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REJECTION LEMMA AND ALMOST SPLIT SEQUENCES

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We study the behavior of almost split sequences and Auslander–Reiten quivers of the orders under rejection of bijective modules as defined in [Yu. A. Drozd and V. V. Kirichenko, *Izv. Akad. Nauk SSSR, Ser. Mat.,* **36**, 328 (1972)]. In particular, we establish the relations for stable categories and almost split sequences of an order A and the order A' obtained from A by the indicated rejection. These results are improved for the Gorenstein and Frobenius cases.

1. Introduction

Bijective moduli and the "rejection lemma" [6] play an important role in the theory of orders and lattices, as well as the Gorenstein (i.e., self-bijective) orders (see, e.g., [6, 7, 11, 12, 17]). Almost split sequences and Auslander–Reiten quivers are also of high importance. In the present paper, we consider the behavior of almost split sequences and Auslander–Reiten quivers under the rejection of bijective modules. In Sec. 2, we recall some general facts about the orders, lattices, and duality but in a more general case because we do not make an assumption that the principal commutative ring is a ring of discrete estimate. However, all basic results of the "classical" theory (as in [4]) remain valid. In Sec. 3, we introduce bijective lattices and Gorenstein orders, prove the rejection lemma in a more general form, and obtain some results connected with it. In particular, we determine the lattices that become projective and injective after rejection (Theorem 3.1). Section 4 is devoted to the *Bass orders*, i.e., the orders all superrings of which are Gorenstein. Theorem 4.1 is the main result of this section. This theorem significantly generalizes the Bass criterion presented in [6]. In Sec. 5, we consider stable categories and connections of a stable category of order A and a stable category of order A' obtained by rejection of a bijective module (Theorem 5.1). In Sec. 6, we study almost split sequences and establish the description of almost split sequences of order A in terms of the A'-modules (Proposition 6.2 and Theorem 6.1). Finally, in Sec. 7, we improve these results in the case of Gorenstein and Frobenius orders.

The present paper is dedicated to the bright memory of my friend, colleague, and many-year coauthor Volodymyr Kyrychenko with whom we enthusiastically studied the structures of modules 50 years ago and were quite happy to discover the rejection lemma.

2. Orders, Lattices, and Duality

In what follows, R denotes a *complete local commutative Noetherian ring without nilpotent ideals of Krull dimension* 1 with the maximal ideal $\mathfrak{m}$, the residue field $\Bbbk = R/\mathfrak{m}$, and the complete ring of quotients K. It follows from [3] that this ring is a Cohen–Macaulay ring. By $R\text{-mod}$ we denote a category of finitely generated R-modules and by $R\text{-lat}$ we denote its complete subcategory that consists of *R-lattices*, i.e., *torsion-free* R-modules M,

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or such that the canonical mapping $M \to K \otimes_R M$ is an immersion. Then we write KM instead of $K \otimes_R M$ and identify M with $1 \otimes M \subseteq KM$. Note that, in this case, the R-lattices are equivalent to the *maximal Cohen–Macaulay modules.* Since R is complete, it has a *canonical module* [3] (Corollary 3.3.8), i.e., an R-lattice ω_R such that $\operatorname{inj.dim}_R \omega_R = 1$ and $\operatorname{Ext}^1_R(\Bbbk, \omega_R) = \Bbbk$. The functor $D\colon M \mapsto \operatorname{Hom}_R(M, \omega_R)$ is the *exact duality* in the category R-lat [3] (Theorem 3.3.10). This implies that if $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ is the exact sequence of lattices, then the sequence $0 \to DL \xrightarrow{D\beta} DM \xrightarrow{D\alpha} DN \to 0$ is also exact and the natural mapping $M \to DDM$ is an isomorphism. Since

$$\operatorname{End}_R(\omega_R) \simeq \operatorname{End}_R R \simeq R \qquad \text{and} \qquad \operatorname{End}_K KM \simeq K \operatorname{End}_R M$$

for each lattice M, we get $K\omega_R \simeq K$ and identify ω_R with its image in K. We also note that K is the direct product of fields:

$$K = \prod_{i=1}^{s} K_i,$$

where K_i is the field of quotients of the ring $R/\mathfrak{p}_i$ and $\mathfrak{p}_i$ runs through the minimal primary ideals of the ring R.

A semiprimary R-algebra A, which is an R-lattice, is called an *R-order* or simply an *order* if R is fixed. Recall that a *semiprimary* ring is a ring that does not contain nilpotent ideals. Then KA is a semisimple K-algebra. We say that A is an *R-order* in KA. By $Z(A)$ we denote the center of A and say that A is *central* if the natural mapping $R \to Z(A)$ is an isomorphism. If A is *connected*, i.e., cannot be decomposed as a ring, then its center is local, and vice versa. By A-mod we denote a category of finitely generated R-modules and by A-lat we denote its complete subcategory of *A-lattices*, i.e., (left) A-modules that are R-lattices. The restriction of the duality functor D to the category A-lat gives the exact duality between A-lat and A^{op}-lat, which is regarded as a category of *right* A-lattices. We set $\omega_A = \operatorname{Hom}_R(A, \omega_R)$. This is an A-bimodule and, moreover, for each (left or right) A-lattice M, its dual lattice DM is identified with $\operatorname{Hom}_A(M, \omega_A)$. Modules of *finite length* are called *finite modules*. The length of a module of this kind is denoted by $\ell_A(M)$. The length $\ell_{KA}(KM)$ is called the *width* of the A-lattice M and denoted by $\operatorname{wd}_A(M)$. It is easy to see that $\operatorname{wd}_A(M)$ is the maximal number m such that M contains the direct sum of m nonzero submodules or, which is the same, a chain of submodules

$$M = M_0 \supset M_1 \supset \ldots \supset M_m$$

all quotients M_i/M_{i+1} of which are lattices. Lattices of width 1 are called *L-irreducible*.[1]

Since the ring R is complete, every *finite R-algebra* (i.e., finitely generated as an R-module) is semiperfect [14]. Hence, the category of finitely generated modules over this algebra A is the Krull–Schmidt category. In particular, every indecomposable projective A-module is isomorphic to the direct summand of A and there exists a bijection between the classes of isomorphism of indecomposable projective modules (called *principal* A-modules) and the classes of isomorphism of simple A-modules that associates the principal module P with the module $P/\mathfrak{r}P$, where $\mathfrak{r} = \operatorname{rad} A$. For any finitely generated A-module M, there exists an epimorphism $\pi\colon P \to M$, where P is projective and $\operatorname{Ker}\pi \subseteq \mathfrak{r}P$. Here, the module P is determined to within an isomorphism. It is called a *projective cover* of the module M and denoted by $P_A(M)$. Sometimes, the epimorphism π is also called a projective cover of M despite the fact that it is defined only to within a factor, which is an automorphism of P. It is clear that π induces the isomorphism $P/\mathfrak{r}P \simeq M/\mathfrak{r}M$.

A *superring* of R-order A is defined as an R-order A' such that $A \subseteq A' \subset KA$. Then A'/A is a finite module and A'-lat is a complete subcategory in A-lat. An order is called *maximal* if it does not have proper

[1] Quite often, these lattices are called *irreducible*. However, in what follows, this term is used in a different context.

superrings. A superring of order A, which is the maximal order, is called its *maximal superring*. Similarly, a *supermodule* of A-lattice M is defined as an A-lattice M' such that $M \subseteq M' \subset KM$. If A' is a superring in A and M is an A-lattice regarded as a submodule in KM, then the A'-lattice $A'M$, which is a supermodule of M, is defined.

In seems likely that the result presented below is known. In the case where R is a ring of discrete estimate, it was proved in [4]. The general case can be easily reduced to the indicated case. However, we failed to find the corresponding reference in the literature.

Proposition 2.1.

1. *Each R-order A has a maximal superring.*
2. *The center of maximal order is the product of rings of discrete estimate.*
3. *A connected maximal order has, to within an isomorphism, a unique indecomposable lattice, which is L-irreducible.*
4. *Conversely, if the order has a unique indecomposable lattice, then it is connected and maximal.*

Proof. It is possible to assume that A is connected. Its center $Z(A)$ is complete and local and each superring A is a $Z(A)$-order. Hence, we can assume that $Z(A) = R$. Then $Z(KA) = K$. Let S be the integral closure of R in K. Since R is complete and local, it is a *marvelous* ring [16]. In particular, S is a finitely generated R-module. Since it is completely closed, it is the direct product of rings of discrete estimate. The ring SA is an S-order and a superring of A. It can be decomposed into the direct product of orders whose centers are rings of discrete estimate. Thus, by virtue of Theorem 26.5 in [4], we conclude that SA and, hence, also A have the maximal superring A' and $Z(A') = S$. All other assertions now follow from [4].

Proposition 2.1 is proved.

Since the algebra KA is semisimple, every finitely generated KA-module is embedded in a finitely generated free module. This immediately implies that each A-lattice M is embedded in a free A-module. Hence, the A-lattices are equivalent to the submodules of free modules.

Proposition 2.2. *Let $I \in A\text{-lat}$. Then the following conditions are equivalent:*

(1) $\operatorname{inj.dim}_A I = 1$*;*

(2) $\operatorname{Ext}^1_A(M, I) = 0$ *for all* $M \in A\text{-lat}$*;*

(3) $\operatorname{Ext}^i_A(M, I) = 0$ *for all* $M \in A\text{-lat}$ *and all* $i \geq 1$*;*

(4) any exact sequence $0 \to I \to N \to M \to 0$, *where* $M \in A\text{-lat}$, *splits;*

(5) $I \simeq DP$, *where P is a finitely generated projective A^{op}-module;*

(6) I is the direct summand ω_A^m for some m.

A lattice satisfying these conditions *is called L-injective.* If an L-injective lattice is indecomposable, then it is called *coprincipal.*

Proof. The implications (3) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (4) are obvious.

(2) $\Rightarrow$ (3) because, in the projective resolvent

$$\ldots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

of the module M, all modules $M_i = \operatorname{Im} d_i$ are lattices and $\operatorname{Ext}^i_A(M, I) \simeq \operatorname{Ext}^1_A(M_{i-1}, I)$ for $i > 1$.

(4) $\Rightarrow$ (5). By duality, condition (4) means that every exact sequence $0 \to M \to N \to DI \to 0$ splits. Since the indicated sequence with projective module N always exists, this implies that $P = DI$ is projective and $I \simeq DP$.

(5) $\Rightarrow$ (6). Since the projective module P is the direct summand of the free module A^m, the module $I = DP$ is the direct summand $D(A^m) = \omega_A^m$.

(6) $\Rightarrow$ (2). Let M be an A-lattice. Consider the exact sequence $0 \to N \to P \to M \to 0$ with projective module P. Since all these modules are lattices, the induced sequence

$$0 \to \operatorname{Hom}_A(M, \omega_A) \to \operatorname{Hom}_A(P, \omega_A) \to \operatorname{Hom}_A(N, \omega_A) \to 0$$

is also exact, which implies that $\operatorname{Ext}^1_A(M, \omega_A) = 0$. The same is also true for the module ω_A^m and its direct summand I.

