VECTOR BUNDLES ON SINGULAR PROJECTIVE CURVES

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Abstract. In this survey article we report on recent results known for vector bundles on singular projective curves (see (Drozd and Greuel; Drozd, Greuel and Kashuba; Yudin). We recall the description of vector bundles on tame and finite configurations of projective lines using the combinatorics of matrix problems. We also show that this combinatorics allows us to compute the cohomology groups of a vector bundle, the dual bundle of a vector bundle, the tensor product of two vector bundles, the dimension of the homomorphism spaces between two vector bundles, and finally to classify simple vector bundles.

Key words: tame and wild representation type, matrix problems, vector bundles on curves.

Mathematics Subject Classification (2000): 14H60, 16G60, 16G50.

1. Introduction

Let X be a projective curve over an algebraically closed field k. For any two coherent sheaves (in particular vector bundles) $\mathcal{E}$ and $\mathcal{F}$ we have

$$\dim_k(\mathrm{Hom}(\mathcal{E}, \mathcal{F})) < \infty.$$

This implies that in the category of vector bundles the generalized Krull-Schmidt theorem holds:

$$\mathcal{E} \cong \bigoplus_{i=1}^{s} \mathcal{F}_i^{m_i},$$

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where the vector bundles $\mathcal{F}_i$ are indecomposable and m_i, $\mathcal{F}_i$ are uniquely determined.
Our aim is to describe all indecomposable vector bundles on X.

What is known about the classification of indecomposable vector bundles on smooth projective curves?

1. Let $X = \mathbb{P}^1_k$. Then indecomposable vector bundles are just the line bundles $\mathcal{O}_{\mathbb{P}^1}(n), n \in \mathbb{Z}$ (Grothendieck).
2. Let X be an elliptic curve. The indecomposable vector bundles are described by two discrete parameters r, d, rank and degree and one continuous parameter (point of the curve X), see (Atiyah).
3. It is well-known that with the growth of the genus g of the curve the moduli spaces of vector bundles become bigger and bigger. For smooth curves of genus $g \geq 2$ it was shown (Drozd and Greuel; Scharlau) that the classification problem of vector bundles is wild. "Wild" means

   a) "geometrically": we have n-parameter families of indecomposable non-isomorphic vector bundles for arbitrary large n;
   b) "algebraically": for every finite-dimensional k-algebra Λ there is an exact functor $(\Lambda\text{-mod}) \longrightarrow \mathrm{VB}_X$ from the category of Λ-modules to the category of vector bundles on X mapping non-isomorphic objects to non-isomorphic and indecomposable to indecomposable ones

   Note that the implication (b) $\Longrightarrow$ (a) is easy, while the equivalence of (a) and (b) appeared to be hard, see (Drozd and Greuel).

   Moreover, in (Drozd and Greuel) the following trichotomy was proved:

1. The category VB_X is finite (indecomposable objects are described by discrete parameters) if X is a configuration of projective lines of type A_n (in this case indecomposable vector bundles are just line bundles)

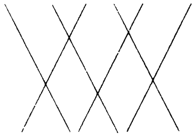

Figure 1.

2. VB_X is tame (intuitively this means that indecomposable objects are parametrized by one continuous parameter and several discrete parameters, see (Drozd and Greuel) for a precise definition) if

   a) X is an elliptic curve

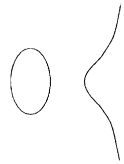

Figure 2.

b) X is a rational curve with one simple node

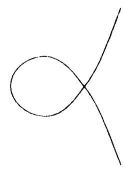

Figure 3.

c) X is a configuration of projective lines of type $\tilde{A}_n$

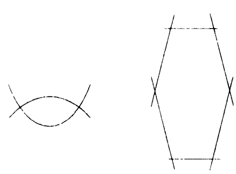

Figure 4.

3. VB_X is wild in all other cases.

2. Category of triples

Let X be a singular curve, $\tilde{X} \xrightarrow{\pi} X$ its normalization, $\tilde{\mathcal{O}} = \pi_*(\mathcal{O}_{\tilde{X}})$. Since π is affine, the categories $\mathrm{Coh}_{\tilde{X}}$ and $\mathrm{Coh}(\tilde{\mathcal{O}}-\mathrm{mod})$ are equivalent, see (Hartshorne).

Let $\mathcal{J} = \mathrm{Ann}_{\mathcal{O}}(\tilde{\mathcal{O}}/\mathcal{O})$ be the conductor. It is the biggest common ideal sheaf of the sheaves of the rings $\mathcal{O}$ and $\tilde{\mathcal{O}}$ such that $\mathcal{J}\mathcal{O} = \mathcal{J}\tilde{\mathcal{O}}$. The usual way to deal with vector bundles on a singular curve is to lift them up to the normalization, and then work on the smooth curve. Surely we lose some information, since non-isomorphic vector bundles can have isomorphic inverse images. To avoid this problem we introduce the following definition:

DEFINITION 2.1. The category of triples, *denoted by* T_X, *is defined as follows*

1. *objects are the triples* $(\tilde{\mathcal{F}}, \mathcal{M}, i)$, *where*
   $\tilde{\mathcal{F}}$ *is a locally free* $\tilde{\mathcal{O}}$*-module,*
   $\mathcal{M}$ *is a locally free* $\mathcal{O}/\mathcal{J}$*-module and*

$i : \mathcal{M} \longrightarrow \tilde{\mathcal{F}} \otimes_{\tilde{O}} \tilde{O}/\mathcal{J}$ is an inclusion of $O/\mathcal{J}$-modules, which induces an isomorphism
$\tilde{i} : \mathcal{M} \otimes_{O} \tilde{O}/\mathcal{J} \longrightarrow \tilde{\mathcal{F}}/\mathcal{J}\tilde{\mathcal{F}}$.

