DERIVED CATEGORIES FOR NODAL RINGS AND PROJECTIVE CONFIGURATIONS

IGOR BURBAN AND YURIY DROZD

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Introduction

This paper is devoted to recent results on explicit calculations in derived categories of modules and coherent sheaves. The idea of this approach is actually not new and was effectively used in several questions of module theory (cf. e.g. [10, 12, 13, 7]). Nevertheless it was somewhat unexpected and successful that the same technique could be applied to derived categories, at least in the case of rings and curves with "simple singularities." We present here two cases: nodal rings and configurations of projective lines of types A and $\tilde{A}$, when these calculations can be carried out up to a result, which can be presented in more or less distinct form, though it involves rather intricate combinatorics of a special sort of matrix problems, namely "bunches of semi-chains" [4] (or, equivalently, "clans" [8]). In Sections 1 and 4 we give a general construction of "categories of triples," which are a connecting link between derived categories and matrix problems, while in Sections 2 and 5 this construction is applied to nodal rings and configurations of types $\tilde{A}$. Section 3 contains examples of calculations for concrete rings and Section 5 also presents those for nodal cubic. We tried to choose typical examples, which allow to better understand the general procedure of passing from combinatorial data to complexes. Section 6 contains an application to

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Cohen–Macaulay modules over surface singularities, which was in fact the origin of investigations of vector bundles over projective curves in [13].

More detailed exposition of these results can be found in [5, 6, 14].

1. Backström rings

We consider a class of rings, which generalizes in a certain way local rings of *ordinary multiple points* of algebraic curves. Following the terminology used in the representations theory of orders, we call them *Backström rings.* Since in the first three sections we are investigating a *local* situation, all rings there are supposed to be *semi-perfect* [3] and *noetherian.* We denote by $\mathbf{A}$-mod the category of finitely generated $\mathbf{A}$-modules and by $D(\mathbf{A})$ the derived category $D^-(\mathbf{A}\text{-mod})$ of right bounded complexes over $\mathbf{A}$-mod. As usually, it can be identified with the homotopy category $K^-(\mathbf{A}\text{-pro})$ of (right bounded) complexes of (finitely generated) projective $\mathbf{A}$-modules. Moreover, since $\mathbf{A}$ is semi-perfect, each complex from $K^-(\mathbf{A}\text{-pro})$ is homotopic to a *minimal* one, i.e. to such a complex $C_\bullet = (C_n, d_n)$ that $\operatorname{Im} d_n \subseteq \operatorname{rad} C_{n-1}$ for all n. If $C_\bullet$ and $C'_\bullet$ are two minimal complexes, they are isomorphic in $D(\mathbf{A})$ if and only if they are isomorphic *as complexes*; moreover, any morphism $C_\bullet \to C'_\bullet$ in $D(\mathbf{A})$ can be presented by a morphism of complexes, and f is an isomorphism if and only if the latter one is.

Definition 1.1. A ring $\mathbf{A}$ is called a *Backström ring* if there is a hereditary ring $\mathbf{H} \supseteq \mathbf{A}$ (also semi-perfect and noetherian) and a (two-sided) $\mathbf{H}$-ideal $\mathbf{I} \subset \mathbf{A}$ such that both $\mathbf{R} = \mathbf{H}/\mathbf{I}$ and $\mathbf{S} = \mathbf{A}/\mathbf{I}$ are semi-simple.

For Backström rings there is a convenient approach to the study of derived categories. Recall that for a hereditary ring $\mathbf{H}$ every object $C_\bullet$ from $D(\mathbf{H})$ is isomorphic to the direct sum of its homologies. Especially, any indecomposable object from $D(\mathbf{H})$ is isomorphic to a shift $N[n]$ for some $\mathbf{H}$-module N, or, the same, to a "short" complex $0 \to P' \xrightarrow{\alpha} P \to 0$, where P and P' are projective modules and α is a monomorphism with $\operatorname{Im} \alpha \subseteq \operatorname{rad} P$ (maybe $P' = 0$). Thus it is natural to study the category $D(\mathbf{A})$ using this information about $D(\mathbf{H})$ and the functor $T : D(\mathbf{A}) \to D(\mathbf{H})$ mapping $C_\bullet$ to $\mathbf{H} \otimes_{\mathbf{A}} C_\bullet$.[1]

Consider a new category $\mathcal{T} = \mathcal{T}(\mathbf{A})$ (the *category of triples*) defined as follows:

- Objects of $\mathcal{T}$ are triples $(A_\bullet, B_\bullet, \iota)$, where
  - $A_\bullet \in D(\mathbf{H})$;
  - $B_\bullet \in D(\mathbf{S})$;
  - ι is a morphism $B_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ from $D(\mathbf{S})$ such that the induced morphism $\iota^R : \mathbf{R} \otimes_{\mathbf{S}} B_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ is an isomorphism in $D(\mathbf{R})$.
- A morphism from a triple $(A_\bullet, B_\bullet, \iota)$ to a triple $(A'_\bullet, B'_\bullet, \iota')$ is a pair (Φ, ϕ), where
  - $\Phi : A_\bullet \to A'_\bullet$ is a morphism from $D(\mathbf{H})$;
  - $\phi : B_\bullet \to B'_\bullet$ is a morphism from $D(\mathbf{S})$;

[1] Of course, we mean here the left derived functor of $\otimes$, but when we consider complexes of projective modules, it restricts indeed to the usual tensor product.

– the diagram

$$\begin{array}{ccc} B_\bullet & \xrightarrow{\iota} & \mathbf{R}\otimes_{\mathbf{H}} A_\bullet \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle 1\otimes\Phi} \\ B'_\bullet & \xrightarrow{\iota'} & \mathbf{R}\otimes_{\mathbf{H}} A'_\bullet \end{array} \tag{1.1}$$

commutes in $D(\mathbf{S})$.

One can define a functor $\mathbf{F} : D(\mathbf{A}) \to \mathcal{T}(\mathbf{A})$ setting $\mathbf{F}(C_\bullet) = (\mathbf{H}\otimes_{\mathbf{A}} C_\bullet, \mathbf{S}\otimes_{\mathbf{A}} C_\bullet, \iota)$, where $\iota : \mathbf{S}\otimes_{\mathbf{A}} C_\bullet \to \mathbf{R}\otimes_{\mathbf{H}} (\mathbf{H}\otimes_{\mathbf{A}} C_\bullet) \simeq \mathbf{R}\otimes_{\mathbf{A}} C_\bullet$ is induced by the embedding $\mathbf{S}\to\mathbf{R}$. The values of $\mathbf{F}$ on morphisms are defined in an obvious way.

Theorem 1.2. *The functor* $\mathbf{F}$ *is a* full representation equivalence*, i.e. it is*

- dense*, i.e. every object from* $\mathcal{T}$ *is isomorphic to an object of the form* $\mathbf{F}(C_\bullet)$*;*
- full*, i.e. each morphism* $\mathbf{F}(C_\bullet)\to\mathbf{F}(C'_\bullet)$ *is of the form* $\mathbf{F}(\gamma)$ *for some* $\gamma : C_\bullet \to C'_\bullet$*;*
- conservative*, i.e.* $\mathbf{F}(\gamma)$ *is an isomorphism if and only if so is* γ*;*

As a consequence, $\mathbf{F}$ *maps non-isomorphic objects to non-isomorphic and indecomposable to indecomposable.*

Note that in general $\mathbf{F}$ is not *faithful*: it is possible that $\mathbf{F}(\gamma)=0$ though $\gamma\neq 0$ (cf. Example 3.1.3 below).

Sketch of the proof. Consider any triple $T=(A_\bullet, B_\bullet, \iota)$. We may suppose that $A_\bullet$ is a minimal complex from $K^-(\mathbf{A}\text{-pro})$, while $B_\bullet$ is a complex with zero differential (since $\mathbf{S}$ is semi-simple) and the morphism ι is a usual morphism of complexes. Note that $\mathbf{R}\otimes_{\mathbf{H}} A_\bullet$ is also a complex with zero differential. We have an exact sequence of complexes

$$0 \longrightarrow \mathbf{I}A_\bullet \longrightarrow A_\bullet \longrightarrow \mathbf{R}\otimes_{\mathbf{H}} A_\bullet \longrightarrow 0.$$

Together with the morphism $\iota : B_\bullet \to \mathbf{R}\otimes_{\mathbf{H}} A_\bullet$ it gives rise to a commutative diagram in the *category of complexes* $\mathrm{Com}^-(\mathbf{A}\text{-mod})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{I}A_\bullet & \longrightarrow & A_\bullet & \longrightarrow & \mathbf{R}\otimes_{\mathbf{H}} A_\bullet & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\iota} & & \\ 0 & \longrightarrow & \mathbf{I}A_\bullet & \longrightarrow & C_\bullet & \longrightarrow & B_\bullet & \longrightarrow & 0, \end{array}$$

where $C_\bullet$ is the preimage in $A_\bullet$ of $\operatorname{Im}\iota$. The lower row is also an exact sequence of complexes and α is an embedding. Moreover, since ι^R is an isomorphism, $\mathbf{I}A_\bullet = \mathbf{I}C_\bullet$. It implies that $C_\bullet$ consists of projective $\mathbf{A}$-modules and $\mathbf{H}\otimes_{\mathbf{A}} C_\bullet \simeq A_\bullet$, wherefrom $T\simeq \mathbf{F}C_\bullet$.