(3) $\Leftrightarrow$ (1). It is known that

$$\text{inj.dim}\, I = \sup \left\{ i \mid \operatorname{Ext}^i_A(A/L, I) \neq 0 \ \text{ for some left ideal } \ L \right\}$$

$$= \sup \left\{ i \mid \operatorname{Ext}^{i-1}_A(L, I) \neq 0 \ \text{ for some left ideal } \ L \right\}.$$

Since each ideal is a lattice, we conclude that (3) $\Rightarrow$ (1). Conversely, if condition (1) is satisfied and M is a lattice, then we embed it in the projective module P. Thus,

$$\operatorname{Ext}^i_A(M, I) = \operatorname{Ext}^{i+1}_A(P/M, I) = 0 \quad \text{for} \quad i \geq 1,$$

i.e., condition (3) is satisfied.

Proposition 2.2 is proved.

The category A-lat becomes *exact* in a sense of [13] if ordinary short exact sequences, i.e., triples

$$N \xrightarrow{\alpha} M \xrightarrow{\beta} L, \quad \text{where} \quad \alpha = \operatorname{Ker} \beta \quad \text{and} \quad \beta = \operatorname{Cok} \alpha,$$

are regarded as *exact pairs (conflations)*. Thus, in this category, *deflations* are epimorphisms of modules and *inflations* are monomorphisms with kernels without torsion (we often use this terminology). This exact category has sufficiently many projective and injective objects, namely, their projective objects are finitely generated projective modules and their injective objects are L-injective lattices. To construct the conflation $M \to I \to N$ with L-injective I, it suffices to dualize the exact sequence $0 \to L \to P \to DM \to 0$ with projective P.

For lattices M and N, we write $M \searrow N$ (resp., $N \nearrow M$) if there exists a deflation $M^r \to N$ (resp., an inflation $N \to M^r$) for some r. In particular, $A \searrow M$ and, dually, $M \nearrow \omega_A$ for any lattice M. We write $N \Subset M$ if N is the direct summand of M^r for some r and $M \bowtie N$ if both $M \Subset N$ and $N \Subset M$. Since A-lat is a Krull–Schmidt category, the notation $N \Subset M$ for the indecomposable lattice N means that N is the direct summand of M and $M \bowtie N$ means that M and N have the same set of indecomposable direct summands. Note that the relations $\searrow$, $\nearrow$, and $\Subset$ are transitive and that $\bowtie$ is the equivalence relation.

Definition 2.1. *Let M be an A-lattice, let $E = \operatorname{End}_A M$, and let $O(M) = \operatorname{End}_E M$. If the natural mapping $A \to O(M)$ is an isomorphism, then M is called a strict A-lattice. It is clear that M is an exact module.*

It is clear that $O(M)$ is a superring of order $A/\operatorname{Ann}_A M$. By the Burnside density theorem [8] (Theorem 2.6.7), $O(M)$ can be identified with a subset $\{a \in KA/\operatorname{Ann} KM \mid aM \subseteq M\}$. In particular, the exact A-lattice M is strict if and only if

$$\{a \in KA \mid aM \subseteq M\} = A.$$

If the lattice N is exact and $M \searrow N$ or $N \nearrow M$, then M is also exact and $O(N) \supseteq O(M)$.

Proposition 2.3. *For each A-lattice M, there exists an exact sequence*

$$0 \to O(M) \to M^n \to M^m \tag{2.1}$$

for some m and n. In particular, M is strict if and only if there exists an exact sequence

$$0 \to A \xrightarrow{\alpha} M^n \xrightarrow{\beta} M^m, \tag{2.2}$$

i.e., $A \nearrow M$.

Proof. If $E = \operatorname{End}_A M$, then there exists an exact sequence of E-modules $E^m \to E^n \to M \to 0$. By using the functor $\operatorname{Hom}_E(\_, M)$, we arrive at the exact sequence (2.1). If M is strict, then it coincides with (2.2). Conversely, if $A \nearrow M$, then, as indicated above, $A = O(A) \supseteq O(M)$. This yields $O(M) = A$.

Proposition 2.3 is proved.

Corollary 2.1. *An A-lattice M is strict if and only if the exact sequence*

$$M^m \to M^n \to \omega_A \to 0 \tag{2.3}$$

exists, i.e., $M \searrow \omega_A$.

We also use one more duality similar to the Matlis duality [15].

Theorem 2.1. *Let $T_R = K\omega_R/\omega_R$. Denote $\hat{M} = \operatorname{Hom}_R(M, T_R)$. The functor $M \mapsto \hat{M}$ induces the exact duality between the categories of Noetherian and Artinianian R-modules.*

Proof. *Step 1.* By γ_M we denote a natural mapping $M \to \hat{\hat{M}}$. Each KR-module V is an injective R-module and

$$\operatorname{Hom}_R(V, M) = 0 = \operatorname{Hom}_R(L, V)$$

for any Noetherian module M and for any periodic R-module L. Since $\operatorname{inj.dim}_R \omega_R = 1$, T_R is also an injective R-module. Hence, the functor $M \mapsto \hat{M}$ is exact. If the R-module L is periodic, then we apply the functor $\operatorname{Hom}_R(L, \_)$ to the exact sequence $0 \to \omega_R \to K\omega_R \to T_R \to 0$. As a result, we conclude that

$$\hat{L} \simeq \operatorname{Ext}^1_R(L, \omega_R).$$

In particular, $\hat{\hat{R}} = \hat{T}_R \simeq \operatorname{Ext}^1_R(T_R, \omega_R)$. We apply the functor $\operatorname{Hom}_R(\_, \omega_R)$ to the same exact sequence and obtain

$$R = \operatorname{Hom}_R(\omega_R, \omega_R) \simeq \operatorname{Ext}^1_R(T_R, \omega_R) = \hat{T}_R.$$

Thus, γ_R and γ_{T_R} are isomorphisms. Therefore, by using the exact sequence $R^m \to R^n \to M \to 0$, we conclude that γ_M is an isomorphism for each Noetherian R-module M.

Step 2. We show that the module $N = \hat{M}$ is Artinianian if M is Noetherian. Indeed, if $N_1 \subset N$, then this immersion induces a surjection $M = \hat{N} \xrightarrow{\alpha} \hat{N}_1$ and, moreover, $\operatorname{Ker}\alpha \simeq \widehat{N/N_1}$. In addition, if $N_2 \subset N_1$, then we obtain the surjections $M \xrightarrow{\alpha} \hat{N}_1 \xrightarrow{\beta} \hat{N}_2$ such that $\operatorname{Ker}\beta\alpha \supset \operatorname{Ker}\alpha$. Thus, every decreasing chain of submodules of the module $\hat{M}$ gives an increasing chain of submodules of the module M. Hence, infinite decreasing chains of submodules do not exist in $\hat{M}$. In particular, the module $T_R = \hat{R}$ is Artinianian.

Step 3. Now let the module N be Artinianian. It contains a simple submodule U. Since $\operatorname{Hom}_R(U, T_R) \neq 0$ and T_R is injective, there exists a nonzero homomorphism $\alpha_0 \colon N \to T_R$. Since $\operatorname{Ker}\alpha_0$ is also Artinianian, there exists a nonzero homomorphism $\operatorname{Ker}\alpha_0 \to T_R$ that can be is extended to the homomorphism $\alpha' \colon N \to T_R$. Let

$$\alpha_1 = \begin{pmatrix} \alpha_0 \\ \alpha' \end{pmatrix} \colon N \to T_R^2.$$

Then $\operatorname{Ker}\alpha_1 \subset \operatorname{Ker}\alpha_0$. Repeating this procedure, we arrive at the homomorphisms $\alpha_k \colon N \to T_R^k$ such that

$$\operatorname{Ker}\alpha_{k+1} \subset \operatorname{Ker}\alpha_k \quad \text{if} \quad \operatorname{Ker}\alpha_k \neq 0.$$

Since N is Artinianian, at a certain step, we arrive at the immersion $\beta \colon N \to T_R^m$. Since $\operatorname{Cok}\beta$ is also Artinianian, we get the exact sequence $0 \to N \to mT_R \to nT_R$. The fact that the mapping γ_{T_R} is an isomorphism now implies that γ_N is also an isomorphism. Further, reasoning as in Step 2, we conclude that the module $\hat{N}$ is Noetherian.

Theorem 2.1 is proved.

It is clear that the application of this duality to A-modules gives a duality between the categories of left (right) Noetherian modules and right (left) Artinianian A-modules. It is easy to see that, in this case, the category of lattices is mapped onto a category of Artinian modules without finite quotient modules.

The duality $M \mapsto \hat{M}$ is closely connected with the duality D.

Proposition 2.4. *Let $0 \to M \xrightarrow{\alpha} N \to L \to 0$ be an exact sequence of A-modules, where M and N are lattices, and let L be a finite module. There exists an exact sequence $0 \to DN \xrightarrow{D\alpha} DM \to \hat{L} \to 0$. In particular, if M is a maximal submodule in N, then DN is a minimal supermodule of the module DM, and vice versa.*

Proof. In Step 1 of the previous proof, it was established that $\hat{L} \simeq \operatorname{Ext}^1_A(L, \omega_A)$. We also note that

$$\operatorname{Hom}_A(L, \omega_A) = 0.$$

Applying the functor $\operatorname{Hom}_A(\_, \omega_A)$ to this exact sequence, we get the required result.

Let M be an A-lattice and let $\mathfrak{r} = \operatorname{rad} A$. Since $(DM)\mathfrak{r}$ is the intersection of maximal submodules of the module DM, its dual module $M^{\mathfrak{r}} = D((DM)\mathfrak{r})$ is the sum of minimal supermodules M. If $\pi \colon P \xrightarrow{\pi} DM$ is a projective cover of DM, then the dual homomorphism $D\pi \colon M \to DP$ is the inflation $\iota \colon M \to I$ such that

I is an L-injective lattice and ι induces an isomorphism $I^{\mathfrak{r}}/I \to M^{\mathfrak{r}}/M$. We say that I (and sometimes also the mapping ι) is an *L-injective hull* of the module M. We also consider iterated supermodules $M^{\mathfrak{r}*k}$ by setting

$$M^{\mathfrak{r}*1} = M^{\mathfrak{r}} \qquad \text{and} \qquad M^{\mathfrak{r}*(k+1)} = (M^{\mathfrak{r}*k})^{\mathfrak{r}}.$$

It is clear that $M^{\mathfrak{r}*k} = D((DM)\mathfrak{r}^k)$. Since the principal A-module P has a unique maximal submodule $\mathfrak{r}P$, the coprincipal A-lattice I has a unique minimal supermodule $I^{\mathfrak{r}}$.

3. Bijective Lattices and Gorenstein Orders

Let A be an R-order and let $\mathfrak{r} = \operatorname{rad} A$. In this section, we assume that the order A *is connected.*

Definition 3.1. *The A-lattice B is called bijective [6] if it is projective and L-injective.*

The most important property of bijective lattices is the so-called *rejection lemma* [6] (Lemma 2.9).