2. *morphisms $(\tilde{\mathcal{F}}_1, \mathcal{M}_1, i_1) \xrightarrow{(\Phi,\varphi)} (\tilde{\mathcal{F}}_2, \mathcal{M}_1, i_2)$ are the pairs (Φ,φ), with*
   $\tilde{\mathcal{F}}_1 \xrightarrow{\Phi} \tilde{\mathcal{F}}_2$ a morphism of $\tilde{O}$-modules,
   $\mathcal{M}_1 \xrightarrow{\varphi} \mathcal{M}_2$ a morphism of $O/\mathcal{J}$-modules, such that the following diagram is commutative

$$\begin{array}{ccccc} \tilde{\mathcal{F}}_1 & \longrightarrow & \tilde{\mathcal{F}}_1 \otimes_{\tilde{O}} \tilde{O}/\mathcal{J} & \xleftarrow{i_1} & \mathcal{M}_1 \\ \downarrow \Phi & & \downarrow \bar{\Phi} & & \downarrow \varphi \\ \tilde{\mathcal{F}}_2 & \longrightarrow & \tilde{\mathcal{F}}_2 \otimes_{\tilde{O}} \tilde{O}/\mathcal{J} & \xleftarrow[i_2]{} & \mathcal{M}_2 . \end{array}$$

Now we formulate the main theorem of this section

THEOREM 2.2. *The functor*

$$\mathrm{VB}_X \xrightarrow{\mathbf{F}} \mathrm{T}_X$$
$$\mathcal{F} \longrightarrow (\tilde{\mathcal{F}}, \mathcal{M}, i),$$

where $\tilde{\mathcal{F}} := \mathcal{F}_{\otimes_O}\tilde{O}$, $\mathcal{M} := \mathcal{F}/\mathcal{J}\mathcal{F}$, and $i : \mathcal{F}/\mathcal{J}\mathcal{F} \longrightarrow \mathcal{F} \otimes_O \tilde{O}/\mathcal{J}(\mathcal{F} \otimes_O \tilde{O})$ is an equivalence of categories. $\tilde{\mathcal{F}}$ is called the normalization *of $\mathcal{F}$.*

PROOF. We construct the quasi-inverse functor

$$\mathrm{T}_X \xrightarrow{\mathbf{G}} \mathrm{VB}_X$$

as follows. Let $(\tilde{\mathcal{F}}, \mathcal{M}, i)$ be some triple. Consider the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\ & & \downarrow id & & \downarrow & & \downarrow i & & \\ 0 & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}} & \longrightarrow & \tilde{\mathcal{F}} & \xrightarrow{\pi} & \tilde{\mathcal{F}}/\mathcal{J}\tilde{\mathcal{F}} & \longrightarrow & 0 \end{array}$$

in the category of O-modules. Since the pull-back is functorial, we get a functor $\mathrm{T}_X \xrightarrow{\mathbf{G}} \mathrm{Coh}_X$. One has to show that

1. the pull-back of π and i is a vector bundle
2. the functors Φ and Ψ are quasi-inverse.

We refer to (Drozd and Greuel) for details of the proof. □

3. Vector bundles on a rational curve with one node

Since the idea to reduce the classification of vector bundles on singular curves to a matrix problem seems to be new and since the technique of matrix problems appear to be unfamiliar to algebraic geometers and, moreover, since the general procedure is not so easy to understand, we consider in this section a rather simple case. We treat it in some detail in order to clarify the ideas.

Let X be a plane curve, given by the equation $zy^2 - x^3 - zx^2 = 0$. Then its normalization is $\mathbb{P}^1 = \tilde{X} \xrightarrow{\pi} X$. Without loss of generality we may suppose that the pre-images of the singular point are $0 = (0:1)$ and $\infty = (1:0)$.

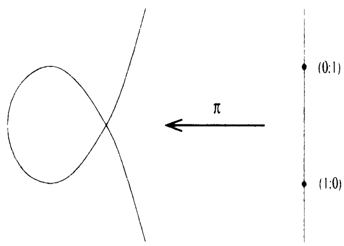

Figure 5.

What does the result of the previous section mean? A vector bundle $\mathcal{F}$ on the curve X is uniquely determined by some triple $(\tilde{\mathcal{F}}, \mathcal{M}, i)$. $\tilde{\mathcal{F}}$ is a locally free $\tilde{O}$-module, or equivalently, a locally free $O_{\mathbb{P}^1}$-module. By the theorem of Grothendieck $\tilde{\mathcal{F}} \cong \bigoplus_{n \in Z} \tilde{O}(n)^{m_n}$.

Since $O/\mathcal{J} = k$, $\mathcal{M}_0$ is nothing but a k-vector space and $i : \mathcal{M}_0 \longrightarrow (\tilde{\mathcal{F}}/\mathcal{J})_0$ can be viewed as a k-linear map. But $(\tilde{\mathcal{F}}/\mathcal{J})_0$ is a $k \times k$-module, hence the map i is given by two matrices $i(0:1)$ and $i(1:0)$. The canonical map $(O/\mathcal{J})_0 \longrightarrow (\tilde{O}/\mathcal{J})_0$ is the diagonal map $k \longrightarrow k \times k$, so the condition that $\tilde{i} : \mathcal{M} \otimes_{O/\mathcal{J}} \tilde{O}/\mathcal{J} \longrightarrow \tilde{\mathcal{F}}/\mathcal{J}$ is an isomorphism means that both matrices $i(0:1)$ and $i(1:0)$ are invertible square matrices.

