COHOMOLOGIES OF REGULAR LATTICES OVER THE KLEINIAN 4-GROUP

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ABSTRACT. We calculate explicitly cohomologies of the lattices over the Kleinian 4-group belonging to the regular components of the Auslander–Reiten quiver as well as of their dual modules. The result is applied to the classification of some crystallographic and Chernikov groups.

CONTENTS

The aim of this paper is to apply the results on cohomologies of the Kleinian 4-group [9] to the classification of crystallographic and Chernikov groups. For this purpose it is important to have an explicit presentation of 2-cocycles. We find such presentation for a special class of lattices, called *regular*, and for their dual modules. Moreover, we describe the orbits of the action of automorphisms of modules on cohomologies. From this results we obtain a complete description of crystallographic and Chernikov groups with the Kleinian top and regular base.

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1. Lattices over the Kleinian 4-group.

In what follows K denotes the Kleinian 4-group, $K = \langle a, b \mid a^2 = b^2 = 1,\ ab = ba \rangle$, We study cohomologies of this group with the values in K-*lattices*, i.e. K-modules M such that the additive group of M is free abelian of finite rank, and in their *duals*, i.e. the modules $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Let $R = \mathbb{Z}K$. We embed it into $R^\sharp = \mathbb{Z}^4$ identifying a withthe the quadruple $(1, 1, -1, -1)$ and b with $(1, -1, 1, -1)$. Note that $R^\sharp$ is the integral closure of R in $\mathbb{Q} \otimes_{\mathbb{Z}} R$. Let $\mathbb{Z}_p$ be the ring of p-adic integers, $M_p = M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for every abelian group M. Then $R_p = \mathbb{Z}_p^4$ for $p \neq 2$ and $R_2 \supseteq 4\mathbb{Z}_2^4$. It follows from [5, Th. 3.7] that two K-lattices M, N are isomorphic if and only if they are *in the same genus*, i.e. $M_p \simeq N_p$ for all p. Moreover, if $p \neq 2$, the R_p-lattice M_p is uniquely defined by the rational envelope $\mathbb{Q} \otimes_{\mathbb{Z}} M$. Therefore, a K-lattice M is uniquely detremined by its 2-adic completion, which we denote by $\hat{M}$. We denote by R-lat the category of R-lattices and by $\hat{R}$-lat the category of $\hat{R}$-*lattices*, i.e. $\hat{R}$-modules which are finitely generated and torsion free (hence free) as $\mathbb{Z}_2$-modules. The functor $M \mapsto \hat{M}$ is a *representation equivalence* between the categories R-lat and $\hat{R}$-lat, i.e. it maps non-isomorphic modules to non-isomorphic, indecomposable to indecomposable and every $\hat{R}$-lattice is isomorphic to $\hat{M}$ for a uniquely defined R-lattice M.

Since $4\hat{H}^n(K, M) = 0$ for any K-module M [1, Prop. XII.2.5], $\hat{H}^n(K, M) \simeq \hat{H}^n(K, \hat{M})$. Let $\mathbb{D} = \mathbb{Q}_2/\mathbb{Z}_2$, where $\mathbb{Q}_2$ is the field of p-adic numbers. It is the group *of type* p^∞, i.e. the direct limit $\varinjlim_n \mathbb{Z}/2^n\mathbb{Z}$ of finite cyclic 2-groups with respect to the natural embeddings $\mathbb{Z}/2^n\mathbb{Z} \to \mathbb{Z}/2^{n+1}\mathbb{Z}$. We call K-modules of the form $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{D}) \simeq \mathrm{Hom}_{\mathbb{Z}_2}(\hat{M}, \mathbb{D})$, where M is a K-lattice, K*-colattices.*

The ring R is *Gorenstein*, i.e. $\mathrm{inj.dim}_R R = 1$. Since R_p is a maximal order for $p \neq 2$, [7, Lem. 2.9] implies that R has a unique minimal overring A and every indecomposable R-lattice, except R itself, is an A-lattice. Actually, A coincides with the subring of $R^\sharp = \mathbb{Z}^4$ consisting of all quadruples (z_1, z_2, z_3, z_4) such that $z_1 \equiv z_2 \equiv z_3 \equiv z_4 \pmod 2$. By ,

Let $\mathfrak{m}$ be the ideal of A consisting of all quadruples (z_1, z_2, z_3, z_4) such that $z_1 \equiv z_2 \equiv z_3 \equiv z_4 \equiv 0 \pmod 2$. Then $\hat{\mathfrak{m}} = \mathrm{rad}\,\hat{A} = \mathrm{rad}\,\hat{R}^\sharp$. So $\hat{A}$ is a *Backström order* in the sense of [12]. Therefore, according to [12], $\hat{A}$-lattices, hence also A-lattices, are classified by the representations of the quiver

$$\Gamma = \bullet \begin{array}{l} \nearrow\ ++ \\ \nearrow\ +- \\ \searrow\ -+ \\ \searrow\ -- \end{array}$$

Recall the corresponding construction (on the level of A-lattices). For any K-module M let $M_{\alpha\beta}$, where $\alpha, \beta \in \{+, -\}$, be the submodule $\{u \in$

$M \mid au = \alpha u, bu = \beta u\}$. If M is an A-lattice, set $M^\sharp = R^\sharp M$. Then $M^\sharp = \bigoplus_{\alpha\beta} M^\sharp_{\alpha\beta}$. and $\mathfrak{m}M = \mathfrak{m}M^\sharp = 2M^\sharp$. Let $V_\bullet = M/\mathfrak{m}M$ and $V_{\alpha\beta} = M^\sharp_{\alpha\beta}/2M^\sharp_{\alpha\beta}$. Taking for $f_{\alpha\beta}$ the natural maps $V_\bullet \to V_{\alpha\beta}$, we obtain a representation of the quiver Γ, which we denote by $\Phi(V)$. Thus we define a functor Φ from the category A-lat of A-lattices to the category $\operatorname{rep}\Gamma$ of represenations of the quiver Γ over the field $\Bbbk = \mathbb{Z}/2\mathbb{Z}$. We have the following result analohous to that of [12].

Theorem 1.1. *Let $\mathcal{R}$ be the category of representations*

$$V = V_\bullet \quad \begin{cases} \xrightarrow{f_{++}} V_{++} \\ \xrightarrow{f_{+-}} V_{+-} \\ \xrightarrow{f_{-+}} V_{-+} \\ \xrightarrow{f_{--}} V_{--} \end{cases} \tag{1.1}$$

of the quiver Γ over $\Bbbk$ such that all maps $f_{\alpha\beta}$ are surjective and the induced map $f_\oplus : V_\bullet \to V_\oplus = \bigoplus_{\alpha\beta} V_{\alpha\beta}$ is injective. The functor Φ is a representation equivalence A-lat $\to \mathcal{R}$ such that all induced maps $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_\Gamma(\Phi(M),\Phi(N))$ are surjective.