Lemma 3.1. *Suppose that B is a bijective A-lattice. Either there exists a unique superring A' such that each A-lattice M is isomorphic to $B' \oplus M'$, where M' is an A'-lattice and $B' \mathrel{\underline{\in}} B$, or A is hereditary and $A \mathrel{\underline{\in}} B$ (then $M \mathrel{\underline{\in}} B$ for each A-lattice M).*

It is said that A' is obtained from A by *rejecting* B. This is denoted by $A^-(B)$. It is clear that if B is indecomposable, then $A^-(B)$ is a minimal superring of order A.

Remark 3.1. In view of duality, DB is also a bijective (right) A-lattice and each right A-lattice N is isomorphic to $B' \oplus N'$, where $B' \mathrel{\underline{\in}} DB$, and N' is a right A'-lattice.

Proof. If $M \mathrel{\underline{\in}} B$, then M is projective. Hence, if $M \mathrel{\underline{\in}} B$ for each A-lattice M, then A is hereditary. Thus, we can assume that there exist A-lattices M such that $M \not\mathrel{\underline{\in}} B$. In this case, it is clear that exact lattices with this property also exist. If M is a strict A-lattice, then $A \nearrow M$. Since B is projective, $B \nearrow M$, which implies that $B \mathrel{\underline{\in}} M$ because B is L-injective. Let

$$A' = \bigcap_M O(M),$$

where M runs over all exact A-lattices that do not have direct summands $B' \mathrel{\underline{\in}} B$. There exists a finite set of lattices $M_1, M_2, \ldots, M_n$ such that $A' = O(N)$, where

$$N = \bigoplus_{i=1}^{n} M_i.$$

If N is strict, then $B \mathrel{\underline{\in}} N$, which is impossible. Hence, $A' \supset A$ and each exact A-lattice M without direct summands $B' \mathrel{\underline{\in}} B$ is an A'-lattice. Let M be an arbitrary A-lattice that does not have direct summands $B' \mathrel{\underline{\in}} B$ and let $U_1, U_2, \ldots, U_s$ be all pairwise nonisomorphic KA-modules. If M is not exact, then one of these modules, say, U_1, is not a direct summand of KM. We now show that there exists an A-lattice $L \subset U_1$ such that $L \not\mathrel{\underline{\in}} B$. Replacing M with $M \oplus L$ and continuing this procedure, we arrive at the exact A-lattice M' without direct summands $B' \mathrel{\underline{\in}} B$ such that M is its direct summand. Therefore, M' and, hence, M are also A'-lattices.

Assume that $L \mathrel{\underline{\in}} B$ for each A-lattice $L \subset U_1$. Let C be a simple component of the algebra KA such that U_1 is a C-module and let A_1 be a projection of A onto C. If M is an arbitrary A_1-lattice, then it has a chain of submodules all factors of which are submodules of U_1. Thus, it is projective and A_1 is hereditary and a direct factor of A. Since A was assumed to be connected, we have $A_1 = A$ and $KA = C$ is a simple K-algebra and, hence, $M \Subset B$ for each A-lattice.

Lemma 3.1 is proved.

To describe the structure of the order $A^-(B)$, we need several simple lemmas.

Lemma 3.2.

1. *Let P be a principal A-module. If all modules $\mathfrak{r}^i P$ are indecomposable and projective, then A is hereditary and every indecomposable A-lattice is isomorphic to some $\mathfrak{r}^i P$.*
2. *Let I be a coprincipal A-module. If all modules $I^{\mathfrak{r}*i}$ are indecomposable and L-injective, then A is hereditary and every indecomposable A-lattice is isomorphic to some $I^{\mathfrak{r}*i}$.*
3. *Let P be a principal A-module. If $\mathfrak{r}P \simeq P$, then the order A is maximal and P is a unique indecomposable A-lattice.*
4. *Let I be a coprincipal A-module. If $I^{\mathfrak{r}} \simeq I$, then the order A is maximal and I is a unique indecomposable A-lattice.*

Proof. 1. Under this condition, $\mathfrak{r}^{i+1}P$ is a unique maximal submodule in $\mathfrak{r}^i P$. Hence, every submodule P coincides with some $\mathfrak{r}^i P$, i.e., it is projective and indecomposable. Therefore, KP is a simple KA-module. Thus, there exists a simple component C of the algebra KA such that KP is a KA-module. If V is an arbitrary C-module, then it is divisible by KP. Therefore, if $M \subset V$ is a lattice, then it has a chain of submodules all factors of which are submodules of KP. This implies that M is projective. In particular, the projection A_1 of order A onto C is projective, i.e., it is the direct summand of A as an A-module. In this case, it is clear that A_1 is the direct factor of A and, hence, $A = A_1$.

The second assertion of the lemma is dual to the first assertion.

3. If $\mathfrak{r}P \simeq P$, then $\mathfrak{r}^k P \simeq P$ for all k. Thus, all these quantities are principal. As in Assertion 1, this implies that the algebra A is simple and P is a unique indecomposable A-lattice. In particular, A is a maximal order.

The fourth assertion of the lemma is dual to the third assertion.

Lemma 3.2 is proved.

Lemma 3.3. *Suppose that the order A is not hereditary. Let B be an indecomposable bijective A-lattice and let $A' = A^-(B)$. Then $B^{\mathfrak{r}} \not\simeq B$, $\mathfrak{r}B \not\simeq B$, $B^{\mathfrak{r}}$ is projective, and $\mathfrak{r}B$ is an L-injective A'-lattice.*

Proof. By Lemma 3.2, $B^{\mathfrak{r}} \not\simeq B$ and $\mathfrak{r}B \not\simeq B$. Therefore, they are A'-lattices and $A'B = B^{\mathfrak{r}}$. The principal A-module B is the direct summand of A and, hence, $A \simeq B \oplus M$ for some M. Then

$$A' = A'A \simeq A'B \oplus A'M = B^{\mathfrak{r}} \oplus A'M$$

and, therefore, $B^{\mathfrak{r}}$ is projective over A'. By the duality, $\mathfrak{r}B$ is injective over A'.

Lemma 3.4.

1. *Suppose that P is a principal A-module and M is its minimal supermodule. Then M is either indecomposable or decomposes as $M_1 \oplus M_2$, where M_1 and M_2 are indecomposable. In the second case, $\mathfrak{r}P = \mathfrak{r}M_1 \oplus \mathfrak{r}M_2$ and neither M_1, nor M_2 are projective.*

2. *Suppose that I is a coprincipal A-module and M is its maximal submodule. Then either M is indecomposable or it decomposes in the form $M_1 \oplus M_2$, where M_1 and M_2 are indecomposable. In the second case, $I^{\mathfrak{r}} = M_1^{\mathfrak{r}} \oplus M_2^{\mathfrak{r}}$ and neither M_1, nor M_2 are L-injective.*

3. *Suppose that B is an indecomposable bijective A-lattice. Its maximal submodule and minimal supermodule are simultaneously decomposed. Moreover, if $\mathfrak{r}B$ is L-injective, then $B^{\mathfrak{r}}$ is projective, and vice versa.*

Proof. 1. Since $P \supseteq \mathfrak{r}M \supseteq \mathfrak{r}P$, we get $\ell_A(M/\mathfrak{r}M) \le 2$. Hence, M is either indecomposable or decomposes as $M_1 \oplus M_2$, where M_1 and M_2 are indecomposable. In the last case, $\ell_A(M_1/\mathfrak{r}M_1) = 1$. Therefore, $N = \mathfrak{r}M_1 \oplus M_2 \ne P$ is a maximal submodule in M, $N \cap P = \mathfrak{r}P$, and $M_1/\mathfrak{r}M_1 \simeq M/N \simeq P/\mathfrak{r}P$. Since $M_1 \not\simeq P$, it cannot be projective. The same is true for M_2. In addition, in this case, $\ell_A(M/\mathfrak{r}M) = 2$. This yields $\mathfrak{r}P = \mathfrak{r}M = \mathfrak{r}M_1 \oplus \mathfrak{r}M_2$.

By virtue of duality, the second assertion of the lemma follows from the first assertion.

3. According to the first and second assertions of the lemma, if $B^{\mathfrak{r}}$ is indecomposable, then $\mathfrak{r}B$ is also indecomposable, and vice versa. Assume that $\mathfrak{r}B$ is L-injective. Then it is indecomposable and, hence, $B = (\mathfrak{r}B)^{\mathfrak{r}}$ is a unique minimal supermodule of $\mathfrak{r}B$. Therefore, B is also a unique maximal submodule in $B^{\mathfrak{r}}$. Thus, there exists an epimorphism $\pi\colon P \to B^{\mathfrak{r}}$, where P is projective. If $P \simeq B$, then π is an isomorphism. If $P \not\simeq B$, then it is an A'-module, where $A' = A^-(B)$. By Lemma 3.3, $B^{\mathfrak{r}}$ is a projective A'-module. In this case, π splits and, hence, is an isomorphism. In both cases, $B^{\mathfrak{r}}$ is projective over A.

The converse assertion is obtained by duality.

Lemma 3.4 is proved.

Definition 3.2. *Let B be a bijective B-lattice.*

1. *A B-link is a set of indecomposable lattices $\{B_1, B_2, \ldots, B_l\}$ such that*

   $B_i \Subset B$ for all $i = 1, \ldots, l$,

   $B_i = \mathfrak{r}B_{i-1}$ for $i = 2, \ldots, l$ (or, equivalently, $B_{i-1} = B_i^{\mathfrak{r}}$),

   $\mathfrak{r}B_l \not\Subset B$ and $B_1^{\mathfrak{r}} \not\Subset B$.

2. *For an indecomposable A-lattice M, $M^{\pm,B}$ is defined as follows:*

   if $M \not\Subset B$, then $M^{\pm,B} = M$;

   if $M \in \{B_1, B_2, \ldots, B_l\}$, where $\{B_1, B_2, \ldots, B_l\}$ is a B-link, then $M^{+,B} = B_1^{\mathfrak{r}}$ and $M^{-,B} = \mathfrak{r}B_l$.

By ι_M^B we denote the immersion $M^{-,B} \to M^{+,B}$.

Theorem 3.1. *Suppose that the order A is not hereditary, B is a bijective A-lattice, and $A' = A^-(B)$. If $A = \bigoplus_{i=1}^n P_i$, where P_i are indecomposable, then $A' = \bigoplus_{i=1}^n P_i^{+,B}$. In particular, all modules $P_i^{+,B}$ are projective as A'-modules and each principal A'-module is isomorphic to a direct summand of some $P_i^{+,B}$.*

Remark 3.2. By duality, if $\omega_A = \bigoplus_{i=1}^n I_i$, where I_i are indecomposable, then $\omega_{A'} = \bigoplus_{i=1}^n I_i^{-,B}$. In particular, all modules $I^{-,B}$ are L-injective as A'-modules and each coprincipal A'-module is isomorphic to a direct summand of some $I_i^{-,B}$.

Proof. We write P_i' instead of $P_i^{+,B}$. Clearly, it is possible to assume that $B = \bigoplus_{j=1}^m B_i$, where all B_i are indecomposable and nonisomorphic. We proceed by induction on m. Let $m = 1$, i.e., B is indecomposable.

