Matrix problems, small reduction and representations of a class of mixed Lie groups

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In representation theory of Lie groups the cases of reductive and solvable groups are highly elaborated (cf. [K]). Much less seems to be known about mixed groups, i.e. those neither reductive nor solvable. Moreover, the simplest examples (cf. §1) show that in a sense the complete description of their representations is a hopeless problem. Nevertheless, in some cases it turns out to be possible to describe "almost all" of them in a rather appropriate way (cf. theorem 1.2), namely, they behave just like representations of a reductive group.

This lecture is an account of the author's joint work with A. Timoshin [DT] where we managed to apply to the investigation of representations of some mixed groups the method of so-called "matrix problems" which enabled us to obtain such an answer.

Thus, the lecture splits into two parts. The first (§§1-3) contains the formulation of the main theorem (theorem 3.1) with necessary preliminaries and its reduction to a matrix problem. The second part (§§4-6) is devoted to matrix problems which we treat in terms of representations of bocses (cf. [R]) and culminates in §6 with the proof of the main theorem. The most technical, but to my mind also the most important, is §5 where we present an algorithm elaborated in [BGOR]. I think that the importance of this algorithm (called "small reduction") is still far from being properly appreciated.

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§1 Mackey Theorem. An Example.

In the calculation of representations of mixed groups a theorem of Mackey concerning representations of group extensions is widely used (cf. [K]). Recall it in the simplest case of a semi-direct product with normal abelian subgroup. From now on, for a locally compact group G we denote by $\hat{G}$ its dual space, i.e. the space of isomorphism classes of irreducible unitary representations (cf. [K]).

Theorem 1.1. *Suppose that a locally compact group is a semi-direct product, $G = H \ltimes N$, of a closed normal abelian subgroup N and a closed subgroup H and that all orbits of H on $\hat{N}$ are locally closed. Then there exists a surjection $p : G \to \hat{N}/H$ such that $p^{-1}(x^H) \cong \hat{H}_x$ for any orbit x^H, where $H_x = \{h \in H \mid x^h = x\}$.*

Here H acts on $\hat{N}$ naturally: $x^h(n) = x(hnh^{-1})$. The representation of G corresponding to the representation T of H_x, is $\mathrm{Ind}(G, G_x, xT)$, where $G_x = H_x N$ and $xT(hn) = x(n)T(h)$ for any $h \in H_x$, $n \in N$.

Thus in order to calculate representations of G we have to find orbits of H on $\hat{N}$, then to define the stabilizers H_x and finally to calculate the representations of each H_x. Moreover, we can obtain the Plancherel measure on $\hat{G}$ from those on $\hat{N}$ and $\hat{H}_x$ as is shown in [KL]. Sometimes it is possible to iterate this procedure, i.e. to decompose H_x into a similar semi-direct product and to apply Mackey's theorem again. But as a rule such iterations become more and more complicated as the following simple example shows.

Let K be a locally compact field and $G = G(m, n)$ be the group of invertible K-matrices of the form:

$$\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}$$

with an $m \times m$ block A_1, an $n \times n$ block A_2 and an $m \times n$ block A_{12}. Clearly, $G = H \ltimes N$ with H consisting of the "diagonal" matrices (such that $A_{12} = 0$) and N of the "unipotent" ones (such that A_1 and A_2 are identities). Then $N = \mathrm{Mat}(m \times n)$, hence $\hat{N} = \mathrm{Mat}(n \times m)$ under the pairing $(x, n) = e(\mathrm{tr}(xn))$, e being a non-trivial character of the additive group of K (cf. [W]) and one easily obtains $x^h = A_2^{-1} x A_1$ for $h = \mathrm{diag}(A_1, A_2)$. Of course, its orbits are well-known: each of them contains a unique matrix x of the form:

$$x = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

(I being an identity matrix). A simple calculation shows that H_x consists of all $h = \mathrm{diag}(A_1, A_2)$ with

$$A_1 = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} B_2 & B_{23} \\ 0 & B_3 \end{pmatrix}$$

(the size of B_2 coincides with that of I in x).

Now again $H_x = H' \ltimes N'$ with H' "diagonal" and N' "unipotent". If k_i is the size of B_i, then

$$N' = \mathrm{Mat}(k_1 \times k_2) \times \mathrm{Mat}(k_2 \times k_3), \quad \text{hence}$$
$$\hat{N}' = \mathrm{Mat}(k_2 \times k_1) \times \mathrm{Mat}(k_3 \times k_2).$$

Moreover, if $h = \mathrm{diag}(B_1, B_2, B_3) \in H'$, $x = (x_1, x_2) \in \hat{N}'$, then one can check that $x^h = (B_2^{-1} x B_1, B_3^{-1} x B_2)$. This means that elements of $\hat{N}'$ can be viewed as representations of the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

and orbits correspond to isomorphism classes of these representations (cf. [G]). It is known then that any orbit contains a unique pair $x = (x_1, x_2)$ with

$$x_1 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the size of columns in x_1 being the same as that of rows in x_2. One can can calculate the stabilizer H'_x of such a pair: it consists of all triples (B_1, B_2, B_3) of the form:

$$B_1 = \begin{pmatrix} C_1 & C_{12} & \underline{C_{13}} \\ 0 & C_2 & C_{23} \\ 0 & 0 & C_3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} C_2 & C_{23} & C_{24} & \underline{C_{25}} \\ 0 & C_3 & 0 & C_{35} \\ 0 & 0 & C_4 & C_{45} \\ 0 & 0 & 0 & C_5 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} C_3 & C_{35} & \underline{C_{36}} \\ 0 & C_5 & C_{56} \\ 0 & 0 & C_6 \end{pmatrix}.$$

Let M be the normal subgroup of H'_x consisting of all triples with $C_i = I$ and $C_{ij} = 0$ except maybe for $(ij) = (13)$, (25) or (36) (the matrices underlined above). Write $F = H'_x/M$. Then again $F = H'' \ltimes N''$ with H'' "diagonal" and N'' "unipotent". But in this case the action of H'' on $\hat{N}''$ is described by the quiver:

$$\begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \\ & & \downarrow & & \downarrow & & \\ & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

which is known to be wild (cf. [N]), i.e. the classification of its representations (or orbits of our action) contains the well-known unsolved problem of classification of pairs of linear operators on a finite-dimensional space. Thus we have no hope of obtaining a precise description of representations of groups $G(m, n)$.

