphi_{ik}\varphi_{kj} \ \text{ if } i,j \in \mathbf{S}^\pm . \end{aligned}$$

The choice of signs in the last formula guarantees the condition $\partial^2 = 0$. Remark that this box is not free though has a free kernel.

Following the usual definition, we can consider the representations of the box $\mathcal{A}(\mathbf{S})$ in some category S, which we call S-*representations of the bisposet* $\mathbf{S}$. Namely, a S-representation is a set $M = \{\, M_0, M_i, f_i \mid i \in \mathbf{S} \,\}$, where M_0 and M_i are objects of $\mathsf{add}\,\mathsf{S}$, while f_i are morphisms: $f_i : M_i \to M_0$ if $i \in \mathbf{S}^-$ and $f_i : M_0 \to M_i$ if $i \in \mathbf{S}^+$, such that $f_i f_j = 0$ if $j \in \mathbf{S}^-, i \in \mathbf{S}^+$ and $j < i$. If $N = \{\, N_0, N_i, g_i \,\}$ is another S-representation, then a morphism $\Phi : M \to N$ is a set of morphisms from S:
$$\left\{\, \Phi_i \mid i \in \hat{\mathbf{S}} \,\right\} \bigcup \{\, \Phi_{ij} \mid i < j \ \text{ and either both } i,j \in \mathbf{S}^- \text{ or both } i,j \in \mathbf{S}^+ \,\} ,$$
where $\Phi_i : M_i \to N_i$ and $\Phi_{ij} : M_j \to N_i$, such that:
$$\begin{aligned} \Phi_0 f_i &= g_i \Phi_i - \sum_{j<i} g_j \Phi_{ji} \ \text{ for each } i \in \mathbf{S}^- ; \\ g_i \Phi_0 &= \Phi_i f_i + \sum_{j>i} \Phi_{ij} f_j \ \text{ for each } i \in \mathbf{S}^+ . \end{aligned}$$

The set of all such morphisms will be denoted by $\hom_{\mathbf{S}}(M,N)$. The category of representations of a bisposet $\mathbf{S}$ in a category S will be denoted by $\mathsf{rep}\,(\mathbf{S},\mathsf{S})$. If $\mathsf{S} = \mathbf{k}$, we often omit its name and write $\mathsf{rep}\,(\mathbf{S})$. In this case all M_i are finite dimensional vector spaces over $\mathbf{k}$, while f_i are linear mappings. Remark that if $\mathbf{S}^+ = \emptyset$ (hence $\mathbf{S}^- = \mathbf{S}$), we practically return to the original definition of Nazarova–Roiter [8] for the representations of the poset $\mathbf{S}$ over the field $\mathbf{k}$.

We always suppose that the set $\mathbf{S}$ is finite. Define then the *group of dimension vectors* $\mathrm{D} = \mathrm{D}(\mathbf{S})$ for the representations of a bisposet $\mathbf{S}$ to be $\mathbb{Z}^{n+1}$, where $n = |\mathbf{S}|$, with coordinates $(x_i \mid i \in \hat{\mathbf{S}})$. Denote by

D^+ its sub-semigroup consisting of all vectors with non-negative coordinates. If S is a ring such that every (finitely generated) projective S-module is free of unique rank (e.g. S is a field), we can identify the objects of $\mathsf{add}\,\mathsf{S}$ with free S-modules and define the *dimension vector of a representation* $M \in \mathsf{rep}\,(\mathbf{S},\mathsf{S})$ to be the vector

$$\mathbf{dim}\,M = (\operatorname{rank} M_i \mid i \in \hat{\mathbf{S}}) \in D^+ .$$

Moreover, if $\mathbf{x} \in D^+$ is any vector, we can consider the set $\mathsf{rep}_\mathbf{x}(\mathbf{S},\mathsf{S})$ of all representations of $\mathbf{S}$ over S of the dimension vector $\mathbf{x}$. To do this, we have only to fix some "standard representatives" for free S-modules of each rank r (say, S^r). In particular, if $\mathsf{S} = \mathbf{k}$, we can consider $\mathsf{rep}_\mathbf{x}(\mathbf{S})$ as an affine algebraic variety over $\mathbf{k}$. Moreover, the isomorphism classes of the representations from $\mathsf{rep}_\mathbf{x}(\mathbf{S})$ are just the orbits of certain algebraic group $\mathsf{G}_\mathbf{x}(\mathbf{S})$ acting on this variety. Namely, the group $\mathsf{G}_\mathbf{x}(\mathbf{S})$ consists of all sets of linear mappings $\Phi = \{\,\Phi_i, \Phi_{ij}\,\}$ as described above (taking into account that $M_i = N_i = \mathbf{k}^{x_i}$), such that all Φ_i are invertible. The composition of two such sets Φ and Ψ is given by the formulae:

$$\begin{aligned} (\Phi\Psi)_i &= \Phi_i\Psi_i \text{ for all } i \in \hat{\mathbf{S}}\,; \\ (\Phi\Psi)_{ij} &= \Phi_{ij}\Psi_j + \Phi_i\Psi_{ij} \pm \sum_{j<k<i} \Phi_{ik}\Psi_{kj} \text{ if } i,j \in \mathbf{S}^\pm . \end{aligned}$$

(cf. [5] or [12]). If $\mathbf{S}$ is fixed, we often omit it and write $\mathsf{rep}_\mathbf{x}$ and $\mathsf{G}_\mathbf{x}$.

As usually, we can assign to the bisposet $\mathbf{S}$ an integral quadratic form $Q = Q_\mathbf{S} : D(\mathbf{S}) \to \mathbb{Z}$, namely, for each $\mathbf{x} \in D$,

$$Q_\mathbf{S}(\mathbf{x}) = x_0^2 + \sum_{j \le i} x_i x_j - \sum_{i \in \mathbf{S}} x_0 x_i .$$

(of course, writing $j \le i$ we assume that $i,j \in \mathbf{S}$). Remark that both the group D and the form Q depend only on the poset $\mathbf{S}$ itself and not on its bisection and coincide with those defined in [3]. This form has an evident geometric meaning. For two elements $j \le i$ from $\mathbf{S}$ write $j \le^- i$ if $j \in \mathbf{S}^-$, $i \in \mathbf{S}^+$ and $j \le^+ i$ otherwise. Then the "minus-form"

$$Q^-(\mathbf{x}) = \sum_{i \in \mathbf{S}} x_0 x_i - \sum_{j \le^- i} x_i x_j$$

gives a lower bound for the dimensions of the irreducible components of the variety $\mathsf{rep}_\mathbf{x}$, while the "plus-form"

$$Q^+(\mathbf{x}) = x_0^2 + \sum_{j \le^+ i} x_i x_j$$

gives the dimension of the group $\mathsf{G}_{\mathbf{x}}$, hence the upper bound for the dimensions of orbits. Of course, $\mathrm{Q} = \mathrm{Q}^{+} - \mathrm{Q}^{-}$, so simple geometric observations (cf. e.g. [4]) lead to the following fact.