By Lemma 3.3, $B^{\mathfrak{r}} \not\simeq B$ and, hence, $B' = B^{\mathfrak{r}}$ is an A'-lattice and, moreover, $A'B = B'$. If P is a principal module and $P \not\simeq B$, then $P' = P$ is an A'-lattice, i.e., $A'P = P$. Thus,

$$A' = A'A = \bigoplus_{i=1}^{n} P_i'.$$

Assume that the theorem is true for the $(m-1)$th summand. If $B_i^{\mathfrak{r}} \Subset B$ for all i, then $B_1^{\mathfrak{r}*k} \Subset B$ for all k. Hence, by Lemma 3.2, A is hereditary, which contradicts the condition. Therefore, we can assume that $B_1^{\mathfrak{r}} \notin B$. Denote $A_1 = A^-(B_1)$ and $\mathfrak{r}_1 = \operatorname{rad} A_1$. Then $A^-(B) = A_1^-(B')$, where $B' = \bigoplus_{i=2}^m B_i$. If $\mathfrak{r}B_1 = B_2 \Subset B$, then B_1 is a unique minimal supermodule of B_2. Since $B_1^{\mathfrak{r}}$ is a unique minimal supermodule of B_1 and B_1 is not an A_1-lattice, we get $B_2^{\mathfrak{r}_1} = B_1^{\mathfrak{r}}$. Thus, $M^{+,B} = M^{+,B'}$ for each A_1-lattice M. If $P_i \simeq B_1$ for $i \le r$ and $P_i \not\simeq B_1$ for $i > r$, then

$$A_1 = A^-(B_1) = \left(\bigoplus_{i=1}^{r} P_i'\right) \oplus \left(\bigoplus_{i=r+1}^{n} P_i\right).$$

Moreover, $P_i'^{+,B'} = P_i'$ for $i \le r$ and $P_i^{+,B'} = P_i'$ for $i > r$. By the induction assumption, we get

$$A^-(B) = \bigoplus_{i=1}^{n} P_i'.$$

Theorem 3.1 is proved.

We now introduce a class of orders that plays an important role both in the analyzed case and, in general, in the theory of orders and lattices. The following result is a direct corollary of Propositions 2.2 and 2.3 and Corollary 2.1:

Proposition 3.1. *Let A be an R-order. Then the following conditions are equivalent:*

(1) A is L-injective as a left A-lattice;

(2) A is L-injective as a right A-lattice;

(3) $A \Subset M$ for every strict A-lattice M;

(4) $\omega_A \Subset M$ for every strict A-lattice M;

(5) if M is a strict A-lattice, then $M \searrow N$ for each A-lattice N;

(6) if M is a strict A-lattice, then $N \nearrow M$ for each A-lattice N;

(7) every projective A-lattice is L-injective;

(8) every L-injective A-lattice is projective.

If these conditions are satisfied, then A is called a Gorenstein order [6].

It is clear that every hereditary order is a Gorenstein order. If A is not hereditary, then, by A^-, we denote the order $A^-(A)$. It is obtained from A by rejecting all bijective (or, which is the same in the analyzed case, projective) modules. Theorem 3.1 can be significantly simplified for the Gorenstein orders due to the following result:

Lemma 3.5. *Suppose that A is a nonhereditary Gorenstein order and B is a principal A-module. Then neither $B^{\mathfrak{r}}$, nor $\mathfrak{r}B$ are projective (or, which is the same, L-injective).*

Proof. Assume that $P = B^{\mathfrak{r}}$ is projective and, hence, also bijective. By Lemma 3.4, it is indecomposable and, hence, $\mathfrak{r}P = B$. Let $N = P^{\mathfrak{r}}$. Then $\mathfrak{r}N \supseteq \mathfrak{r}P = B$. If $\mathfrak{r}N = B$, then $B^{\mathfrak{r}} \supseteq N$, which is impossible. Hence, $\mathfrak{r}N = P$ and, consequently, $N/\mathfrak{r}N$ is a simple module. Therefore, there exists a surjection $P' \to N$, where P' is the principal module, and hence, the surjection $\mathfrak{r}P' \to P$ also exists. Thus, P is the direct summand of $\mathfrak{r}P'$. By Lemma 3.4, $\mathfrak{r}P' \simeq P$. This yields $P' \simeq N$ and, hence, $N = B^{\mathfrak{r}*2}$ is also bijective. Continuing this procedure, we see that all lattices $B^{\mathfrak{r}*k}$ are bijective. By Lemma 3.2, A is hereditary, which contradicts the condition. Thus, $B^{\mathfrak{r}}$ cannot be projective. The assertion for $\mathfrak{r}B$ is obtained by duality.

Lemma 3.5 is proved.

Corollary 3.1. *Suppose that A is a nonhereditary Gorenstein order, $A = \bigoplus_{i=1}^{n} P_i$, where P_i are indecomposable, $P_i' = P_i^{\mathfrak{r}}$, and B is a bijective A-lattice. Assume that $P_i \Subset B$ for $i \leq k$ and $P_i \not\Subset B$ for $i > k$. Then*

$$A^-(B) = \left(\bigoplus_{i=1}^{k} P_i'\right) \oplus \left(\bigoplus_{i=k+1}^{n} P_i\right).$$

Moreover, $\mathfrak{r}P_i$ and $P_i^{\mathfrak{r}}$ are $A^-(B)$-lattices for all i. In particular, $A^- = \bigoplus_{i=1}^{k} P_i'$ and $\mathfrak{r}$ and $A^{\mathfrak{r}}$ are A^--lattices (both left and right).

Proof. The proof directly follows from Theorem 3.1 and Lemma 3.5.

For the Gorenstein orders, the following statement converse to Lemma 3.1 is true:

Proposition 3.2. *If A is a Gorenstein order, then each its minimal superring has the form $A^-(B)$, where B is an indecomposable bijective A-lattice.*

Proof. If every projective (or, equivalently, bijective) A-lattice is indeed an A'-lattice, then $A' = A$. Thus, there exists an indecomposable bijective A-lattice B, which is not an A'-lattice. Then $A' \supseteq A^-(B)$. Since A' is minimal, we conclude that $A' = A^-(B)$.

4. Bass Orders

Recall that an order A is called *Bass* [9] if all its superrings (including A) are Gorenstein. By using the results obtained in the previous section, we get the following criterion [6] (Theorem 3.1):

Proposition 4.1. *The following conditions are equivalent:*

(1) A is a Bass order,

(2) $M \searrow O(M)$ for each A-lattice M,

(3) if $M \searrow N$ for some A-lattices M and N, then $N \nearrow M$,

(4) if $N \nearrow M$ for some A-lattices M and N, then $M \searrow N$.

Thus, if an order is Morita-equivalent to a Bass order, then it is also a Bass order.

Example 4.1.

1. Every hereditary order is a Bass order.
2. If each ideal A has two generatrices, then A is Bass. This follows from [18] in the case where R is a ring of discrete estimate. In the general case, the proof is the same.

3. Let Δ be the maximal order in a body, let $\mathfrak{d} = \operatorname{rad}\Delta$, and let $B(k,\Delta)$ be a subring of $\operatorname{Mat}(2,\Delta)$ formed by matrices (a_{ij}) such that $a_{12} \in \mathfrak{d}^k$. This is a Bass order (hereditary for $k = 1$). We symbolically write

$$B(k,\Delta) = \begin{pmatrix} \Delta & \mathfrak{d}^k \\ \Delta & \Delta \end{pmatrix}.$$

In [9], it was established that every connected Bass order is either hereditary, or Morita-equivalent to a local order each ideal of which has two generatrices, or Morita-equivalent to a certain order $B(k,\Delta)$. We obtain this description as a corollary of the following theorem that generalizes Theorem 3.3 in [6]:

Theorem 4.1. *Suppose that A is a connected nonmaximal order, P is an indecomposable bijective A-lattice, and $A_1 = A^-(P)$. If $P^{\mathfrak{r}} \simeq \mathfrak{r}P$, then the following assertions are true:*

(1) there exist chains of supermodules $P = P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_m$ and superrings $A = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_m$ such that, for each $0 \le i < m$:

- *(a) $P_{i+1} = P_i^{\mathfrak{r}_i} \simeq \mathfrak{r}_i P_i$, where $\mathfrak{r}_i = \operatorname{rad} P_i$;*
- *(b) P_i is an indecomposable bijective A_i-lattice, which is not projective over A_{i-1} (and, hence, also over A) for $i \neq 0$;*
- *(c) A_i is nonmaximal and $A_{i+1} = A_i^-(P_i)$.*

(2) If this chain has the maximal length, then A_m is a hereditary order, has at most two nonisomorphic indecomposable lattices, and each indecomposable A-lattice is isomorphic either to P_i for some $0 \le i < m$ or to the direct summand P_m.

(3) A is Morita-equivalent either to a local Bass order $E = (\operatorname{End}_A P)^{\mathrm{op}}$ or to a Bass order $B(k,\Delta)$ for some k and Δ.

The condition $P^{\mathfrak{r}} \simeq \mathfrak{r}P$ is satisfied if $P^{\mathfrak{r}}$ does not have direct summands L-injective over A but is L-injective as an A_1-lattice or, by duality, if $\mathfrak{r}P$ does not have direct summands projective over A but is projective over A_1.

Note that, by Lemma 3.5, $P^{\mathfrak{r}}$ cannot have L-injective summands if A is Gorenstein.

Proof. First of all, we prove the last assertion. It follows from Theorem 3.1 that the L-injective lattice over A_1 either is L-injective over A or is the direct summand of $\mathfrak{r}P$. If $P^{\mathfrak{r}}$ does not have L-injective summands over A but is L-injective over A_1, then each direct summand of $P^{\mathfrak{r}}$ is isomorphic to the direct summand of $\mathfrak{r}P$. By Lemma 3.4, either $P^{\mathfrak{r}}$ and $\mathfrak{r}P$ are indecomposable or $P^{\mathfrak{r}} = L_1 \oplus L_2$ and $\mathfrak{r}P = \mathfrak{r}L_1 \oplus \mathfrak{r}L_2$, where L_1, L_2, $\mathfrak{r}L_1$, and $\mathfrak{r}L_2$ are indecomposable. This implies that $P^{\mathfrak{r}} \simeq \mathfrak{r}P$.

Let $P_1 = P^{\mathfrak{r}} \simeq \mathfrak{r}P$. Since A is not maximal, by Lemma 3.2, we get $P_1 \not\simeq P$. Therefore, the chains of supermodules and superrings with properties (a)–(c) exist: e.g., $P = P_0 \subset P_1 = P^{\mathfrak{r}}$ and $A = A_0 \subset A_1 = A^-(P)$. Since there are no infinite chains of superrings, we consider the longest chain with this property. By Lemma 3.3 and Theorem 3.1, we conclude that:

P_i is a bijective A_i-lattice not projective over A_{i-1} (and, hence, also over A) for $i \neq 0$;

if $i < m$, then each indecomposable A-lattice either is isomorphic to one of the modules $P_0, P_1, \ldots, P_i$ or is an A_{i+1}-module;

every principal A_i-module is either projective over A or isomorphic to the direct summand of P_i (and, hence, isomorphic to P_i for $i < m$).