Nevertheless, we can describe "almost all" their representations via the following simple observation. Consider the open dense subset $\tilde{N} \subseteq \hat{N}$ consisting of all matrices of maximal rank $r = \min(m, n)$. This subset forms an orbit of H. Namely, a matrix $x \in \tilde{N}$ is equivalent to one of the form

$$\begin{pmatrix} 0 & I \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ I \end{pmatrix} \quad \text{or} \quad I$$

according as $n < m$ or $m < n$ or $m = n$, respectively. Hence its stabilizer H_x is isomorphic to the group of pairs (A_1, A_2) with

$$A_1 = \begin{pmatrix} B_1 & B_{12} \\ 0 & A_2 \end{pmatrix} \quad \text{or} \quad A_2 = \begin{pmatrix} A_1 & B_{12} \\ 0 & B_2 \end{pmatrix} \quad \text{or} \quad A_1 = A_2,$$

respectively. Thus H_x is isomorphic respectively to $G(m-n,n)$ or $G(m,n-m)$ or $GL(m,K)$. Now the iteration gives us the following

Theorem 1.2. *For $G = G(m,n)$ there exists an open dense subset $G^0 \subseteq \hat{G}$ homeomorphic to $\widehat{GL(d,K)}$, where d is the greatest common divisor of m and n.*

Moreover, using [KL] one can show that under this homeomorphism the restriction to G^0 of the Plancherel measure μ_G on G coincides with the Plancherel measure on $GL(d,K)$ and $\mu_G(G^0) = 1$.

An analogous result was proved in [L] for groups of block-triangular matrices with an arbitrary number of blocks. In this case $G^0 \cong \hat{D}$ for some direct product D of full linear groups over K.

We are going to generalise the latter result to a wider class of linear groups which will be defined in the next paragraph.

§2 Dynkin Algebras

The groups we will consider are linear groups over algebras. These are by definition the groups of the form $G(P,A) = \mathrm{Aut}_A(P)$, where A is a finite-dimensional algebra and P a finitely generated projective A-module. To obtain a good description of their representations we have to restrict the class of algebras to the so-called Dynkin algebras which we are going to define. We need some notions about categories in order to do so.

All categories considered are supposed to be linear over some field K and all functors will be K-linear. We write $\otimes$ and Hom instead of $\otimes_K$ and Hom_K. A module over a category A is by definition a functor $M : A \to \mathrm{Vect}$ (the category of vector spaces over K). Let A-mod denote the category of A-modules. An A-B-bimodule is a K-bilinear bifunctor $V : A^{\mathrm{op}} \times B \to \mathrm{Vect}$. Denote by add A the smallest category which contains A and is completely additive, i.e. additive and with all idempotents split. A category A will be called *skeletal* if its objects are pairwise non-isomorphic and there are no non-trivial idempotents in the endomorphism algebras $A(i,i)$. A *skeleton* of A is defined as a skeletal category equivalent to A. For instance, if A is a finite-dimensional algebra, a skeleton can be chosen as the full subcategory of A-mod consisting of a complete set of pairwise non-isomorphic indecomposable projective A-modules.

Let A be a skeletal category such that all spaces $A(i,j)$ are finite-dimensional and $A(i,i)/\mathrm{rad}\ A(i,i) = K$ for all $i \in \mathrm{Ob}A$ (in this case we call A *split* over K). Write

$$g(i,j) = \dim\ \mathrm{rad}\ A(i,j)/\mathrm{rad}^2\ A(i,j)$$

(in this case rad A can be defined as the two-sided ideal of A consisting of all non-invertible morphisms). Define the *graph* (Gabriel quiver) Q_A of A as the graph having Ob A for the set of points and $g(i,j)$ arrows leading from i to j for all i, j. Call A connected provided Q_A is connected.

If a category A has a skeleton A_0, it is called *split* if A_0 is split and *connected* if A_0 is connected. For instance, a finite-dimensional algebra is split if

$$A/\text{rad } A \cong \Pi\text{Mat}(n_m, K)$$

and is connected if it can be decomposed into a direct product of algebras (cf. [DK]).

Suppose A and $\tilde{A}$ are skeletal categories. Recall that a functor $F : \tilde{A} \to A$ is called a *covering* [BG] provided it is surjective on objects and for any $i \in$ Ob A, $j \in$ Ob A the following mappings induced by F are surjective:

$$\oplus_{Fk=i} \tilde{A}(k,j) \to A(i, Fj);$$
$$\oplus_{Fk=i} \tilde{A}(j,k) \to A(Fj, i).$$

A skeletal category A is called *simply connected* if it is connected and has no non-trivial connected coverings. A category A with a skeleton A_0 is called *simply connected* if A_0 is simply connected.

Now let A be skeletal, split and finite-dimensional over K (that means that Ob A is finite and all spaces $A(i,j)$ finite-dimensional). Denote by C_A its Cartan matrix, i.e. the $n \times n$ integral matrix, where $n = |\text{Ob } A|$, with entries $c_{ij} = \dim A(i,j)$, and by ${}^SC_A$ the symmetrisation of C_A, i.e. the symmetric $n \times n$ matrix with entries ${}^Sc_{ij} = (c_{ij} + c_{ji})/2$. Call A *positive definite* if the matrix ${}^SC_A$ is positive definite. If A is of finite global dimension (e.g. there are no oriented cycles in its graph Q_A), we can reformulate this notion. Namely, consider the n-dimensional real vector space $\mathbb{R}^n$ and define for a finite-dimensional A-module M its *vector-dimension* $\underline{\dim}\, M$ as the vector $(d_1, ..., d_n)$ where $d_i = \dim M(i)$. Then a bilinear (non-symmetric) form can be defined on $\mathbb{R}^n$ such that

$$(\underline{\dim}\, M, \underline{\dim}\, N)_A = \sum_m (-1)^m \dim \text{Ext}_A^m(M, N)$$

for all finite-dimensional A-modules M, N, and hence the quadratic form of A, $q_A(X) = (X, X)_A$ is defined on $\mathbb{R}^n$. One can easily check (cf. [Ri]) that in this case the dimensions of the representable functors $A(i, -)$ form a basis in $\mathbb{R}^n$ and that C_A is just the matrix of the form $(,)_A$ with respect to this basis. Thus A is positive definite if its quadratic form q_A is.

Of course, these definitions may be applied to any category A having a finite dimensional skeleton A_0, so we have for such categories the notions of Cartan matrix, positive definiteness, bilinear and quadratic forms $(,)_A$ and q_A, and vector-dimensions $\underline{\dim}\, M$. In the case of a finite-dimensional algebra all of them coincide with the usual ones as defined, e.g. in [Ri].

Definition. *A finite-dimensional K-algebra A is called* Dynkin *if it is split, positive definite, has no oriented cycles in its Gabriel quiver and all its connected direct factors are simply connected.*

One can verify that this definition is equivalent to that of Happel [H] (we will not use this equivalence).

We give some examples of Dynkin algebras A and linear groups $G(P, A)$ over them:

(i) Take for A the algebra of upper-triangular $n \times n$ matrices over K. Its quiver is

$$\bullet \longrightarrow \bullet \longrightarrow \dots \longrightarrow \bullet$$

(n points) and its quadratic form is just that of the quiver [G], thus known to be positive definite. Thus A is Dynkin. In this case the groups $G(P, A)$ which arise are just the groups of block-triangular matrices with n diagonal blocks, or, which is the same, parabolic subgroups of full linear groups.