Let now $(\Phi,\phi) : \mathbf{F}C_\bullet \to \mathbf{F}C'_\bullet$. We suppose again that both $C_\bullet$ and $C'_\bullet$ are minimal, while $\Phi : \mathbf{H}\otimes_{\mathbf{A}} C_\bullet \to \mathbf{H}\otimes_{\mathbf{A}} C'_\bullet$ and $\phi : \mathbf{S}\otimes_{\mathbf{A}} C_\bullet \to \mathbf{S}\otimes_{\mathbf{A}} C'_\bullet$ are morphisms of complexes. Then the diagram (1.1) is commutative in the category of complexes, so $\Phi(C_\bullet)\subseteq C'_\bullet$ and Φ induces a morphism $\gamma : C_\bullet \to C'_\bullet$. It is evident from the construction that $\mathbf{F}(\gamma)=(\Phi,\phi)$. Moreover, if (Φ,ϕ) is an isomorphism, so are Φ and ϕ (since our complexes are minimal). Therefore $\Phi(C_\bullet)=C'_\bullet$, i.e. $\operatorname{Im}\gamma = C'_\bullet$. But $\ker\gamma = \ker\Phi\cap C_\bullet = 0$, thus γ is an isomorphism too. □

2. Nodal rings

We apply these considerations to the class of rings first considered in [10], where the second author has shown that they are unique pure noetherian rings such that the classification of their modules of finite length is tame (all others being wild).

Definition 2.1. A ring $\mathbf{A}$ (semi-perfect and noetherian) is called a *nodal ring* if it is *pure noetherian*, i.e. has no minimal ideals, and there is a hereditary ring $\mathbf{H} \supseteq \mathbf{A}$, which is semi-perfect and pure notherian such that

1) $\operatorname{rad}\mathbf{A} = \operatorname{rad}\mathbf{H}$; we denote this common radical by $\mathbf{R}$.
2) $\operatorname{length}_{\mathbf{A}}(\mathbf{H} \otimes_{\mathbf{A}} U) \le 2$ for every simple left $\mathbf{A}$-module U and $\operatorname{length}_{\mathbf{A}}(V \otimes_{\mathbf{A}} \mathbf{H}) \le 2$ for every simple right $\mathbf{A}$-module V.

Note that condition 2 must be imposed both on left and on right modules.

It is known that such a hereditary ring $\mathbf{H}$ is Morita equivalent to a direct product of rings $\mathbf{H}(\mathbf{D}, n)$, where $\mathbf{D}$ is a discrete valuation ring (maybe non-commutative) and $\mathbf{H}(\mathbf{D}, n)$ is the subring of $\operatorname{Mat}(n, \mathbf{D})$ consisting of all matrices (a_{ij}) with non-invertible entries a_{ij} for $i < j$. Especially, $\mathbf{H}$ and $\mathbf{A}$ are *semi-prime* (i.e. without nilpotent ideals)

Example 2.2.

1. The first example of a nodal ring is the completion of the local ring of a *simple node* (or a simple double point) of an algebraic curve over a field $\Bbbk$. It is isomorphic to $\mathbf{A} = \Bbbk[[x, y]]/(xy)$ and can be embedded into $\mathbf{H} = \Bbbk[[x_1]] \times \Bbbk[[x_2]]$ as the subring of pairs (f, g) such that $f(0) = g(0)$: x maps to $(x_1, 0)$ and y to $(0, x_2)$. Evidently this embedding satisfies conditions of Definition 2.1.
2. The *dihedral algebra* $\mathbf{A} = \Bbbk\langle\langle x, y \rangle\rangle/(x^2, y^2)$ is another example of a nodal ring. In this case $\mathbf{H} = \mathbf{H}(\Bbbk[[t]], 2)$ and the embedding $\mathbf{A} \to \mathbf{H}$ is given by the rule
$$x \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
3. The "Gelfand problem" is that of classification of diagrams with relations
$$2 \underset{x_-}{\overset{x_+}{\rightleftarrows}} 1 \underset{y_-}{\overset{y_+}{\leftrightarrows}} 3 \qquad x_+x_- = y_+y_-.$$
If we consider the case when x_+x_- is nilpotent (the main part of the problem), such diagrams are just modules over the ring $\mathbf{A}$, which is the subring of $\operatorname{Mat}(3, \Bbbk[[t]])$ consisting of all matrices (a_{ij}) with $a_{12}(0) = a_{13}(0) = a_{23}(0) = a_{32}(0) = 0$. The arrows of the diagram correspond to the following matrices:
$$x_+ \mapsto te_{12}, \quad x_- \mapsto e_{21}, \quad y_+ \mapsto te_{13}, \quad y_- \mapsto e_{31},$$
where e_{ij} are matrix units. It is also a nodal ring with $\mathbf{H}$ being the subring of $\operatorname{Mat}(3, \Bbbk[[t]])$ consisting of all matrices (a_{ij}) with $a_{12}(0) = a_{13}(0) = 0$ (it is Morita equivalent to $\mathbf{H}(\Bbbk[[t]], 2)$).

4. The classification of *quadratic functors*, which play an important role in algebraic topology, reduces to the study of modules over the ring $\mathbf{A}$, which is the subring of $\mathbb{Z}_2^2 \times \mathrm{Mat}(2, \mathbb{Z}_2)$ consisting of all triples

$$\left(a, b, \begin{pmatrix} c_1 & 2c_2 \\ c_3 & c_4 \end{pmatrix} \right) \quad \text{with} \quad a \equiv c_1 (\bmod 2) \text{ and } b \equiv c_4 (\bmod 2),$$

where $\mathbb{Z}_2$ is the ring of p-adic integers [11]. It is again a nodal ring: one can take for $\mathbf{H}$ the ring of all triples as above, but without congruence conditions; then $\mathbf{H} = \mathbb{Z}_2^2 \times \mathbf{H}(\mathbb{Z}_2, 2)$.

Certainly, a nodal ring is always Backström, so Theorem 1.2 can be applied. Moreover, in nodal case the resulting problem belongs to a well-known type. For the sake of simplicity, we consider now the situation, when $\mathbf{A}$ is a $\mathbf{D}$-algebra finitely generated as $\mathbf{D}$-module, where $\mathbf{D}$ is a discrete valuation ring with algebraically closed residue field $\Bbbk$. We denote by $U_1, U_2, \dots, U_s$ indecomposable non-isomorphic projective (left) modules over $\mathbf{A}$ and by $V_1, V_2, \dots, V_r$ those over $\mathbf{H}$. Condition 2 from Definition 2.1 implies that there are three possibilities:

1) $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j$ for some j and V_j does not occur as a direct summand in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \neq i$;
2) $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j \oplus V_{j'}$ $(j \neq j')$ and neither V_j nor $V_{j'}$ occur in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \neq i$;
3) there are exactly two indices $i \neq i'$ such that $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq \mathbf{H} \otimes_{\mathbf{A}} U_{i'} \simeq V_j$ and V_j does not occur in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \notin \{ i, i' \}$.

We denote by H_j the indecomposable projective $\mathbf{H}$-module such that $H_j / \mathbf{R} H_j \simeq V_j$. Since $\mathbf{H}$ is a semi-perfect *hereditary order*, any indecomposable complex from $D(\mathbf{H})$ is isomorphic either to $0 \to H_k \xrightarrow{\phi} H_j \to 0$ or to $0 \to H_j \to 0$ (it follows, for instance, from [9]). Moreover, the former complex is completely defined by either j or k and the length $l = \mathrm{length}_{\mathbf{H}} \operatorname{Coker} \phi$. We shall denote it both by $C(j, -l, n)$ and by $C(k, l, n+1)$, while the latter complex will be denoted by $C(j, \infty, n)$, where n denotes the place of H_j (so the place of H_k is $n+1$). We denote by $\tilde{\mathbb{Z}}$ the set $(\mathbb{Z} \setminus \{ 0 \}) \cup \{ \infty \}$ and consider the ordering $\leq$ on $\tilde{\mathbb{Z}}$, which coincides with the usual ordering separately on positive integers and on negative integers, but $l < \infty < -l$ for any $l \in \mathbb{N}$. Note that for each j the submodules of H_j form a chain with respect to inclusion. It immediately implies the following result.

Lemma 2.3. *There is a homomorphism $C(j, l, n) \to C(j, l', n)$, which is an isomorphism on the n-th components, if and only if $l \leq l'$ in $\tilde{\mathbb{Z}}$. Otherwise the n-th component of any homomorphism $C(j, l, n) \to C(j, l', n)$ is zero modulo $\mathbf{R}$.*

We transfer the ordering from $\tilde{\mathbb{Z}}$ to the set $\mathfrak{E}_{j,n} = \left\{ C(j, l, n) \mid l \in \tilde{\mathbb{Z}} \right\}$, so the latter becomes a chain with respect to this ordering. We also denote by $\mathfrak{F}_{j,n}$ the set $\{ (i, j, n) \mid V_j \text{ is a direct summand of } \mathbf{H} \otimes_{\mathbf{A}} U_i \}$. It has at most two elements. We always consider $\mathfrak{F}_{j,n}$ with trivial ordering. Then a triple $(A_\bullet, B_\bullet, \iota)$ from the category $\mathcal{T}(\mathbf{A})$ is given by homomorphisms $\phi^{ijn}_{jln} : d_{i,j,n} U_i \to r_{j,l,n} V_j$, where $(i, j, n) \in \mathfrak{F}_{jn}$, the left U_i comes from B_n and the right V_j comes from direct summands $r_{j,l,n}C(j,l,n)$ of $A_\bullet$. Note that if both $C(j,-l,n)$ and $C(k,l,n+1)$ correspond to the same complex (then we write $C(j,-l,n)\sim C(k,l,n+1)$), we have $r_{j,-l,n}=r_{k,l,n+1}$. We present ϕ_{jln}^{ijn} by its matrix M_{jln}^{ijn}. Then Lemma 2.3 implies the following

Proposition 2.4. *Two sets of matrices* $\left\{M_{jln}^{ijn}\right\}$ *and* $\left\{N_{jln}^{ijn}\right\}$ *describe isomorphic triples if and only if one of them can be transformed to the other by a sequence of the following "elementary transformations":*