If $i < m$, then $P_{i-1} \neq \mathfrak{r}_i P_i$ because P_{i-1} is not an A_i-lattice but $\mathfrak{r}_i P_i \supseteq \mathfrak{r}_{i-1} P_{i-1}$. If $\mathfrak{r}_i P_i = \mathfrak{r}_{i-1} P_{i-1} \simeq P_i$, then A_i is maximal, which contradicts the condition. Thus, $\mathfrak{r}_i P_i \cap P_{i-1} = \mathfrak{r}_{i-1} P_{i-1}$ and $\mathfrak{r}_i P_i + P_{i-1} = P_i$. This yields

$$P_i/\mathfrak{r}_i P_i \simeq P_{i-1}/\mathfrak{r}_{i-1} P_{i-1} \simeq P_{i-2}/\mathfrak{r}_{i-2} P_{i-2} \simeq \ldots \simeq P/\mathfrak{r}P. \tag{4.1}$$

Since $\mathfrak{r}_i P_i \simeq P_{i+1}$ and $\mathfrak{r}_{i-1} P_{i-1} \simeq P_i$, we also get

$$P_{i+1}/P_i \simeq P_i/P_{i-1} \simeq P_{i-1}/P_{i-2} \simeq \ldots \simeq P_1/P. \tag{4.2}$$

We first assume that P_m can be decomposed: $P_m = L_1 \oplus L_2$, where L_1 and L_2 are indecomposable and not projective over A_{m-1} (and, hence, also over A) by Lemma 3.4. Since $\mathfrak{r}_{i-1} P_m = \mathfrak{r}_{i-1} L_1 \oplus \mathfrak{r}_{i-1} L_2 \simeq L_1 \oplus L_2$ and $\mathfrak{r}_{i-1} L_1$, $\mathfrak{r}_{i-1} L_2$ are indecomposable, either $\mathfrak{r}_{i-1} L_1 \simeq L_1$ and $\mathfrak{r}_{i-1} L_2 \simeq L_2$ or $\mathfrak{r}_{i-1} L_1 \simeq L_2$ and $\mathfrak{r}_{i-1} L_2 \simeq L_1$. In both cases, all submodules of the modules L_1 and L_2 are projective and isomorphic either to L_1 or to L_2. Hence, all indecomposable A_m-lattices are isomorphic either to L_1 or to L_2, A_m is hereditary, and $P_0, P_1, \ldots, P_{m-1}, L_1, L_2$ are all indecomposable A-lattices. Thus, $A_0, A_1, \ldots, A_{m-1}$ are all nonhereditary superrings of A and, therefore, A is Bass. Since P is the unique principal A-module, A is Morita-equivalent to the local Bass order $E = \mathrm{End}_A P$.

Now let P_m be indecomposable. Note that $P_{m-1} \supseteq \mathfrak{r}_{i-1} P_m \supseteq \mathfrak{r}_{i-1} P_{m-1}$. Assume that P_m is projective as an A_{m-1}-module. Then $\mathfrak{r}_{i-1} P_m = P_{m-1}$. Conversely, if $\mathfrak{r}_{i-1} P_m = P_{m-1}$, i.e., $\ell_{A_{m-1}}(P_m/\mathfrak{r}_{i-1} P_m) = 1$, then there exists an epimorphism $\varphi\colon P' \to P_m$, where P' is the principal A_{m-1}-module. If $P' = P_{m-1}$, then φ is an isomorphism because $\mathrm{wd}(P_{m-1}) = \mathrm{wd}(P_m)$. Otherwise, P' is an A_m-module and, hence, $P' \simeq P_m$ because P_m is also projective over A_m. Thus, P_m is projective over A_{m-1} and, hence, also over A. Since $\mathfrak{r}_{m-1} P_m \simeq P_{m-1}$ and $\mathfrak{r}_{m-1} P_{m-1} \simeq P_m$, it follows from Lemma 3.2 that A_{m-1} is hereditary and P_{m-1} and P_m are all its indecomposable modules. We set $\Delta = \mathrm{End}_A P_m$ and $\mathfrak{d} = \mathrm{rad}\,\Delta$. This is the maximal order and, in addition, $\mathrm{End}_A P_{m-1} \simeq \Delta$ [4]. Since $P_m \not\simeq P$, the quotient modules P_m/P_{m-1} and $P/\mathfrak{r}P$ are not isomorphic. It follows from isomorphisms (4.1) and (4.2) that, for each $i < m$, P_{i-1} is a unique maximal submodule in P_i such that $P_i/P_{i-1} \simeq P_m/P_{m-1}$. Therefore, $\varphi(P_{i-1}) \subseteq P_{i-1}$ for each endomorphism $\varphi \in \mathrm{End}_A P_i$ and, hence, $\mathrm{End}_A P_i \simeq \Delta$ for all i. In particular, $\mathrm{End}_A P \simeq \Delta$. Since P and P_m are all principal A-modules, A is Morita-equivalent to the ring

$$\tilde{A} = \big(\mathrm{End}_A(P \oplus P_m)\big)^{\mathrm{op}}.$$

Since each (right or left) Δ-ideal coincides with $\mathfrak{d}^k$ for some k, we get

$$\tilde{A} \simeq \begin{pmatrix} \Delta & \mathfrak{d}^k \\ \mathfrak{d}^l & \Delta \end{pmatrix} \simeq \begin{pmatrix} \Delta & \mathfrak{d}^{k+l} \\ \Delta & \Delta \end{pmatrix} = B(k+l, \Delta)$$

for some k and l.

Now let P_m be indecomposable and not projective over A_{m-1}. Then $\mathfrak{r}_{m-1} P_m = \mathfrak{r}_{m-1} P_{m-1}$ and $P_m \supset \mathfrak{r}_m P_m \supseteq \mathfrak{r}_{m-1} P_{m-1}$. If $\mathfrak{r}_m P_m = \mathfrak{r}_{m-1} P_{m-1} \simeq P_m$, then A_m is a maximal order and P_m is a unique indecomposable A_m-lattice. Hence, $P_0, P_1, \ldots, P_m$ are all indecomposable A-lattices, $A_0, A_1, \ldots, A_m$ are all superrings of A, and A is Bass. Moreover, P is a unique principal A-module and, hence, A is Morita-equivalent to $E = \mathrm{End}_A P$.

If P_m is indecomposable and not projective over A_{m-1} and $P_{m-1} \neq \mathfrak{r}_m P_m \neq \mathfrak{r}_{i-1} P_{i-1}$, then $\mathfrak{r}_m P_m$ is a minimal supermodule $\mathfrak{r}_{m-1} P_{m-1} \simeq P_m$. Hence, $\mathfrak{r}_m P_m \simeq P_m^{\mathfrak{r}_m}$. Therefore, setting $P_{m+1} = P_m^{\mathfrak{r}_m}$ and $A_{m+1} = A_m^-(P_m)$, we obtain longer chains of superrings and supermodules satisfying conditions (a)–(c), which is impossible,

Theorem 4.1 is proved.

Corollary 4.1 ([6], Theorem 3.3)**.** *Let A be a connected Gorenstein order. If at least one of its minimal superrings is also Gorenstein, then A is a Bass order. Moreover, it is either hereditary, or Morita-equivalent to a local Bass order, or Morita-equivalent to an order $B(k, \Delta)$ for some k and Δ.*

Proof. The proof of the corollary follows from Theorem 4.1, Lemma 3.5, and Proposition 3.2.

Corollary 4.2 ([6], Proposition 3.7)**.** *Let A be a local Gorenstein order and let $A' = A^{-}(A)$ be its minimal superring. If A' is not local, then it is hereditary and A is Bass.*

Proof. By Proposition 3.2, $A' = A^{-}(A)$. If A' is not local, then $A' = P_1 \oplus P_2$, where both modules P_i are principal A'-modules and $\mathfrak{r}P_i$ are coprincipal A'-lattices. In particular, $\operatorname{rad} A' = \mathfrak{r}$. Let P_1' be a minimal supermodule of P_1 and let M be a maximal submodule in P_1'. Then $M = P_1$; otherwise, $M \cap P_1 = \mathfrak{r}P_1$, i.e., M is a minimal supermodule of $\mathfrak{r}P_1$, which is impossible because P_1 is the unique minimal supermodule of $\mathfrak{r}P_1$. Thus, P_1 is a unique maximal submodule in P_1'. Hence, there exists an epimorphism $\varphi\colon P \to P_1'$ for some principal A'-module P. If $P = P_1$, then φ is an isomorphism. If $P = P_2$, then φ induces the epimorphism $\varphi'\colon \mathfrak{r}P_2 \to \mathfrak{r}P_1' = P_1$. Since $\mathfrak{r}P_2$ is indecomposable, φ' is an isomorphism and, hence, φ is also an isomorphism. Thus, either $P_1' \simeq P_1$ or $P_1' \simeq P_2$. Similarly, if P_2' is a minimal supermodule of P_2, then either $P_2' \simeq P_1$ or $P_2' \simeq P_2$. By Lemma 3.2, A' is hereditary and A is Bass.

Corollary 4.2 is proved.

5. Stable Categories

Definition 5.1.

1. *Let $\mathcal{C}$ be an additive category and let $\mathfrak{S}$ be a set of its morphisms. By $\langle \mathfrak{S} \rangle$ we denote an ideal in $\mathcal{C}$ generated by $\mathfrak{S}$, i.e., consisting of morphisms of the form $\sum_{i=1}^{k} \alpha_i \sigma_i \beta_i$, where $\sigma_i \in \mathfrak{S}$. By $\mathcal{C}^{\mathfrak{S}}$ we denote the quotient category $\mathcal{C}/\langle \mathfrak{S} \rangle$. Its objects are the same as in $\mathcal{C}$, the set of morphisms from M into N is*

$$\operatorname{Hom}_{\mathcal{C}}^{\mathfrak{S}}(M, N) = \operatorname{Hom}_{\mathcal{C}}(M, N)/\mathfrak{S}(M, N),$$

   where $\mathfrak{S}(M, N) = \langle \mathfrak{S} \rangle \cap \operatorname{Hom}_{\mathcal{C}}(M, N)$.

2. *The category $A\text{-mod}^{\langle 1_A \rangle}$ is denoted by $\underline{A\text{-mod}}$ and its sets of morphisms are denoted by $\underline{\operatorname{Hom}}_A(M, N)$. It is clear that it coincides with $A\text{-mod}^{\mathfrak{P}}$, where $\mathfrak{P} = \{1_{P_1}, 1_{P_2}, \ldots, 1_{P_n}\}$, and $P_1, P_2, \ldots, P_n$ is the complete list of nonisomorphic principal A-modules. If A is an order, then the complete subcategory in $A\text{-mod}^{\langle 1_A \rangle}$ that consists of A-lattices coincides with $A\text{-lat}^{\langle 1_A \rangle}$ and is denoted by $\underline{A\text{-lat}}$. It is called a stable category of order A.*

3. *Similarly, a category $A\text{-lat}^{\langle 1_{\omega_A} \rangle}$ is denoted by $\overline{A\text{-lat}}$ and its sets of morphisms are denoted by*

$$\overline{\operatorname{Hom}}_A(M, N).$$

   It coincides with $A\text{-lat}^{\mathfrak{I}}$, where $\mathfrak{I} = \{1_{I_1}, 1_{I_2}, \ldots, 1_{I_n}\}$ and $I_1, I_2, \ldots, I_n$ is the complete list of nonisomorphic coprincipal A-lattices. This category is called a costable category of order A.