(ii) Let S be a finite partially ordered set and $A = A(S)$ its incidence algebra, i.e. the subalgebra of $\mathrm{Mat}(n, K)$, where $n = |S|$, with a basis formed by matrix units e_{ij} with $i \leq j$ in S. One can check that if q_A is positive definite, this algebra is simply connected. Now as $G(P, A)$ we obtain a class of so-called net subgroups [B] (this was the starting point of the investigation in [DT]).

§3 Main Theorem. Reduction to a matrix problem

Theorem 3.1. *Let $G = G(P, A)$ be a linear group over a Dynkin K-algebra, where K is a locally compact field. Then there exists an open dense subset $G^0 \subseteq \hat{G}$ with $\mu_G(G^0) = 1$ such that $G^0 \cong \hat{D}$ and the restriction $\mu_G|_{G^0}$ coincides with μ_D (via this homeomorphism), where $D = \Pi_m GL(d_m, K)$ for some dimensions d_m (depending on A and P).*

Proof. Choose a complete set $P_1, \dots, P_n$ of pairwise non-isomorphic indecomposable projective A-modules. For each projective A-module P put $|P| = k_1 + \dots + k_n$ if $P = k_1 P_1 \oplus \dots \oplus k_n P_n$. Then an obvious induction reduces our theorem to the following statement.

Lemma 3.2. *Under the assumptions of theorem 3.1 suppose that $G \not\cong \Pi_m GL(d_m, K)$ for any d_m. Then there exists an open dense subset $\tilde{G} \subseteq \hat{G}$ with $\mu_G(\tilde{G}) = 1$ such that $\tilde{G} \cong \hat{G}'$ and $\mu_G|_{\tilde{G}} = \mu_{G'}$, where $G' = G(P', A')$ for some Dynkin algebra A' and some projective A'-module P' with $|P'| < |P|$.*

Proof of the lemma. As there are no cycles in Q_A, we can find an indecomposable projective, say P_1, such that $\mathrm{Hom}_A(P_1, P_i) = 0$ for all $i \neq 1$. Then $P = P^1 \oplus P^2$ with $P^1 = k_1 P_1$ and $P^2 = k_2 P_2 \oplus \dots \oplus k_n P_n$. Then

$\mathrm{Hom}_A(P^1, P^2) = 0$, so an element $g \in G$ can be written as a matrix:

$$g = \begin{pmatrix} g_1 & f \\ 0 & g_2 \end{pmatrix}$$

with $g_i \in G_i = G(P^i, A)$ and $f \in N = \mathrm{Hom}(P^2, P^1)$. Hence $G = H \ltimes N$ where $H = G_1 \times G_2$ and $\hat{N}$ is isomorphic to the dual vector space of N. It can happen, of course, that $N = 0$ and we obtain no real reduction. To avoid this case we need some additional results.

It is convenient to consider instead of A its skeleton, which has the form KQ_A/I, where KQ_A is the category of paths of the graph Q_A and I an ideal of KQ_A contained in J^2 (J is the ideal generated by all arrows). The points of this category, which we will also denote A, are just the indices $1, ..., n$ and $A(i,j) = \mathrm{Hom}_A(P_i, P_j)$. Let $c_{ij} = \dim A(i,j)$ (the entries of the Cartan matrix of A) and put $B = \mathrm{End}_A(P_2 \oplus ... \oplus P_n)$. Then B can be viewed as the full subcategory of A consisting of objects $2, ..., n$.

Call two objects i, j of A *joint* if either $A(i,j) \neq 0$, or $A(j,i) \neq 0$, and *disjoint* otherwise.

Lemma 3.3. *(i)* $c_{ij} \leq 1$ *for all* i, j.
(ii) There exists no chain $i_1, i_2, ..., i_m$ *with* m *even, each* i_k *joint with* i_{k+1}, i_m *joint with* i_1 *and all other pairs* i_k, i_l *disjoint.*
(iii) If i, j, k *are pairwise different and disjoint, then there exists at most one object joint to each of them.*
(iv) If $a \in A(i,1)$ *and* $b \in A(j,i)$ *are non-zero but* $ab = 0$, *then* $A(j,1) = 0$.
Proof. *(i)-(iii)* follow from the positive definiteness of A. *(iv)* will be proved in §6.

Lemma 3.4. *The algebra* B *is also Dynkin.*
Proof. Clearly, we have only to prove that each component of B is simply connected. For the sake of simplicity, suppose B is connected (it plays practically no rôle in the proof). Let $F : \tilde{B} \to B$ be a non-trivial connected covering. Consider the set S of all objects i of B such that there exists an arrow in Q_A leading from i to 1. It follows from lemma 3.3 (iv) that objects of S are pairwise disjoint. Define an equivalence relation $\vee$ on $F^{-1}(S)$ as the weakest one such that $i \vee j$ provided there exists an object k of $\tilde{B}$ and non-zero morphisms $b : k \to i$ and $c : k \to j$ with $a_{F(i)}F(b)$ and $a_{F(j)}F(c)$ both non-zero. Here, for any $s \in S$ we denote by a_s the only arrow leading from s to 1. Lemma 3.3 implies that $F(i) \neq F(j)$ for distinct equivalent $i, j \in F^{-1}(S)$. Let $S/\vee$ be the set of equivalence classes and s_T, for $s \in S$, $T \in S/\vee$, be the unique object in T with $F(s_T) = s$ if such an object exists. Now construct $\tilde{A}$ as the category with object set $\mathrm{Ob}\, B \cup S/\vee$ and arrow set the arrows of $\tilde{B}$ together with new arrows $a_{sT} : s_T \to T$ for each $s \in S$, $T \in S/\vee$ such that s_T exists. Extend F to $\tilde{A}$ by putting $F(T) = 1$ and $F(a_{sT}) = a_s$ whenever s_T exists. One can easily extend also all relations

from I to $\tilde{A}$ and hence obtain a non-trivial connected covering $F : \tilde{A} \to A$ which contradicts simple connectedness of A, q.e.d.

Now if $N = 0$, we can consider the group G_2 which is again a linear group over the Dynkin algebra B. So without loss of generality we may suppose that $N \neq 0$.

Put $A_i = \mathrm{End}_A(P^i)$. Then N is an A_1-A_2-bimodule and $\hat{N}$ is an A_2-A_1-bimodule. Now G_i is the group of invertible elements of A_i and if $h = (g_1, g_2) \in H$, $x \in \hat{N}$, we have $x^h = g_2^{-1} x g_1$ via this bimodule structure.

Define $\mathrm{End}(x)$ to be the subalgebra of $A_1 \times A_2$ consisting of all pairs (a_1, a_2) for which $a_2 x = x a_1$. Then H_x is just the group of invertible elements of $\mathrm{End}(x)$. So the last step will be:

Lemma 3.5. *If $N \neq 0$, there exists an open dense orbit x^H in $\hat{N}$ such that $\mathrm{End}(x) = A'$ is also a Dynkin algebra and $|A'| < |P|$.*

We then put $\tilde{G} = p^{-1}(x^H)$ (cf. theorem 1.1) and obtain $\tilde{G} \cong \hat{H}_x$. But $H_x = G(A', A')$ has just the necessary form.