Proof. Obviously, always $\Phi(M) \in \mathcal{R}$. Let $V \in \mathcal{R}$, $d_{\alpha\beta} = \dim V_{\alpha\beta}$ and $d_\bullet = \dim V_\bullet$. Denote by $\mathbb{Z}_{\alpha\beta}$ the K-module $\mathbb{Z}$, where a acts as $\alpha 1$ and b acts as $\beta 1$. Thus $R^\sharp = \bigoplus_{\alpha\beta} \mathbb{Z}_{\alpha\beta}$. Set $M^\sharp = \bigoplus_{\alpha\beta} \mathbb{Z}^{d_{\alpha\beta}}_{\alpha\beta}$ and define $M(V)$ as the preimage of $\operatorname{Im} f_\oplus$ in $M^\sharp$ under the epimorphism $M^\sharp \to M^\sharp/2M^\sharp \simeq V_\oplus$. Then $M(V)$ is an A-lattice such that $\Phi(M(V)) \simeq V$ and $M(V)^\sharp = M^\sharp$. It is also evident that $M(\Phi(M)) \simeq M$. Hence Φ is a representation equivalence.

A morphism $\phi : V \to V'$, where

$$V' = V'_\bullet \quad \begin{cases} \xrightarrow{f'_{++}} V'_{++} \\ \xrightarrow{f'_{+-}} V'_{+-} \\ \xrightarrow{f'_{-+}} V'_{-+} \\ \xrightarrow{f'_{--}} V'_{--} \end{cases}$$

is given by a quintuple of homomorphisms $\{\phi_\bullet, \phi_{++}, \phi_{+-}, \phi_{-+}, \phi_{--}\}$ such that $\phi_{\alpha\beta} f_\bullet = f'_{\alpha\beta}\phi_\bullet$ for all α, β. If $V = \Phi(M)$ and $V' = \Phi(N)$, these homomorphisms give a homomorphism $\tilde{\phi} : M^\sharp/2M^\sharp \to N^\sharp/2N^\sharp$ such that

$\tilde{\phi} f_{\oplus} = f'_{\oplus}\tilde{\phi}$. If we lift $\tilde{\phi}$ to a homomorphism $\psi^{\sharp} : M^{\sharp} \to N^{\sharp}$, it implies that $\psi^{\sharp}(M) \subseteq N$, so we obtain a homomorphism $\psi : M \to N$ such that $\Phi(\psi) = \phi$. □

We call the quintuple $(d_{\bullet}, d_{\alpha\beta})$ $(\alpha, \beta \in \{+,-\})$ the *dimension* of the representation V or of the corresponding lattice $M = M(V)$, denote it by $\mathbf{dim}\, V$ or $\mathbf{dim}\, M$ and usually present it in the form

$$\boxed{\begin{matrix} & d_{++} \\ & d_{+-} \\ d_{\bullet} & d_{-+} \\ & d_{--} \end{matrix}}$$

We also denote, if necessary, $d_{\bullet} = d_{\bullet}(M)$, $d_{\alpha\beta} = d_{\alpha\beta}(M)$ and $d_{\oplus} = d_{\oplus}(M) = \sum_{\alpha\beta} d_{\alpha\beta}(M)$. Note that the rank of M as of $\mathbb{Z}$-module equals $\sum_{\alpha\beta} d_{\alpha\beta}$.

We also need analogues of some results from [13]. For this purpose we establish a lemma. For any lattice M we set $\bar{M} = M/2M$ and if $\alpha : M \to N$ is a homomorphism of lattices, we denote by $\bar{\alpha}$ the induced map $\bar{M} \to \bar{N}$.

Lemma 1.2. *Any exact sequence* $0 \to \bar{M} \xrightarrow{\bar{\alpha}} \bar{N} \xrightarrow{\bar{\beta}} \bar{L} \to 0$ *can be lifted to an exact sequence* $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} L \to 0$.

Proof. We choose bases in M, N, L and the corresponing bases in $\bar{M}, \bar{N}, \bar{L}$ and identify $\bar{\alpha}$ and $\bar{\beta}$ with their matrices with respect to these bases. There are invertible matrices $\bar{S}, \bar{T}$ of appropriate sizes over the field $\Bbbk$ such that $\bar{S}^{-1}\bar{\alpha}\bar{T} = \left(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right)$, where I is the unit matrix. Then $\bar{\beta}\bar{S} = \begin{pmatrix} 0 & \bar{C} \end{pmatrix}$ for some invertible matrix $\bar{C}$. Since the maps $\mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \Bbbk)$ is surjective, we can lift $\bar{S}, \bar{T}, \bar{C}$ to invertible matrices S, T, C over $\mathbb{Z}$. Then the homomorphisms $M \to N$ and $N \to L$ given, respectively, by the matrices $\alpha = S \left(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right) T^{-1}$ and $\beta = \begin{pmatrix} 0 & C \end{pmatrix} S^{-1}$ are the necessary liftings of $\bar{\alpha}$ and $\bar{\beta}$. □

Corollary 1.3. *Let* $0 \to V' \xrightarrow{\bar{\alpha}} V \xrightarrow{\bar{\beta}} V'' \to 0$ *be an exact sequence of representations from* $\mathcal{R}$. *It can be lifted to an exact sequence of* A*-lattices* $0 \to M(V') \xrightarrow{\alpha} M(V) \xrightarrow{\beta} M(V'') \to 0$.[1]

Proof. We denote $\tilde{V} = \bigoplus_{\alpha\beta} V_{\alpha\beta}$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V'_{\bullet} & \xrightarrow{\tilde{\alpha}} & V_{\bullet} & \xrightarrow{\tilde{\beta}} & V''_{\bullet} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f'_{V'}} & & \Big\downarrow{\scriptstyle f_V} & & \Big\downarrow{\scriptstyle f''_{V''}} & & \\ 0 & \longrightarrow & \tilde{V}' & \xrightarrow{\tilde{\alpha}} & \tilde{V} & \xrightarrow{\tilde{\beta}} & \tilde{V}'' & \longrightarrow & 0 \end{array} \tag{1.2}$$

By Lemma 1.2, the second row can be lifted to an exact sequence

$$0 \to M(V')^{\sharp} \xrightarrow{\alpha^{\sharp}} M(V)^{\sharp} \xrightarrow{\beta^{\sharp}} M(V'')^{\sharp} \to 0.$$

[1] This result does not follow directly from [13, Lem. 4], where the case of algebras over comlete discrete valuation rings is considered. Moreover, it highly depends on Lemma 1.2, hence on the "smallness" of the residue field $\Bbbk$.