1) *For any given values of* i,n*, simultaneously* $M_{jln}^{ijn}\mapsto M_{jln}^{ijn}S$ *for all* j,l *such that* $(ijn)\in\mathfrak{F}_{j,n}$*, where* S *is an invertible matrix of appropriate size.*
2) *For any given values of* j,l,n*, simultaneously* $M_{jln}^{ijn}\mapsto S'M_{jln}^{ijn}$ *for all* $(i,j,n)\in\mathfrak{F}_{jn}$ *and* $M_{k,-l,n-\operatorname{sgn} l}^{i,k,n-\operatorname{sgn} l}\mapsto S'M_{k,-l,n-\operatorname{sgn} l}^{i,k,n-\operatorname{sgn} l}$ *for all* $(i,k,n-\operatorname{sgn} l)\in\mathfrak{F}_{k,n-\operatorname{sgn} l}$*, where* S' *is an invertible matrix of appropriate size and* $C(j,l,n)\sim C(k,-l,n-\operatorname{sgn} l)$*. If* $l=\infty$*, it just means* $M_{j\infty n}^{ijn}\mapsto SM_{j\infty n}^{ijn}$*.*
3) *For any given values of* $j,l'<l,n$*, simultaneously* $M_{jln}^{ijn}\mapsto M_{jln}^{ijn}+RM_{jl'n}^{ijn}$ *for all* $(i,j,n)\in\mathfrak{F}_{j,n}$*, where* R *is an arbitrary matrix of appropriate size. Note that, unlike the preceding transformation, this one does not touch the matrices* $M_{k,-l,n-\operatorname{sgn} l}^{i,k,n-\operatorname{sgn} l}$ *such that* $C(j,l,n)\sim C(k,-l,n-\operatorname{sgn} l)$*.*

This sequence must contain finitely many transformations for every fixed values of j *and* n*.*

Therefore we obtain *representations of the bunch of semi-chains* $\mathfrak{E}_{jn},\mathfrak{F}_{jn}$ in the sense of [4], so we can deduce from this paper a description of indecomposables in $D(\mathbf{A})$. We arrange it in terms of *strings and bands*, often used in representation theory.

Definition 2.5. 1. We define the *alphabet* $\mathfrak{X}$ as the set $\bigcup_{j,n}(\mathfrak{E}_{j,n}\cup\{(j,n)\})$. We define symmetric relations $\sim$ and $-$ on $\mathfrak{X}$ by the following exhaustive rules:
 (a) $C(j,l,n)-(j,n)$ for all $l\in\mathbb{Z}$;
 (b) $C(j,-l,n)\sim C(k,l,n+1)$ defined as above;
 (c) $(j,n)\sim(k,n)$ $(k\neq j)$ if $V_j\oplus V_k\simeq\mathbf{H}\otimes_{\mathbf{A}}U_i$ for some i;
 (d) $(j,n)\sim(j,n)$ if $V_j\simeq\mathbf{H}\otimes_{\mathbf{A}}U_i\simeq\mathbf{H}\otimes_{\mathbf{A}}U_{i'}$ for some $i'\neq i$.

2. We define an $\mathfrak{X}$-*word* as a sequence $w=x_1r_1x_2r_2x_3\dots r_{m-1}x_m$, where $x_k\in\mathfrak{X},\ r_k\in\{-,\sim\}$ such that
 (a) $x_kr_kx_{k+1}$ in $\mathfrak{X}$ for $1\le k<m$;
 (b) $r_k\neq r_{k+1}$ for $1\le k<m-1$.
 We call x_1 and x_m the *ends of the word* w.
3. We call an $\mathfrak{X}$-word w *full* if
 (a) $r_1=r_{m-1}=-$
 (b) $x_1\not\sim y$ for each $y\neq x_1$;
 (c) $x_m\not\sim z$ for each $z\neq x_m$.
 Condition (a) reflects the fact that ι^R must be an isomorphism, while conditions (b,c) come from generalities on bunches of semi-chains [4].

4. A word w is called *symmetric*, if $w = w^*$, where $w^* = x_m r_{m-1} x_{m-1} \dots r_1 x_1$ (the *inverse word*), and *quasisymmetric*, if there is a shorter word v such that $w = v \sim v^* \sim \dots \sim v^* \sim v$.
5. We call the end x_1 (x_m) of a word w *special* if $x_1 \sim x_1$ and $r_1 = -$ (respectively, $x_m \sim x_m$ and $r_{m-1} = -$). We call a word w
   (a) *usual* if it has no special ends;
   (b) *special* if it has exactly one special end;
   (c) *bispecial* if it has two special ends.
   Note that a special word is never symmetric, a quasisymmetric word is always bispecial, and a bispecial word is always full.
6. We define a *cycle* as a word w such that $r_1 = r_{m-1} =\sim$ and $x_m - x_1$. Such a cycle is called *non-periodic* if it cannot be presented in the form $v - v - \dots - v$ for a shorter cycle v. For a cycle w we set $r_m = -$, $x_{qm+k} = x_k$ and $r_{qm+k} = r_k$ for any $q, k \in \mathbb{Z}$.
7. A (k-th) *shift* of a cycle w, where k is an even integer, is the cycle $w^{[k]} = x_{k+1} r_{k+1} x_{k+2} \dots r_{k-1} x_k$. A cycle w is called *symmetric* if $w^{[k]} = w^*$ for some k.
8. We also consider *infinite words* of the sorts $w = x_1 r_1 x_2 r_2 \dots$ (with one end) and $w = \dots x_0 r_0 x_1 r_1 x_2 r_2 \dots$ (with no ends) with restrictions
   (a) every pair (j, n) occurs in this sequence only finitely many times;
   (b) there is an n_0 such that no pair (j, n) with $n < n_0$ occurs.
   We extend to such infinite words all above notions in the obvious manner.

Definition 2.6 (String and band data)**.** 1. *String data* are defined as follows:
   (a) a *usual string datum* is a full usual non-symmetric $\mathfrak{X}$-word w;
   (b) a *special string datum* is a pair (w, δ), where w is a full special word and $\delta \in \{0, 1\}$;
   (c) a *bispecial string datum* is a quadruple $(w, m, \delta_1, \delta_2)$, where w is a bispecial word that is neither symmetric nor quasisymmetric, $m \in \mathbb{N}$ and $\delta_1, \delta_2 \in \{0, 1\}$.
2. A *band datum* is a triple (w, m, λ), where w is a non-periodic cycle, $m \in \mathbb{N}$ and $\lambda \in \Bbbk^*$; if w is symmetric, we also suppose that $\lambda \neq 1$.

The results of [4, 8] imply

Theorem 2.7. *Every string or band datum* $\mathbf{d}$ *defines an indecomposable object* $C_\bullet(\mathbf{d})$ *from* $D(\mathbf{A})$*, so that*
1) *Every indecomposable object from* $D(\mathbf{A})$ *is isomorphic to* $C_\bullet(\mathbf{d})$ *for some* $\mathbf{d}$.
2) *The only isomorphisms between these complexes are the following:*
   (a) $C(w) \simeq C(w^*)$;
   (b) $C(w, m, \delta_1, \delta_2) \simeq C(w^*, m, \delta_2, \delta_1)$;
   (c) $C(w, m, \lambda) \simeq C(w^{[k]}, m, \lambda) \simeq C(w^{*[k]}, m, 1/\lambda)$ *if* $k \equiv 0 \pmod 4$;
   (d) $C(w^*, m, \lambda) \simeq C(w^{[k]}, m, 1/\lambda) \simeq C(w^{*[k]}, m, \lambda)$ *if* $k \equiv 2 \pmod 4$.
3) *Every object from* $D(\mathbf{A})$ *uniquely decomposes into a direct sum of indecomposable objects.*

The construction of complexes $C_\bullet(\mathbf{d})$ is rather complicated, especially in the case, when there are pairs (j, n) with $(j, n) \sim (j, n)$ (e.g. special ends are involved). So we only show several examples arising from simple node, dihedral algebra and Gelfand problem.

3. Examples

3.1. **Simple node.** In this case there is only one indecomposable projective $\mathbf{A}$-module ($\mathbf{A}$ itself) and two indecomposable projective $\mathbf{H}$-modules H_1, H_2 corresponding to the first and the second direct factors of the ring $\mathbf{H}$. We have $\mathbf{H} \otimes_{\mathbf{A}} \mathbf{A} \simeq \mathbf{H} \simeq H_1 \oplus H_2$. So the $\sim$-relation is given by:

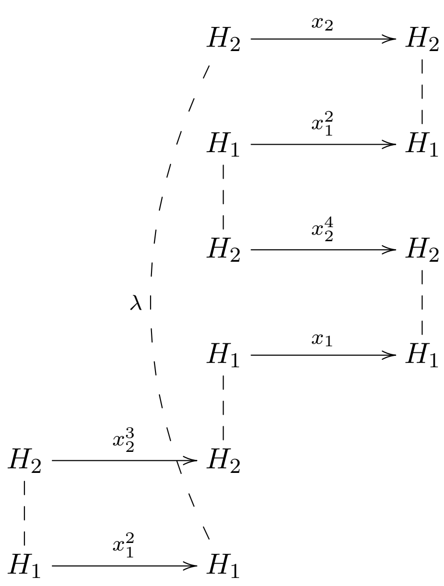
1) $(1,n) \sim (2,n)$;
2) $C(j,l,n) \sim C(j,-l,n-\operatorname{sgn} l)$ for any $l \in \mathbb{Z} \setminus \{0\}$.

Therefore there are no special ends at all. Moreover, any end of a full string must be of the form $C(j,\infty,n)$. Note that the homomorphism in the complex corresponding to $C(j,-l,n)$ and $C(j,l,n+1)$ ($l \in \mathbb{N}$) is just multiplication by x_j^l. Consider several examples of strings and bands.