Proposition 1.1. *If a constructive subset $Z \subseteq \mathsf{rep}_{\mathbf{x}}$ meat each isomorphism class from some component of $\mathsf{rep}_{\mathbf{x}}$, then $\dim Z > -\mathrm{Q}(\mathbf{x})$. In particular, if $\mathbf{S}$ is of finite representation type, then the form $\mathrm{Q}_{\mathbf{S}}$ is weakly positive, and if $\mathbf{S}$ is tame, then $\mathrm{Q}_{\mathbf{S}}$ is weakly non-negative.*

Remind that Q is said to be *weakly positive* (*weakly non-negative*) if $\mathrm{Q}(\mathbf{x}) > 0$ (resp. $\mathrm{Q}(\mathbf{x}) \geq 0$) for each non-zero vector $\mathbf{x} \in \mathrm{D}^{+}$. We shall mark these properties by writing $\mathrm{Q} \rhd 0$ (resp. $\mathrm{Q} \unrhd 0$).

If $\mathbf{X}$ is any set of representations from $\mathsf{rep}\,(\mathbf{S})$, denote by $[\mathbf{X}]$ the ideal of the category $\mathsf{rep}\,(\mathbf{S})$ consisting of all morphisms, which factorize through the direct sums of representations from $\mathbf{X}$. Consider the *trivial* (or *simple*) representations T^{i} for each point $i \in \hat{\mathbf{S}}$. Namely, $T^{i}_{i} = \mathbf{k}$, while $T^{i}_{j} = 0$ for all $j \neq 0$. Put $\mathbf{I}_{i} = [T^{i}]$ and $\mathsf{rep}^{(i)}(\mathbf{S}) = \mathsf{rep}\,(\mathbf{S})/bI_{s}$. Of course, the isomorphism classes of objects from $\mathsf{rep}^{(i)}(\mathbf{S})$ are in one-to-one correspondence with those from $\mathsf{rep}\,(\mathbf{S})$ with no direct summand isomorphic to T^{i}. We need the following obvious fact.

Lemma 1.2. *A representation M of a bisposet $\mathbf{S}$ has no direct summand isomorphic to T^{i} if and only if the following conditions hold:*

1. *If $i \in \mathbf{S}^{-}$, then $f_{i}^{-1}(\sum_{j<i} \mathrm{Im}\, f_{j}) = 0$.*
2. *If $i \in \mathbf{S}^{+}$, then $f_{i}(\bigcap_{j>i} \ker f_{j}) = M_{i}$.*
3. *If $i = 0$, then $\bigcap_{i \in \mathbf{S}^{+}} \ker f_{i} \subseteq \sum_{j \in \mathbf{S}^{-}} \mathrm{Im}\, f_{i}$.*

(As usually, we suppose that the sum of an empty family of subspaces is 0, while its intersection equals the whole space.)

Proof. Let $i \in \mathbf{S}^{-}$ (the other cases can be treated in a just analogous way). Suppose that $f_{i}^{-1}(\sum_{j<i} \mathrm{Im}\, f_{j}) = 0$, Φ is a morphism $T^{i} \to M$ and u a non-zero element of T^{i}_{i}. Then, by definition, $f_{i}\Phi_{i}(u) = \sum_{j<i} f_{j}\Phi_{ji}(u)$, whence $\Phi_{i}(u) = 0$. But then also $\Psi_{i}\Phi_{i} = 0$ for each $\Psi : M \to T^{i}$, thus T^{i} cannot be a direct summand of M.

Suppose now that $v \in M_{i}$ is a non-zero element such that $f_{i}(v) = \sum_{j<i} f_{j}(v_{j})$ for some $v_{j} \in M_{j}$. Include v in some basis of M_{i}. Then we can find some linear mappings $\Phi_{ij} : M_{i} \to M_{j}$ $(j < i)$ such that $\Phi_{ij}(v) = v_{j}$ and $\Phi_{ij}(v') = 0$ for all basic vectors $v' \neq v$. Now define a representation $\overline{M} = (\overline{M}_0, \overline{M}_j, \overline{f}_j)$ as follows:

$$\overline{M}_0 = M_0 \text{ and } \overline{M}_j = M_j \text{ for all } j;$$
$$\overline{f}_j = f_j \text{ for all } j \neq i;$$
$$\overline{f}_i(v) = 0 \text{ and } \overline{f}_i(v') = f_i(v') \text{ for all basic vectors } v' \neq v.$$

Put also $\Phi_j = 1$ for all objects $j \neq i$ and $\Phi_{jk} = 0$ for $k \neq i$. Then one can easily check that Φ is an isomorphism of the representations M and $\overline{M}$. But the latter obviously contains a direct summand $T \simeq T^i$, where $T_i = \mathbf{k}v$ and $T_j = 0$ for $j \neq i$. □

These conditions obviously imply the following useful result.

Corollary 1.3. *For each dimension vector* $\mathbf{x} \in \mathrm{D}^+(\mathbf{S})$ *the subset* $\mathsf{rep}_{\mathbf{x}}^{(i)} \subseteq \mathsf{rep}_{\mathbf{x}}$ *consisting of all representations, which have no direct summand isomorphic to* T^i *is open.*