Since the diagram (1.2) is commutative, $\alpha^\sharp(M(V')) \subseteq M(V)$ and $\beta^\sharp(M(V)) \subseteq M(V'')$. So we obtain the necassary lifting $0 \to M(V') \xrightarrow{\alpha} M(V) \xrightarrow{\beta} M(V'') \to 0$. It is exact by the 3×3 lemma, □

We say that a monomorphism of lattices $\phi : M \to N$ is *strict* if $\operatorname{Coker}\phi$ is torsion free (hence, also a lattice).

Corollary 1.4. *Every epimorphism (monomorphism, isomorphism) $V \to V'$ of representations from $\mathcal{R}$ can be lifted to an epimorphism (respectively, strict monomorphism, isomorphism) of A-lattices $\Phi(V) \to \Phi(V')$.*

Corollary 1.5. *Given a chain of subrepresentations*

$$V = V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_{m-1} \supset V_m = 0,$$

there is a chain of sublattices in $M = M(V)$

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_{m-1} \supset M_m = 0$$

such that $M_k \simeq M(V_k)$ and $M_k/M_{k+l} \simeq M(V_k/V_{k+l})$ for all possible values of k, l.

2. Regular lattices

Recall the structure of the Auslander–Reiten quiver $\mathcal{Q}$ of the category $\hat{A}$-lat [11]. According to [13], it is obtained from the Auslander–Reiten quiver of the category $\operatorname{rep}\Gamma$ by gluing the preprojective and the preinjective components into one component. The resulting *preprojective-preinjective component* is shown at Figure 1. Here all lattices are uniquely determined by their dimensions. The Auslander–Reiten transpose τ of the category $\hat{R}$-lat acts on this component as the shift to the left. Note that, for every A-lattice M, $\tau\hat{M} \simeq \Omega\hat{M}$, the syzygy of $\hat{M}$ as of $\hat{R}$-module [9, Prop. 1.1]. Hence, $\Omega M \simeq N$ if $\tau\hat{M} = \hat{N}$.

The other components, called *regular*, are *tubes*. They are parametrized by the projective line $\mathbb{P}^1$ over the field $\Bbbk$, which consists of unital irreducible polynomials and the symbol ∞. We denote the tube corresponding to the polynomial $f(t)$ by $\mathcal{T}^f$ and the tube corresponding to ∞ by $\mathcal{T}^\infty$. We also write $\mathcal{T}^0$ instead of $\mathcal{T}^t$ and $\mathcal{T}^1$ instead of $\mathcal{T}^{t-1}$. When describing tubes, we substitute A-lattices for their completions and say that T belongs to a tube $\mathcal{T}^f$ if $\hat{T}$ belongs to this tube. Then we call T a *regular K-lattice*. We call a K-lattice M is *regular* if all its indecomposable direct summands are regular.

All tubes except $\mathcal{T}^\lambda$ ($\lambda \in \{0, 1, \infty\}$) are *homogeneous*, i.e. $\tau M = M$ for all M from this tube. They are of the form

$$T_1^f \rightleftarrows T_2^f \rightleftarrows T_3^f \rightleftarrows \cdots \tag{2.1}$$

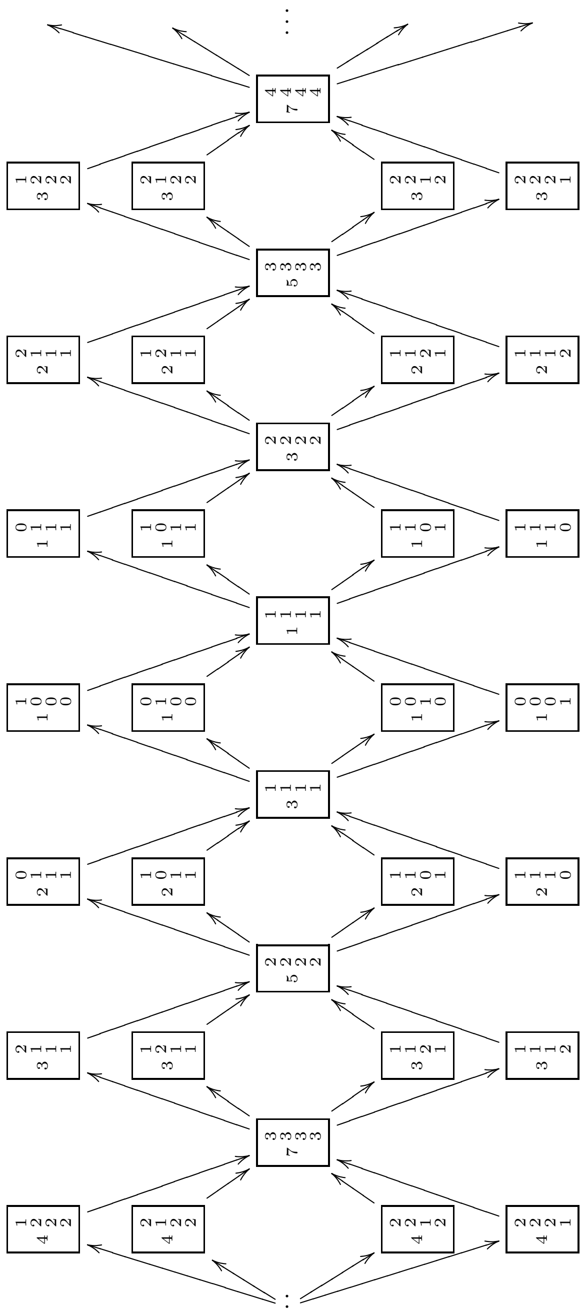
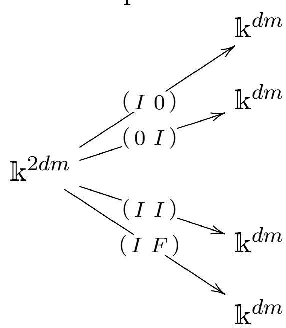


FIGURE 1. Preprojective-preinjective component

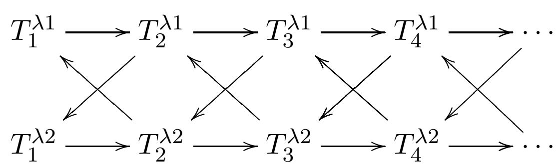
$\mathbf{dim}\, T^f_m = \boxed{\begin{smallmatrix} & dm \\ 2dm & dm \\ & dm \\ & dm \end{smallmatrix}}$, where $d = \deg f(t)$. Actually, $T^f_m = M(V)$, where V is the following representation of the quiver Γ:



(2.2)

Here I is the $dm \times dm$ unit matrix and F is the Frobenius matrix with the characteristic polynomial $f(t)^m$.