Example 3.1. 1. Let w be the cycle

$$\begin{aligned} C(2,1,1) \sim C(2,-1,0) - (2,0) \sim (1,0) - C(1,-2,0) \sim C(1,2,1)-\\ - (1,1) \sim (2,1) - C(2,4,1) \sim C(2,-4,0) - (2,0) \sim (1,0)-\\ - C(1,-1,0) \sim C(1,1,1) - (1,1) \sim (2,1) - C(2,-3,1) \sim C(2,3,2)-\\ - (2,2) \sim (1,2) - C(1,2,2) \sim C(1,-2,1) - (1,1) \sim (2,1) \end{aligned}$$

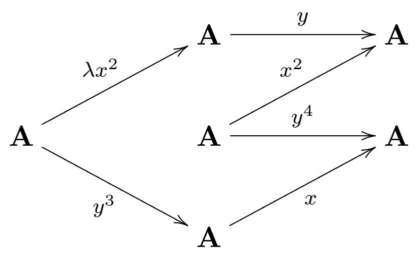
Then the band complex $C_\bullet(w,1,\lambda)$ is obtained from the complex of $\mathbf{H}$-modules



by gluing along the dashed lines (they present the $\sim$ relations $(1,n) \sim (2,n)$). All glueings are trivial, except the last one marked with 'λ';

the latter must be twisted by λ. It gives the $\mathbf{A}$-complex

(3.1)



Here each column presents direct summands of a non-zero component C_n (in our case $n = 2, 1, 0$) and the arrows show the non-zero components of the differential. According to the embedding $\mathbf{A} \to \mathbf{H}$, we have to replace x_1 by x and x_2 by y. Gathering all data, we can rewrite this complex as

$$A \xrightarrow{\begin{pmatrix}\lambda x^2\\ 0\\ y^3\end{pmatrix}} \mathbf{A} \oplus \mathbf{A} \oplus \mathbf{A} \xrightarrow{\begin{pmatrix} y & 0\\ x^2 & y^4\\ 0 & x\end{pmatrix}} \mathbf{A} \oplus \mathbf{A}\,,$$

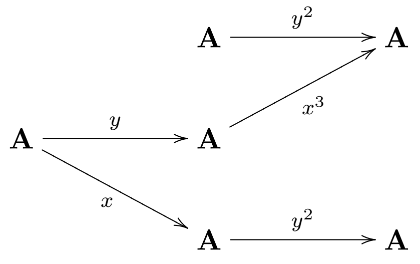
though the form (3.1) seems more expressive, so we use it further. If $m > 1$, one only has to replace $\mathbf{A}$ by $m\mathbf{A}$, each element $a \in \mathbf{A}$ by aE, where E is the identity matrix, and λa by $aJ_m(\lambda)$, where $J_m(\lambda)$ is the Jordan $m \times m$ cell with eigenvalue λ. So we obtain the complex

$$m\mathbf{A} \xrightarrow{\begin{pmatrix} x^2 J_m(\lambda)\\ 0\\ y^3E\end{pmatrix}} m\mathbf{A} \oplus m\mathbf{A} \oplus m\mathbf{A} \xrightarrow{\begin{pmatrix} yE & 0\\ x^2E & y^4E\\ 0 & xE\end{pmatrix}} m\mathbf{A} \oplus m\mathbf{A}\,.$$

2. Let w be the word

$$\begin{aligned} &C(1,\infty,1) - (1,1) \sim (2,1) - C(2,2,1) \sim C(2,-2,0) - (2,0) \sim\\ &\sim (1,0) - C(1,-3,0) \sim C(1,3,1) - (1,1) \sim (2,1) - C(2,-1,1) \sim\\ &\sim C(2,1,2) - (2,2) \sim (1,2) - C(1,1,2) \sim C(1,-1,1) - (1,1) \sim\\ &\sim (2,1) - C(2,2,1) \sim C(2,-2,0) - (2,0) \sim (1,0) - C(1,\infty,0)\end{aligned}$$

Then the string complex $C_\bullet(w)$ is



Note that for string complexes (which are always usual in this case) there are no multiplicities m and all glueings are trivial.

3. Set $a = x + y$. Then the factor $\mathbf{A}/a\mathbf{A}$ is represented by the complex $\mathbf{A} \xrightarrow{a} \mathbf{A}$, which is the band complex $C_\bullet(w, 1, 1)$, where

$$w = C(1,1,1) \sim C(1,-1,0) - (1,0) \sim (2,0) - \\ - C(2,-1,0) \sim C(2,1,1) - (2,1) \sim (1,1).$$

Consider the morphism of this complex to $\mathbf{A}[1]$ given on the 1-component by multiplication $\mathbf{A} \xrightarrow{x} \mathbf{A}$. It is non-zero in $D(\mathbf{A})$, but the corresponding morphism of triples is $(\Phi, 0)$, where Φ arises from the morphism of the complex $\mathbf{H} \xrightarrow{a} \mathbf{H}$ to $\mathbf{H}[1]$ given by multiplication with x_1. But Φ is homotopic to 0: $x_1 = e_1 a$, where $e_1 = (1,0) \in \mathbf{H}$, thus $(\Phi, 0) = 0$ in the category of triples.

4. The string complex $C_\bullet(\mathbf{l}, 0)$, where w is the word

$$C(1,\infty,0) - (1,0) \sim (2,0) - C(2,-1,0) \sim C(2,1,1) - (2,1) \sim \\ \sim (1,1) - C(1,-2,1) \sim C(1,1,2) - (1,2) \sim (2,2) - C(2,-1,2) \sim \\ \sim C(2,1,3) - (2,3) \sim (1,3) - C(1,-2,3) \sim C(1,2,4) - \dots,$$

is

$$\dots \ \mathbf{A} \xrightarrow{x^2} \mathbf{A} \xrightarrow{y} \mathbf{A} \xrightarrow{x^2} \mathbf{A} \xrightarrow{y} \mathbf{A} \longrightarrow 0.$$

Its homologies are not left bounded, so it does not belong to $D^b(\mathbf{A}\text{-mod})$.

3.2. **Dihedral algebra.** This case is very similar to the preceding one. Again there is only one indecomposable projective $\mathbf{A}$-module ($\mathbf{A}$ itself) and two indecomposable projective $\mathbf{H}$-modules H_1, H_2 corresponding to the first and the second columns of matrices from the ring $\mathbf{H}$, and we have $\mathbf{H} \otimes_{\mathbf{A}} \mathbf{A} \simeq \mathbf{H} \simeq H_1 \oplus H_2$. The main difference is that now the unique maximal submodule of H_j is isomorphic to H_k, where $k \neq j$. So the $\sim$-relation is given by:

1) $(1, n) \sim (2, n)$;
2) $C(j, l, n) \sim C(j, -l, n - \operatorname{sgn} l)$ if $l \in \mathbb{Z} \setminus \{0\}$ is even, and $C(j, l, n) \sim C(j', -l, n - \operatorname{sgn} l)$, where $j' \neq j$, if $l \in \mathbb{Z} \setminus \{0\}$ is odd.

Again there are no special ends. The embeddings $H_k \to H_j$ are given by right multiplications with the following elements from $\mathbf{H}$:

$$\begin{aligned} H_1 \to H_1 \ &- \ \text{by} \ \ t^r e_{11} \quad (\text{colength } 2r), \\ H_1 \to H_2 \ &- \ \text{by} \ \ t^r e_{12} \quad (\text{colength } 2r-1), \\ H_2 \to H_1 \ &- \ \text{by} \ \ t^r e_{21} \quad (\text{colength } 2r+1), \\ H_2 \to H_2 \ &- \ \text{by} \ \ t^r e_{22} \quad (\text{colength } 2r). \end{aligned}$$

When gluing $\mathbf{H}$-complexes into $\mathbf{A}$-complexes we have to replace them respectively

$$\begin{aligned} t^r e_{11} \ &- \ \text{by} \ \ (xy)^r, \\ t^r e_{22} \ &- \ \text{by} \ \ (yx)^r, \\ t^r e_{12} \ &- \ \text{by} \ \ (xy)^{r-1}x, \\ t^r e_{21} \ &- \ \text{by} \ \ (yx)^r y. \end{aligned}$$

The glueings are quite analogous to those for simple node, so we only present the results, without further comments.

Example 3.2. 1. Consider the band datum $(w, 1, \lambda)$, where

$$\begin{aligned} w = C(1,-2,0) \sim C(1,2,1) - (1,1) \sim (2,1) - C(2,-5,1) \sim \\ \sim C(1,5,2) - (1,2) \sim (2,2) - C(2,4,2) \sim C(2,-4,1) - (2,1) \sim \\ \sim (1,1) - C(1,3,1) \sim C(2,-3,0) - (2,0) \sim (1,0). \end{aligned}$$

The corresponding complex $C_\bullet(w, m, \lambda)$ is

$$\begin{array}{ccccccc} & & m\mathbf{A} & \xrightarrow{xyE} & m\mathbf{A} \\ & \nearrow^{(xy)^2xE} & & \nearrow^{xyxJ_m(\lambda)} & \\ m\mathbf{A} & \xrightarrow{(yx)^2E} & m\mathbf{A} & & \end{array}$$

2. Let w be the word

$$\begin{aligned} C(2,\infty,0)-(2,0) \sim (1,0)-C(1,-1,0) \sim C(2,1,1)-(2,1) \sim (1,1)-C(1,3,1) \sim \\ \sim C(2,-3,0)-(2,0) \sim (1,0)-C(1,-3,0) \sim C(2,3,1)-(2,1) \sim (1,1)-C(1,\infty,1). \end{aligned}$$

Then the string complex $C_\bullet(w)$ is

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{e_{21}} & \mathbf{A} \\ & \searrow^{t^2e_{12}} & \\ \mathbf{A} & \xrightarrow{te_{21}} & \mathbf{A} \end{array}$$

3. The factor $\mathbf{A}/\mathbf{R}$ is described by the infinite string complex $C_\bullet(w)$

$$\begin{array}{ccccccc} \cdots & \xrightarrow{e_{21}} & \mathbf{A} & \xrightarrow{te_{12}} & \mathbf{A} & \xrightarrow{e_{21}} & \mathbf{A}. \\ & & & & & \nearrow^{te_{12}} & \\ \cdots & \xrightarrow{te_{12}} & \mathbf{A} & \xrightarrow{e_{21}} & \mathbf{A} & & \end{array}$$