2. Reflection Functors

If $\mathbf{S}$ is a bisposet and $\mathcal{A} = (\mathsf{A}, \mathsf{V})$ the corresponding box, we define two subsets in $\hat{\mathbf{S}}$ called the sets of *sources* and *sinks*. Namely, if $\mathbf{S}^- \neq \emptyset$, the set $\sigma(\mathbf{S})$ of sources consists of all points maximal in $\mathbf{S}^-$, while if $\mathbf{S}^- = \emptyset$, we put $\sigma(\mathbf{S}) = \{0\}$. Dually, if $\mathbf{S}^+ \neq \emptyset$, the set $\tau(\mathbf{S})$ of sinks consists of all elements minimal in $\mathbf{S}^+$, while if $\mathbf{S}^+ = \emptyset$, we put $\tau(\mathbf{S}) = \{0\}$. Note that an object is a source if there are no arrows (both solid and dotted) in the box $\mathcal{A}$ leading from any another object to that one and it is a sink if there are no arrows leading from it to any other object. If s is a source, we define a new bisposet $\mathbf{S}_s$ in the following way. Its underlying poset is the same as for $\mathbf{S}$ but the bisection changes. Namely, if s is a maximal element from $\mathbf{S}^-$, then $\mathbf{S}_s^- = \mathbf{S}^- \setminus \{s\}$ and $\mathbf{S}_s^+ = \mathbf{S}^+ \cup \{s\}$, while if $s = 0$ (hence $\mathbf{S}^- = \emptyset$), then $\mathbf{S}_s^- = \mathbf{S}$ and $\mathbf{S}_s^+ = \emptyset$. Dually, if s is a sink, we also define a new bisposet $\mathbf{S}_s$ with the same underlying poset and a new bisection. Namely, if s is a minimal element of $\mathbf{S}^+$, put $\mathbf{S}_s^+ = \mathbf{S}^+ \setminus \{s\}$ and $\mathbf{S}_s^- = \mathbf{S}^- \cup \{s\}$, while if $s = 0$, put $\mathbf{S}_s^+ = \mathbf{S}$ and $\mathbf{S}_s^- = \emptyset$. The box corresponding to $\mathbf{S}_s$ is also denoted by $\mathcal{A}_s$. Obviously, if s was a source, it becomes a sink in $\mathbf{S}_s$ (and vice versa) and $\mathbf{S}_{ss} = \mathbf{S}$. Moreover, the points of $\mathbf{S}^-$ can be placed into such sequence $s_1, s_2, \dots, s_m$, where $m = \#(\mathbf{S}^-)$, that any s_{k+1} is a source in $\mathbf{S}_{s_1 s_2 \dots s_k}$ (in particular, s_1 is a source in $\mathbf{S}$) and $\mathbf{S}^-_{s_1 s_2 \dots s_m} = \emptyset$. The same claim is also true for $\mathbf{S}^+$ and sinks. Therefore, if $\mathbf{S}$ and $\mathbf{S}'$ are two bisposets with the same underlying posets, we can place the elements of $\hat{\mathbf{S}}$ into such sequence $s_1, s_2, \dots, s_{n+1}$ that each s_{k+1} is a source in $\mathbf{S}_{s_1 s_2 \dots s_k}$ and $\mathbf{S}' = \mathbf{S}_{s_1 s_2 \dots s_{n+1}}$ (the same for sinks).

For $i \in \mathbf{S}$ put $i^{\pm} = \{ j \in \mathbf{S}^{\pm} \mid j \text{ is comparable with } i \}$. Put also $0^{\pm} = \mathbf{S}^{\pm}$ and introduce the following notations for a vector $\mathbf{x} \in \mathrm{D} = \mathrm{D}_{\mathbf{S}}$ and for a representation $M \in \mathsf{rep}\,(\mathbf{S})$:

Notations 2.1.

- $x_i^{\pm} = \sum_{j \in i^{\pm}} x_j$;
- $x_i' = x_0 - x_i^+ - x_i^-$ *if* $i \in \mathbf{S}$ *and* $x_0' = x_0^+ + x_0^- - x_0$;
- $M_i^{\pm} = \bigoplus_{j \in i^{\pm}} M_j$;
- f_i^+ *is the mapping* $M_0 \to M_i^+$ *with the components* f_j *and* f_i^- *is the mapping* $M_i^- \to M_0$ *with the components* f_j *for suitable indices* j .

We are now going to define the *reflection functors*

$$F_s : \mathsf{rep}^{(s)}(\mathbf{S}) \to \mathsf{rep}^{(s)}(\mathbf{S}_s)\,,$$

where s is a source or a sink. Let first s be a maximal point of $\mathbf{S}^-$. Then $\operatorname{Im} f_i^- \subseteq \ker f_i^+$ for each representation M of $\mathbf{S}$. Hence we are able to consider the factor-space $M_s' = \ker f_s^+ / \operatorname{Im} f_s^-$. Choose some retraction $\eta_M : M_0 \to \ker f_s^+$ and denote by π_M the natural mapping $\ker f_s^+ \to M_s'$. Now define a representation M' of the bisposet $\mathbf{S}_s$ by the rules:

$$M_i' = M_i \text{ and } f_i' = f_i \text{ for all } i \neq s\,;$$
$$M_s' \text{ is as defined above and } f_s' = \pi_M \eta_M\,.$$

Suppose that N is another representation of $\mathbf{S}$, η_N a retraction of N_0 onto $\ker g_s^+$ and N' the corresponding representation of $\mathbf{S}_s$. Remark that f_s^+ induces an injection $(1-\eta_M)(M_0) \to M_s^+$. Therefore there exists a linear mapping $\xi : M_s^+ \to M_0$ such that $\xi f_s^+ = 1 - \eta_M$. Choose such ξ and define for each morphism $\Phi : M \to N$ a morphism $\Phi' = \Phi_\xi' : M' \to N'$ by the rules:

$$\Phi_0' = \Phi_0 \text{ and } \Phi_i' = \Phi_i \text{ for all } i \neq s\,;$$
$$\Phi_s'(v + \operatorname{Im} f_s^-) = \Phi_0(v) + \operatorname{Im} g_s^-, \text{ where } v \in \ker f_s^+\,;$$
$$\Phi_{ij}' = \Phi_{ij} \text{ if } j \neq s\,;$$
$$\Phi_{is}' = g_s' \Phi_0 \xi_i \text{ for all } i > s\,,$$

where $\xi_i : M_i \to M_0$ the i-th component of ξ. Remark that the relations for the components of Φ imply that Φ_0 maps $\ker f_s^+$ to $\ker g_s^+$ and $\operatorname{Im} f_s^-$ to $\operatorname{Im} g_s^-$. One can easily check that Φ_ξ' is indeed a morphism from M' to N'.