The duality D induces the duality between the categories $\underline{A\text{-lat}}$ and $\overline{A^{\mathrm{op}}\text{-lat}}$. If A is Gorenstein, then stable and costable categories coincide.

We see that all R-modules $\underline{\mathrm{Hom}}_A(M, N)$ and $\overline{\mathrm{Hom}}_A(M, N)$ are finite. Moreover, it is possible to estimate their annihilators.

Lemma 5.1. *Suppose that* A_0 *is a hereditary (e.g., maximal) superring of order* A, $\mathfrak{c} = \mathrm{Ann}_R(A_0/A)$. *Then*

$$\mathfrak{c}^2\, \underline{\mathrm{Hom}}_A(M, N) = \mathfrak{c}^2\, \overline{\mathrm{Hom}}_A(M, N) = 0$$

for any A*-lattices.*

Proof. Let M and N be A-latices and let $\lambda, \mu \in \mathfrak{c}$. Consider $A_0M \subset KM$. Then $\lambda A_0 M \subseteq M$. Since A_0 is hereditary, A_0M is a projective A_0-module. Therefore, A_0M is the direct summand of the free A_0-module F' that can be identified with A_0F, where F is a free A-module. Every homomorphism $f\colon M \to N$ can be extended to the homomorphism $A_0M \to A_0N$ and, hence, also to the homomorphism $g\colon F' \to A_0N$. Moreover, $F \supseteq \lambda F' \supseteq \lambda M$ and $\mathrm{Im}(\mu g) \subseteq \mu A_0 N \subseteq N$. Therefore, the homomorphism $\lambda\mu f$ can be regarded as a composition

$$M \xrightarrow{\lambda} \lambda M \hookrightarrow F \xrightarrow{\mu g|_F} N.$$

Thus, the homomorphism $\lambda\mu f$ factors through the projective module and its image in $\underline{\mathrm{Hom}}_A(M, N)$ is zero. By duality, the same is also true for $\overline{\mathrm{Hom}}_A(M, N)$.

Lemma 5.1 is proved.

Note that two important functors are defined on stable categories. Let $\pi\colon P \to M$ be a projective cover of a finitely generated A-module M and let $\Omega M = \mathrm{Ker}\,\pi$. Also note that ΩM is always an A-lattice nonzero if M is not projective. If M is a nonprojective lattice, then ΩM is not L-injective (otherwise, π splits). If $\pi'\colon P' \to M'$ is a projective cover of M', then any homomorphism $\alpha\colon M \to M'$ rises to the homomorphism $P \to P'$ and, hence, induces the homomorphism $\gamma\colon \Omega M \to \Omega M'$. If γ' originates from another rise α, then we can easily verify that $\gamma - \gamma'$ factors through P. Hence, the class γ in the stable category $\underline{A\text{-mod}}$ or $\underline{A\text{-lat}}$ is uniquely defined and Ω can be regarded as an endofunctor on a stable category. By using L-injective shells, we obtain a similar functor Ω' on the costable category $\overline{A\text{-lat}}$. If A is Gorenstein, then the projective cover M is simultaneously an L-injective shell ΩM. Hence, Ω' is quasiinverse to the functor Ω and both these quantities are automorphisms of a stable category.

Now let $P_1 \xrightarrow{\psi} P_0 \xrightarrow{\varphi} M \to 0$ be the *minimal projective representation* of a finitely generated A-module M, i.e., an exact sequence in which the modules P_0 and P_1 are projective, $\mathrm{Ker}\,\varphi \subseteq \mathfrak{r}P_0$, and $\mathrm{Ker}\,\psi \subseteq \mathfrak{r}P_1$. By applying the functor $^\vee = \mathrm{Hom}_A(\_\,, A)$ to this sequence, we get the following exact sequence of right modules:

$$0 \to M^\vee \xrightarrow{\varphi^\vee} P_0^\vee \xrightarrow{\psi^\vee} P_1^\vee \to \mathrm{tr}\, M \to 0, \tag{5.1}$$

where $\mathrm{tr}\, M = \mathrm{Cok}\,\psi^\vee$. Moreover, we can easily verify that, in fact, we obtain the functor

$$\mathrm{tr}\colon (\underline{A\text{-mod}})^{\mathrm{op}} \to \underline{A^{\mathrm{op}}\text{-mod}}.$$

Since the natural mapping $P \to P^{\vee\vee}$ is an isomorphism for any finitely generated projective module P, there exists an isomorphism of functors $\mathbf{1}_{\underline{A\text{-mod}}} \simeq \mathrm{tr}^2$. Note that even if M is a lattice, $\mathrm{tr}\, M$ may be not a lattice.

There exists a natural homomorphism $M^\vee \otimes_A N \to \mathrm{Hom}_A(M, N)$ that maps uv into the homomorphism $x \mapsto u(x)v$. We see that its image coincides with $\mathfrak{P}(M, N)$ [2]. It follows from the exact sequence (5.1) that

$$\mathrm{Tor}_1^A(\mathrm{tr}\, M, N) \simeq \underline{\mathrm{Hom}}_A(M, N).$$

Consider the behavior of the categories $\underline{A\text{-lat}}$ and $\overline{A\text{-lat}}$ under the rejection of bijective lattices.

Lemma 5.2. *Suppose that the order A is not maximal. Let B be an indecomposable bijective A-lattice, let $A' = A^-(B)$, and let M and N be some A'-lattices.*

1. *The restrictions $\gamma_+ : \mathrm{Hom}_A(B^{\mathfrak{r}}, M) \to \mathrm{Hom}_A(B, M)$ and $\gamma_- : \mathrm{Hom}_A(M, \mathfrak{r}B) \to \mathrm{Hom}_A(M, B)$ are bijective mappings.*
2. *The homomorphism $\alpha : M \to N$ factors through B if and only if it factors through the immersion $\mathfrak{r}B \to B^{\mathfrak{r}}$.*

Proof. 1. Since $B/\mathfrak{r}B$ is a finite module, the mapping γ_- is injective. Since M does not contain B as the direct summand, $\mathrm{Im}\,\alpha \subseteq \mathfrak{r}B$ for any $\alpha : M \to B$. Hence, γ_- is bijective. The assertion for γ_+ is dual.

The second assertion of the lemma is an obvious corollary of the first assertion.

Theorem 5.1. *Suppose that A is a nonhereditary order, B is a bijective A-lattice, $P_1, P_2, \ldots, P_n$ is the complete list of nonisomorphic principal A-modules, $I_1, I_2, \ldots, I_n$ is the complete list of nonisomorphic coprincipal A-lattices, and $A' = A^-(B)$. Let*

$$\mathfrak{P}^B = \left\{\iota^B_{P_i} \mid 1 \le i \le n\right\} \quad \text{and} \quad \mathfrak{I}^B = \left\{\iota^B_{I_i} \mid 1 \le i \le n\right\}.$$

Then $\underline{A\text{-lat}} \simeq A'\text{-lat}^{\mathfrak{P}_B}$ and $\overline{A\text{-lat}} \simeq A'\text{-lat}^{\mathfrak{I}_B}$.

Indeed, this means that, in the definition of $\underline{A\text{-lat}}$ (resp., $\overline{A\text{-lat}}$), A can be replaced with A' and, for each B-link $B_1, B_2, \ldots, B_l$, all mappings 1_{B_i}, $1 \le i \le l$, in $\mathfrak{P}$ (resp., in $\mathfrak{I}$) can be replaced by the immersions $\mathfrak{r}B_l \to B_1^{\mathfrak{r}}$.

Proof. If B is not hereditary, then this follows from Lemma 5.2. The general case is obtained by induction on the number of nonisomorphic indecomposable direct summands of the lattice B with the use of Theorem 3.1.

Corollary 5.1. *Let A be a nonhereditary Gorenstein order, let $P_1, P_2, \ldots, P_n$ be the complete list of nonisomorphic principal A-modules, let ι_i be the immersion $\mathfrak{r}P_i \to P_i^{\mathfrak{r}}$, and let $A' = A^-(A)$. Then $\underline{A\text{-lat}} \simeq A'\text{-lat}^{\mathfrak{P}'}$, where $\mathfrak{P}' = \{\iota_1, \iota_2, \ldots, \iota_n\}$.*

Proof. The proof of the corollary follows from Theorem 5.1 and Lemma 3.5.

6. Almost Split Sequences

We now recall some definitions and results (see [2]). Let A be an order and let $\alpha : N \to M$ and $\beta : M \to N$ be homomorphisms of A-lattices, where M is indecomposable.

Definition 6.1.

1. *The homomorphism α is called almost right split if the following conditions are satisfied:*
   - *(a) α is not a split epimorphism;*
   - *(b) each homomorphism $\xi : X \to M$, which is not a split epimorphism, factors through α;*
   - *(c) if $\varphi : N \to N$ is such that $\alpha\varphi = \alpha$, then φ is an isomorphism.*

   Note that if conditions (a) and (b) are satisfied, then either condition (c) is also satisfied or $N = N_0 \oplus N_1$, where $N_0 \subset \mathrm{Ker}\,\alpha$, and the restriction of α to N_1 is almost right split.

2. *The homomorphism β is called almost left split if the following conditions are satisfied:*

   (a) β is not a split inflation;

   (b) each homomorphism $\xi\colon X \to M$, which is not a split monomorphism, factors through β;

   (c) if $\varphi\colon N \to N$ is such that $\varphi\beta = \beta$, then φ is an isomorphism.

   Note that if conditions (a) and (b) are satisfied, then either condition (c) is also satisfied or $N = N_0 \oplus N_1$, where $\operatorname{Im}\beta \subset N_1$ and β is almost left split (if it is regarded as the homomorphism $M \to N_1$).

3. *A nonsplit exact sequence of A-lattices $\varepsilon\colon 0 \to L \xrightarrow{\beta} N \xrightarrow{\alpha} M \to 0$, where M and L are indecomposable, is called an almost split sequence if the following conditions are satisfied:*

   (a) α is almost right split;

   (b) β is almost left split;

   (c) for each homomorphism $\xi\colon X \to M$, which is a nonsplit epimorphism, the exact sequence $\varepsilon\xi$ can be split;

   (d) for each homomorphism $\eta\colon L \to X$, which is a nonsplit monomorphism, the exact sequence $\eta\varepsilon$ can be split.

Here, $\varepsilon\xi$ (resp., $\eta\varepsilon$) is the rise of the exact sequence ε along ξ (resp., the lowering of ε along η).