Fix a direct sum decomposition $\tilde{\mathcal{F}} \cong \bigoplus_{n \in Z} \tilde{O}(n)^{m_n}$ and choose trivializations of $\tilde{\mathcal{F}}$ at the points 0 and ∞. They induce a basis of the $k \times k$-module $\tilde{\mathcal{F}}/\mathcal{J}\tilde{\mathcal{F}}$. Choose also a basis of $\mathcal{M}_0$. With respect to these choices i is given by some matrix, divided into horizontal blocks, as in Figure 6.

Now we have to answer the main question: when do two triples $(\tilde{\mathcal{F}}_1, \mathcal{M}_1, i_1)$ and $(\tilde{\mathcal{F}}_2, \mathcal{M}_2, i_2)$ define isomorphic vector bundles? Surely, we have to require $\tilde{\mathcal{F}}_1 \cong \tilde{\mathcal{F}}_2$, $\mathcal{M}_1 \cong \mathcal{M}_2$. But what condition should be satisfied by the matrices defining i_1 and i_2 in order to give isomorphic vector bundles? The answer follows from the definition of the morphism in the category of triples. Namely, there should be isomorphisms $\Phi : \tilde{\mathcal{F}}_1 \longrightarrow \tilde{\mathcal{F}}_2$, $\varphi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ such that $\bar{\Phi} i_1 = i_2 \varphi$.

Let $\tilde{\mathcal{F}} = \bigoplus_{n \in Z} \tilde{O}(n)^{m_n}$. An endomorphism of $\tilde{\mathcal{F}}$ can be written in a matrix form: $\Phi = (\Phi)_{ij}$, where Φ_{ij} is a $n_j \times n_i$-matrix with coefficients in the vector space $\mathrm{Hom}(\tilde{O}(j), \tilde{O}(i))$. But since $\mathrm{Hom}(\tilde{O}(i), \tilde{O}(j)) = k[x_0, x_1]_{i-j}$, our matrix is lower

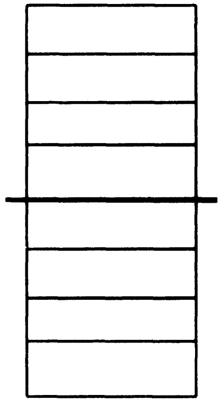

Figure 6.

triangular. Moreover, the diagonal $n_i \times n_i$-matrices Φ_{ii} are just matrices over k. Φ is an isomorphism if and only if all Φ_{ii} are invertible. What is a map $\bar{\Phi} : \tilde{\mathcal{F}}/\mathcal{J} \longrightarrow \tilde{\mathcal{F}}/\mathcal{J}$? Let $N = \operatorname{rank}(\mathcal{F})$. Then $\bar{\Phi} : k^{2N} \longrightarrow k^{2N}$ is given by the diagonal block matrix $\operatorname{diag}(\Phi(0:1), \Phi(1:0))$. Note, that the matrices $\Phi_{ij}(1:0)$ and $\Phi_{ij}(0:1)$, $(i > j)$ can be *arbitrary*. As a result we get a matrix problem:

We have two matrices $i(1:0)$ and $i(0:1)$. We require them to be square and nondegenerate. Each of them is divided into horizontal blocks labeled by integer numbers (they are called sometimes weights). Blocks of $i(0:1)$ and $i(1:0)$, labeled by the same integer, have to have the same size. We can perform the following transformations:

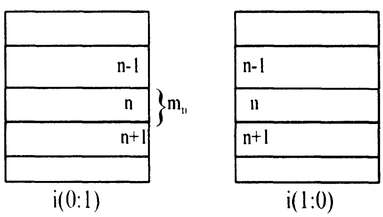

Figure 7.

1. We can simultaneously do any elementary transformations of columns of $i(0:1)$ and $i(1:0)$.
2. We can simultaneously do any invertible elementary transformations of rows inside of the conjugated horisontal blocks.
3. We can independentely add in each of the matrices $i(0:1)$ and $i(1:0)$ a scalar multiple of any row with lower weight to any row with higher weight.

These types of matrix problems are well-known in representation theory. First they appeared in the work of Nazarova and Roiter (1969) about the classification of $k[[x,y]]/(xy)$-modules. They are sometimes called Gelfand problems in honour of I. M. Gelfand, who formulated a conjecture at the International Congress of Mathematics in Nice (1970) about the structure of Harish-Chandra modules at the singular point of $SL_2(\mathbb{R})$. This problem was reduced to some matrix problem of this type in (Nazarova and Roiter, 1973). The idea to apply this technique is that we can write the matrix i in some canonical form which is quite analogous to the Jordan normal form.

EXAMPLE 3.1. *The following data define an indecomposable vector bundle of rank 2 on X: the normalization $\tilde{\mathcal{O}} \oplus \tilde{\mathcal{O}}(n), n \neq 0$, together with the matrices shown in figure 8.*

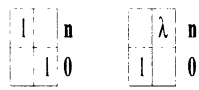

Figure 8.

A Gelfand matrix problem can be coded by some partially ordered set, together with some equivalence relation on it. For example, the problem of classifying vector bundles on a rational curve with one node, corresponds to the following partially ordered set. There are two infinite sets $E_0 = \{E_0(i) | i \in \mathbb{Z}\}$ and $E_\infty = \{E_\infty(i) | i \in \mathbb{Z}\}$ with total order induced by the order on $\mathbb{Z}$, and two one-point sets F_0 and F_∞. On the set

$$\mathbf{E} \bigcup \mathbf{F} = (E_0 \cup E_\infty) \bigcup (F_0 \cup F_\infty)$$

we introduce an equivalence relation: $E_0(i) \sim E_\infty(i), i \in \mathbb{Z}, F_0 \sim F_\infty$.