If we choose in the last construction another mapping ξ', then the only non-zero components of the difference $\Delta = \Phi_\xi' - \Phi_{\xi'}'$ can be Δ_{is}

for $i > s$. But such a morphism evidently factorizes through mT^s, where $m = \dim N'_s$. Namely, $\Delta = \Delta^1\Delta^0$, where $\Delta^0 : M' \to mT^i$ has all zero components except of $\Delta^0_{is} = \Delta_{is}$ $(i > s)$, while $\Delta^1 : mT^s \to N'$ has all zero components except of $\Delta^1_s = \Delta_s$. All relations, which we have to check in order to prove that Δ^0 and Δ^1 are really morphisms, are trivial, except the only one at the point s for Δ^0. But the last relation coincides with the same one for Δ. Hence, inside the category $\mathsf{rep}^{(s)}(\mathbf{S}_s)$ the morphism Φ' does not depend on the choice of ξ. Obviously, this implies that the isomorphism class of M' in the same category does not depend on the choice of the retraction η_M.

By the way, the definition of M' guarantees that it contains no direct summand isomorphic to T^s. On the other hand, if $M = T^s$, then $M' = 0$, hence our construction leads indeed to a functor $F_s : \mathsf{rep}^{(s)}(\mathbf{S}) \to \mathsf{rep}^{(s)}(\mathbf{S}_s)$, which maps M to M' for some prescribed choice of η_M, and any other such choice leads to an isomorphic functor.

Just in the same way we are able to construct the reflection functor $F_s : \mathsf{rep}^{(s)}(\mathbf{S}) \to \mathsf{rep}^{(s)}(\mathbf{S}_s)$, in the case, when s is a minimal element in $\mathbf{S}^+$ (thus a sink). Again we have that $\operatorname{Im} f_s^- \subseteq \ker f_s^+$. So we can put $M'_s = \ker f_s^+ / \operatorname{Im} f_s^-$, choose a section $\eta_M : M_0/\operatorname{Im} f_s^- \to M_0$ and define M' having the same components as M except of M'_s and $f'_s = \eta_M \varepsilon_M$, where ε_M is the embedding $M'_s \to M_0/\operatorname{Im} f_s^-$. Almost the same observations show that in this way we really get the wanted functor.

Suppose now that 0 is a source, i.e. $\mathbf{S}^- = \emptyset$. In this case we put $M'_0 = \operatorname{Cok} f_0^+$ and take for $f'_i : M'_i = M_i \to M'_0$ the i-th component of the natural projection $M_0^+ \to M'_0$. At last, if 0 is a sink, put $M'_0 = \ker f_0^-$ and take for $f'_i : M'_0 \to M'_i = M_i$ the i-th component of the embedding $M'_0 \to M_0^-$. This time we even obtain indeed functors $\mathsf{rep}\,(\mathbf{S}) \to \mathsf{rep}\,(\mathbf{S}_0)$, which evidently map T^0 to zero, thus inducing the functors $F_0 : \mathsf{rep}^{(0)}(\mathbf{S}) \to \mathsf{rep}^{(0)}(\mathbf{S}_0)$.

Certainly, our notations for the reflection functors are rather ambiguous. More accurate were to write, for instance, $F_s^{\pm}(\mathbf{S})$, with "$+$" for sinks and "$-$" for sources. Nevertheless, it seems no doubts that one could never mix up, which construction is used in each considered case.

Though F_s cannot be considered as a *functor* $\mathsf{rep}\,(\mathbf{S}) \to \mathsf{rep}\,(\mathbf{S}_s)$, the above construction defines F_sM, where $M \in \mathsf{rep}\,(\mathbf{S})$, up to isomorphism. Let now $s_1, s_2, \dots, s_m$ be a sequence of elements of $\mathbf{S}$ (not necessarily distinct) such that each s_{k+1} is a source or a sink in $\mathbf{S}_{s_1 s_2 \dots s_k}$ (in particular, s_1 is a source or a sink). We call such sequence an *admissible sequence* (in $\mathbf{S}$). Moreover, if each s_{k+1} is a source (a sink) in $\mathbf{S}_{s_1 s_2 \dots s_k}$, call it a *source sequence* (resp. a *sink sequence*).

In this case, for any representation $M \in \mathsf{rep}\,(\mathbf{S})$, the representation $F_{s_m\dots s_2s_1}M = F_{s_m}\dots F_{s_2}F_{s_1}M$ is well-defined. Denote

$$\mathbf{I}_{s_1s_2\dots s_k} = [\,T^{s_1}, F_{s_1}T^{s_2}, F_{s_1s_2}T^{s_3}, \dots, F_{s_1s_2\dots s_{k-1}}T^{s_k}\,],$$
$$\mathsf{rep}^{(s_1s_2\dots s_k)}(\mathbf{S}) = \mathsf{rep}\,(\mathbf{S})/\mathbf{I}_{s_1s_2\dots s_k}.$$

Here T^{s_2} is considered as representation of S_{s_1}, T^{s_3} as that of $S_{s_1s_2}$, ..., T^{s_k} as that of $\mathbf{S}_{s_1s_2\dots s_{k-1}}$. Then $F_{s_k\dots s_2s_1}$ can be considered as a functor:

$$F_{s_k\dots s_2s_1} : \mathsf{rep}^{(s_1s_2\dots s_k)}(\mathbf{S}) \to \mathsf{rep}^{(s_k\dots s_2s_1)}(\mathbf{S}_{s_1s_2\dots s_k}).$$

In particular, the functor $F_{ss} : \mathsf{rep}^{(s)}(\mathbf{S}) \to \mathsf{rep}^{(s)}(\mathbf{S})$ is defined (we do not still dare to write F_s^2 here).

Theorem 2.2. *If s is a source or a sink, then $F_{ss} \simeq \mathbf{1}$, the identity functor of the category $\mathsf{rep}^{(s)}(\mathbf{S})$. Hence, the functors F_s establish an equivalence of the categories $\mathsf{rep}^{(s)}(\mathbf{S})$ and $\mathsf{rep}^{(s)}(\mathbf{S}_s)$.*