In order to prove lemma 3.5 we have to elaborate some technical tools which allow us to investigate some kinds of "matrix problems", e.g. those arising from actions of linear groups on bimodules as above.

§4 Representations of bocses

Recall ([R], [D]) that a *bocs* is a pair $\underline{A} = (A, V)$ where A is a category and V an A-coalgebra, i.e. an A-bimodule supplied with bimodule homomorphisms $m : V \to V \otimes_A V$ (comultiplication) and $e : V \to A$ (counit) satisfying the usual rules (cf. [M]). We always suppose that the bocs is *normal*, i.e. for each object i there is an element $u_i \in V(i,i)$ such that $e(u_1) = 1_i$ and $m(u_i) = u_i \otimes u_i$. Let $\overline{V} = \mathrm{Ker}\ e$, the *kernel* of the bocs.

For any $a \in A(i,j)$ and $v \in \overline{V}(i,j)$ let $Da = au_i - u_j a$ and $Dv = m(v) - vu_i - u_j v$ (here and later we write xy instead of $x \otimes y$ for elements x, y of the bimodule V). One easily sees that $Da \in \overline{V}$ and $Dv \in \overline{V} \otimes_A \overline{V}$. The mapping D is called the *differential* of the bocs (it really induces a differential on the (graded) tensor category of the bimodule $\overline{V}$). It is clear that knowing the kernel and the differential we can reconstruct the bocs.

A *representation* of a bocs $\underline{A}$ is defined as a functor $M : A \to \mathrm{Vect}$ (the category of finite-dimensional vector spaces). Given another representation N, we define $\mathrm{Hom}_{\underline{A}}(M, N)$ as the set of all bimodule homomorphisms $V \to (M, N)$, where (M, N) is the A-bimodule such that $(M, N)(i,j) = \mathrm{Hom}(M(i), N(j))$ and left (right) multiplication by a is defined as left (right) multiplication by $N(a)$ (resp. by $M(a)$). If $f : M \to N$ and $g : N \to L$ are two such homomorphisms, their product gf is defined as composition:

$$V \xrightarrow{m} V \otimes_A V \xrightarrow{g \otimes f} (N, L) \otimes_A (M, N) \xrightarrow{\mathrm{mult}} (M, L)$$

the last arrow being induced by the usual multiplication of mappings. One can prove that in this way we obtain the category $R(\underline{A})$ of representations of the bocs $\underline{A}$.

A morphism from a bocs $\underline{A} = (A, V)$ to another bocs $\underline{B} = (B, W)$ is by definition a pair $F = (F_0, F_1)$ consisting of a functor $F_0 : A \to B$ and a homomorphism of A-bimodules $F_1 : V \to W$ compatible with the comultiplications and counits of V and W ([D]). Such a morphism induces a natural functor $F^* : R(\underline{B}) \to R(\underline{A})$.

Suppose given a bocs $\underline{A} = (A, V)$, a category B and a functor $F : A \to B$ such that any object of B is isomorphic to a direct summand of some Fi. Construct a new bocs $\underline{A}^F = (B, B \otimes_A V \otimes_A B)$ and a morphism $(F, F_1) : \underline{A} \to \underline{A}^F$ where $F_1(v) = 1_{Fi} \otimes v \otimes 1_{Fj}$ for $v \in V(i,j)$ (we denote this morphism also by F). The restriction imposed on F involves the surjectivity of the counit in $\underline{A}^F$ and then we obtain easily from general nonsense

Proposition 4.1. *In the above situation the induced functor $F^* : R(\underline{A}^F) \to R(\underline{A})$ is fully faithful and its image consists of those representations $M : A \to$ Vect which can be factored through F.*

A bocs $\underline{A} = (A, V)$ is called *free* provided A is a free category, i.e. the path category KQ of some graph Q and the kernel $\overline{V}$ is a free A-bimodule. Such a bocs is usually described by its bigraph $Q_{\underline{A}} = (Q_0, S_0, S_1)$ whose vertex set is the same as that of Q but as well as the set $S_0 = Q_1$ of arrows of Q (free generators of A) there is an additional set of arrows S_1 corresponding to free generators of the A-bimodule $\overline{V}$: if such a generator lies in $V(i,j)$, the corresponding arrow goes from i to j (usually the arrows of S_0 are called *solid* and those of S_1 *dashed*). To define $\underline{A}$ we need also to know its differential D and of course it has to be defined only for arrows, i.e. for free generators. The set $S = S_0 \cup S_1$ is called the set of free generators of $\underline{A}$.

Notice that in general we can change the set of free generators and a good choice of it sometimes plays an important rôle (e.g. this is the case in the definition of triangularity below). Nevertheless, the bigraph $Q_{\underline{A}}$ does not depend on this choice. We call $\underline{A}$ *connected* if its bigraph is connected. If $i, j \in \text{Ob } A$, let $S(i,j) = \{s \in S \mid s : i \to j\}$ and similarly for $S_0(i,j)$, $S_1(i,j)$.

A system $S = S_0 \cup S_1$ of free generators is said to be *triangular* provided there exists a function $h : S \to \mathbb{N}$ such that $h(Ds) < h(s)$ for any $s \in S$ where $h(Ds)$ denotes the maximum of $h(b)$ for all $b \in S$ which occur when we express Ds via generators from S (as $\underline{A}$ is free, such an expression is unique). A free bocs possessing a triangular system of free generators is called *triangular*. It is known (cf. [RK]) that if $\underline{A}$ is triangular, then any idempotent in $R(\underline{A})$ splits and a homomorphism $f \in \text{Hom}_{\underline{A}}(M, N)$ is invertible if and only if $f(u_i)$ is invertible for each $i \in \text{Ob } A$.

The notion of coverings can also be defined for bocses. Namely, a morphism

$$F = (F_0, F_1) : \underline{\tilde{A}} = (\tilde{A}, \tilde{V}) \to \underline{A} = (A, V)$$

is called a *covering* if F_0 is a covering of categories and for all objects $i \in \mathrm{Ob}\, A$, $j \in \mathrm{Ob}\, \tilde{A}$, F_1 induces bijections:

$$\bigoplus_{Fk=i} \tilde{V}(k,j) \to V(i,Fj) \quad \text{and} \quad \bigoplus_{Fk=i} \tilde{V}(j,k) \to V(Fj,i).$$

A connected bocs (free, triangular) will be called *simply connected* if it possesses no non-trivial connected covering.