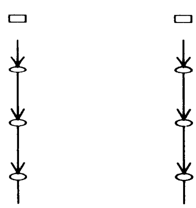

Figure 9.

From this picture we can easily recover the corresponding matrix problem: $E_0 \cup F_0$ and $E_\infty \cup F_\infty$ correspond to $i(0:1)$ and $i(1:0)$ respectively. $F_0 \sim F_\infty$ means that we have to do elementary transformations with columns of $i(0:1)$ and $i(1:0)$ simultaneously, the partial order on E_0 and E_∞ implies the division of matrices into horizontal blocks, where each block has some weight. $E_0(i) \sim E_\infty(i), i \in \mathbb{Z}$, means the conjugation of blocks: they have the same number of rows and elementary transformations inside of them should be done simultaneously.

Let now X be a configuration of projective lines of type A_n or $\tilde{A}_n$. We can proceed with constructing the category of triples in the similar way as we have done for a rational curve with one node. The matrix problem we get is coded by the Bondarenko's partially ordered set as follows: consider the set of pairs $\{(L,a)\}$, where L is an irreducible component of X, $a \in L$ a singular point. To each such pair corresponds a totally ordered set $E_{(L,a)} = \{E_{(L,a)}(i) | i \in \mathbb{Z}\}$ and a one-point set $F_{(L,a)}$. On the set

$$\mathbf{E} \bigcup \mathbf{F} = \bigcup_{(L,a)} (E_{(L,a)} \cup F_{(L,a)})$$

we introduce an equivalence relation:

1. $F_{(L,a)} \sim F_{(L',a)}$,
2. $E_{(L,a)}(i) \sim F_{(L,a')}(i), i \in \mathbb{Z}$.

This means that we have a set of matrices $M(L,a)$, where (L,a) runs through all possible pairs $(L,a), a \in L$, and each of the matrices is divided into horizontal blocks with respect to the partial order on $E_{(L,a)}$. The principle of conjugation of blocks is the same as for a rational curve with one node.

What is the combinatorics of indecomposable objects in this case? A Gelfand problem has two types of indecomposable objects: *bands* and *strings* (see (Bondarenko, 1992) and Appendix A in (Drozd and Greuel)). If X is a configuration of projective lines of type A_n, then each indecomposable vector bundle has to be a line bundle. Let X be either a rational curve with one node or a configuration of projective lines of type $\tilde{A}_n$. The condition on matrices $M(L,a)$ to be square and nondegenerate implies that vector bundles correspond to band representations.

DEFINITION 3.2. *Let X be either a rational projective curve with one node or a cycle of s projective lines. Let $\{a_1, a_2, \dots, a_s\}$ be the set of singular points of X, $\tilde{X} \xrightarrow{\pi} X$ the normalization of X, i.e. $\tilde{X}$ is the disjoint union of s copies $L_1, \dots, L_s$ of the projective line, and $\{a'_i, a''_i\} = \pi^{-1}(a_i)$. Suppose that $a'_i, a''_{i+1} \in L_i$, where $a''_{s+1} = a''_1$.*

A band $\mathcal{B}(\mathbf{d}, m, \lambda) = (\tilde{\mathcal{B}}, \mathcal{M}, i)$ is defined by the following parameters:

1. *$\mathbf{d} = d_1 d_2 \dots d_s d_{s+1} d_{s+2} \dots d_{2s} \dots d_{rs-s+1} d_{rs-s+2} \dots d_{rs}$ is a sequence of degrees on the normalized curve $\tilde{X}$. This sequence should not be of the form $\mathbf{e}^l$, where $\mathbf{e}$ is another sequence (i. e. $\mathbf{d}$ is not a self-concatenation of some other sequence).*
2. *m is the size of the elementary block of the matrix defining the glueing. The first two properties mean that the resriction of the normalized vector bundle on the l-th component of $\tilde{X}$ is*

$$\bigoplus_{i=1}^{r} \mathcal{O}_{L_l}(d_{l+is})^m.$$

3. *$\lambda \in k^*$ is a continuous parameter.*

We have 2s matrices $M(L_i, a'_i)$ and $M(L_i, a''_{i+1}), i = 1, \dots, s$, occurring in the triple, corresponding to $\mathcal{B}(\mathbf{d}, m, \lambda)$. Each of them has size $mr \times mr$. Divide these matrices into $m \times m$ square blocks.

Consider then sequences $\mathbf{d}(i) = d_i d_{i+s} \dots d_{i+(r-1)s}$ and label the horizontal strips of $M(L_i, a'_i)$ and $M(L_i, a''_{i+1})$ with respect to occurence of integers in the sequence $\mathbf{d}(i)$. If some integer d occured k times in $\mathbf{d}(i)$ then the horizontal strip corresponding to d consists of k substrips with m rows each. Recall now an algorithm of writing the components of the matrix i in normal form:

1. *Write a sequence*

$$(L_1, a''_2) \xrightarrow{1} (L_2, a'_2) \xrightarrow{1} (L_2, a''_3) \xrightarrow{1} \dots$$
$$\xrightarrow{1} (L_1, a'_1) \xrightarrow{1} (L_1, a''_1) \xrightarrow{2} \cdots \xrightarrow{r} (L_1, a'_1).$$