Proof. We consider the case, when s is a source in $\mathbf{S}$, as the dual case is rather similar and the case $s = 0$ is quite obvious. Let M be any representation of $\mathbf{S}$ not containing T^s as a direct summand, $M' = F_sM$ and $M'' = F_sM'$. Then all components of M' and M'' coincide with those of M except of $M'_s = \ker f_s^+/\operatorname{Im} f_s^-$, $f'_s = \pi_M\eta_M$ and $M''_s = \ker f'^+_s/\operatorname{Im} f'^-_s$, $f''s = \eta_{M'}\varepsilon_{M'}$. But, by definition,

$$\ker f'^+_s = \ker f_s^+ \cap \ker \pi_M\eta_M = \operatorname{Im} f_s^-,$$

as $v \in \ker f_s^+(v)$ implies that $\eta_M(v) = v$, whence $\pi_M(v) = 0$ if and only if $v \in \operatorname{Im} f_s^-$. Therefore, $M''_s = \operatorname{Im} f_s^-/\sum_{j<i}\operatorname{Im} f_j$, so we are able to define the natural mapping $\Theta_s : M_s \to M''_s$ as the composition of f_s and the projection $\operatorname{Im} f_s^- \to M''_s$. Now Lemma 1.2 implies evidently that $\operatorname{Im} f_s^- \simeq f_s(M_s) \oplus \sum_{j<s}\operatorname{Im} f_j$ and f_s is a monomorphism, whence $M''_s \simeq M_s$. Moreover, as we are free in choosing a section $\eta_{M'} : M_0/\operatorname{Im} f'^-_s \to M_0$, do it in such way that its restriction onto $f_s(M_s)$ were identical. Then we can define an isomorphism $\Theta = \Theta_M : M \simeq M''$ putting $\Theta_i = 1$ for all $i \in \hat{\mathbf{S}}$ and $\Theta_{ij} = 0$ for all i, j. It is quite evident that this construction is really functorial (modulo the ideal $\mathbf{I}_s$), thus we get an isomorphism of functors $\Theta : \mathbf{1} \to F_{ss}$. □

Corollary 2.3. *For any admissible sequence $s_1, s_2, \dots, s_m$ the functors $F_{s_1s_2\dots s_m}$ and $F_{s_m\dots s_2s_1}$ define an equivalence of the categories $\mathsf{rep}^{(s_m\dots s_2s_1)}(\mathbf{S}_{s_1s_2\dots s_m})$ and $\mathsf{rep}^{(s_1s_2\dots s_m)}(\mathbf{S})$.*

Corollary 2.4. *Let $s_1, s_2, \dots, s_m$ be an admissible sequence and M be such indecomposable representation of $\mathbf{S}$ that $F_{s_m\dots s_2s_1}M \neq 0$. Then $M \simeq F_{s_1s_2\dots s_m}F_{s_m\dots s_2s_1}M$.*

Remark that the isomorphism classes of indecomposable objects of the category $\mathsf{rep}^{(s_1 s_2 \dots s_m)}(\mathbf{S})$ coincide with those of the category $\mathsf{rep}\,(\mathbf{S})$ except the classes of the representations $T^{s_1}, F_{s_1}T^{s_2}, F_{s_1s_2}T^{s_3}, \dots, F_{s_1s_2\dots s_{m-1}}T^{s_m}$.

Corollary 2.5. *If bisposets* $\mathbf{S}$ *and* $\mathbf{S}'$ *have the same underlying poset, they are of the same representation type. In particular, a bisposet is of finite representation type, tame or wild if and only if so is its underlying poset.*

3. Finite representation type

Now we are able to prove the main theorem on bisposets of finite representation type, which generalizes and precises that of [3]. First, it inverts the claim of Proposition 1.1 concerning the finite representation type and, moreover, gives an exhausting description of the representations in this case. Just as for usual posets, call a representation M of a bisposet $\mathbf{S}$ *sincere* if $M_i \neq 0$ for all $i \in \hat{\mathbf{S}}$.

Theorem 3.1. *Suppose that the quadratic form* $\mathrm{Q} = \mathrm{Q}_{\mathbf{S}}$ *is weakly positive. Then:*

1. *The bisposet* $\mathbf{S}$ *is of finite representation type.*
2. *A vector* $\mathbf{x} \in \mathrm{D}^+ = \mathrm{D}^+(\mathbf{S})$ *is a dimension vector of an indecomposable representation of* $\mathbf{S}$ *if and only if* $\mathrm{Q}(\mathbf{x}) = 1$. *Moreover, in this case there is only one (up to isomorphism) indecomposable representation of dimension vector* $\mathbf{x}$.
3. $\hom_{\mathbf{S}}(M,M) = \mathbf{k}$ *for each indecomposable representation* M *of* $\mathbf{S}$.
4. *The orbit of each indecomposable representation* M *of* $\mathbf{S}$ *is open and dense in* $\mathsf{rep}_{\mathbf{x}}$, *where* $\mathbf{x} = \mathbf{dim}\, M$.
5. *If* $\mathrm{Q}(\mathbf{x}) = 1$, *then the variety* $\mathsf{rep}_{\mathbf{x}}$ *is irreducible and* $\dim \mathsf{rep}_{\mathbf{x}} = \mathrm{Q}^-(\mathbf{x})$.
6. *For each indecomposable representation* M *of* $\mathbf{S}$ *there exists a source sequence (as well as a sink sequence)* $s_1, s_2, \dots, s_m$ *such that* $M \simeq F_{s_m \dots s_2 s_1} L$ *for some non-sincere representation of the bisposet* $\mathbf{S}_{s_1 s_2 \dots s_m}$. (Possibly $m = 0$ if M is non-sincere itself.)

Proof. Up to the end of the proof we suppose that $\mathrm{Q} \triangleright 0$. We also go on using Notations 2.1. If $\mathbf{S} = \emptyset$, all claims are trivial. So we prove them by induction on $|\mathbf{S}|$ and suppose that they hold for all proper sub-bisposets $\mathbf{S}'$ of $\mathbf{S}$ or, the same, for all non-sincere representations. Remark that the quadratic forms corresponding to these sub-bisposets are also weakly positive. Prove first the following lemmas, the second one rather alike to the key lemma of [3].

Lemma 3.2. *Let the assertion 4 of Theorem 3.1 be valid for some indecomposable representation M of the bisposet $\mathbf{S}$. Then for each $i \in \mathbf{S}$:*

- *if $x'_i \geq 0$, the mapping f_i^+ is an epimorphism and the mapping f_i^- is a monomorphism,*
- *if $x'_i \leq 0$, we have that* $\dim \operatorname{Im} f_i^- \geq \dim \ker f_i^+$.