It is clear that if an almost right (left) split morphism exists, then it is unique to within an automorphism of the module N. Similarly, if an almost split sequence with fixed term M (or L) exists, then it is unique to within an isomorphism of the term L (resp., M). Indeed, in the category A-lat, this sequence exists for any nonprojective indecomposable lattice M, as in the case of each indecomposable lattice L, which is not L-injective. The proof of this fact exactly repeats the proof of Proposition 1.1 in [1]. Hence, we only recall its main steps.

The functor

$$\tau_A = D\Omega\,\mathrm{tr}\,:\ \underline{A\text{-lat}} \to \overline{A\text{-lat}}$$

is called an *Auslander–Reiten translation*. As in [1] (Proposition 1.1), we can prove that

$$\mathrm{Ext}^1_A(N, \tau_A M) \simeq \widehat{\underline{\mathrm{Hom}}_A(M}, N).$$

Let M be an indecomposable nonprojective A-lattice. Then the ring $\Lambda = \underline{\mathrm{Hom}}_A(M, M)$ is local. By duality, $\widehat{\underline{\mathrm{Hom}}_A(M}, M)$ has a unique minimal Λ-submodule U. If u is a nonzero element from U, then $u(\lambda) = 0$ for each noninvertible element $\lambda \in \Lambda$. If $\xi\colon X \to M$ is not a split epimorphism, then $\xi\varphi$ is not invertible for each $\varphi\colon M \to X$, whence it follows that $(u\xi)\varphi = u(\xi\varphi) = 0$, i.e., $u\xi = 0$. Then the same is true for the corresponding extension $\varepsilon \in \mathrm{Ext}^1_A(M, \tau_A M)$. Hence,

$$\varepsilon\colon\ 0 \to \tau_A M \xrightarrow{\beta} E \xrightarrow{\alpha} M \to 0 \tag{6.1}$$

is an almost split sequence. Note that if $0 \to L \to N \to M \to 0$ is an almost split sequence, then the dual sequence $0 \to DM \to DN \to DL \to 0$ is also almost split. Hence, if $L = \tau_A M$, then $DM \simeq \tau_A DL$ and $M \simeq D\tau_A DL \simeq \Omega\,\mathrm{tr}\,DL$. Thus, the functor τ_A has the quasiinverse functor

$$\tau_A^{-1} = \Omega\,\mathrm{tr}\,D\colon\ \overline{A\text{-lat}} \to \underline{A\text{-lat}}.$$

Let $M = \bigoplus_j M_j$ and $N = \bigoplus_i N_i$, where M_j and N_i are indecomposable A-lattices. By $\mathrm{Rad}_A(M, N)$ we denote a set of homomorphisms $\varphi\colon M \to N$ such that each component $\varphi_{ij}\colon M_j \to N_i$ is not an isomorphism.

Clearly, we get an ideal of category A-lat called its *radical*. We can consider its powers Rad_A^n, $n \in \mathbb{N}$, and

$$\mathrm{Rad}_A^\infty = \bigcap_{n=1}^{\infty} \mathrm{Rad}_A^n .$$

The homomorphisms from $\mathrm{Rad}_A(M,N) \setminus \mathrm{Rad}_A^2(M,N)$ are called *irreducible*. The quotient module

$${}_N V_M = \mathrm{Rad}_A(M,N)/\,\mathrm{Rad}_A^2(M,N)$$

is a finite-dimensional vector space over the residue field $\Bbbk$. In particular, if the lattice M is indecomposable, then $F_M = {}_M V_M$ is a body and, for every lattice N, both ${}_N V_M$ and ${}_M V_N$ are finite-dimensional vector spaces over F_M (resp., right and left). Let A-ind be the set of classes of isomorphisms of indecomposable A-lattices. A collection $\{F_M,\, {}_N V_M \mid M,\, N \in A\text{-ind}\}$ is called an *AP-type* of order A and denoted by AR_A. This is indeed a *type* in a sense of [5] because all F_M are bodies and ${}_N V_M$ is an F_N-F_M-bimodule. If the residue field $\Bbbk$ is algebraically closed, then $F_M = \Bbbk$ for each indecomposable lattice M. This type is usually regarded as a quiver whose vertices are the lattices $M \in A$-ind. Moreover, there are d_{NM} arrows, where $d_{NM} = \dim_{\Bbbk}({}_N V_M)$ passing from the vertex M to the vertex N. This object is called an *Auslander–Reiten quiver* of order A. It is clear that the AP-type of order A^{op}-lat is $(F_M^{\mathrm{op}}, {}_M V_N)$. Thus, in particular, in the Auslander–Reiten quiver, it is only necessary to change the directions of all arrows into the opposite.

If the lattice M is indecomposable and not projective, then, by the definition of an almost split sequence, each homomorphism from $\mathrm{Rad}_A(N,M)$, just as each homomorphism from $\mathrm{Rad}_A(\tau_A M, N)$ factors through the term E of sequence (6.1). Thus, if $E = \bigoplus_{i=1}^r E_i$, where all E_i are indecomposable, then ${}_M V_N = 0 = {}_N V_{\tau_A M}$ if $N \not\simeq E_i$ for all $1 \le i \le r$, and ${}_M V_{E_i}$ and ${}_{E_i} V_{\tau_A M}$ are all nonzero. In particular, in the Auslander–Reiten quiver, all arrows go only from each E_i to M and from $\tau_A M$ to each E_i. We also note that if α_i are components of the homomorphism α and β_i are components of the homomorphism β from sequence (6.1), then

$$\sum_{i=1}^{r} \alpha_i \beta_i = 0.$$

If the module P is principal, then the image of each homomorphism $N \to P$ that is not a split epimorphism is contained in $\mathfrak{r}P$. Thus, if $\mathfrak{r}P = \bigoplus_{i=1}^r E_i$, where all E_i are indecomposable, then only ${}_P V_{E_i}$ spaces are nonzero among the spaces ${}_P V_N$ in the AP-type. By duality, if the lattice I is coprincipal and $I^{\mathfrak{r}} = \bigoplus_{i=1}^r E_i$ with indecomposable E_i, then only ${}_{E_i} V_I$ spaces are nonzero among the spaces ${}_N V_I$.

If the lattices M and N are not projective, then each homomorphism from $\mathfrak{P}(M,N)$ belongs to $\mathrm{Rad}_A^2(M,N)$. Hence, we can consider a *stable AP-type* (or *a stable Auslander–Reiten quiver*) $\underline{\mathrm{AR}}_A$, which is a part of AR_A in which M and N run only through the nonprincipal indecomposable lattices. The *costable AP-type* (or *a costable Auslander–Reiten quiver*) $\overline{\mathrm{AR}}_A$ is defined by duality. In this type, M and N run through indecomposable lattices that are not coprincipal. The functor τ_A induces the *Auslander–Reiten translation* $\underline{AR}_A \xrightarrow{\sim} \overline{AR}_A$. In the Gorenstein case, stable and costable types or quivers also coincide.

In what follows, we use the following result for irreducible morphisms between indecomposable lattices, which is, most likely, known but we failed to find it in the literature:

Proposition 6.1. *Let M and N be indecomposable lattices and let $\alpha\colon N \to M$ be an irreducible morphism. There are two possible cases:*

(1) α is a monomorphism and its image is the direct summand of a maximal submodule M;

(2) α is an epimorphism of N onto the direct summand of a certain quotient module N/L, where L is an L-irreducible sublattice in N such that N/L is a lattice.

Proof. Let $M' = \operatorname{Im}\alpha$, let ι be the immersion $M' \to M$, and let π be the projection $N \to M'$. If $M' = \bigoplus_{i=1}^{m} M_i$, where M_i are indecomposable, then let ι_i and π_i be the components of ι and π for this decomposition. Thus, $\alpha = \sum_{i=1}^{m} \iota_i \pi_i$. Since α is irreducible, at least one of the morphisms ι_i or π_i must be invertible. Assume that one of ι_i is invertible. Then $m = 1$ and α is an epimorphism. Let L be an irreducible nonzero sublattice in $\operatorname{Ker}\alpha$ such that $\operatorname{Ker}\alpha/L$ and, hence, N/L is also a lattice (if $\operatorname{Ker}\alpha$ is L-irreducible, then $L = \operatorname{Ker}\alpha$). Thus, $\alpha = \xi\eta$, where η is the epimorphism $N \to N/L$ and $\xi\colon N/L \to M$. If $\xi = \alpha\gamma$, then $\alpha = \alpha\gamma\eta$. Since α is irreducible and N is indecomposable, $\gamma\eta$ must be an isomorphism, which is impossible. Hence, ξ does not factor through α and, therefore, it is a split epimorphism, i.e., defines M as the direct summand N/L. Thus, we get Case 2.

If some π_i is invertible, then $m = 1$ and α is a monomorphism. If M' is a maximal submodule in M that contains $\operatorname{Im}\alpha$, then α factors through the immersion $\operatorname{Im}\alpha \to M'$. Thus, it must split and, hence, we get Case 1.

Proposition 6.1 is proved.

We now study the behavior of these structures in the case of rejection of bijective lattices. First, we prove the following assertion:

Proposition 6.2. *Let B be a bijective A-lattice, let $A' = A^-(B)$, and let M, N, and L be A'-lattices.*

1. *If $\alpha\colon N \to M$ is almost right split in A'-lat, then it also has this property in A-lat.*
2. *If $\beta\colon M \to N$ is almost left split in A'-lat, then it also has this property in A-lat.*
3. *If $0 \to L \to M \to N \to 0$ is an almost split sequence in A'-lat, then it also has the same property in A-lat.*

Proof. 1. Let X be an A-lattice and let $\xi \in \operatorname{Hom}_A(X, M)$ be a nonsplit epimorphism. If $X \notin B$, then it is an A'-lattice and, hence, ξ factors through α. If $X \Subset B$, then it is projective and ξ also factors through α.

The second assertion is true by duality.

The third assertion follows either from the first assertion or form the second assertion.

The following theorem describes the position of new projective modules over the order $A^-(B)$ in almost split sequences of the category A-lat. A similar result was presented in [17].

Theorem 6.1. *Suppose that B is an indecomposable bijective A-lattice and $A' = A^-(B)$. Assume that $B^{\mathfrak{r}}$ is not projective over A (or, equivalently, $\mathfrak{r}B$ is not L-injective over A).*

1. *If $B^{\mathfrak{r}}$ decomposes, i.e., $B^{\mathfrak{r}} = M_1 \oplus M_2$, then there exist almost split sequences*

$$0 \to \mathfrak{r}M_1 \to B \to M_2 \to 0,$$

$$0 \to \mathfrak{r}M_2 \to B \to M_1 \to 0.$$

In particular, $\tau_A M_1 = \mathfrak{r}M_2$ and $\tau_A M_2 = \mathfrak{r}M_1$.

2. *If $B^{\mathfrak{r}}$ is indecomposable, then $B^{\mathfrak{r}}$ has a maximal submodule $X \neq B$ and there exists an almost split sequence*

$$0 \to \mathfrak{r}B \to B \oplus X \xrightarrow{\alpha} B^{\mathfrak{r}} \to 0. \tag{6.2}$$

In particular, $\tau_A B^{\mathfrak{r}} = \mathfrak{r}B$.