The tube $\mathcal{T}^\lambda$ for $\lambda \in \{0, 1, \infty\}$ is of the form



(2.3)

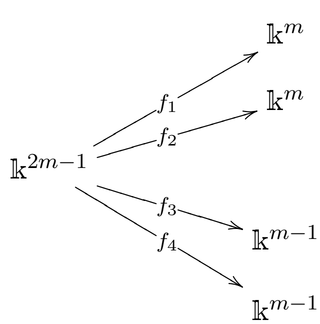
Here $\tau T_n^{\lambda 1} = T_n^{\lambda 2}$ and $\tau T_n^{\lambda 2} = T_n^{\lambda 1}$. For $\lambda = 1$ we have

$$\mathbf{dim}\, T^{1j}_{2m} = \boxed{\begin{smallmatrix} & m \\ 2m & m \\ & m \\ & m \end{smallmatrix}} \quad \text{for both } j = 1 \text{ and } j = 2,$$

$$\mathbf{dim}\, T^{11}_{2m-1} = \boxed{\begin{smallmatrix} & m \\ 2m{-}1 & m \\ & m{-}1 \\ & m{-}1 \end{smallmatrix}},$$

$$\mathbf{dim}\, T^{12}_{2m-1} = \boxed{\begin{smallmatrix} & m{-}1 \\ 2m{-}1 & m{-}1 \\ & m \\ & m \end{smallmatrix}}.$$

Actually, $T^{11}_{2m} = M(V)$, where V is of the form (2.2), where $d = 1$ and $F = J_1$ is the Jordan $m \times m$ matrix with eigenvalue 1. $T^{11}_{2m-1} = M(V')$, where V' is of the form



Here

$$f_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$f_2 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$f_3 = \begin{pmatrix} I & I & 0 \end{pmatrix},$$
$$f_4 = \begin{pmatrix} I & J_1 & \mathrm{e} \end{pmatrix}, \text{ with } \mathrm{e} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The lattices T_n^{12} are obtained from T_n^{11} by the permutations of the 1st with the 3rd rows and of the 2nd with the 4th rows.

The tubes $\mathcal{T}^0$ and $\mathcal{T}^\infty$ are obtained from the tube $\mathcal{T}^1$ by pernutations, respectively, of the 2nd with the 4th rows and of the 2nd with the 3rd rows.

Note that an indecomposable lattice M belongs to a tube if and only if

$$2d_\bullet(M) = \textstyle\sum_{\alpha\beta} d_{\alpha\beta}(M). \tag{2.4}$$

In this case

$$d_\bullet(\Omega M) = d_\bullet(M) \text{ and } d_{\alpha\beta}(\Omega M) = d_\bullet(M) - d_{\alpha\beta}(M). \tag{2.5}$$

The structure of the representations of quivers belonging to tubes is described in [3, 4] (see also [6, Thm. 31 and 36]). Together with Corollary 1.5 it gives the following result for lattices.

Theorem 2.1. *Every module T_m^f or $T_m^{\lambda j}$ has a chain of submodules*

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_{m-1} \supset M_m = \{0\} \tag{2.6}$$

such that

(1) $M_k/M_{k+l} \simeq \begin{cases} T_l^f & \text{if } M = T_m^f, \\ T_l^{\lambda j} & \text{if } M = T_m^{\lambda j} \text{ and } k \text{ is even}, \\ T_l^{\lambda i}, & \text{where } i \neq j, \text{ if } M = T_m^{\lambda j} \text{ and } k \text{ is odd}. \end{cases}$

(2) *The maps $T_m^f \to T_{m+1}^f$ and $T_m^{\lambda j} \to T_{m+1}^{\lambda j}$ in the diagrams, respectively,* (2.1) *and* (2.3) *can be chosen injective, with the quotients, respectively, T_1^f and $T_1^{\lambda j}$.*

(3) *The maps $T_{m+1}^f \to T_m^f$ and $T_{m+1}^{\lambda i} \to T_m^{\lambda j}$ $(i \neq j)$ in the diagrams, respectively,* (2.1) *and* (2.3) *can be chosen surjective, with the kernels, respectively, T_1^f and $T_1^{\lambda i}$.*

(4) *If M and M' belong to different tubes, then $\operatorname{Im}\varphi \subseteq 2N$ for every homomorphism $\varphi : M \to N$.*

One can aslo get a description of endomorphisms of indecomposable lattices belonging to tubes. For an irreducible polynomial $f(t) \in f[t]$ of degree d choose an integer unital polynomial $\tilde{f}[t]$ such that $f[t] = \tilde{f}[t] \bmod 2$. Set $\mathbb{Z}^f = \mathbb{Z}[t]/(\tilde{f}[t])$, $\mathbb{Z}_m^f = \mathbb{Z}^f[r]/(r^m)$ and identify $\mathbb{Z}_m^f$ with its image in

$\mathrm{Mat}(dm,\mathbb{Z})$ obtained when we consider the action of this ring on itself. Let mI_m^f be the image of $\mathbb{Z}_m^f$ under the diagonal embadding $\mathrm{Mat}(dm,\mathbb{Z})$ into $\mathrm{Mat}(dm,\mathbb{Z})^4$. If $d=1$, hence $\mathbb{Z}^f=\mathbb{Z}$, we denote it by $\mathbb{I}_m$. Direct calculations give the following result.

Theorem 2.2. (1) $\mathrm{End}_R\, T_m^f \simeq \mathbb{I}_m^f + \mathrm{Mat}(dm, 2\mathbb{Z})^4$.
(One easily sees that it does not depend on the choice of $\tilde{f}[t]$).
(2) $\mathrm{End}_R\, T_m^{\lambda j} \simeq \mathbb{I}_m + \mathrm{Mat}(m, 2\mathbb{Z})^4$.

3. Cohomologies

We will give an explicit description of the cohomologies $H^n(K,M)$ and $H^n(K,DM)$ for $n>0$ and regular K-lattices M. Obviously, we only have to calculate them for indecomposable lattices.

It follows from [8] that a free resolution $\mathbf{P}$ for the trivial K-module $\mathbb{Z}$ can be chosen as follows: $\mathbf{P}_n$ is the set of homogeneous polynomials of degree n from $R[x,y]$ and

$$d(x^ky^l) = (a+(-1)^k)x^{k-1}y^l + (-1)^k(b+(-1)^l)x^ky^{l-1}.$$

So an n-cocycle γ is given by the values $\gamma(x^my^{n-m})$ $(0\le m\le n)$.

Let M be an indecomposable regular lattice. Set

$$M(n) = \begin{cases} M_{++} & \text{if } n \text{ is even,}\\ M_{-+} & \text{if } n \text{ is odd and } M\notin \mathcal{T}^\infty,\\ M_{+-} & \text{if } n \text{ is odd and } M\in \mathcal{T}^\infty, \end{cases}$$

We define a homomorphism $\xi : M(n) \to H^n(K,M)$ $(n>0)$. It sends an element $v\in M(n)$ to the class of the cocycle ξ_v which is defined as follows.