The advantage of bocses is that they admit plenty of "reduction algorithms" based on proposition 4.1. A general scheme for producing such algorithms is the following one. Suppose that A' is a subcategory of A and $F' : A' \to B'$ is a functor. Consider the amalgamation (or push-out) B of B' and A with respect to F' and the inclusion of A' into A. Then universality of amalgamation and proposition 4.1 imply for the natural functor $F : A \to B$

Proposition 4.2. *The functor $F^* : R(\underline{A}^F) \to R(A)$ is an equivalence between $R(\underline{A}^F)$ and the full subcategory of $R(\underline{A})$ consisting of all representations $F : A \to \mathrm{Vect}$ whose restrictions to A' can be factored through F'.*

In most cases (cf. [RK], [R], [D]) A' and F' are chosen in such a way that any functor $A' \to \mathrm{Vect}$ can be factored through F', so F^* is an equivalence of categories. However for our present purposes an algorithm will be useful to deal with other cases.

Recall also some notions related to representations of bocses. If the category A contains finitely many (say, n) objects, then the vector-dimension $\underline{\dim}\, M = (d_1, ..., d_n)$ is defined as in §2 and we put $|M| = d_1 + ... + d_n$. Suppose now that $\underline{A}$ is free with a finite set of free generators. Denote by s_{ij} the cardinality of $S_0(i,j)$, and by t_{ij} that of $S_1(i,j)$. Then the bilinear form $(,)_{\underline{A}}$ on $\mathbb{R}^n$ is defined as follows:

$$(X,Y)_{\underline{A}} = \sum_i x_i y_i - \sum_{i,j} s_{ij} x_i x_j + \sum_{i,j} t_{ij} x_i y_j.$$

The corresponding quadratic form $q_{\underline{A}}(X) = (X,X)_{\underline{A}}$ is called the *Tits form* of the bocs $\underline{A}$.

Fixing a vector $\underline{d} \in \mathbb{R}^n$, denote by $R_{\underline{d}}(\underline{A})$ the set of all representations of $\underline{A}$ of vector-dimension $\underline{d}$. Fixing bases in all $M(i)$ we can consider these representations as sets of matrices, thus $R_{\underline{d}}(\underline{A})$ as an algebraic variety over K (really, an affine space, as our bocs is free and hence any set of matrices of prescribed dimensions defines a representation of $\underline{A}$).

§5 Small reduction

We are going to use proposition 4.2 in the following situation. Let $\underline{A}$ be a free triangular bocs given by its triangular system S of free generators and its differential D. Suppose there is $a \in S_0(i,j)$ with $i \neq j$ and $Da = 0$ (call such a a *minimal* edge). Denote by A' the subcategory of A consisting of two objects i, j and the single morphism a. Take for B' the trivial category $K\{i,j\}$ (i.e. the category consisting of the objects i and j and scalar multiples of the identity morphisms only) and for the functor $F' : A' \to B'$, that mapping i to $i \oplus j$, j to j and a to the morphism $i \oplus j \to j$ given by the matrix $(\,0 \quad 1_i\,)$ (we really consider rather the additive envelope add B' of the category B'). Now we can construct the amalgamation B and the functor $F : A \to B$ prolonging F' and hence construct a new bocs $\underline{A}^F$ and the functor $F^* : R(\underline{A}^F) \to R(\underline{A})$. Call $\underline{A}^F$ the *small reduction* of $\underline{A}$ via a in direction ij. The small reduction of $\underline{A}$ via a in direction ji is defined similarly: we only have to take for F' the functor mapping i to i, j to $i \oplus j$ and a to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Now proposition 4.2 implies

Proposition 5.1. *If $\underline{A}^F$ is the small reduction of a minimal edge a in direction ij (ji), then F^* establishes an equivalence between $R(\underline{A}^F)$ and the full subcategory of $R(\underline{A})$ consisting of all representations M with* rank $M(a) =$ dim $M(j)$ *(resp.* rank $M(a) =$ dim $M(i)$*).*
Moreover, in this case $\underline{A}^F$ is also a free triangular bocs and if both $M(i)$ and $M(j)$ are non-zero for some $M = F^*(N)$, then $|M| > |N|$.

(The last assertions are proved just as they are in [D] for the usual reduction of a minimal edge.)

Corollary 5.2. *Under the conditions of proposition 5.1 suppose that a vector-dimension $\underline{d}$ of representations of $\underline{A}$ is given such that $d_j \leq d_i$ (resp. $d_i \leq d_j$). Then there is an open and dense (in the Zariski topology) subset $U \subseteq R_{\underline{d}}(\underline{A})$ such that $U \subseteq$ Im F^*.*

We need a precise description of $\underline{A}^F$ given in [BGOR] (it can also be easily obtained using calculations similar to [D]) in terms of its bigraph $Q' = Q_{\underline{A}}F$. Suppose that the minimal edge a was reduced in direction ij (for the ji case the answer is analogous). Then Q' and the new differential D' are as follows. The set of vertices of Q' coincides with that of $Q = Q_{\underline{A}}$. The set of free generators (arrows) S' is obtained from S by deleting a, adding instead of it a new arrow $a' \in S'_1(j,i)$ (note that a' differs from a both in direction and in type: it is dashed while a was solid) and extra arrows b' corresponding to the arrows $b \in S$ with source or target i (for the sake of simplicity suppose that $S(i,i) = \emptyset$ as it is in the only case we need). Namely, if $b \in S_m(i,k)$ $(m = 0,1)$, then $b' \in S'_m(j,k)$ and if $b \in S_m(k,i)$, then $b' \in S'_m(k,j)$. In other words, corresponding to each arrow (other than a) incident on i, we adjoin another incident on j having the same type and direction.

To define D' we introduce some notation. If $w = ...ab_1...ab_2...$ is an (oriented) path in Q where all inclusions of a are shown, put $\tilde{w} = ...b_1'...b_2'...$ (all a are excluded and each right neighbour b of some a replaced by the corresponding b', if a does not occur in w, then $\tilde{w} = w$); if $w = pa$, i.e. a is the last arrow of w, then $\tilde{w} = 0$. For $w = pa$ put $w' = pa'$, otherwise $w' = w$; finally for $w = ap$ put $'w = a'p$, otherwise $'w = w$. Extend these three operations by linearity to all elements of A, $\overline{V}$ and $\overline{V} \otimes_A \overline{V}$. Now we are able to describe D':

- if $b \in S(k,l)$ and $l \neq i$, then $D'b = \widetilde{Db}$;
- if $l = i$, then $D'b = \widetilde{Db} + a'b'$ and $D'b' =' (\widetilde{Db})$;
- if $k = i$, then $D'b' = (\widetilde{Db})' + ba'$;
- $Da' = 0$.

A useful consequence of this description is that we can determine whether a bocs was really obtained from another one by a small reduction of some minimal edge.

Proposition 5.3. *Suppose a bocs $\underline{A}$ is given by its quiver and there is a dashed arrow $a' \in S_1(j,i)$ for $j \neq i$ and there exists an injective mapping $b \mapsto b'$ putting in correspondence to each $b \in S_m(i,k)$ or $S_m(k,i)$ an arrow $b' \in S_m(j,k)$ or $S_m(k,j)$, respectively, such that:*

- *if $b \in S(i,k)$, then $Db' = (Db)' + ba'$;*
- *if $b \in S(k,i)$, then $Db = w + a'b'$ and $Db' = 'w$ for some element w (not necessarily a word);*

where the operations w' and $'w$ are defined as above. Then there exists a bocs $\underline{B}$ (with the same set of objects) and a minimal edge $a : i \to j$ in $\underline{B}$ such that $\underline{A}$ is the small reduction of $\underline{B}$ via a in direction ij.