Proof. $x'_i \geq 0$ means that $x_0 - x_i^+ \geq x_i^- \geq 0$. Then one can easily construct a representation $\tilde{M}$ of $\mathbf{S}$ of dimension vector $\mathbf{x}$ such that $\tilde{f}_i^+$ is an epimorphism and $\tilde{f}_i^-$ is an monomorphism. Indeed, put $\tilde{f}_k = 0$ for all k non-comparable with i, take for $\tilde{f}_i^+$ any epimorphism $M_0 \to M_i^+$ and for $\tilde{f}_i^-$ any monomorphism $M_i^- \to \ker f_i^+$ (they exist due to the given relations between the dimension vectors). But these conditions pick out an open subset $Z \subseteq \mathsf{rep}_{\mathbf{x}}(\mathbf{S})$. Moreover, Z is evidently stable under the action of the group $\mathsf{G}_{\mathbf{x}}$. As the orbit of M is dense, we obtain that $M \in Z$.

If $x'_i \leq 0$, the condition "$\dim \operatorname{Im} f_i^- \geq \ker f_i^+$" also picks out a non-empty open subset $Y \subseteq \mathsf{rep}_{\mathbf{x}}$. Indeed, we may put again $\tilde{f}_k = 0$ for all k non-comparable with i and take for $\tilde{f}_i^+$ any mapping $M_0 \to M_i^+$ of maximal possible rank. If $x_0 \geq x_i^+$, then this mapping is an epimorphism, hence $\dim \ker \tilde{f}_i^+ = x_0 - x_i^+ \leq x_i^-$ and we may take for $\tilde{f}_i^-$ any epimorphism $M_i^- \to \ker \tilde{f}_i^+$. If $x_0 \leq x_i^+$, then $\tilde{f}_i^+$ is a monomorphism, so its kernel is zero. Hence just as above we obtain that $M \in Y$. □

Lemma 3.3. *Suppose that $s \in \mathbf{S}$ is a source or a sink, $M \not\simeq T^s$ is an indecomposable representation of $\mathbf{S}$ and $\mathbf{x} = \mathbf{dim}\, M$. Then:*

1. *If $M_s \neq 0$ or $x'_s \geq 0$, then f_s^+ is an epimorphism and f_s^- is a monomorphism.*
2. *If $M_s = 0$ and $x'_s \leq 0$ then* $\operatorname{Im} f_s^- = \ker f_s^+$.

Proof. If $M_s = 0$, then M may be considered as a representation of the bisposet $\mathbf{S}' = \mathbf{S} \setminus \{ s \}$. By the inductive hypothesis, Theorem 3.1 holds for $\mathbf{S}'$. Hence, in particular, we may apply Lemma 3.2 to M, which gives the sought result. One has only to remark that in this case always $\operatorname{Im} f_s^- \subseteq \ker f_s^+$, thus the inverse inequality for their dimensions implies that they coincide.

Suppose that $M_s \neq 0$. Denote by $\overline{M}$ the restriction of M to $\mathbf{S}' = \mathbf{S} \setminus \{ s \}$ and let $N = (N_i, g_i)$ be an indecomposable direct summand of $\overline{M}$. Again we may apply Lemma 3.2 to N. If we show, for each possible N, that g_s^+ is an epimorphism and g_s^- is a monomorphism, then it is valid for the whole $\overline{M}$ and we come to the sought claim simply by applying Lemma 1.2 to the point s and the representation M. But otherwise $\operatorname{Im} g_s^- = \ker g_s^+$. Consider some retraction $\Pi : \overline{M} \to N$ such that $\Pi\overline{\mathrm{E}} = 1$, the identity morphism of N, where $\overline{\mathrm{E}}$ is the embedding $N \to \overline{M}$. Let s be a source (for a sink the proof is quite analogous). Then $\operatorname{Im} f_s \subseteq \ker f_s^+$, whence $\operatorname{Im} \Pi_0 f_s \subseteq \ker g_s^+ = \operatorname{Im} g_s^-$. Therefore, there exists a linear mapping $\pi : M_s \to N_s^-$ such that $\Pi_0 f_s = -g_s^- \pi$. Denote by Π_{is} $(i < s)$ the components of π. Then we become able to consider Π as a morphism $M \to N$ and can easily check that $\Pi\mathrm{E} = 1$, where E is the embedding of N into M. Hence N is a direct summand of M, which contradicts the assumptions. $\square$

Corollary 3.4. 1. *If $M \not\simeq T^s$ is an indecomposable representation of $\mathbf{S}$, s is a source (a sink) and $M_s \neq 0$, then* $\hom_{\mathbf{S}}(T^s, M) = 0$ *(resp.* $\hom_{\mathbf{S}}(M, T^s) = 0$*).*

2. $\hom_{\mathbf{S}}(M, N) \simeq \hom_{\mathbf{S}_s}(F_s M, F_s N)$ *for each representation N with the same properties.*

Proof. Suppose that s is a source. Then a morphism $\Phi : T^s \to M$ is given by its components $\Phi_s : \mathbf{k} \to M_s$ and $\Phi_{is} : \mathbf{k} \to M_i$, where $i < s$, subject to the relation $f_s\Phi_s = \sum_{i<s} f_i \Phi_{is}$. But as f_s^- is a monomorphism, this relation implies that $\Phi_s = 0$ and all $\Phi_{is} = 0$. In the same way the first claim is proved for sinks. The second one now follows from Theorem 2.2. $\square$

Remind the definition of the *reflections* $w_i : \mathrm{D} \to \mathrm{D}$ with respect to the form Q. Namely, if $(\_,\_)$ denotes the corresponding symmetric bilinear form, then $w_i(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}, \mathbf{e}^i)\mathbf{e}^i$, where $\mathbf{e}^i$ are standard basic vectors (by the way, $\mathbf{e}^i = \mathbf{dim}\, T^i$). Here we use the fact that $\mathrm{Q}(\mathbf{e}^i) = 1$. The definition of Q implies now that if $\mathbf{x}$ has the coordinates x_j, then the only coordinate of $w_i(\mathbf{x})$ different from the corresponding one of $\mathbf{x}$ is the i-th one, which equals just x_i' (using Notations 2.1). This leads us to the following fact.