- If n is even, $M\notin\mathcal{T}^\infty$ and $v\in M_{++}$,
$$\xi_v(x^my^{n-m}) = \begin{cases} v & \text{if } m=n,\\ 0 & \text{otherwise.}\end{cases}$$
- If n is even, $M\in\mathcal{T}^\infty$ and $v\in M_{++}$,
$$\xi_v(x^my^{n-m}) = \begin{cases} v & \text{if } m=0,\\ 0 & \text{otherwise.}\end{cases}$$
- If n is odd, $M\notin\mathcal{T}^\infty$ and $v\in M_{-+}$,
$$\xi_v(x^my^{n-m}) = \begin{cases} v & \text{if } m=n,\\ 0 & \text{otherwise.}\end{cases}$$
- If n is odd, $M\in\mathcal{T}^\infty$ and $v\in M_{+-}$,
$$\xi_v(x^my^{n-m}) = \begin{cases} v & \text{if } m=0,\\ 0 & \text{otherwise.}\end{cases}$$

One easily verifies that ξ_v is indeed a cocycle.

Theorem 3.1. *For every indecomposable regular K-lattice M and every $n > 0$ the map ξ induces an isomorphism $M(n)/2M(n) \xrightarrow{\sim} H^n(K, M)$.*

For the proof we use the following results.

Lemma 3.2. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of regular K-lattices, then the induced sequences*

$$0 \to \Omega M' \to \Omega M \to \Omega M'' \to 0, \tag{3.1}$$

$$0 \to \Omega^{-1} M' \to \Omega^{-1} M \to \Omega^{-1} M'' \to 0 \tag{3.2}$$

are also exact.

Proof. From the properties of syzygies it follows that there is an exact sequence

$$0 \to \Omega M' \to \Omega M \oplus P \to \Omega M'' \to 0$$

for some projective R-module P. But, as $\Omega M = \tau M$, the formulae (2.4) and (2.5) show that $d_{\alpha\beta}(\Omega M) = d_{\alpha\beta}(\Omega M') + d_{\alpha\beta}(\Omega M'')$. Therefore, $P = 0$ and we get the exact sequence (3.1). Then (3.2) follows by duality. □

Corollary 3.3. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of regular K-lattices, the induced sequence of cohomologies*

$$0 \to \hat{H}^n(K, M') \to \hat{H}^n(K, M) \to \hat{H}^n(K, M'') \to 0$$

is also exact for every $n \in \mathbb{Z}$.

Proof. It is known [9, Lem. 2.2] that if M contains no direct summands L_{++}, then $\hat{H}^0(K, M) \simeq \mathbb{Z}_2^{d_{++}(M)}$. It implies the claim for $n = 0$. The general case follows from Lemma 3.2 and the known fact that $\hat{H}^n(K, M) \simeq \hat{H}^{n+1}(K, \Omega M) \simeq \hat{H}^{n-1}(K, \Omega^{-1} M)$. □

Proof of Theorem 3.1. Note that [9, Th. 2.3] shows that $M(n)/2M(n) \simeq H^n(K, M)$. Hence we only have to check that ξ is injective. First, we check the claim for the lattices T_1^f and $T_1^{\lambda j}$. As the calculations are quite similar, we only consider the case of $M = T_1^{\infty 1}$ and n even (it seems the most complicated). Then M is the submodule of $L_{++} \oplus L_{--}$ consisting of the pairs (z, z') such that $z \equiv z' \pmod 2$. Thus the basic element of $M(n)$ is $v = (2, 0)$. We have to check that $\xi_v \neq \partial\gamma$ for any map $\gamma : \mathbf{P}_{n-1} \to M$. Suppose that $\xi_v = \partial\gamma$. Note that if $\gamma(x^{n-1}) = (z, z')$, then $\partial\gamma(x^n) = (2z, 0)$, whence $z = 1$. Let $\gamma(x^{n-k-1}y^k) = (z_k, z'_k)$ $(0 < k < n)$. Then

$$\partial\gamma(x^{n-1}y) = (a-1)(z_1, z'_1) - (b-1)(z, z') = (0, -2z' + 2z'_1) = (0, 0),$$

hence $z'_1 = z' \equiv 1 \pmod 2$;

$$\partial\gamma(x^{n-2}y^2) = (a+1)(z_2, z'_2) + (b+1)(z_1, z'_1) = (2z_2 + 2z_1, 0) = (0, 0),$$

hence $z_2 = -z_1 \equiv 1 \pmod 2$. Repeating this process, we obtain that all $z_k \equiv 1 \pmod 2$, so $z_k \neq 0$. Then $\partial\gamma(y^n) = (2z_{n-1}, 0) \neq 0 = \xi_v(y^n)$ and we get a contradiction.

For the lattice $M = T_m^f$ or $T_m^{\lambda j}$ $(m > 1)$ we have an exact sequence $0 \to M' \to M \to M'' \to 0$, where $M' \simeq T_1^f$ or $T_1^{\lambda j}$ and $M'' \simeq T_{m-1}^f$ or $T_{m-1}^{\lambda j}$. It gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'(n) & \longrightarrow & M(n) & \longrightarrow & M''(n) & \longrightarrow & 0 \\ & & \downarrow \xi & & \downarrow \xi & & \downarrow \xi & & \\ 0 & \longrightarrow & H^n(K, M') & \longrightarrow & H^n(K, M) & \longrightarrow & H^n(K, M'') & \longrightarrow & 0 \end{array}$$

Using induction, we can suppose that the first and the third vertical maps satisfy the assertion of the theorem. Then the same is true for the second vertical map. □

Analogous considerations give an explicit description of the cohomologies for *regular colattices*, i.e. the dual modules of regular lattices. For an indecomposable regular colattice $N = DM$ set $\bar{N} = \{ u \in N \mid 2u = 0 \}$ and

$$N(n) = \begin{cases} \bar{N}_{++} & \text{if } n \text{ is odd} \\ \bar{N}_{-+} & \text{if } n \text{ is even and } M \notin T^\infty, \\ \bar{N}_{+-} & \text{if } n \text{ is even and } M \in T^\infty. \end{cases}$$

We define a homomorphism $\eta : N(n) \to H^n(K, N)$ $(n > 0)$. It sends an element $u \in N(n)$ to the class of the cocycle η_u which is defined as follows.

- If n is even, $M \notin \mathcal{T}^\infty$ and $u \in \bar{N}_{-+}$,

$$\eta_u(x^m y^{n-m}) = \begin{cases} u & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

- If n is even, $M \in \mathcal{T}^\infty$ and $u \in \bar{N}_{+-}$,

$$\eta_u(x^m y^{n-m}) = \begin{cases} u & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

- If n is odd, $u \in \bar{N}_{++}$,

$$\eta_u(x^m y^{n-m}) = \begin{cases} u & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

One easily verifies that η_u is indeed a cocycle.