(Of course, the analogous result is valid for a small reduction in direction ji). We shall call such an arrow a' *integrable* and the bocs $\underline{B}$ *integrated* from $\underline{A}$ via a' in direction ij (or ji as the case may be).

Recall also the algorithm of "regularisation" (cf. [RK], [D]). An arrow $a \in S_0(i,j)$ is called *irregular* provided $Da \in S_1(i,j)$ (this depends on the choice of free generators). Define A' and B' as for the algorithm of small reduction but put now $F'(i) = i$, $F'(j) = j$ and $F'(a) = 0$ (notice that now it is possible that $i = j$). Again we are able to construct the amalgamation B and the functor $F : A \to B$ prolonging F', but in this case any representation of $\underline{A}$ can easily be shown to be equivalent to some M with $M(a) = 0$, thus proposition 4.2 implies

Proposition 5.4. *In the above situation $F^* : R(\underline{A}^F) \to R(\underline{A})$ is an equivalence of categories.*

Small reductions and regularisation are compatible with the Tits forms of bocses.

Proposition 5.5. *If a bocs $\underline{B}$ is obtained from another one $\underline{A}$ by small reduction or regularisation, then their bilinear forms $(,)_{\underline{A}}$ and $(,)_{\underline{B}}$ are equivalent (hence so are also their Tits forms $q_{\underline{A}}$ and $q_{\underline{B}}$).*
Proof. For regularisation it is obvious that even $(,)_{\underline{A}} = (,)_{\underline{B}}$. For small reduction, say in direction ij, one can easily check using the above description of $\underline{B}$ that $(,)_{\underline{B}}$ is obtained from $(,)_{\underline{A}}$ if we change, in the standard basis $\{e_i\}$ of $\mathbb{R}^n$, the vector e_j to $e_j + e_i$.

Small reduction and regularisation are also compatible with coverings of bocses.

Proposition 5.6. *If a bocs $\underline{B}$ is a small reduction of another bocs $\underline{A}$, then $\underline{A}$ and $\underline{B}$ are either both simply connected or both not.*
Proof. Clearly, either both $\underline{A}$ and $\underline{B}$ are connected or neither is. Suppose they are connected and $F : \tilde{\underline{A}} \to \underline{A}$ is a non-trivial connected covering. Let $\underline{B}$ be the small reduction of a minimal edge $a : i \to j$, say in direction ij. Then one can choose for each object i' of $\tilde{\underline{A}}$ such that $Fi' = i$ an object j' with $Fj' = j$ and an arrow $a' : i' \to j'$ with $Fa' = a$ (hence $Da' = 0$). But then we can apply to $\tilde{\underline{A}}$ the small reduction of all a' (each in direction $i'j'$) and obtain a commutative diagram:

$$\begin{array}{ccc} \tilde{\underline{A}} & \to & \tilde{\underline{B}} \\ F\downarrow & & G\downarrow \\ \underline{A} & \to & \underline{B} \end{array}$$

in which G is again a covering (for more details, cf. [DOF]). The description of small reduction shows immediately that $\tilde{\underline{B}}$ is connected, thus $\underline{B}$ is not simply connected.

Conversely, let $G : \tilde{\underline{B}} \to \underline{B}$ be a non-trivial connected covering of $\underline{B}$ and $a' : j \to i$ the new dashed arrow in $\underline{B}$, hence integrable in the sense of proposition 5.3. Then again for any object i' of $\tilde{\underline{B}}$ such that $Gi' = i$ one can choose j' with $Gj' = j$ and an integrable arrow $a'' : j' \to i'$. Thus we can integrate all a'' and obtain a covering $F : \tilde{\underline{A}} \to \underline{A}$ which is also non-trivial and connected.

Analogous arguments give

Proposition 5.7. *If $\underline{B}$ was obtained from $\underline{A}$ by regularisation and $\underline{A}$ was simply connected, then each component of $\underline{B}$ is also simply connected.*

Of course, in this case $\underline{B}$ need not be connected. Moreover, easy examples show that $\underline{B}$ can be simply connected even if $\underline{A}$ was not.

§6 Proof of the main theorem

Recall that in order to accomplish the proof of Theorem 3.1, we have only to prove lemma 3.5 and lemma 3.3 (iv). To do this, we need a special type of bocs related to bimodules (cf. [D]).

Suppose given two categories A_1, A_2 and an A_2-A_1-bimodule U. It is convenient here to consider A_1 and A_2 additive but with finite-dimensional skeletons with object sets I_1 and I_2 respectively. Define the category $C(U)$ of "elements of the bimodule U" which has for objects the elements of all $U(i,j)$ with $i \in \text{Ob } A_2$, $j \in \text{Ob } A_1$. A homomorphism from $u \in U(i,j)$ to $u' \in U(i',j')$ is defined as a pair (a,b) with $a : i \to i'$ and $b : j \to j'$ such that $bu = u'a$.

Categories of type $C(U)$ coincide with those of bocs representations, as the following theorem, proved in [D], shows.

Theorem 6.1. *Suppose that, in the above situation, the categories A_1, A_2 are split and all spaces $U(i,j)$ are finite-dimensional. Then there exists a free triangular bocs $\underline{A}$ such that categories $C(U)$ and $R(\underline{A})$ are equivalent.*

We will need also the precise construction of the bocs $\underline{A}$ which will be given in terms of its bigraph $Q_{\underline{A}}$ and differential D. Namely, the vertex set of $Q = Q_{\underline{A}}$ will be $I_1 \cup, I_2$. All solid arrows go from some vertex $i \in I_2$ to some $j \in I_1$ and the solid arrows $a : i \to j$ are in 1-1 correspondence with a basis of the dual vector space $U(i,j)^*$. All dashed arrows are going from some i to j where i and j are both either in I_1 or in I_2 and the dashed arrows $b : i \to j$ are in 1-1 correspondence with a basis of $\text{rad}(i,j)^*$, where rad denotes the radical of the appropriate category. The differential D is defined on a solid arrow $a : i \to j$ as $l^*(a) - r^*(a)$ and on a dashed arrow $b : i \to j$ as $m^*(b)$ where l^*, r^*, m^* are the mappings dual to l, r, m respectively, where

$$l : \bigoplus_k \text{rad}(k,j) \otimes U(i,k) \to U(i,j)$$

$$r : \bigoplus_k U(k,j) \otimes \text{rad}(i,k) \to U(i,j)$$

$$m : \bigoplus_k \text{rad}(k,j) \otimes \text{rad}(i,k) \to \text{rad}(i,j)$$

are generated by multiplications.