The corresponding word w is

$$\begin{aligned} \cdots - C(2,1,2) \sim C(1,-1,1) - (1,1) \sim (2,1)- \\ - C(2,1,1) \sim C(1,-1,0) - (1,0) \sim (2,0) - C(2,-1,0) \sim \\ \sim C(1,1,1) - (1,1) \sim (2,1) - C(2,-1,1) \sim C(1,1,2) - \dots \end{aligned}$$

3.3. **Gelfand problem.** In this case there are 2 indecomposable projective $\mathbf{H}$-modules H_1 (the first column) and H_2 (both the second and the third columns). There are 3 indecomposable $\mathbf{A}$-projectives A_i $(i = 1,2,3)$; A_i correspond to the i-th column of $\mathbf{A}$. We have $\mathbf{H} \otimes_{\mathbf{A}} A_1 \simeq H_1$ and $\mathbf{H} \otimes_{\mathbf{A}} A_2 \simeq \mathbf{H} \otimes_{\mathbf{A}} A_3 \simeq H_2$. So the relation $\sim$ is given by:

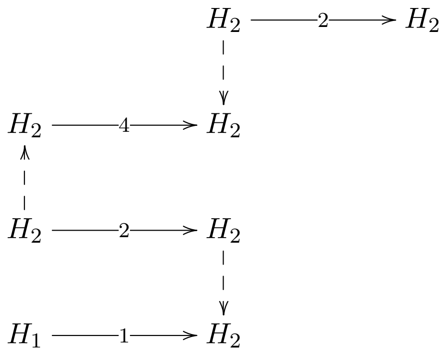
1) $(2,n) \sim (2,n)$;
2) $C(j,l,n) \sim C(j,-l,n-\operatorname{sgn} l)$ if l is even;
3) $C(j,l,n) \sim C(j',-l,n-\operatorname{sgn} l)$ $(j' \neq j)$ if l is odd.

So a special end is always $(2,n)$.

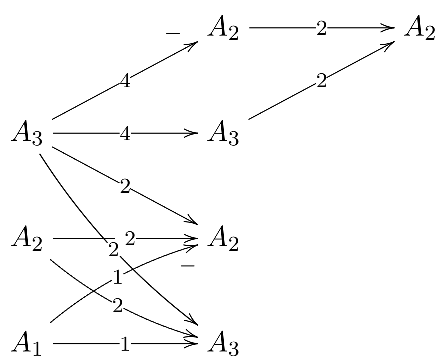
Example 3.3. 1. Consider the special word w:

$$(2,0) - C(2,-2,0) \sim C(2,2,1) - (2,1) \sim (2,1) - C(2,-4,1) \sim \\ \sim C(2,4,2) - (2,2) \sim (2,2) - C(2,2,2) \sim C(2,-2,1) - \\ - (2,1) \sim (2,1) - C(2,-1,1) \sim C(1,1,2) - (1,2)$$

The complex $C_\bullet(w,0)$ is obtained by gluing from the complex of **H**-modules



Here the numbers inside arrows show the colengths of the corresponding images. We mark dashed lines defining glueings with arrows going from the bigger complex (with respect to the ordering in $\mathfrak{E}_{j,n}$) to the smaller one. When we construct the corresponding complex of **A**-modules, we replace each H_2 by A_2 and A_3 starting with A_2 (since $\delta = 0$; if $\delta = 1$ we start from A_3). Each next choice is arbitrary with the only requirement that every dashed line must touch both A_2 and A_3. (Different choices lead to isomorphic complexes: one can see it from the pictures below.) All horizontal mappings must be duplicated by slanting ones, carried along the dashed arrow from the starting point or opposite the dashed arrow with the opposite sign from the ending point (the latter procedure will be marked by '$-$' near the duplicated arrow). So we get the **A**-complex

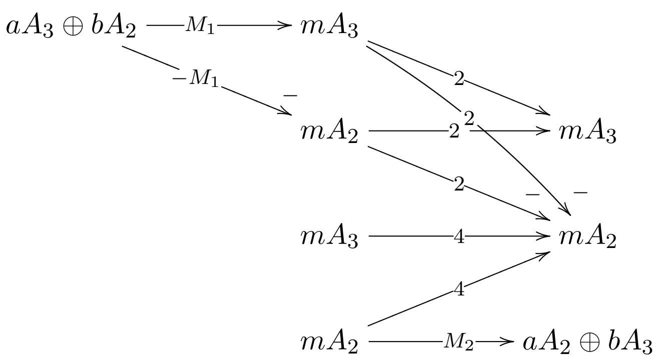


All mappings are uniquely defined by the colengths in the **H**-complex, so we just mark them with 'l.'

2. Let w be the bispecial word

$$\begin{aligned}(2,2)-C(2,2,2)\sim C(2,-2,1)-(2,1)\sim (2,1)-C(2,2,1)\sim\\ \sim C(2,-2,0)-(2,0)\sim (2,0)-C(2,-4,0)\sim C(2,4,1)-\\ -(2,1)\sim(2,1)-C(2,6,1)\sim C(2,-6,0)-(2,0)\end{aligned}$$

The complex $C_\bullet(w,m,1,0)$ is the following one:



where $a=[(m+1)/2]$, $b=[m/2]$, so $a+b=m$. (The change of δ_1,δ_2 transpose A_2 and A_3 at the ends.) All arrows are just $\alpha_l E$, where α_l is defined by the colength l, except of the "end" matrices M_i. To calculate the latter, write $\alpha_l E$ for one of them (say, M_1) and $\alpha_l J$ for anothher one (say, M_2), where J is the Jordan $m\times m$ cell with eigenvalue 1, then put the odd rows or columns into the first part of M_i and the even ones to its second part. In our example we get

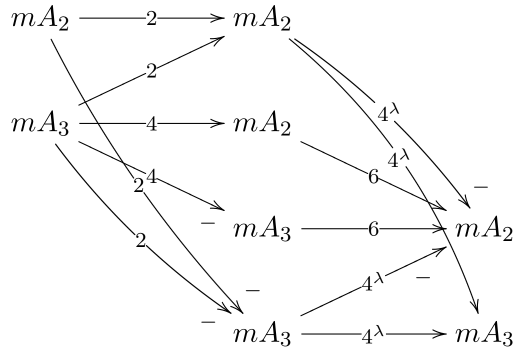
$$M_1=\alpha_2\begin{pmatrix}\begin{array}{ccc|cc}1&0&0&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\end{array}\end{pmatrix},\qquad M_2=\alpha_6\begin{pmatrix}\begin{array}{ccccc}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&1\\\hline 0&1&1&0&0\\0&0&0&1&1\end{array}\end{pmatrix}.$$

(We use columns for M_1 and rows for M_2 since the left end is the source and the right end is the sink of the corresponding mapping.)

3. The band complex $C_\bullet(w,1,\lambda)$, where w is the cycle

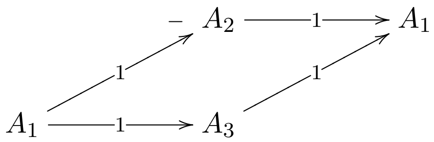
$$\begin{aligned}(2,1)\sim(2,1)-C(2,-2,1)\sim C(2,2,2)-(2,2)\sim(2,2)-\\ -C(2,4,2)\sim C(2,-4,1)-(2,1)\sim(2,1)-C(2,6,1)\sim\\ \sim C(2,-6,0)-(2,0)\sim(2,0)-C(2,-4,0)\sim C(2,4,1)\end{aligned}$$

is



Superscript 'λ' denotes that the corresponding mapping must be twisted by $J_m(\lambda)$.

4. The projective resolution of the simple **A**-module U_1 is



It coincides with the usual string complex $C_\bullet(w)$, where w is

$$(1,0)-C(1,-1,0) \sim C(2,1,1)-(2,1) \sim (2,1)-C(2,-1,1) \sim C(1,1,2)-(1,2).$$

The projective resolution of U_2 (U_3) is $A_1 \to A_2$ (respectively $A_1 \to A_3$), which is the special string complex $C_\bullet(w,0)$ (respectively $C_\bullet(w,1)$), where

$$w = (2,0) - C(2,-1,0) \sim C(1,1,1) - (1,1).$$

Note that $\operatorname{gl.dim} \mathbf{A} = 2$.

4. Projective configurations

We can "globalize" the results of the preceding sections. The simplest way is to consider the so called *projective configurations*, which are a sort of global analogues of Backström rings.

Definition 4.1. Let X be a projective curve over $\Bbbk$, which we suppose reduced, but possibly reducible. We denote by $\pi : \tilde{X} \to X$ its *normalization*; then $\tilde{X}$ is a disjoint union of smooth curves. We call X a *projective configuration* if all components of $\tilde{X}$ are rational curves (i.e. of genus 0) and all singular points p of X are *ordinary*. The latter means that if $\pi^{-1}(p) = \{ y_1, y_2, \dots, y_m \}$, the image of $\mathcal{O}_{X,p}$ in $\prod_{i=1}^m \mathcal{O}_{\tilde{X},y_i}$ contains $\prod_{i=1}^m \mathfrak{m}_i$, where $\mathfrak{m}_i$ is the maximal ideal of $\mathcal{O}_{\tilde{X},y_i}$.