Theorem 3.4. *For every indecomposable regular colattice N and every $n > 0$ the map η induces an isomorphism $N(n) \xrightarrow{\sim} H^n(K, N)$.*

We omit the proof since it is quite analogous to that of Theorem 3.1 (and even easier).

4. Action of automorphisms

4.1. **Lattices.** We also need to know how automorphisms of lattices and of the group act on cohomologies. Let $M = T^f_m$ or $M = T^{\lambda j}_m$. Consider the chain of submodules $M_k \subset M$ from Theorem 2.1. We denote by $E^f_{k,m,n}$ or, respectively, by $E^{\lambda j}_{k,m,n}$, where $0 \le k < m$, the set $M_k(n) \setminus \big(2M_k(n) + M_{k+1}(n)\big)$. Note that $E^{\lambda j}_{k,m,n} \ne \emptyset$ if and only if $k < m$ and

$$k \equiv \begin{cases} j & \text{if } n \text{ is even} \\ j+1 & \text{if } n \text{ is odd} \end{cases} \pmod 2 \tag{4.1}$$

Theorems 2.1 and 2.2 easily imply the following result.

Theorem 4.1. *Let $e \in E^f_{k,m,n}$ or $e \in E^{\lambda j}_{k,m,n}$ and $e' \in E^f_{k',m',n}$ or, respectively, $e' \in E^{\lambda j'}_{k',m',n}$. There is a homomorphism $\theta : T^f_m \to T^f_{m'}$ or, respectively, $\theta : T^{\lambda j}_m \to T^{\lambda j'}_{m'}$ such that $\theta(e) = e'$ if and only if either $m \ge m'$ and $k \le k'$ or $m \le m'$ and $k \le k' - m' + m$. If $m = m'$ and $k = k'$, θ can be chosen as an isomorphism.*

Definition 4.2. (1) We fix for every quadruple (f, m, k, n), where $k < m$, an element $e^f_{m,k,n} \in E^f_{m,k,n}$ and for every quintuple (λ, j, m, k, n), where $k < m$ and k, j satisfy the condition (4.1), an element $e^{\lambda j}_{m,k,n} \in E^{\lambda j}_{m,k,n}$.

(2) For a homogeneous tube $\mathcal{T}^f$ we call a *standard sequence* a sequence $\sigma = (m_i, k_i)$ $(1 \le i \le s)$, where $m_1 > m_2 > \dots > m_s$, $1 \le k_i < m_i$ and $k_{i'} < k_i < k_{i'} + m_i - m_{i'}$ for $i < i'$. We set $M^f_\sigma = \bigoplus_{i=1}^s T^f_{m_i}$ and $e^f_{\sigma,n} = \sum_{i=1}^s e^f_{m_i,k_i,n}$.

(3) For a special tube $\mathcal{T}^\lambda$ we call a *standard sequence* a sequence $\sigma = (j_i, m_i, k_i)$ $(1 \le i \le s)$, where $j_i \in \{1, 2\}$, $m_1 > m_2 > \dots > m_s$, $1 \le k_i < m_i$ and $k_{i'} < k_i < k_{i'} + m_i - m_{i'}$ for $i < i'$. We set $M^\lambda_\sigma = \bigoplus_{i=1}^s T^{\lambda j_i}_{m_i}$. We call such sequence
- *even* if $k_i \equiv j_i \pmod 2$ for all i,
- *odd* if $k_i \equiv j_i + 1 \pmod 2$ for all i.

For an even (odd) standard sequence and even (respectively, odd) n we set $e^\lambda_{\sigma,n} = \sum_{i=1}^s e^{\lambda j_i}_{m_i,k_i,n}$.

(4) We define *standard data* as a pair $\Delta = (\Sigma, S)$, where $\Sigma = \{\mathcal{T}^{f_q}\}$ $(1 \le q \le r)$ is a set of different tubes and $S = \{\sigma_q\}$ $(1 \le q \le r)$ is a set of standard sequences σ_q for each tube T^{f_q}. We call such data *special* if at least one of the tubes $\mathcal{T}^{f_q}$ is special. Special standard data are said to be even or odd if all standard sequences σ_q for special tubes $\mathcal{T}^{f_q}$ are so. We set $M_\Delta = \bigoplus_{q=1}^r M^{f_q}_{\sigma_q}$ and $e_{\Delta,n} = \sum_{q=1}^s e^{f_q}_{\sigma_q,n}$. In the latter definition we suppose that, if Δ is special, it is even if

n is even and it is odd if n is odd (otherwise the element $e_{\Delta,n}$ is not defined).

If the tube $\mathcal{T}^\infty$ occurs in Σ, say, $f_k = \infty$, we denote by $e^\infty_{\Delta,n} = e^\infty_{\sigma_k,n}$ and by $e^0_{\Delta,n}$ the rest of the sum defining $e_{\Delta,n}$. Of course, it is possible that $e^\infty_{\Delta,n} = 0$ or $e^0_{\Delta,n} = 0$.

Theorem 4.1 implies the following result.

Theorem 4.3. *Let M be a regular R-lattice and $\varepsilon \in H^n(K,M)$ $(n > 0)$. There are standard data Δ and an isomorphism $\theta : M \xrightarrow{\sim} M_0 \oplus M_\Delta$ such that the projection of $\theta(\varepsilon)$ onto $H^n(K,M_0)$ is zero and the projection of $\theta(\varepsilon)$ onto $H^n(K,M_\Delta)$ equals $\xi(e_{\Delta,n})$* (see page 9 for the definition of ξ).

If $\varepsilon = 0$, $M_\Delta = 0$.

In particular, we obtain a description of orbits of automorphisms of indecomposable regular lattices on cohomologies.

Corollary 4.4. *Let M be an indecomposable regular lattice. Consider the chain* (2.6) *of its submodules and denote by $H^n_k(K,M)$ the image in $H^n(K,M)$ of $H^n(K,M_k)$. Then the orbits of $\operatorname{Aut}_K M$ on $H^n(K,M)$ $(n > 0)$ are $H^n_k(K,M) \setminus H^n_{k+1}(K,M)$ $(0 \le k < m)$ and $\{0\}$.*

The group of automorphisms of the group K is the symmetric group S_3: it just permutes the elements a, b and $c = ab$. Its generators are the transpositions $\tau_2 : a \leftrightarrow b$ and $\tau_3 : a \leftrightarrow c$. They permute the $+-$ component of the diagram (1.1), respectively, with the $-+$ component and with the $--$ component. Thus τ_2 permutes $\mathcal{T}^1$ and $\mathcal{T}^0$, while τ_3 permutes $\mathcal{T}^1$ and $\mathcal{T}^\infty$. Rather simple matrix calculations show that τ_2 permutes $\mathcal{T}^f$ with $\mathcal{T}^{f^{(2)}}$, while τ_3 permutes $\mathcal{T}^f$ with $\mathcal{T}^{f^{(3)}}$, where

$$\begin{aligned} f^{(2)}(t) &= f(1)^{-1}(t-1)^d f\left(\frac{t}{t-1}\right), \\ f^{(3)}(t) &= (-1)^d f(1-t), \end{aligned}$$

where $d = \deg f$. It induces the action of S_3 on the set of standard data. Note that, if $\psi \in \operatorname{Aut} K$, there is an automorphism $\varphi \in \operatorname{Aut} M_{\tau\Delta}$ such that $\psi\xi(e_{\Delta,n}) = \varphi\xi(e_{\psi\Delta,n})$.