It is convenient to consider elements of U as matrices with entries in $U(i,j)$. Namely, let $v \in U(x,y)$. Decompose $x = \oplus_{d_j} j$ with $j \in I_1$, $y = \oplus_{d_j} i$ with $i \in I_2$. Then v will be considered as a block matrix (v_{ij}), where v_{ij} is a matrix of size $d_i \times d_j$ with coefficients from $U(i,j)$. Then we obtain a representation of the bocs $\underline{A}$ corresponding to v by mapping each vertex $i \in I_1 \cup I_2$ to a d_i-dimensional vector space and any solid arrow

$a : i \to j$, i.e. an element of $U(i,j)^*$, to the linear map defined by the matrix $a(v_{ij})$ (applying a to a matrix component-wise). Thus the dimension of this representation is the vector $\underline{d} = (d_i)$.

Returning to the proof of the main theorem, recall that in lemma 3.5 we are considering the A_2-A_1-bimodule $\hat{N} = \mathrm{Hom}_A(P^2, P^1)^*$, where $A_m = \mathrm{End}_A(P^m)$, $P^1 = k_1 P_1$ and $P^2 = k_2 P_2 \oplus ... \oplus k_n P_n$; $P_1, ..., P_n$ are all non-isomorphic, indecomposable projective modules over a Dynkin algebra A and $\mathrm{Hom}_A(P_1, P_i) = 0$ if $i \neq 1$. Of course, the categories A_1 and A_2 are not additive, but we are able to replace them by additive ones, say, that of all modules of the form $d_1 P_1$ and that of all modules of the form $d_2 P_2 \oplus ... \oplus d_n P_n$ respectively. So we can define the bimodule U with $U(X,Y) = \mathrm{Hom}_A(Y,X)^*$ and obtain $N = U(P^1, P^2)$. Clearly, isomorphic elements of N in the category $C(U)$ are just those lying in one H-orbit, and the endomorphism ring $\mathrm{End}(x)$ of an element $x \in N$ defined before the lemma 3.5 is in fact its endomorphism ring in the category $C(U)$.

Applying theorem 6.1, we construct the corresponding bocs $\underline{A}$. Its vertices correspond to modules $P_1, ..., P_n$ (and will be denoted by $1, ..., n$). The only solid arrows are those from 1 to some $i \neq 1$ corresponding to a basis of $A(i,1) = \mathrm{Hom}_A(P_i, P_1) = U(P_1, P_i)^*$. Since $A(1,1) = K$ (there are no oriented cycles in Q_A), there are no dashed arrows $1 \to 1$. If $i, j \neq 1$, the dashed arrows $b : i \to j$ correspond to a basis of $\mathrm{rad}(i,j)^*$. Hence, there are no loops and at most one arrow $i \to j$ if $i \neq j$ (it exists if $A(i,j) \neq 0$). The differential of $\underline{A}$ arises from multiplication in A. Namely, using the above description, one can verify that for an arrow $a : i \to j$ it has the form: $Da = \sum b_k c_k$, where the sum is taken over all vertices k such that there are arrows $c_k : i \to k$, $b_k : k \to j$ for which the corresponding products in A are non-zero (if a is solid, hence $i = 1$, we need $A(j,1)A(k,j) \neq 0$ and if a is dashed, then $A(k,j)A(i,k) \neq 0$). As the algebra A can be reconstructed from the bocs $\underline{A}$, it is quite obvious that any covering of one of them provides a covering for the other, whence we have

Proposition 6.2. *The bocs $\underline{A}$ is simply connected.*

Consider the bilinear form $(,)_{\underline{A}}$ of the bocs $\underline{A}$. By definition, it is:

$$(X,Y)_{\underline{A}} = \sum_i x_i y_i - \sum_{i \neq 1} s_{1i} x_1 y_i + \sum_{i,j \neq 1} s_{ij} x_i y_j$$

where s_{ij} is the number of arrows from i to j (in our case the type of an arrow is prescribed by its ends). But $s_{1i} = \dim A(i,1) = c_{1i}$ and, for $i \neq 1$, $s_{ij} = \dim A(i,j) = c_{ij}$, where the c_{ij} are the entries of the Cartan matrix C_A of the algebra A. Then an easy calculation shows that $(,)_{\underline{A}}$ is equivalent to the bilinear form with Cartan matrix C_A. So we have

Proposition 6.3. *The bilinear forms* $(\,,\,)_{\underline{A}}$ and $(\,,\,)_A$ *are equivalent. Thus the Tits form* $q_{\underline{A}}$ *is positive definite.*

The next lemma is the key to the whole proof. Define a *road* in the bigraph $Q = Q_{\underline{A}}$ as a path in the graph obtained from Q by adding a new arrow a^{-1} for each a of Q (no matter of what type) going in the opposite direction. In other words, a road is a non-oriented chain of arrows in contrast with paths which are always supposed oriented. If the source of a road w coincides with its target, call w a *circle* (again in contrast to cycles which are supposed oriented). A road (in particular, a circle) will be called *clean* if all arrows with their sources and targets in the vertices of the road are themselves in the road. In particular, if a path p of Q occurs in Da for an arrow a (with non-zero coefficient, of course), then $a^{-1}p$ is a circle in Q and is called an *active circle* with *marked arrow* a. All other circles will be called *passive*.

Lemma 6.4. *If* $\underline{A}$ *is a simply connected bocs with positive definite Tits form* $q = q_{\underline{A}}$*, then every clean circle in its bigraph* Q *is active.*

Leaving the proof of this lemma (which is rather cumbersome) to the end of the section, we now show how it implies all the results needed. First of all, since the conclusion of the lemma is, of course, sufficient for $\underline{A}$ to be simply connected, we obtain

Corollary 6.5. *Any bocs* $\underline{B}$ *obtained from* $\underline{A}$ *by deleting some vertices (together with all arrows which they are ends of) is also simply connected.*

Proof of lemma 3.3 (iv). (Cf. the notation of this lemma.) If $ab = 0$, then, of course, $A(j,1)A(j,i) = 0$ as these spaces are 1-dimensional. Now if $A(j,1) \neq 0$, there are arrows $1 \to i$, $1 \to j$ and $j \to i$ in $\underline{A}$, which form a clean passive circle, so we have a contradiction to proposition 6.2.