Corollary 3.5. *If $M \not\simeq T^s$ is an indecomposable representation of $\mathbf{S}$ with $M_s \neq 0$, where s is a source or a sink, then $\mathbf{dim}\, F_s M = w_s(\mathbf{dim}\, M)$. In particular, $w_s(\mathbf{dim}\, M) \in \mathrm{D}^+$.*

Now we are able to accomplish the proof of Theorem 3.1. Place all elements of $\hat{\mathbf{S}}$ into a source (or sink) sequence: $s_1, s_2, \dots, s_{n+1}$, where $n = |\mathbf{S}|$, and put

$$\phi = w_{s_{n+1}} \dots w_{s_2} w_{s_1}.$$

This is a *Coxeter transformation* for the form Q (cf. [3]). Remind that for any vector $\mathbf{x} \in \mathrm{D}^+$ there is some integer k such that $\phi^k(\mathbf{x}) \notin \mathrm{D}^+$. In view of Corollary 3.5, this means that there is a source (resp. sink) sequence $s_1, s_2, \dots, s_m$ such that the representation $L = F_{s_m \dots s_2 s_1} M$ is non-sincere, and it is the shortest such a sequence. As $M \simeq F_{s_1 s_2 \dots s_m} L$ by Corollary 2.4, we get the assertion *6* of Theorem 3.1. Remark also that the vector $\mathbf{y} = \mathbf{dim}\, L$ depends only on $\mathbf{x} = \mathbf{dim}\, M$ and not on the choice of indecomposable M in $\mathsf{rep}_\mathbf{x}$. Applying the inductive hypothesis to the set $\mathbf{S} \setminus \{ s \}$, we get the following properties:

1. $\mathrm{Q}(\mathbf{y}) = 1$, hence also $\mathrm{Q}(\mathbf{x}) = 1$ as Q is invariant under reflections.
2. L is a unique (up to isomorphism) indecomposable representation of dimension vector $\mathbf{y}$, hence M is also a unique indecomposable representation of dimension vector $\mathbf{x}$.
3. $\mathrm{hom}_{\mathbf{S}_{s_1 s_2 \dots s_m}}(L, L) = \mathbf{k}$, hence also $\mathrm{hom}_\mathbf{S}(M, M) = \mathbf{k}$ in view of Corollary 3.4,

Moreover, if $\mathbf{x} \in \mathrm{D}^+$ is any such vector that $\mathrm{Q}(\mathbf{x}) = 1$, $x_i \neq 0$ and $\mathbf{x} \neq \mathbf{e}^i$, then $x_i - x_i' \leq 1$: otherwise $\mathrm{Q}(\mathbf{x} - \mathbf{e}^i) \leq 0$. So choose a shortest possible source (or sink) sequence $s_1, s_2, \dots, s_m$ such that $\mathbf{y} = w_{s_m} \dots w_{s_2} w_{s_1}(\mathbf{x})$ has a zero coordinate (possibly, $m = 0$). Then, again by the inductive hypothesis, there exists an indecomposable representation L of $\mathbf{S}_{s_1 s_2 \dots s_m}$ of dimension vector $\mathbf{y}$. Hence $M = F_{s_1 s_2 \dots s_m} L$ is an indecomposable representation of $\mathbf{S}$ of dimension vector $\mathbf{x}$. So we have proved the assertions *2* and *3* of Theorem 3.1. As the equation $\mathrm{Q}(\mathbf{x}) = 1$ has only finitely many solutions in D^+ (cf. [3]), it implies also the assertion *1*.

Let now $\mathbf{x} \in \mathrm{D}^+$ be such that $\mathrm{Q}(\mathbf{x}) = 1$. Each irreducible component C of $\mathsf{rep}_\mathbf{x}$ has the dimension at least $\mathrm{Q}^-(\mathbf{x})$. As it contains only finitely many G-orbits, one of them is open and dense in C. But the dimension of such an orbit equals $\mathrm{Q}^+(\mathbf{x}) - \dim \mathrm{hom}_\mathbf{S}(M, M)$, where M is the representation lying on it, whence

$$\dim \mathrm{hom}_\mathbf{S}(M, M) = \mathrm{Q}^+(\mathbf{x}) - \dim C \leq \mathrm{Q}(\mathbf{x}) = 1 .$$

Therefore $\dim C = \mathrm{Q}^-(\mathbf{x})$ and $\dim \mathrm{hom}_\mathbf{S}(M, M) = 1$, which implies that M is indecomposable. As there exists only one indecomposable representation of dimension vector $\mathbf{x}$ (up to isomorphism), $C = \mathsf{rep}_\mathbf{x}$, which proves the assertions *4* and *5*. □

Corollary 3.6. *The dimension vectors of indecomposable representations of a bisposet of finite representation type depend only on the underlying poset and not on its bisection. Moreover, they are the same for any such poset* $\mathbf{S}$ *and its dual poset* $\mathbf{S}^*$ *(as* $\mathrm{Q}_\mathbf{S} = \mathrm{Q}_{\mathbf{S}^*}$ *).*

4. Remarks and questions

4.1. Remind that in [3] two transformations were defined for representations of any (usual) poset $\mathbf{S}$. The first one, denoted by σ, corresponds to the reflection w_0 for dimension vectors, while the second one, denoted by ρ, corresponds to the composition $w_{s_n} \dots w_{s_2} w_{s_1}$, where $s_1, s_2, \dots, s_n$ is some source sequence containing all points of $\mathbf{S}$. Both map representations of $\mathbf{S}$ to those of the dual poset $\mathbf{S}^*$. They were not defined as functors though one could do it using some factor-categories of $\mathsf{rep}\,(\mathbf{S})$ as we have done above. They are really closely related to our functors F_s. Namely, σ coincides with DF_0, while ρ coincides with $DF_{s_n \dots s_2 s_1}$, where D means the standard duality of vector spaces. The use of duality was indispensable as one can see now: otherwise instead of representations of a usual poset $\mathbf{S}$ (identified with the bisposet having $\mathbf{S}^+ = \emptyset$) one would get representations of what we denote by $\mathbf{S}_0$, which is the same poset but with $\mathbf{S}^+ = \mathbf{S}$.

4.2. Let $\mathbf{S}'$ be a sub-bisposet of $\mathbf{S}$. Then one can consider $\mathsf{rep}\,(\mathbf{S}')$ as a full subcategory in $\mathsf{rep}\,(\mathbf{S})$. Suppose that t is a source or a sink in $(\mathbf{S}')^*$. Then the functor F_t is defined on $\mathsf{rep}^{(t)}(\mathbf{S}')$ though not on $\mathsf{rep}^{(t)}(\mathbf{S})$. We consider it as a sort of "*partial reflection functor*". Now Theorem 3.1 evidently implies that, if $\mathbf{S}$ is of finite representation type, any of its indecomposable representations can be obtained from a trivial one by applying such partial reflection functors. All examples show that it is so even if we use only the "full" reflection functors defined above. Nevertheless, we cannot prove it now, as Corollary 3.5 may be wrong for representations M with $M_s = 0$. Hence, for such representations there is no evident relations between the reflection functors and the reflections of their dimension vectors.