We denote by $S = \{ p_1, p_2, \dots, p_s \}$ the set of singular points of X and by $\tilde{S} = \{ y_1, y_2, \dots, y_r \}$ its preimage in $\tilde{X}$. We also put $\mathcal{O} = \mathcal{O}_X$, $\tilde{\mathcal{O}} = \mathcal{O}_{\tilde{X}}$ and denote by $\mathcal{J}$ the conductor of $\tilde{\mathcal{O}}$ in $\mathcal{O}$, i.e. the maximal sheaf of $\pi_*\tilde{\mathcal{O}}$-ideals contained in $\mathcal{O}$. Set $\mathbf{S} = \mathcal{O}/\mathcal{J}$ and $\mathbf{R} = \pi_*\tilde{\mathcal{O}}/\mathcal{J} \simeq \tilde{\mathcal{O}}/\pi^{-1}\mathcal{J}$. Both these sheaves have 0-dimensional support S, so we may (and shall) identify them with the algebras of their global sections. In the case of projective configurations both these algebras are semi-simple, namely $\mathbf{S} = \prod_{i=1}^{s} \Bbbk(p_i)$ and $\mathbf{R} = \prod_{i=1}^{r} \Bbbk(y_i)$.

Let $D(X) = D^-(\operatorname{Coh} X)$ be the right bounded derived category of coherent sheaves over X. As X is a projective variety, it can be identified with the category of fractions $K^-(\operatorname{VB} X)[Q^{-1}]$, where $K^-(\operatorname{VB} X)$ is the category of right bounded complexes of vector bundles (or, the same, locally free coherent sheaves) over X modulo homotopy and Q is the set of quasi-isomorphisms in $K^-(\operatorname{VB} X)$. So we always present objects from $D(X)$ and from $D(\tilde{X})$ as complexes of vector bundles. We denote by $T : D(X) \to D(\tilde{X})$ the left derived functor $L\pi^*$. Again if $\mathcal{C}_\bullet$ is a complex of vector bundles, $T\mathcal{C}_\bullet$ coincides with $\pi^*\mathcal{C}_\bullet$.

Just as in Section 1, we define the *category of triples* $\mathcal{T} = \mathcal{T}(X)$ as follows:

- Objects of $\mathcal{T}$ are triples $(\mathcal{A}_\bullet, B_\bullet, \iota)$, where
  - $\mathcal{A}_\bullet \in D(\tilde{X})$;
  - $B_\bullet \in D(\mathbf{S})$;
  - ι is a morphism $B_\bullet \to \mathbf{R} \otimes_{\tilde{\mathcal{O}}} \mathcal{A}_\bullet$ from $D(\mathbf{S})$ such that the induced morphism $\iota^R : \mathbf{R} \otimes_{\mathbf{S}} B_\bullet \to \mathbf{R} \otimes_{\tilde{\mathcal{O}}} \mathcal{A}_\bullet$ is an isomorphism in $D(\mathbf{R})$.
- A morphism from a triple $(\mathcal{A}_\bullet, B_\bullet, \iota)$ to a triple $(\mathcal{A}'_\bullet, B'_\bullet, \iota')$ is a pair (Φ, ϕ), where
  - $\Phi : \mathcal{A}_\bullet \to \mathcal{A}'_\bullet$ is a morphism from $D(\tilde{X})$;
  - $\phi : B_\bullet \to B'_\bullet$ is a morphism from $D(\mathbf{S})$;
  - the diagram

$$\begin{array}{ccc} B_\bullet & \xrightarrow{\ \iota\ } & \mathbf{R} \otimes_{\tilde{\mathcal{O}}} \mathcal{A}_\bullet \\ \phi\big\downarrow & & \big\downarrow 1\otimes\Phi \\ B'_\bullet & \xrightarrow{\ \iota'\ } & \mathbf{R} \otimes_{\tilde{\mathcal{O}}} \mathcal{A}'_\bullet \end{array} \tag{4.1}$$

    commutes in $D(\mathbf{S})$.

We define a functor $\mathbf{F} : D(X) \to \mathcal{T}(X)$ setting $\mathbf{F}(\mathcal{C}_\bullet) = (\pi^*\mathcal{C}_\bullet, \mathbf{S} \otimes_{\mathcal{O}} \mathcal{C}_\bullet, \iota)$, where $\iota : \mathbf{S} \otimes_{\mathcal{O}} \mathcal{C}_\bullet \to \mathbf{R} \otimes_{\tilde{\mathcal{O}}} (\pi^*\mathcal{C}_\bullet) \simeq \mathbf{R} \otimes_{\mathcal{O}} \mathcal{C}_\bullet$ is induced by the embedding $\mathbf{S} \to \mathbf{R}$. Just as in Section 1 the following theorem holds (with almost the same proof, see [6]).

Theorem 4.2. *The functor* $\mathbf{F}$ *is a representation equivalence, i.e. it is dense and conservative.*

Remark. We do not now whether it is *full*, though it seems to be true.

5. Configurations of type A and $\tilde{A}$

As it was shown in [13], even classification of vector bundles is wild for almost all projective curves. Among singular curves the only exceptions are projective configurations of type A and $\tilde{A}$. These curves only have ordinary *double* points (so no three components have a common point). Moreover, in A case irreducible components $X_1, X_2, \dots, X_s$ and singular points $p_1, p_2, \dots, p_{s-1}$ can be so arranged that $p_i \in X_i \cap X_{i+1}$, while in $\tilde{A}$ case the components $X_1, X_2, \dots, X_s$ and the singular points $p_1, p_2, \dots, p_s$ can be so arranged that $p_i \in X_i \cap X_{i+1}$ for $i < s$ and $p_s \in X_s \cap X_1$. Note that in A case $s > 1$, while in $\tilde{A}$ case $s = 1$ is possible: then there is one component with one ordinary double point (a nodal plane cubic). These projective configurations are global analogues of nodal rings, and the calculations according Theorem 4.2 are quite similar to those of Section 2. We present here the $\tilde{A}$ case and add remarks explaining which changes should be done for A case.

If $s > 1$, the normalization of X is just a disjoint union $\bigsqcup_{i=1}^{s} X_i$; for uniformity, we write $X_1 = \tilde{X}$ if $s = 1$. We also denote $X_{qs+i} = X_i$. Note that $X_i \simeq \mathbb{P}_1$ for all i. Every singular point p_i has two preimages p_i', p_i'' in $\tilde{X}$; we suppose that $p_i' \in X_i$ corresponds to the point $\infty \in \mathbb{P}^1$ and $p_i'' \in X_{i+1}$ corresponds to the point $0 \in \mathbb{P}^1$. Recall that any indecomposable vector bundle over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d)$ for some $d \in \mathbb{Z}$. So every indecomposable complex from $D(\tilde{X})$ is isomorphic either to $0 \to \mathcal{O}_i(d) \to 0$ or to $0 \to \mathcal{O}_i(-lx) \to \mathcal{O}_i \to 0$, where $\mathcal{O}_i = \mathcal{O}_{X_i}$, $d \in \mathbb{Z}$, $l \in \mathbb{N}$ and $x \in X_i$. The latter complex corresponds to the indecomposable sky-scraper sheaf of length l and support $\{ x \}$. We denote this complex by $C(x, -l, n)$ and by $C(x, l, n+1)$. The complex $0 \to \mathcal{O}_i(d) \to$ is denoted by $C(p_i', d\omega, n)$ and by $C(p_{i-1}'', d\omega, n)$. As before, n is the unique place, where the complex has non-zero homologies. We define the symmetric relation $\sim$ for these symbols setting $C(x, -l, n) \sim C(x, l, n+1)$ and $C(p_i', d\omega, n) \sim C(p_{i-1}'', d\omega, n)$.

Let $\mathbb{Z}^\omega = (\mathbb{Z} \oplus \{ 0 \}) \cup \mathbb{Z}\omega$, where $\mathbb{Z}\omega = \{ d\omega \mid d \in \mathbb{Z} \}$. We introduce an ordering on $\mathbb{Z}^\omega$, which is natural on $\mathbb{N}$, on $-\mathbb{N}$ and on $\mathbb{Z}\omega$, but $l < d\omega < -l$ for each $l \in \mathbb{N}$, $d \in \mathbb{Z}$. Then an analogue of Lemma 2.3 can be easily verified.

Lemma 5.1. *There is a morphism of complexes $C(x, z, n) \to C(x, z', n)$ such that its* nth *component induces a non-zero mapping on $\mathcal{C}_n(x)$ if and only if $z \le z'$ in $\mathbb{Z}^\omega$.*

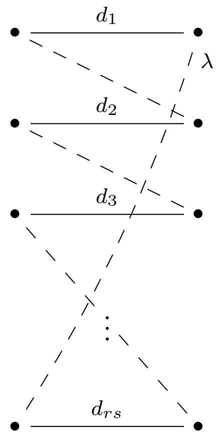
We introduce the ordered sets $\mathfrak{E}_{x,n} = \{ C(x, z, n) \mid z \in \mathbb{Z}^\omega \}$ with the ordering inherited from $\mathbb{Z}^\omega$, We also put $\mathfrak{F}_{x,n} = \{ (x, n) \}$ and $(p_i', n) \sim (p_{i-1}'', n)$ for all i, n. Lemma 5.1 shows that the category of triples $\mathcal{T}(X)$ can be again described in terms of the bunch of chains $\{ \mathfrak{E}_{x,n}, \mathfrak{F}_{x,n} \}$. Thus we can describe indecomposable objects in terms of strings and bands just as for nodal rings. We leave the corresponding definitions to the reader; they are quite analogous to those from Section 2. If we consider a configuration of type A, we have to exclude the points p_s', p_s'' and the corresponding symbols $C(p_s', z, n)$, $C(p_s'', z, n)$, (p_s', n), (p_s'', n). Thus in this case $C(p_{s-1}'', d\omega, n)$ and $C(p_1', d\omega, n)$ are not in $\sim$ relation with any symbol. It makes possible finite or one-side infinite full strings, while in $\tilde{A}$ case only two-side infinite strings are full. Note that an infinite word must contain a finite set of symbols (x, n) with any fixed n; moreover there must be n_0 such that $n \ge n_0$ for all entries (x, n) that occur in this word.