2. *We unroll the sequence $\mathbf{d}$. This means that we write over each (L_i, a) the corresponding term of the subsequence $\mathbf{d}(i)$ together with the number as often as this term already occured in $\mathbf{d}(i)$:*

$$(L_1, a''_2)^{(d_1,1)} \xrightarrow{1} (L_2, a'_2)^{(d_2,1)} \xrightarrow{1} (L_2, a''_3)^{(d_2,1)} \xrightarrow{1} \dots$$
$$\xrightarrow{r} (L_s, a''_1)^{(d_{rs},*)} \xrightarrow{r} (L_1, a'_1)^{(d_1,1)}.$$

3. *We are ready to fill the entries of the matrices $M(L,a)$:*
   a) *Consider each arrow above: $(L,a)^{(d,i)} \xrightarrow{k}$. Then we put the matrix I_m in the block $((d,i),k)$ of the matrix $M(L,a)$ (the block which is an intersection of the i-th substrip of the horizontal strip with label d' and the k-th vertical strip).*
   b) *We put on the $((d_1,1),r)$-th place of $M(L_1, a'_1)$ the Jordan block $J_m(\lambda)$ (our continous parameter λ appears at this moment).*

EXAMPLE 3.3. *Let X be a union of two projective lines intersecting transversally it two different points ($\tilde{A}_1$-configuration), $d = 01131$–2. Then $\mathcal{B}(\mathbf{d}, m, \lambda)$ is a vector bundle of rank $3m$ given by its normalization*

$$(\mathcal{O}_{L_1}^m \oplus \mathcal{O}_{L_1}(1)^{2m}) \oplus (\mathcal{O}_{L_2}(-2)^m \oplus \mathcal{O}_{L_2}(1)^m \oplus \mathcal{O}_{L_2}(3)^m)$$

and the matrices in figure 10.

EXAMPLE 3.4. *A vector bundle from example 3.1 is just $\mathcal{B}(0n, 1, \lambda)$.*

REMARK 3.5. *One can ask the following question: if bands corresponds to vector bundles, what do correspond to strings that are degenerated series in the Gelfand problem? As one can guess, some of the strings correspond to torsion-free sheaves which are not locally free. For this purpose one should modify a little bit the*

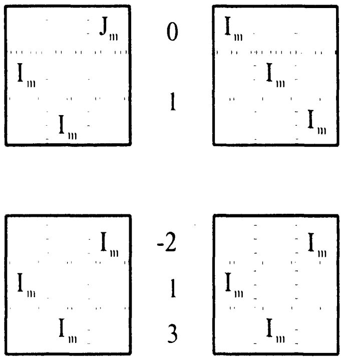

Figure 10.

definition of the category of triples. For example, if X is a rational curve with one node, then the triple $(\tilde{O}_{\mathbb{P}^1}, k^2, i)$, where i is given by two matrices

$$0\ \boxed{1 \;\; 0} \qquad\qquad 0\ \boxed{0 \;\; 1}$$

defines $\tilde{O} = \pi_*(O_{\tilde{X}})$.

In what follows we shall see that this way of representing of vector bundles on rational curves is indeed a convenient one. It allows in particular to determine the dual bundle, the decomposition of tensor products and the computation of cohomology groups.

4. Dual vector bundle

THEOREM 4.1. *Let X be either a projective curve with one node or a configuration of projective lines of type $\tilde{A}_n$, let $\mathcal{B} = \mathcal{B}(\mathbf{d}, m, \lambda)$ be a vector bundle on X. Then $\mathcal{B}^\vee \cong \mathcal{B}(-\mathbf{d}, m, \lambda^{-1})$.*

PROOF. Let $\mathcal{B}$ be a vector bundle on X. By the adjoint property we have:

$$\mathcal{H}om_{\tilde{O}}(\mathcal{B} \otimes_O \tilde{O}, \tilde{O}) \cong \mathcal{H}om_O(\mathcal{B}, \mathcal{H}om_{\tilde{O}}(\tilde{O}, \tilde{O})) \cong \mathcal{H}om_O(\mathcal{B}, \tilde{O}).$$

But we have a canonical map $\mathcal{H}om_O(\mathcal{B}, O) \otimes_O \tilde{O} \longrightarrow \mathcal{H}om_O(\mathcal{B}, \tilde{O})$, which is an isomorphism in case $\mathcal{B}$ is locally-free. Hence, if the normalization of $\mathcal{B}$ is $\tilde{\mathcal{B}}$ then the normalization of $\mathcal{B}^\vee$ is $\tilde{\mathcal{B}}^\vee$. Now, let $\mathcal{B}$ be given by a triple $(\tilde{\mathcal{B}}, \mathcal{M}, i)$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}\tilde{\mathcal{B}} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle id} & & \big\downarrow & & \big\downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathcal{J}\tilde{\mathcal{B}} & \longrightarrow & \tilde{\mathcal{B}} & \xrightarrow{\pi} & \tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}} & \longrightarrow & 0\,.
\end{array}$$

Apply the functor $\mathcal{H}om_{\mathcal{O}}(\ ,\mathcal{O})$ to obtain the glueing matrices of $\mathcal{H}om_{\mathcal{O}}(\mathcal{B},\mathcal{O})$. Then we get

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}},\mathcal{O}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}}(\mathcal{J}\tilde{\mathcal{B}},\mathcal{O}) & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}},\mathcal{O}) & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{B}},\mathcal{O}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 \longrightarrow \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}},\mathcal{O}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}}(\mathcal{B},\mathcal{O}) & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{B},\mathcal{O}) & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{B}},\mathcal{O}) \end{array}$$

But since X has nodal singularities, $\mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{O}},\mathcal{O}) = 0$, and we get, as a corollary, $\mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{B}},\mathcal{O}) = 0$. Moreover, since our curve X is Gorenstein, $\mathcal{K}/\mathcal{O}$ is an injective $\mathcal{O}$-module ($\mathcal{K}$ denotes a sheaf of rational functions). Hence,