Proof. The lattice $B^{\mathfrak{r}}$ is projective and $\mathfrak{r}B$ is L-injective over A' by Lemma 3.3. Let M be the direct summand of $B^{\mathfrak{r}}$, let $N = \tau_A M$, and let $0 \to N \to E \to M \to 0$ be an almost split sequence in $A\text{-lat}$. If N is not L-injective as an A'-lattice, then $A'\text{-lat}$ contains an almost split sequence $0 \to N \to E' \to M' \to 0$. By Proposition 6.2, it is also almost split in $A\text{-lat}$. This implies that $M' \simeq M$, which is impossible because M is projective over A'. Thus, $\tau_A M$ is L-injective as an A'-lattice but not as an A-lattice. Hence, it is the direct summand of $\mathfrak{r}B$. In particular, if $B^{\mathfrak{r}}$ is indecomposable, then $\tau_A B^{\mathfrak{r}} = \mathfrak{r}B$.

Since the irreducible morphism $B \to M$ exists, B must be the direct summand of E, i.e., $E = B \oplus X$. If $B^{\mathfrak{r}} = M_1 \oplus M_2$, then the exact sequence $0 \to \mathfrak{r}M_1 \to B \to M_2 \to 0$ exists and, since $KB \simeq KM_1 \oplus KM_2$, we get $X = 0$. If $B^{\mathfrak{r}}$ is indecomposable, then $KX \simeq KB$. By Proposition 6.1, in the almost split sequence (6.2), the restriction of α to X is an isomorphism onto the maximal submodule in $B^{\mathfrak{r}}$, which cannot coincide with B.

Remark 6.1.

1. It is possible that $M_1 \simeq M_2$ in Case 1 and $X \simeq B$ in Case 2. If $X \not\simeq B$, then this is an A'-lattice and $X = \mathfrak{r}' B^{\mathfrak{r}}$, where $\mathfrak{r}' = \operatorname{rad} A'$. At the same time, if $X \simeq B$, then $\mathfrak{r}' B^{\mathfrak{r}} = \mathfrak{r} B^{\mathfrak{r}}$.
2. By Lemma 3.5, the condition that "$B^{\mathfrak{r}}$ is not projective" is always satisfied if A is connected, Gorenstein, and not hereditary.

7. Gorenstein and Frobenius Cases

If A is a Gorenstein order, then the functor ${}^{\vee}\colon M \mapsto M^{\vee} = \operatorname{Hom}_A(M,A)$ is the exact duality $A\text{-lat} \to A^{\mathrm{op}}\text{-lat}$. Combining it with the duality $D\colon\ A^{\mathrm{op}}\text{-lat} \to A\text{-lat}$, we arrive at the *Nakayama equivalence*

$$\mathcal{N} = D^{\vee}\colon\ A\text{-lat} \to A\text{-lat}.$$

It maps projective modules into projective modules. Hence, it can be regarded as a functor on the stable category $\underline{A\text{-lat}} \to \underline{A\text{-lat}}$. The following result is an analog of Proposition IV.3.6 in [2]:

Proposition 7.1. *If the order A is Gorenstein, then the functors τ_A, $\Omega\mathcal{N}$, and $\mathcal{N}\Omega$ are isomorphic.*

Proof. Let M be a nonprojective A-lattice. Consider an exact sequence

$$0 \to N \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0 \xrightarrow{\gamma} M \to 0,$$

where $P_1 \xrightarrow{\beta} P_0 \xrightarrow{\gamma} M \to 0$ is the minimal projective mapping of M. It gives the exact sequence

$$0 \to M^{\vee} \xrightarrow{\gamma^{\vee}} P_0^{\vee} \xrightarrow{\beta^{\vee}} P_1^{\vee} \xrightarrow{\alpha^{\vee}} N^{\vee} \to 0.$$

Thus, $N^{\vee} \simeq \operatorname{tr} M$ and $\Omega \operatorname{tr} M \simeq \operatorname{Im} \beta^{\vee}$. Hence, the exact sequence

$$0 \to D(\operatorname{Im} \beta^{\vee}) \to P_0^{\vee\vee} \to DM^{\vee} \to 0$$

shows that $\tau_A M \simeq D(\operatorname{Im} \beta^{\vee}) \simeq \Omega\mathcal{N}M$. It is clear that this structure is functorial with respect to M and therefore, establishes an isomorphism $\tau_A \simeq \Omega\mathcal{N}$. Since $\mathcal{N}$ is exact and maps projective modules into projective modules, it commutes with Ω, i.e., $\Omega\mathcal{N} \simeq \mathcal{N}\Omega$.

Proposition 7.1 is proved.

Let $A \simeq \bigoplus_{i=1}^{s} P_i^{m_i}$, where $P_1, P_2, \ldots, P_s$ are all pairwise nonisomorphic principal left A-modules. Then, in addition, $A \simeq \bigoplus_{i=1}^{s} (P_i^\vee)^{m_i}$ as a right A-module, $DA \simeq \bigoplus_{i=1}^{s} (DP_i^\vee)^{m_i}$ as a left A-module, and

$$DP_1^\vee, DP_2^\vee, \ldots, DP_s^\vee$$

are all pairwise nonisomorphic coprincipal left A-modules. Thus, A is Gorenstein if and only if there exists a permutation ν such that $P_i \simeq DP_{\nu i}^\vee$ for all $i = 1, 2, \ldots, s$. The permutation ν is called the *Nakayama permutation*.

Definition 7.1. *The order A is called Frobenius if $A \simeq DA$ as a left A-module and symmetric if $A \simeq DA$ as an A-bimodule.*

It is clear that this definition is left/right symmetric and A is Frobenius if and only if it is Gorenstein and $m_i = m_{\nu i}$ for all $i = 1, 2, \ldots, s$, where ν is the Nakayama permutation.

Definition 7.2. *Let M be a left A-module and let σ be an automorphism of A. By ${}^\sigma M$ we denote a left A-module that coincides with M as a group but, for each $a \in A$ and $x \in M$, the product ax in ${}^\sigma M$ is equal to the product $\sigma(a)x$ in M. Similarly, we define N^σ for the right A-module N and ${}^\rho M^\sigma$ for the A-bimodule M, where ρ is also an automorphism of A. If ρ or σ is identical, then we reject it and write M^σ or ${}^\rho M$, respectively.*

It is easy to see that the mappings $x \mapsto \rho^{-1}(x)$ and $x \mapsto \sigma^{-1}(x)$ are isomorphisms of the A-bimodules ${}^\rho A^\sigma \simeq A^{\rho^{-1}\sigma}$ and ${}^\rho A^\sigma \simeq {}^{\sigma^{-1}\rho}A$, respectively.

Proposition 7.2. *A is Frobenius if and only if there exists an automorphism $\sigma \in \operatorname{Aut} A$ such that $DA \simeq A^\sigma$ as an A-bimodule. Moreover, there exists an inverse element $s \in KA$ such that $\sigma(a) = s^{-1}as$ for all $a \in A$.*

Proof. It is clear that if this automorphism exists, then A is Frobenius. Assume that A is Frobenius and $\varphi\colon A \xrightarrow{\sim} \Delta$ is an isomorphism of left A-modules, where $\Delta = DA$. It induces an isomorphism of left KA-modules $K\varphi\colon KA \xrightarrow{\sim} K\Delta$. Since KA is semisimple, it is symmetric as a K-algebra [4] (9.8), i.e., there exists an isomorphism of KA-bimodules $\theta\colon KA \xrightarrow{\sim} K\Delta$. The composition $\theta^{-1}{\cdot}K\varphi$ is an automorphism of KA as a left KA-module. Hence, there exists an inverse element $s \in KA$ such that $\theta^{-1}K\varphi(x) = xs$ for each $x \in KA$. In particular, $\varphi(x) = \theta(xs)$ for each $x \in A$, whence it follows that $\Delta = \theta(As)$. This implies that $As = \theta^{-1}(\Delta)$ is a two-sided A-module, i.e., $sA \subseteq As$ and $sAs^{-1} \subseteq A$. Thus, $sAs^{-1} = A$ and $s^{-1}As = A$. Moreover,

$$\varphi(xa) = \theta(xas) = \theta(xss^{-1}as) = \theta(xs)s^{-1}as = \varphi(x)s^{-1}as.$$

Therefore, φ is an isomorphism of A-bimodules $A^\sigma \xrightarrow{\sim} \Delta$, where $\sigma(a) = s^{-1}as$.

Proposition 7.2 is proved.

It can be proved that the element s is determined to within a factor of the form $q\lambda$, where q and λ are invertible elements from A and from the center KA, respectively.

Corollary 7.1. *Let A be a Frobenius order, let $\sigma \in \operatorname{Aut} A$ be an automorphism from Proposition 7.2, and let $\mathcal{N}$ be the Nakayama equivalence. The following functorial isomorphisms exist:*

$DM \simeq (M^\vee)^\sigma$ for each left A-lattice M and $DN \simeq {}^{\sigma^{-1}}(N^\vee)$ for each right A-lattice N;

$\mathcal{N}M \simeq {}^{\sigma^{-1}}M$ and $\tau_A M \simeq \Omega({}^{\sigma^{-1}}M) \simeq {}^{\sigma^{-1}}(\Omega M)$ for each left A-lattice M.

In particular, if A is symmetric, then $\mathcal{N} \simeq \operatorname{Id}$ and $\tau_A \simeq \Omega$.

Proof. The proof of the corollary is obvious.

Corollary 7.2. *Let A be Gorenstein, let $\mathfrak{r} = \operatorname{rad} A$, let $P_1, P_2, \dots, P_s$ be the complete list of nonisomorphic principal A-modules, and let $\omega_i = DP_i^\vee$ (then $\omega_1, \dots, \omega_s$ is the complete list of nonisomorphic coprincipal modules). Also let $A' = A^-(A)$, $P_i' = P_i^{\mathfrak{r}}$, and $\omega_i' = \mathfrak{r}\omega_i$. Then $\tau_A P_i' \simeq \omega_{\nu i}'$, where ν is the Nakayama permutation.*

Proof. The proof of the corollary follows from Theorem 6.1.

Corollary 7.3. *Let G be a finite group and let A be a block of its group ring $\mathbb{Z}_p G$. This is a symmetric $\mathbb{Z}_p$-order. Also let $A' = A^-(A)$. Then, for each nonprojective A-lattice M (or, equivalently, for each A'-lattice M),*

$$\hat{H}^n(G, M) \simeq \hat{H}^{n+1}(G, \tau_A M) \simeq \hat{H}^{n-1}(G, \tau_A^{-1} M).$$

Proof. The proof of the corollary follows from Corollary 7.1 and Proposition 6.2.

Note that $\tau_A M = \tau_{A'} M$ if M is not projective over A'. Otherwise, $\tau_A M$ is determined by Corollary 7.2. In some cases, the structure of the AP-type of $AR_{A'}$ can be efficiently calculated. This gives the values of cohomologies. An example, where G is Klein's four-group, was presented in [10].

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