If $x \notin S$ and $z \notin \mathbb{Z}\omega$, the complex $C(x, z, n)$ vanishes after tensoring by $\mathbf{R}$, so gives no essential input into the category of triples. It gives rise to the n-th shift of a sky-scraper sheaf with support at the regular point x. Therefore in the following examples we only consider complexes $C(x, z, n)$ with $x \in S$. Moreover, we confine most examples to the case $s = 1$ (so X is a nodal cubic). If $s > 1$, one must distribute vector bundles in the pictures below among the components of $\tilde{X}$.

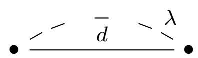
Example 5.2. 1. First of all, even a classification of vector bundles is non-trivial in $\tilde{A}$ case. They correspond to bands concentrated at 0 place, i.e. such that the underlying cycle w is of the form

$$\begin{aligned}(p'_s,0)\sim(p''_s,0)-C(p''_s,d_1\omega,0)\sim C(p'_1,d_1\omega,0)-\\ -(p'_1,0)\sim(p''_1,0)-C(p''_1,d_2\omega,0)\sim C(p'_2,d_2\omega,0)-\\ -(p'_2,0)\sim(p''_2,0)-C(p''_2,d_3\omega,0)\sim\dots\sim C(p'_s,d_{rs}\omega,0)\end{aligned}$$

(obviously, its length must be a multiple of s, and we can start from any place p'_k,p''_k). Then $\mathcal{C}_\bullet(w,m,\lambda)$ is actually a vector bundle, which can be schematically described as the following gluing of vector bundles over $\tilde{X}$.



Here horizontal lines symbolize line bundles over X_i of the superscripted degrees, their left (right) ends are basic elements of these bundles at the point ∞ (respectively 0), and the dashed lines show which of them must be glued. One must take m copies of each vector bundle from this picture and make all glueings trivial, except one going from the uppermost right point to the lowermost left one (marked by 'λ'), where the gluing must be performed using the Jordan $m\times m$ cell with eigenvalue λ. In other words, if $e_1,e_2,\dots,e_m$ and $f_1,f_2,\dots,f_m$ are bases of the corresponding spaces, one has to identify f_1 with λe_1 and f_k with λe_k+e_{k-1} if $k>1$. We denote this vector bundle over X by $\mathcal{V}(\mathbf{d},m,\lambda)$, where $\mathbf{d}=(d_1,d_2,\dots,d_{rs})$; it is of rank mr and of degree $m\sum_{i=1}^r d_i$. If $r=s=1$, this picture becomes



If $r=m=1$, we obtain all line bundles: they are $\mathcal{V}((d_1,d_2,\dots,d_s),1,\lambda)$ (of degree $\sum_{i=1}^s d_i$). Thus the Picard group is $\mathbb{Z}^s\times\Bbbk^*$.

In A case there are no bands concentrated at 0 place, but there are finite strings of this sort:

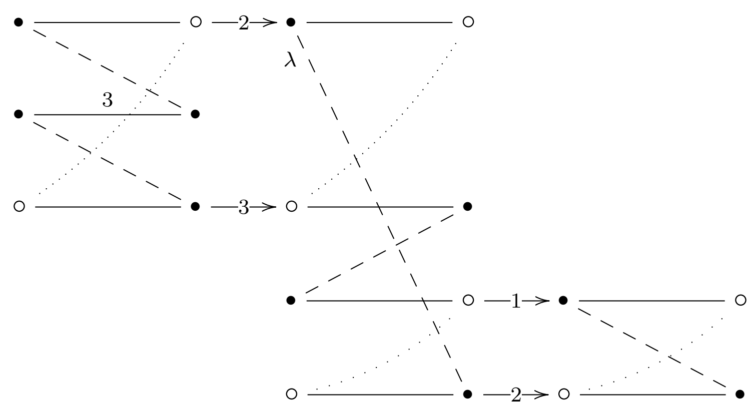
$$\begin{aligned}C(p''_1,d_1\omega,0)-(p'_1,0)\sim(p''_1,0)-C(p''_1,d_2\omega,0)\sim\\ \sim C(p'_2,d_2,0)-(p'_2,0)\sim(p''_2,0)-C(p''_2,d_3,0)\sim\\ \dots\sim C(p'_{s-1},d_{s-1}\omega,0)-(p'_{s-1},0)\sim(p''_{s-1},0)-C(p''_{s-1},d_s\omega,0)\end{aligned}$$

So vector bundles over such configurations are in one-to-one correspondence with integral vectors $(d_1, d_2, \dots, d_s)$; in particular, all of them are line bundles and the Picard group is $\mathbb{Z}^s$. In the picture above one has to set $r = 1$ and to omit the last gluing (marked with 'λ').

2. From now on $s = 1$, so we write p instead of p_1. Let w be the cycle

$$\begin{aligned}(p'', 1) \sim (p', 1) - C(p', -2, 1) \sim C(p', 2, 2) - (p', 2) \sim (p'', 2)-\\ - C(p'', 3\omega, 2) \sim C(p', 3\omega, 2) - (p', 2) \sim (p'', 2) - C(p'', 3, 2) \sim\\ \sim C(p'', -3, 1) - (p'', 1) \sim (p', 1) - C(p', 1, 1) \sim C(p', -1, 0)-\\ - (p', 0) \sim (p'', 0) - C(p'', -2, 0) \sim C(p'', 2, 1).\end{aligned}$$

Then the band complex $\mathcal{C}_\bullet(w, m, \lambda)$ can be pictured as follows:

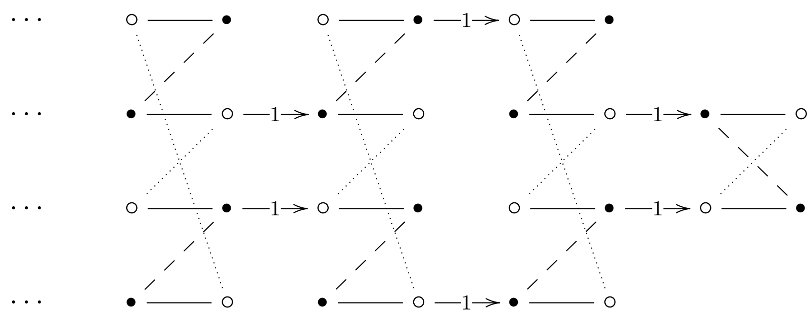


Again horizontal lines describe vector bundles over $\tilde{X}$. Bullets and circles correspond to the points ∞ and 0; circles show those points, where the corresponding complex gives no input into $\mathbf{R} \otimes_{\tilde{\mathcal{O}}} \mathcal{A}_\bullet$. Horizontal arrows show morphisms in $\mathcal{A}_\bullet$; the numbers l inside give the lengths of factors. Dashed and dotted lines describe glueings. Dashed lines (between bullets) correspond to mandatory glueings arising from relations $(p', n) \sim (p'', n)$ in the word w, while dotted lines (between circles) can be drawn arbitrarily; the only conditions are that each circle must be an end of a dotted line and the dotted lines between circles sitting at the same level must be parallel (in our picture they are between the 1st and 3rd levels and between the 4th and 5th levels). The degrees of line bundles in complexes $C(x, z, n)$ with $z \in \mathbb{N} \cup (-\mathbb{N})$ (they are described by the levels containing 2 lines) can be chosen as $d - l$ and d with arbitrary d (we set $d = 0$), otherwise (in the second row) they are superscripted over the line. Thus the resulting complex is

$$\mathcal{V}((-2, 3, -3), m, 1) \longrightarrow \mathcal{V}((0, 0, -1, -2), m, \lambda) \longrightarrow \mathcal{V}((0, 0), m, 1)$$

(we do not precise mappings, but they can be easily restored).

3. If $s = 1$, the sky-scraper sheaf $\Bbbk(p)$ is described by the complex



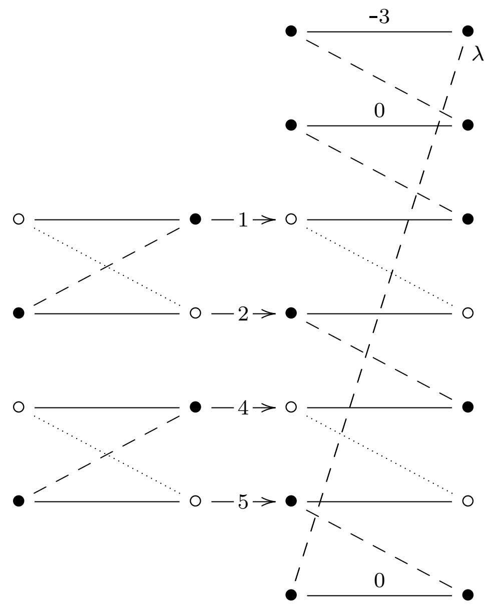
which is the string complex corresponding to the word

$$\dots C(p',-1,2) - (p',2) \sim (p'',2) - C(p'',1,2) \sim C(p'',-1,1) - \\ - (p'',1) \sim (p',1) - C(p',1,1) \sim C(p',-1,0) - (p',0) \sim \\ \sim (p'',0) - C(p'',-1,0) \sim C(p'',1,1) - (p'',1) \sim (p',1) - \\ - C(p',-1,1) \sim C(p',1,2) - (p',2) \sim (p'',2) - C(p'',-1,2) \dots$$