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{O} \longrightarrow 0$$

is an injective resolution of $\mathcal{O}$. If $\mathcal{N}$ is a skyscraper sheaf, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{N},\mathcal{K}) = 0$ and $\mathcal{E}xt^1_{\mathcal{O}}(\mathcal{N},\mathcal{O})$ is naturally isomorphic to $\mathcal{H}om_{\mathcal{O}}(\mathcal{N},\mathcal{K}/\mathcal{O})$. So, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}},\mathcal{O}) & \xrightarrow{\cong} & \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}},\tilde{\mathcal{O}}/\mathcal{O}) \\ \uparrow & & \uparrow \\ \mathcal{E}xt^1_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{B},\mathcal{O}) & \xrightarrow{\cong} & \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{B},\tilde{\mathcal{O}}/\mathcal{O}) \end{array}$$

where $\mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{B},\tilde{\mathcal{O}}/\mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}},\tilde{\mathcal{O}}/\mathcal{O})$ is the map induced from the exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{i} \tilde{\mathcal{B}}/\mathcal{J} \longrightarrow \tilde{\mathcal{B}}/\mathcal{B} \longrightarrow 0.$$

Let us see what are $\mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}},\mathcal{O})$ and $\mathcal{H}om_{\mathcal{O}}(\mathcal{J}\tilde{\mathcal{B}},\mathcal{O})$.

1. We have canonical $\mathcal{O}$-module homomorphisms

$$\mathcal{J}\mathcal{H}om_{\tilde{\mathcal{O}}}(\tilde{\mathcal{B}},\tilde{\mathcal{O}}) \longrightarrow \mathcal{H}om_{\tilde{\mathcal{O}}}(\tilde{\mathcal{B}},\mathcal{J}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}},\mathcal{O}).$$

   Since all modules occurring in the sequence are coherent $\mathcal{O}$-modules, we can check that these homomorphisms are isomorphisms just looking at stalks. For the stalks howewer, this is true since all singular points of X are nodes.
2. We have canonical $\mathcal{O}$-module homomorphisms

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{J}\tilde{\mathcal{B}},\mathcal{O}) \longrightarrow \mathcal{H}om_{\tilde{\mathcal{O}}}(\mathcal{J}\tilde{\mathcal{B}},\mathcal{J}) \longleftarrow \mathcal{H}om_{\tilde{\mathcal{O}}}(\tilde{\mathcal{B}},\tilde{\mathcal{O}})$$

   which are isomorphisms on stalks.

Finally we get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}\mathcal{H}om_{\tilde{\mathcal{O}}}(\tilde{\mathcal{B}},\tilde{\mathcal{O}}) & \longrightarrow & \mathcal{H}om_{\tilde{\mathcal{O}}}(\tilde{\mathcal{B}},\tilde{\mathcal{O}}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}},\tilde{\mathcal{O}}/\mathcal{O}) & \longrightarrow & 0 \\ & & {\scriptstyle id}\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{J}\mathcal{H}om_{\tilde{\mathcal{O}}}(\tilde{\mathcal{B}},\tilde{\mathcal{O}}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}}(\mathcal{B},\mathcal{O}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{B}}/\mathcal{B},\tilde{\mathcal{O}}/\mathcal{O}) & \longrightarrow & 0. \end{array}$$

But this is a diagram we are looking for. It remains only to describe a map giving an inclusion of the kernel of the map of $\mathcal{O}/\mathcal{J}$-modules

$$\mathcal{H}om_{\mathcal{O}/\mathcal{J}}(\tilde{\mathcal{B}}/\mathcal{B},\tilde{\mathcal{O}}/\mathcal{O}) \xrightarrow{i^*} \mathcal{H}om_{\mathcal{O}/\mathcal{J}}(\mathcal{M},\tilde{\mathcal{O}}/\mathcal{O}).$$

Let x be a singular point, $\{u_1, u_2, \dots, u_n\}$ a basis of $\mathcal{M}_x$, $\{v_1, v_2, \dots, v_n; w_1, w_2, \dots, w_n\}$ a basis of $(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}})_x$ and i_x is given by

$$\begin{cases} i_x(v_1) &= \sum a_{i1} v_i + \sum b_{i1} w_i \\ i_x(v_2) &= \sum a_{i2} v_i + \sum b_{i2} w_i \\ & \vdots \\ i_x(v_n) &= \sum a_{in} v_i + \sum b_{in} w_i. \end{cases}$$

Let $(\tilde{\mathcal{O}}/\mathcal{J})_x = \langle \alpha, \beta \rangle$, $(\mathcal{O}/\mathcal{J})_x = \langle \gamma \rangle$. Since $\tilde{\mathcal{O}}/\mathcal{J}$ is supplied with an $\mathcal{O}/\mathcal{J}$-module structure by the diagonal mapping, $(\tilde{\mathcal{O}}/\mathcal{O})_x = \langle \alpha, \beta \rangle / \langle \gamma, \gamma \rangle$. So we may suppose that the isomorphism $(\tilde{\mathcal{O}}/\mathcal{O})_x \longrightarrow k$ is given by $[\alpha] \mapsto 1$, $[\beta] \mapsto -1$.