Now, as $\hat{N}$ was identified with the set of representations of $\underline{A}$ of dimension $(k_1, ..., k_n)$, the proof of lemma 3.5 follows from the more general result:

Lemma 6.6. *Let* $\underline{A}$ *be a simply connected bocs with positive definite Tits form, and* $\underline{d} = (d_1, ..., d_n)$ *be a fixed dimension for representations of* $\underline{A}$*. Then in the set of all representations of* $\underline{A}$ *of dimension* $\underline{d}$ *there is an open dense subset consisting of isomorphic representations and, if* M *is one of them, then* $\mathrm{End}_A(M)$ *is a Dynkin algebra. Moreover, if there exists a solid arrow* $a : j \to i$ *with* $d_i d_j \neq 0$*, then* $|\mathrm{End}_{\underline{A}}(M)| < |\underline{d}|$ *if* a *is regular.*

Proof. Corollary 6.5 allows us to assume that all $d_i \neq 0$. Now use induction on $d = d_1 + ... + d_n$. If $\underline{A}$ has no solid arrows at all, then its representation is completely defined by its dimension, so there exists, up to isomorphism, only one representation M of dimension $\underline{d}$. Its endomorphism ring is, by definition,

$\mathrm{Hom}_{A\text{-}A}(V,(M,M))$. Since the category A is trivial, this endomorphism ring is isomorphic to:

$$\bigoplus_{i,j} \mathrm{Hom}(V(i,j),\mathrm{Hom}(M(i),M(j))) = \bigoplus_{i,j} M(j) \otimes V(i,j)^* \otimes M(i)^*.$$

Denote the last expression simply as $M \otimes V^* \otimes M^*$. One can check that the multiplication of $\underline{A}$-homomorphisms corresponds via this isomorphism to the composition

$$M \otimes V^* \otimes M^* \otimes M \otimes V^* \otimes M^* \xrightarrow{\mathrm{ev}} M \otimes V^* \otimes V^* \otimes M^* \xrightarrow{1\otimes m^*\otimes 1} M \otimes V^* \otimes M^*,$$

where m is the comultiplication in V and ev is generated by the evaluation map $M^* \otimes M \to K$. But $m^* : V^* \otimes V^* \to V^*$ turns V^* into an algebra. Since $\underline{A}$ was simply connected, this algebra is easily shown to be simply connected too and one can check that the Cartan matrix of V^* is just that of the bilinear form $(,)_{\underline{A}}$. Hence V^* is a Dynkin algebra. To each vertex i of the bigraph $Q_{\underline{A}}$ corresponds a projective indecomposable V^*-module P_i' and if we put $P' = d_1P_1' \oplus ... \oplus d_nP_n'$, then $M \otimes V^* \otimes M^*$ may be identified with $\mathrm{End}_{V^*}(P')$ which is therefore also Dynkin. Note that in this case $|\mathrm{End}_{\underline{A}}(M)| = |P'| = |\underline{d}|$.

If there are solid arrows in $\underline{A}$, take one of them, say $a : i \to j$ with $h(a)$ a minimal value of the function h used in the definition of triangular bocses. Then either $Da = 0$ or $Da = b$ for some $b \in S_1$. Note that in the first case $i = j$ is impossible as $\underline{A}$ is simply connected with positive Tits form. Thus we may apply to a either small reduction or regularisation. The latter does not change the dimension but reduces the number of arrows. As for the former, proposition 5.1 and corollary 5.2 allow us to choose the direction of reduction in such a way that all representations of $R_{\underline{d}}(\underline{A})$ lying in some open dense subset have the form $F^*(M)$ for some representation M of the reduced bocs $\underline{A}^F$ with dimension $\underline{d}'$ where $|\underline{d}'| < |\underline{d}|$. Propositions 5.5 and 5.6 show that the new bocs is also simply connected with positive definite Tits form, so an obvious induction completes the proof.

The proof of lemma 6.4 requires some topological methods, though these are rather simple and available from any elementary course, say [AB].

First of all define the fundamental group of the bocs $\underline{A}$ to be the group consisting of equivalence classes of circles containing some chosen vertex i under the equivalence relation

$$a^{-1}a \sim aa^{-1} \sim 1 \quad \text{for any arrow } a,$$
$$w \sim 1 \quad \text{for any active circle } w.$$

(Group multiplication is, of course, composition of circles.) If this group is non-trivial, then one can build a non-trivial covering of $\underline{A}$ using practically the same construction as in [BG]. But it is quite obvious that the group defined above coincides with the fundamental group of the cell complex Q^* obtained by attaching a 2-dimensional cell to each active circle of the bigraph Q. I am going to show that, if $\underline{A}$ has positive definite Tits form and contains a passive circle, then even $H_1(Q^*) \neq 0$.

On the contrary, suppose $H_1(Q^*) = 0$. Choose a minimal subcomplex S of Q^* such that: (i) $H_1(S) = 0$; (ii) S contains a clean passive circle w_0; (iii) with any two points, S contains all arrows connecting them, if such arrows exist in Q.

Recall that if X, Y are two subcomplexes of Q^*, there is a Mayer-Vietoris exact sequence for homologies:

$$\begin{aligned} 0 \longrightarrow & H_2(X \cap Y) \longrightarrow H_2(X) \oplus H_2(Y) \longrightarrow H_2(X \cup Y) \xrightarrow{d_2} \\ & H_1(X \cap Y) \longrightarrow H_1(X) \oplus H_1(Y) \longrightarrow H_1(X \cup Y) \xrightarrow{d_1} \\ & H_0(X \cap Y) \longrightarrow H_0(X) \oplus H_0(Y) \longrightarrow H_0(X \cup Y) \longrightarrow 0. \end{aligned} \qquad \text{(MV)}$$

We use it to show that $H_2(S) = 0$. Otherwise take a 2-dimensional cycle z on S and a cell X which really occurs in z. We can suppose that $z = X + \ldots$. Put $Y = S - \overset{\circ}{X}$ (where $\overset{\circ}{X}$ is the interior of X). Then in (MV)

$$H_1(X \cap Y) = \mathbf{Z} = \operatorname{Im} d_2, \quad X \cup Y = S,$$

whence $H_1(Y) = 0$ in contradiction to the minimality of S.

We prove now that S contains only one passive circle. If there were two of them, w and w', then there would be an edge a belonging to w' but not to w. Let X be the closure of a 2-dimensional cell containing a. Put $Y = S - (\overset{\circ}{X} \cup \overset{\circ}{a})$. Then $X \cup Y = S$ and $X \cap Y = B - \overset{\circ}{a}$, where B is the boundary of X. Thus (MV) implies $H_1(Y) = 0$ and, since Y contains w, this contradicts the minimality of S. Thus w_0 is the only passive circle in S.

Now it is evident that each arrow a belonging to S lies either on w_0 or on some active circle such that the corresponding cell is in S (we shall say then that this circle is active in S). Otherwise apply (MV) for $X = a$, $Y = S - \overset{\circ}{a}$. Again $X \cup Y = S$, so $d_1 = 0$ while $X \cap Y$ consists of 2 points, thus $H_0(X \cap Y) = \mathbf{Z} \oplus \mathbf{Z}$ whence also $H_0(Y) = \mathbf{Z} \oplus \mathbf{Z}$. This means that Y consists of 2 connected components and we can diminish S replacing it by the component of Y containing w_0.

Call a circle of Q *even* if it contains an even number of dashed arrows and *odd* otherwise. I claim that Q contains no even clean circles. To see this, let w be such a circle. Consider the quadratic form q_w obtained from q by putting to 0 all variables except those corresponding to the vertices of w. It has the form:

$$q_w(x_1, \ldots, x_m) = \sum_i x_i^2 + \sum_i (\pm x_i x_{i+1}), \qquad (x_{m+1} = x_1)$$