4. The band complex $\mathcal{C}(w, m, \lambda)$, where w is the cycle

$$(p',0) \sim (p'',0) - C(p'',-3\omega,0) \sim C(p',-3\omega,0) - \\ - (p',0) \sim (p'',0) - C(p'',0\omega,0) \sim C(p',0\omega,0) - (p',0) \sim \\ \sim (p'',0) - C(p'',-1,0) \sim C(p'',1,1) - (p'',1) \sim (p',1) - \\ - C(p'',2,1) \sim C(p',-2,0) - (p',0) \sim (p'',0) - C(p'',-4,0) \sim \\ \sim C(p'',4,1) - (p'',1) \sim (p',1) - C(p',5,1) \sim C(p',-5,0) - \\ - (p',0) \sim (p'',0) - C(p'',0\omega,0) \sim C(p',0\omega,0)$$

describes the complex

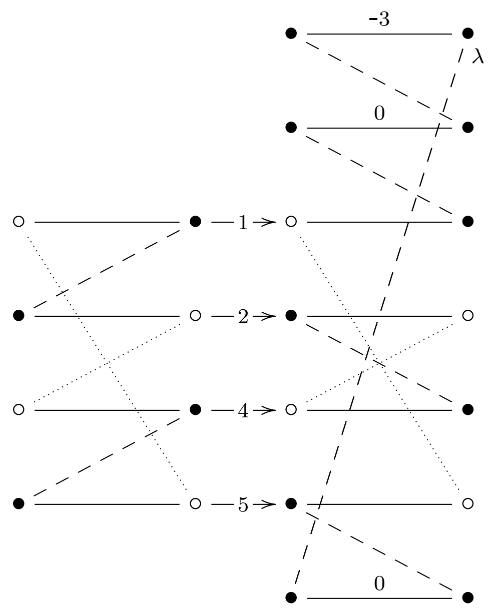


or

$$\mathcal{V}((0,0),m,1) \oplus \mathcal{V}((0,0),m,1) \longrightarrow \mathcal{V}((-3,0,1,2,4,5,0),m,\lambda).$$

Its homologies are zero except the place 0, so it correspond to a coherent sheaf. One can see that this sheaf is a "mixed" one (neither torsion free nor sky-scraper).

Note that this time we could trace dotted lines another way, joining the first free end with the last one and the second with the third.



It gives an isomorphic object in $D(X)$

$$\mathcal{V}((0,0,0,0),m,1) \longrightarrow \mathcal{V}((\text{-}3,0,1,5,0),m,\lambda) \oplus \mathcal{V}((2,4),m,1).$$

Remark. In [6] we used another encoding of strings and bands for projective configurations, which is equivalent, but uses more specifics of the situation. In this paper we prefer to use a uniform encoding, which is the same both for nodal rings and for projective configurations.

6. Application: Cohen–Macaulay modules over surface singularities

The results on vector bundles over projective configurations can be applied to study Cohen–Macaulay modules over normal surface singularities. Recall some related notions. Let $\mathbf{A}$ be a noetherian local complete domain of Krull dimension 2, which is *normal* (i.e. integrally closed in its field of fractions), $X = \operatorname{Spec} \mathbf{A}$ and o be the unique closed points of X (corresponding to the maximal ideal $\mathfrak{m}$ of $\mathbf{A}$). We call $\mathbf{A}$ or X a *normal surface singularity.* A *resolution* of this singularity is a morphism of schemes $\pi : Y \to X$ such that

- Y is smooth;
- π is projective and birational;
- the restriction of π onto $\check{Y} = Y \setminus \pi^{-1}(o)$ is an isomorphism $\check{Y} \to \check{X} = X \setminus \{ o \}$.

We denote by $E = \pi^{-1}(o)_{\mathrm{red}}$ and call it the *exceptional curve* of the resolution. It is indeed a projective curve. Let $E_1, E_2, \dots, E_s$ be its irreducible components. We call *effective cycles* non-zero divisors on Y of the form $Z = \sum_{i=1}^{s} k_i E_i$ with $k_i \geq 0$ and consider such a cycle as a projective curve (non-reduced if some $k_i > 1$), namely the subscheme of Y defined by the sheaf of ideals $\mathcal{O}_Y(-Z)$. Obviously $Z_{\mathrm{red}} = \bigcup_{k_i > 0} E_i$. In [17] C. Kahn established a one-to-one correspondence between Cohen–Macaulay modules over $\mathbf{A}$ and some vector bundles over a special effective cycle Z, called a *reduction cycle.* We shall not present here his result in full generality, but only in the case, when the singularity is *minimally elliptic*, which means, by definition, that $\mathbf{A}$ is Gorenstein and $\dim_{\Bbbk} \mathrm{H}^1(Y, \mathcal{O}_Y) = 1$ [19]. We also suppose that the resolution $\pi : Y \to X$ is *minimal*, i.e. cannot be factored through any other non-isomorphic resolution. Then Kahn's result can be stated as follows

Theorem 6.1 ([17])**.** *Let $\mathbf{A}$ be a minimally elliptic surface singularity and Z be the* fundamental cycle *of its minimal resolution, i.e. the smallest effective cycle such that $(Z.E_i) \leq 0$ for all i. There is one-to-one correspondence between Cohen–Macaulay modules over $\mathbf{A}$ and vector bundles $\mathcal{F}$ over Z such that $\mathcal{F} \simeq \mathcal{G} \oplus n\mathcal{O}_Z$, where*

1) *$\mathcal{G}$ is* generically spanned*, i.e. global sections from $\Gamma(E, \mathcal{G})$ generate $\mathcal{G}$ everywhere, except maybe finitely many closed points;*
2) $\mathrm{H}^1(E, \mathcal{G}) = 0$*;*
3) $n \geq \dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{G}(Z))$.

Especially, indecomposable Cohen–Macaulay $\mathbf{A}$-modules correspond to vector bundles $\mathcal{F} \simeq \mathcal{G} \oplus n\mathcal{O}_Z$, where either $\mathcal{G} = 0$, $n = 1$ or $\mathcal{G}$ is indecomposable, satisfies the above conditions (a,b) *and $n = \dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{G}(Z))$. (The vector bundle $\mathcal{O}_Z$ corresponds to the regular $\mathbf{A}$-module, i.e. $\mathbf{A}$ itself.)*

Kahn himself deduced from this theorem and the results of Atiyah [1] a description of Cohen–Macaulay modules over *simple elliptic* singularities, i.e. such that E is an elliptic curve (smooth curve of genus 1). Using the results of Section 5, one can obtain an analogous description for *cusp singularities*, i.e. such that E is a projective configuration of type $\tilde{A}$. Briefly, one gets the following theorem (for more details see [14]).

Theorem 6.2. *There is a one-to-one correspondence between indecomposable Cohen–Macaulay modules over a cusp singularity* $\mathbf{A}$*, except the regular module* $\mathbf{A}$*, and vector bundles* $\mathcal{V}(\mathbf{d}, m, \lambda)$*, where* $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$ *satisfies the following conditions*[2]*:*

- $\mathbf{d} > \mathbf{0}$*, i.e.* $d_i \geq 0$ *for all* i *and* $\mathbf{d} \neq (0, 0, \dots, 0)$*;*
- *no shift of* $\mathbf{d}$*, i.e. a sequence* $(d_{k+1}, \dots, d_{rs}, d_1, \dots, d_k)$*, contains a subsequence* $(0, 1, 1, \dots, 1, 0)$*, in particular* $(0, 0)$*;*
- *no shift of* $\mathbf{d}$ *is of the form* $(0, 1, 1, \dots, 1)$.

Moreover, from Theorem 6.1 and the results of [13] one gets the following

Theorem 6.3 ([14])**.** *If a minimally elliptic singularity* $\mathbf{A}$ *is neither simple elliptic nor cusp, it is Cohen–Macaulay wild, i.e. the classification of Cohen–Macaulay* $\mathbf{A}$*-modules includes the classification of representations of all finitely generated* $\Bbbk$*-algebras.*

As a consequence of Theorem 6.2 and the Knörrer periodicity theorem [18, 20], one also obtains a description of Cohen–Macaulay modules over hypersurface singularities of type T_{pqr}, i.e. factor-rings

$$\Bbbk[[x_1, x_2, \dots, x_n]]/(x_1^p + x_2^q + x_3^r + \lambda x_1 x_2 x_3 + Q) \quad (n \geq 3,\ 1/p + 1/q + 1/r \leq 1),$$

where Q is a non-degenerate quadratic form of $x_4, \dots, x_n$, and over curve singularities of type T_{pq}, i.e. factor-rings

$$\Bbbk[[x, y]]/(x^p + y^q + \lambda x^2 y^2) \quad (1/p + 1/q \leq 1/2).$$

The latter fills up a flaw in the result of [12], where one has only proved that the curve singularities of type T_{pq} are Cohen–Macaulay tame, but got no explicit description of modules.

Recall that a normal surface singularity $\mathbf{A}$ is Cohen–Macaulay finite, i.e. has only a finite number of non-isomorphic indecomposable Cohen–Macaulay modules, if and only if it is a *quotient singularity*, i.e. $\mathbf{A} \simeq \Bbbk[[x, y]]^G$, where G is a finite group of automorphisms [2, 15]. Just in the same way one can show that all singularities of the form $\mathbf{A} = \mathbf{B}^G$, where $\mathbf{B}$ is either simple elliptic or cusp, are Cohen–Macaulay tame, and obtain a description of Cohen–Macaulay modules in this case. We call such singularities *elliptic-quotient*. There is an evidence that all other singularities are Cohen–Macaulay wild, so Table 1 completely describes Cohen–Macaulay types of isolated singularities (we mark by '?' the places, where the result is still a conjecture).

[2]There was a mistake in the preprint [14], where we claimed that $\mathbf{d} > \mathbf{0}$ is enough for $\mathcal{V}(\mathbf{d}, m, \lambda)$ to satisfy Kahn's conditions. It has been improved in the final version. We are thankful to Igor Burban who has noticed this mistake.

Table 1.

Cohen–Macaulay types of singularities

CM type	curves	surfaces	hypersurfaces
finite	dominate A-D-E	quotient	simple (A-D-E)
tame	dominate T_{pq}	elliptic-quotient (only ?)	T_{pqr} (only ?)
wild	all other	all other ?	all other ?

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Kyiv Taras Shevchenko University, University of Kaiserslautern and Institute of Mathematics of the National Academy of Sciences of Ukraine
E-mail address: burban@mathematik.uni-kl.de
E-mail address: yuriy@drozd.org