The space $\mathcal{H}om_{\mathcal{O}/\mathcal{J}}(\tilde{\mathcal{B}}/\mathcal{B},\tilde{\mathcal{O}}/\mathcal{O})$ has a basis $v_1^*, v_2^*, \dots, v_n^*; w_1^*, w_2^*, \dots, w_n^*$, where $v_i^*(v_j) = \delta_{ij}[\alpha]$, $v_i^*(w_j) = 0$, $w_i^*(v_j) = 0$, $w_i^*(w_j) = \delta_{ij}[\beta]$. Therefore we get

$$i_x^*(v_i^*)(u_j) = a_{ij}[\alpha] = a_{ij}$$

and on the other hand

$$i_x^*(w_i^*)(u_j) = b_{ij}[\beta] = -b_{ij}.$$

So, if i_x was $\binom{A}{B}$, then i_x^* is given by $(A^T, -B^T)$. Now we may suppose that the matrix i has a canonical form. Then we can easily compute the matrix giving an embedding of the kernel of i^*. As a corollary we get the claim of the theorem. □

5. Cohomology groups, tensor products and homomorphism spaces

5.1. COMPUTATION OF COHOMOLOGY GROUPS OF VECTOR BUNDLES

Let $\mathcal{B} = \mathcal{B}(\mathbf{d}, m, \lambda)$. The developed technique allows us to compute the cohomology groups of $\mathcal{B}$ in terms of the combinatorics of the sequence $\mathbf{d}$. Let $(\tilde{\mathcal{B}}, \mathcal{M}, i)$ be a triple corresponding to $\mathcal{B}$. We have an exact sequence

$$0 \longrightarrow \mathcal{B} \longrightarrow \tilde{\mathcal{B}} \longrightarrow \tilde{\mathcal{B}}/\mathcal{B} \longrightarrow 0.$$

After taking cohomology we get the long exact sequence:

$$0 \longrightarrow H^0(\mathcal{B}) \longrightarrow H^0(\tilde{\mathcal{B}}) \xrightarrow{0} H^0(\tilde{\mathcal{B}}/\mathcal{B}) \longrightarrow H^1(\mathcal{B}) \longrightarrow H^1(\tilde{\mathcal{B}}).$$

The map $f : H^0(\tilde{\mathcal{B}}) \longrightarrow H^0(\tilde{\mathcal{B}}/\mathcal{B})$ can be computed explicitly; it is just the composition

$$H^0(\tilde{\mathcal{B}}) \longrightarrow H^0(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}}) \longrightarrow H^0(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}})/H^0(\mathcal{B}/\mathcal{J}\mathcal{B}).$$

But $\mathcal{B}/\mathcal{J}\mathcal{B} \cong \mathcal{M}$, and the embedding $H^0(\mathcal{B}/\mathcal{J}\mathcal{B}) \longrightarrow H^0(\tilde{\mathcal{B}}/\mathcal{J}\tilde{\mathcal{B}})$ is given by the matrix i. So we can compute cohomology groups as the kernel and cokernel of the map f. We refer to (Drozd, Greuel and Kashuba) for more details and just give the result:

$$\dim_k H^0(\mathcal{E}) = m(\sum_{i=1}^{rs}(d_i+1)^+ - \theta(\mathbf{d})) + \delta(\mathbf{d},\lambda),$$

$$\dim_k H^1(\mathcal{E}) = m(\sum_{i=1}^{rs}(d_i+1)^- + rs - \theta(\mathbf{d})) + \delta(\mathbf{d},\lambda),$$

where $\delta(\mathbf{d},\lambda) = 1$ if $\mathbf{d} = (0,\dots,0)$, $\lambda = 1$ and 0 otherwise; $k^+ = k$ if $k > 0$ and zero otherwise, $k^- = k^+ - k$. Call a subsequence $p = (d_{k+1},\dots,d_{k+l})$, where $0 \le k < rs$ and $1 \le l \le rs$, a *positive part* of d if all $d_{k+j} \ge 0$ and either $l = rs$ or both $d_k < 0$ and $d_{k+l+1} < 0$. For such a positive part put $\theta(p) = l$ if either $l = rs$ or $p = (0,\dots,0)$ and $\theta(p) = l+1$ otherwise. Then $\theta(\mathbf{d}) = \sum\theta(p)$, where we take a sum over all positive subparts of d.

5.2. TENSOR PRODUCT OF VECTOR BUNDLES

All the results of this subsection are taken from (Yudin). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two indecomposable vector bundles either on a rational curve with one node or on a configuration of projective lines of type $\tilde{A}_n$. What is $\mathcal{B}_1 \otimes_{\mathcal{O}} \mathcal{B}_2$?

Let $(\tilde{\mathcal{B}}_1, \mathcal{M}_1, i_1)$ and $(\tilde{\mathcal{B}}_2, \mathcal{M}_2, i_2)$ be triples corresponding to $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Then it is not difficult to see that $\mathcal{B}_1 \otimes_{\mathcal{O}} \mathcal{B}_2$ corresponds to $(\tilde{\mathcal{B}}_1 \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{B}}_2, \mathcal{M}_1 \otimes_{\mathcal{O}/\mathcal{J}} \mathcal{M}_2, i_1 \otimes i_2)$. Moreover $i_1 \otimes i_2$ is given by the Kronecker product of matrices, but the problem is that we have to care of the horizontal block division of these matrices.

Let us suppose for simplicity that $\mathrm{char}(k) = 0$ (for fields of prime characteristic an answer is analogous but more sophisticated). The first two steps in describing the decoposition of the tensor product are the following lemmas

LEMMA 5.1. *Let s be the number of components of X, $\mathbf{0} = 00\dots0$ a sequence of 0's of length s. Then*

$$\mathcal{B}(\mathbf{d},m,\lambda) \cong \mathcal{B}(\mathbf{d},1,\lambda) \otimes_{\mathcal{O}} \mathcal{B}(\mathbf{0},m,1).$$

LEMMA 5.2. *Moreover*

$$\mathcal{B}(\mathbf{0},m,1) \otimes_{\mathcal{O}} \mathcal{B}(\mathbf{0},n,1) \cong \bigoplus_{j=1}^{m} \mathcal{B}(\mathbf{0},n-m+1+2j,1).$$

It remains to describe the tensor product of vector bundles of type $\mathcal{B}(\mathbf{d},1,\lambda)$. For simplicity let us use the following notation. Let $\mathbf{d}^l = \mathbf{dd}\dots\mathbf{d}$ (l times), then

$$\mathcal{B}(\mathbf{d}^l,m,\lambda) = \bigoplus_{i=1}^{l} \mathcal{B}(\mathbf{d},m,\xi^i\sqrt{\lambda}),$$

where ξ is a l-th primitive root of 1.