4.3. We also do not know, whether Corollary 3.6 remains valid, say, for tame bisposets. At least, Lemma 3.3 and thus Corollary 3.5 remain no more valid for them as the following example shows. Let $\mathbf{S}$ consist of 4 non-comparable elements a_1, a_2, a_3, a_4 and one more element b such that $b > a_1$ and $b > a_2$. As usual poset (i.e. with $\mathbf{S}^+ = \emptyset$) it has a sincere indecomposable representation M given by the matrices:

$$\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & B \\ \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} \end{array}$$

Here $\operatorname{Im} f_1 = \operatorname{Im} f_2$ and $\mathbf{dim}\, F_b M = (2,1,1,1,1,0)$, while $w_b(\mathbf{dim}\, M) = (2,1,1,1,\ 1,-1)$. In the same way $F_{bb} M \simeq M$, while the vector $\mathbf{dim}\, F_b M$ is stable under all reflections.

Nevertheless, Corollary 2.5 together with the well-known Nazarova's criterion for tameness of posets [9] shows that $\mathbf{S}$ is tame if and only if $Q_{\mathbf{S}} \trianglerighteq 0$.

4.4. This technique can be also applied to the "peak posets" considered by Kasjan and Simson [7]. Namely, we can introduce a "*multisected poset*" $\mathbf{M}$ as consisting of a poset $\mathbf{S}$, a set of *peaks* $\mathbf{P}$ and for each $p \in \mathbf{P}$ given two non-intersecting subsets p^+ and p^- of $\mathbf{S}$ such that if $i < j$ and $j \in p^+ \cup p^-$, then also $i \in p^+ \cup p^-$. Moreover, in this case $i \in p^+$ implies $j \in p^+$, while $j \in p^-$ implies $i \in p^-$. One can define for each multisected poset $\mathbf{M}$ the corresponding box $\mathcal{A}(\mathbf{M})$ just in the same way as it has been done above for bisposets. Thus we obtain the category $\mathsf{rep}\,(\mathbf{M})$ of representations of the multisected poset $\mathbf{M}$. Moreover, following the definitions and proofs given above, one can easily get for multisected posets the notions of sinks and sources, the construction of reflection functors and quite the same results as we have obtained for representations of bisposets. Having many peaks instead of one (as in the case of bisposets) produces no troubles, though is not very pleasant to deal with. That is why we have preferred to give all constructions in the simplest case, when $\mathbf{P}$ consists of one element. Remark that we do not need to suppose $\mathbf{S} = \bigcup_{p \in \mathbf{P}} (p^+ \cup p^-)$: if there is a point i not belonging to the latter union, then the only indecomposable representation non-zero at this point is the trivial one T^i. Just in the same way we need not to suppose that $p^+ \cup p^- \neq \emptyset$.

We may also suppose that the same considerations could be applied to the class of matrix problems considered by Golovaschuk and Ovsienko [6] in the case, when they are simply connected. But at the moment we lack techniques to do it properly.

4.5. As the box $\mathcal{A}(\mathbf{S})$ corresponding to a bisected poset $\mathbf{S}$ is finite dimensional, the category of its representations $\mathsf{rep}\,(\mathbf{S})$ has almost split sequences (cf., for instance, [1]). Hence, one can consider the Auslander-Reiten quiver $\Gamma(\mathbf{S})$ of this category. It seems very probable that this quiver possesses a preprojective component, just as in the case of ordinary posets (cf. [11]). Moreover, we hope that the techniques used in [11] could also be used for bisected posets, though some details are still not evident.

4.6. One can also apply the analogous technique to representations of $\mathbf{k}$-*structures* considered by Dlab and Ringel (cf. [2]). In this case there are also practically no troubles with carefully following the procedure of our paper obtaining the evident generalization of all results.

References

[1] W. L. Burt and M. C. R. Butler, *Almost split sequences for bocses*, In: Representations of Finite Dimensional Algebras: Proceedings o0f the Tsukuba International Conference (Fifth ICRA), CMS Conference Proc., **11** (1991) 89–121.

[2] V. Dlab and C. M. Ringel, *On algebras of finite representation type*, J. Algebra, **33** (1975), 306–394.

[3] Yu. A. Drozd, *Coxeter transformations and representations of partially ordered sets*, Funk. Anal. Prilozh., **8**:3 (1974), 34–42.

[4] Yu. A. Drozd, *On tame and wild matrix problems*, In: Matrix Problems, Kyiv, 1977, 104–114.

[5] Yu. A. Drozd, *Tame and wild matrix problems*, In: Representations and quadratic forms, Kyiv, 1979, 39–74 (English translation in: AMS Translations, **128** (1986), 31–55).

[6] N. S. Golovaschuk, *Coverings of certain class of matrix problems*, Thesis, Kyiv, 1991 (revised version: N. S. Golovaschuk, S. A. Ovsienko, *Coverings of bimodule problems*, to appear).

[7] S. Kasjan and D. Simson, *Varieties of poset representations and minimal posets of wild prinjective type*, In: Representations of Algebras and Related Topics (Ottawa, 1992), CMS Conference Proc. **14** (1993), 245-284.

[8] L. A. Nazarova and A. V. Roiter, *Representations of partially ordered sets*, Zapiski Nauch. Semin. LOMI, **28** (1972), 5–31 (English translation in: J. Soviet Math., **3** (1975), 585–606).

[9] L. A. Nazarova, *Partially ordered sets of infinite type*, Izv. Akad. Nauk SSSR, Ser. Mat., **39** (1975), 963–991.

[10] S. A. Ovsienko, *Representations of quivers with relations*, In: Matrix Problems, Kyiv, 1977, 88–103.

[11] J. A. de la Peña and D. Simson, *Prinjective modules, reflection functors, quadratic forms and Auslander-Reiten sequences*, Trans. Amer. Math. Soc., **329** (1992), 733–753.

[12] A. V. Roiter, *Matrix problems and representations of BOCS's*, In: Representation Theory I, Springer Lecture Notes in Math., **831** (1980), 288–324.

Department of Mechanics and Mathematics, Kiev Taras Shevchenko University, 252033 Kiev, Ukraine

E-mail address: drozd@uni-alg.kiev.ua