4.2. **Colattices.** Let now $N = DM$, where M is a K-lattice. If M is regular, we call N regular too. If $N = DM$, where $M = T^f_m$ or $M = T^{\lambda j}_m$, there is a chain of submodules, dual to the chain (2.6) from Theorem 2.1

$$0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_{m-1} \subset N_m = N, \tag{4.2}$$

where $N_k = M_k^\perp$ and

$$N_{k+l}/N_k \simeq \begin{cases} DT^f_l & \text{if } N = DT^f_m, \\ DT^{\lambda j}_l & \text{if } N = DT^{\lambda j}_m \text{ and } k \text{ is even}, \\ DT^{\lambda i}_l, & \text{where } i \ne j, \text{ if } N = DT^{\lambda j}_m \text{ and } k \text{ is odd}. \end{cases}$$

We denote by $Z^f_{k,m,n}$ or, respectively, by $Z^{\lambda j}_{k,m,n}$ the set $N_{k+1}(n)\setminus N_k$. Again, $Z^{\lambda j}_{k,m,n} \neq \emptyset$ if and only if $k < m$ and the condition (4.1) holds.

The duality gives analogues of Theorems 4.1 and 4.3.

Theorem 4.5. *Let $z \in Z^f_{k,m,n}$ or $z \in Z^{\lambda j}_{k,m,n}$ and $z' \in Z^f_{k',m',n}$ or, respectively, $z' \in Z^{\lambda j'}_{k',m',n}$. There is a homomorphism $\theta : DT^f_m \to DT^f_{m'}$ or, respectively, $\theta : DT^{\lambda j}_m \to DT^{\lambda j'}_{m'}$ such that $\theta(z) = z'$ if and only if either $m \le m'$ and $k \ge k'$ or $m \ge m'$ and $k \ge k' - m' + m$. If $m = m'$ and $k = k'$, θ can be chosen as an isomorphism.*

Definition 4.6. (1) We fix for every quadruple (f, m, k, n), where $k < m$, an element $z^f_{m,k,n} \in Z^f_{m,k,n}$ and for every quintuple (λ, j, m, k, n), where $k < m$ and k, j satisfy the condition (4.1), an element $z^{\lambda j}_{m,k,n} \in Z^{\lambda j}_{m,k,n}$.

(2) For a homogeneous tube $\mathcal{T}^f$ we call a *costandard sequence* a sequence $\sigma = (m_i, k_i)$ $(1 \le i \le s)$, where $m_1 < m_2 < \dots < m_s$, $1 \le k_i < m_i$ and $k_{i'} > k_i > k_{i'} + m_i - m_{i'}$ for $i < i'$. We set $N^f_\sigma = \bigoplus_{i=1}^s P^f_{m_i}$ and $z^f_{\sigma,n} = \sum_{i=1}^s e^f_{m_i,k_i,n}$.

(3) For a special tube $\mathcal{T}^\lambda$ we call a *costandard sequnce* a sequence $\sigma = (j_i, m_i, k_i)$ $(1 \le i \le s)$, where $j_i \in \{1, 2\}$, $m_1 < m_2 < \dots < m_s$, $1 \le k_i < m_i$, $1 \le k_i \le m_i$ and $k_i < k_{i'} < k_i + m_{i'} - m_i$ for $i' < i$. We set $N^\lambda_\sigma = \bigoplus_{i=1}^s T^{\lambda j_i}_{m_i}$. We call such sequence
- *even* if $k_i \equiv j_i \pmod 2$ for all i,
- *odd* if $k_i \equiv j_i + 1 \pmod 2$ for all i.

For an even (odd) costandard sequence and even (respectively, odd) n we set $z^\lambda_{\sigma,n} = \sum_{i=1}^s z^{\lambda j_i}_{m_i,k_i,n}$.

(4) We define *costandard data* as a pair $\Delta = (\Sigma, S)$, where $\Sigma = \{\mathcal{T}^{f_q}\}$ $(1 \le q \le r)$ is a set of different tubes and $S = \{\sigma_q\}$ $(1 \le q \le r)$ is a set of costandard sequences σ_q for each tube T^{f_q}. We call such data *special* if at least one of the tubes $\mathcal{T}^{f_q}$ is special. Special costandard data are said to be even or odd if all costandard sequences σ_q for special tubes $\mathcal{T}^{f_q}$ are so. We set $N_\Delta = \bigoplus_{q=1}^r M^{f_q}_{\sigma_q}$ and $z_{\Delta,n} = \sum_{q=1}^s z^{f_q}_{\sigma_q,n}$. In the latter definition we suppose that, if Δ is special, it is even if n is even and it is odd if n is odd (otherwise the elements $z^{f_q}_{\sigma,q,n}$ and hence $z_{\Delta,n}$ are not defined).

If the tube $\mathcal{T}^\infty$ occurs in Σ, say, $f_k = \infty$, we denote by $z^\infty_{\Delta,n} = z^\infty_{\sigma_k,n}$ and by $z^0_{\Delta,n}$ the rest of the sum defining $z_{\Delta,n}$. Of course, it is possible that $z^\infty_{\Delta,n} = 0$ or $z^0_{\Delta,n} = 0$.

Theorem 4.7. *Let $N = DM$, where M is a regular R-lattice, $\varepsilon \in H^n(K, N)$ $(n > 0)$. There are costandard data Δ and an isomorphism $\theta : N \xrightarrow{\sim} N_0 \oplus N_\Delta$*

such that the projection of $\theta(\varepsilon)$ onto $H^n(K,N_0)$ is zero and the projection of $\theta(\varepsilon)$ onto $H^n(K,N_\Delta)$ equals $\eta(z_{\Delta,n})$ (see page 11 for the definition of η).

If $\varepsilon = 0$, $M_\Delta = 0$.