Let $\mathcal{B}(\mathbf{d},1,\lambda)$ and $\mathcal{B}(\mathbf{e},1,\mu)$ be two vector bundles of rank k and l respectively. The tensor product of this two bundles has to be of rank kl. Let $\mathbf{d}=\vec{d}_1\vec{d}_2\dots\vec{d}_k$ (each $\vec{d}_i=d_{(i-1)s+1}d_{(i-1)s+2}\dots d_{is}$ defines a sequence of degrees on $L_1,L_2,\dots,L_s$; one should not mix $\vec{d}_i$ with $\mathbf{d}(i)$ from section 2), and $\mathbf{e}=\vec{e}_1\vec{e}_2\dots\vec{e}_l$. Write

$$\tilde{\mathbf{d}}=\underbrace{\vec{d}_1\vec{d}_2\dots\vec{d}_k}_{1}\underbrace{\vec{d}_1\vec{d}_2\dots\vec{d}_k}_{2}\dots\underbrace{\vec{d}_1\vec{d}_2\dots\vec{d}_k}_{l},\quad \tilde{\mathbf{e}}=\underbrace{\vec{e}_1\vec{e}_2\dots\vec{e}_l}_{1}\underbrace{\vec{e}_1\vec{e}_2\dots\vec{e}_l}_{2}\dots\underbrace{\vec{e}_1\vec{e}_2\dots\vec{e}_l}_{k}.$$

We have (k,l) sequences $\mathbf{f}_1,\ \mathbf{f}_2,\dots,\mathbf{f}_{(k,l)}$ (where (k,l) is the greatest common divisor of k and l):

$$\mathbf{f}_1=\vec{d}_1+\vec{e}_1,\vec{d}_2+\vec{e}_2,\dots,\vec{d}_k+\vec{e}_l$$

(the lenght of $\mathbf{f}_1$ is $[k,l]$, the smallest common multiple of k and l),

$$\mathbf{f}_2=\vec{d}_1+\vec{e}_2,\vec{d}_2+\vec{e}_3,\dots,\vec{d}_k+\vec{e}_1,$$

$$\mathbf{f}_{(k,l)}=\vec{d}_1+\vec{e}_{(k,l)},\vec{d}_2+\vec{e}_{(k,l)+1},\dots,\vec{d}_k+\vec{e}_{(k,l)-1}.$$

Then

$$\mathcal{B}(\mathbf{d},1,\lambda)\otimes_{\mathcal{O}}\mathcal{B}(\mathbf{e},1,\mu)\cong\bigoplus_{i=1}^{(k,l)}\mathcal{B}(\mathbf{f}_i,1,\lambda^{\frac{l}{k,l}}\mu^{\frac{k}{k,l}}).$$

5.3. COMPUTATION OF EXTENSION AND HOMOMORPHISM SPACES

Now we are able to compute the dimension of the homomorphism space and of the first Ext-group between two vector bundles,

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{F})\cong H^0(\mathcal{E}^\vee\otimes_{\mathcal{O}}\mathcal{F}),$$

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{E},\mathcal{F})\cong H^1(\mathcal{E}^\vee\otimes_{\mathcal{O}}\mathcal{F}).$$

We have formulas for computing the dual of a vector bundle and for decomposing a tensor product into a direct sum of indecomposables.

Let us describe the simple vector bundles (i.e. those bundles $\mathcal{E}$, for which $\mathrm{End}_{\mathcal{O}}(\mathcal{E})=k$. In other words, each automorphism of $\mathcal{E}$ is a scalar multiple of the identity automorphism). This question is motivated by the recent work of (A. Polishchuk), which relates spherical objects (Seidel and Thomas) with solutions of the classical Yang-Baxter equation. Suppose $\mathcal{E}=\mathcal{B}(\mathbf{d},m,\lambda)$ a simple vector bundle. Since an automorphism of a Jordan block defines also an automorphism of $\mathcal{B}(\mathbf{d},m,\lambda)$, we can conclude that $m=1$. Further, it is easy to see that each $\mathbf{d}(i)$ should not contain degrees with difference greater than one (otherwise there will be an automorphism of $\tilde{\mathcal{E}}$ which induces the identity map modulo conductor). Since it is enough to describe simple bundles modulo the action of the Picard group, we may suppose that each $\mathbf{d}_i(i=1,2,\dots,s)$ consists of 0's and 1's. Now we apply the machinery developed above and get

THEOREM 5.3. *$\mathcal{B}(\mathbf{d},m,\lambda)$ is a simple vector bundle if and only if*

1. *$m = 1$;*
2. *the difference between degrees of any two vector bundles on the same component of the normalized curve is at most 1;*
3. *consider all possible differences $\mathbf{d} - \mathbf{d}[t]$, where t is a shift of the sequence $\mathbf{d}$ ($\mathbf{d}[1] = d_{s+1}d_{s+2}\dots d_{rs}d_1d_2\dots d_s$); then each of these differences does not contain a subsequence of type $10\dots01$.*

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