Corollary 4.8. *Let N be an indecomposable regular colattice. Consider the chain* (4.2) *of its submodules and denote by $H^n_k(K,N)$ the image in $H^n(K,N)$ of $H^n(K,N_k)$. Then the orbits of $\operatorname{Aut}_K N$ on $H^n(K,N)$ $(n>0)$ are $H^n_k(K,N)\setminus H^n_{k-1}(K,N)$ $(0<k\le m)$ and $\{0\}$.*

5. Applications

5.1. **Crystallographic groups.** Recall that a *crystallographic group G* is a discontinuous group of isometries of an Euclidean space having a compact fundamental domain [14]. Equivalently, G contains a maximal commutative subgroup M of finite index, which is normal and is a free abelian group of finite rank. Then the group $\Gamma = \Gamma/M$ acts on M by the rule ${}^g v = \bar g v \bar g^{-1}$, where $\bar g$ is a preimage of g in G, and G is given by a class $\varepsilon \in H^2(\Gamma, M)$. One easily sees that actually M is a unique maximal abelian subgroup of G of finite index. We call the group Γ the *top* and the Γ-module M the *base* of the crystallographic group G. If $\varphi : G \xrightarrow{\sim} G'$, where G' is another crystallographic group, then $M' = \varphi(M)$ is the maximal commutative subgroup of G', i.e. the base of G'. Hence $\Gamma' = G'/M'$ is the top of G' and we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1\,, \\ & & \big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\psi} & & \\ 1 & \longrightarrow & M' & \longrightarrow & G' & \longrightarrow & \Gamma' & \longrightarrow & 1 \end{array}$$

where θ and ψ are isomorphisms. Let ${}^\psi M'$ be the group M' considered as G-module by the rule ${}^g u = {}^{\psi(g)} u$ for $g \in G$, $u \in M'$. Then θ is an isomorphism $M \xrightarrow{\sim} {}^\psi M'$ and the cohomology class defining the group G' is $\theta\varepsilon\psi^{-1}$. Therefore, isomorphism classes of crystallographic groups with the top Γ and the base M are in one-to-one correspondence with the orbits of the action of the group $\operatorname{Aut} G \times \operatorname{Aut}_G M$ on $H^2(G,M)$.

Therefore, Theorem 4.3 implies a classification result for crystallographic groups with the Kleinian top and regular base.

Definition 5.1. Let Δ be standard data, even if they are special. We call the group $\operatorname{Cr}(\Delta)$ that is the extension of K with the kernel M corresponding to the cohomology class $\varepsilon = \xi(e_{\Delta,2})$ a *standard crystallographic group.*

Note that $\mathrm{Cr}(\Delta)$ is generated by the group M and two elements $\bar{a}$ and $\bar{b}$ subject to the relations

$$\begin{aligned}
&\bar{a}w = ({}^{a}w)\bar{a} \textit{ for every } w \in M,\\
&\bar{b}w = ({}^{b}w)\bar{b} \textit{ for every } w \in M,\\
&\bar{a}\bar{b} = \bar{b}\bar{a},\\
&\bar{a}^2 = e^{0}_{\Delta,2},\\
&\bar{b}^2 = e^{\infty}_{\Delta,2}.
\end{aligned}$$

Theorem 5.2. *Let G be a crystallographic group with the Kleinian top K such that its base M is a regular K-lattice.*

(1) *There are standard data Δ, a direct decomposition $M \simeq M_0 \oplus M_\Delta$ and a semidirect decomposition $G \simeq M_0 \rtimes \mathrm{Cr}(\Delta)$, where $\mathrm{Cr}(\Delta)$ acts on M_0 as its quotient $\mathrm{Cr}(\Delta)/M_\Delta \simeq K$.*
(2) *If $G \simeq M_0' \rtimes \mathrm{Cr}(\Delta')$ is another such decomposition, there is an automorphism ψ of the group K such that $M_0' \simeq {}^{\psi}M_0$ and $\Delta' = \psi\Delta$.*

Remark 5.3. G is crystallographic if and only if $M_{\alpha\beta} \neq 0$ for at least two of the pairs $(+-), (-+), (--)$. For a regular K-lattice M it means that it is not a multiple of some lattice $T_1^{\lambda 1}$ $(\lambda \in \{0, 1, \infty\})$.

5.2. **Chernikov groups.** Recall that a *Chernikov group* is a locally finite group with minimality comdition on subgroups [2]. Such a group G has a maximal divisible subgroup N which is a finite direct sum of quasicyclic groups and N is normal in G with the finite quotient $\Gamma = G/N$. We consider the case when G is a 2-group. Then N is a direct sum of groups $\mathbb{D}$ of type 2^∞ and Γ is a finite 2-group. It is known that $\mathrm{End}\,\mathbb{D} \simeq \mathbb{Z}_2$. Therefore, if $N = \mathbb{D}^d$, then $\mathrm{Aut}_{\mathbb{Z}} N \simeq \mathrm{GL}(d, \mathbb{Z}_2)$. Hence $N \simeq DM$ for some Γ-lattice M. The group Γ and the G-module N are defined up to an isomorphism. We call Γ the *top* and N the *base* of the Chernikov group G. Again, the isomorphism classes of Chernikov groups with the top Γ and the base N are in one-to-one correspondence with the orbits of the group $\mathrm{Aut}\,K \times \mathrm{Aut}_K N$ on the cohomology group $H^2(\Gamma, N)$.

Theorem 4.7 implies the following description of Chernikov groups with the Kleinian top and regular bottom.

Definition 5.4. Let Δ be costandard data, even if they are special. We call the group $\mathrm{Ch}(\Delta)$ that is the extension of K with the kernel N corresponding to the cohomology class $\varepsilon = \eta(z_{\Delta,2})$ a *standard Chernikov group.*

Note that $\mathrm{Ch}(\Delta)$ is generated by the group N and two elements $\bar{a}$ and $\bar{b}$ subject to the relations

$$\begin{aligned}
&\bar{a}w = ({}^a w)\bar{a} \textit{ for every } w \in N,\\
&\bar{b}w = ({}^b w)\bar{b} \textit{ for every } w \in N,\\
&\bar{a}\bar{b} = \bar{b}\bar{a},\\
&\bar{a}^2 = z^0_{\Delta,2}\\
&\bar{b}^2 = z^\infty_{\Delta,2}.
\end{aligned}$$

Theorem 5.5. *Let G be a Chernikov group with the Kleinian top K such that its base N is a regular K-colattice.*

(1) *There are costandard data Δ, a direct decomposition $N = N_0 \oplus N_\Delta$ and a semidirect decomposition $G \simeq N_0 \rtimes \mathrm{Ch}(\Delta)$, where $\mathrm{Ch}(\Delta)$ acts on N_0 as its quotient $\mathrm{Ch}(\Delta)/N_\Delta \simeq K$.*
(2) *If $G \simeq N_0' \rtimes \mathrm{Ch}(\Delta')$ is another such decomposition, there is an automorphism ψ of the group K such that $N_0' \simeq {}^\psi N_0$ and $\Delta' = \psi\Delta$.*

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