ON GENERA OF POLYHEDRA

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Abstract. We consider the stable homotopy category $\mathscr{S}$ of *polyhedra* (finite cell complexes). We say that two polyhedra X, Y are *in the same genus* and write $X \sim Y$ if $X_p \simeq Y_p$ for all prime p, where X_p denotes the image of X in the localized category $\mathscr{S}_p$. We prove that it is equivalent to the stable isomorphism $X \vee B_0 \simeq Y \vee B_0$, where B_0 is the wedge of all spheres S^n such that $\pi_n^S(X)$ is infinite. We also prove that a stable isomorphism $X \vee X \simeq Y \vee X$ implies a stable isomorphism $X \simeq Y$.

Genera, i.e. classes of modules which have isomorphic localizations, are widely studied and applied in the theory of integral representations, see [3, 4]. On the other hand, the notion of genera is natural in other categories, for instance, in the *stable homotopy category* [2], which plays an important role in algebraic topology. This paper is an attempt to study genera in the stable homotopy category. If we consider finite cell complexes ("polyhedra"), such a study can be reduced to integral representations of orders in semisimple algebras. Therefore we can apply a deep theory developed for genera of integral representations. In particular, we obtain the following results:

1. *Two polyhedra X, Y are in the same genus if and only if $X \vee B_0$ is stably isomorphic to $Y \vee B_0$, where B_0 is the wedge of all spheres S^n such that the stable homotopy group $\pi_n^S(X)$ is infinite* (*or, the same, not torsion*).

2. *If $X \vee X$ and $Y \vee X$ are stably isomorphic, then X and Y are stably isomorphic too.*

It seems very plausible that other results on genera of integral representations also have analogues in the stable homotopy category. In particular, we conjecture that the number of stable isomorphism classes in a genus is bounded when we consider polyhedra of a prescribed dimension. We also give some examples of calculating genera for polyhedra of small dimensions.

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1. Stable category and genera

First recall the definitions concerning *stable homotopy category*. We consider the category of connected topological spaces X with fixed points $*_X$. The *homotopy category* Hot has such spaces as objects, while $\mathrm{Hot}(X,Y)$ is the set of homotopy classes of continuous maps (preserving fixed points). We denote by $X \vee Y$ the *wedge* (one-point union) of the spaces X and Y, i.e. the subspace $(X \times *_Y) \cup (*_X \times Y) \subset X \times Y$, and by $X \wedge Y$ their *smash product*, i.e. $X \times Y/X \vee Y$. The *suspension of* X is, by definition, the smash product $SX = S^1 \wedge Y$. We denote by S^nX the n-th iterated suspension (isomorphic to $S^n \wedge X$). The suspension induces a functor $S : \mathrm{Hos} \to \mathrm{Hos}$. On the subcategory of simply connected spaces this functor is *conservative*, i.e. Sf is an isomorphism if and only if so is f. Note that the suspensions are *cogroups* in the homotopy category and their iterations S^nX are commutative cogroups. Therefore, all spaces $\mathrm{Hos}(S^nX,Y)$ have a natural group structure, commutative if $n > 1$. The suspension induces group homomorphisms $\mathrm{Hos}(S^nX,S^nY) \to \mathrm{Hos}(S^{n+1}X,S^{n+1}Y)$ (see [10] for details). Now we define the *group of stable maps* $\mathrm{Hos}(X,Y)$ as the direct limit $\varinjlim_n \mathrm{Hot}(S^nX,S^nY)$. It is a commutative group. If $\alpha \in \mathrm{Hot}(S^nX,S^nY)$, $\beta \in \mathrm{Hot}(S^mY,S^mZ)$, the product $S^n\beta \circ S^m\alpha$ is defined and belongs to $\mathrm{Hot}(S^{m+n}X,S^{m+n}Z)$. Its class in $\mathrm{Hos}(X,Z)$ is, by definition, the product of the classes of α and β. Thus we obtain the *stable homotopy category* Hos. It is an additive category, where the wedge plays role of the direct sum. Moreover, it is *fully additive*, i.e. any idempotent $e \in \mathrm{Hos}(X,X)$ splits as $e = \iota\pi$ for some morphisms $\pi : X \to Y$ and $\iota : Y \to X$ such that $\pi\iota = 1_Y$ (see [2, page 86]). If $1 - e = \iota'\pi'$, where $\pi' : X \to Y'$, $\iota' : Y' \to X$ and $\pi'\iota' = 1_{Y'}$, the morphisms ι,π,ι',π' define a decomposition $X \simeq Y \vee Y'$. We denote by $\mathrm{Es}(X)$ the endomorphism ring $\mathrm{Hos}(X,X)$ of X in the stable homotopy category, and by kX the wedge of k copies of X.

Consider the full subcategory $\mathsf{CW} \subset \mathrm{Hot}$ consisting of *polyhedra*, i.e. finite cell complexes. For such polyhedra the direct limit in the definition of Hos actually stabilizes at a finite level. It follows from the Generalized Freudenthal Theorem [2, Theorem 1.21].

Theorem 1.1. *If* $\dim X \le m$ *and* Y *is* $(n-1)$*-connected* (*i.e.* $\pi_k(Y) = 0$ *for* $k < n$) *then the suspension map* $\mathrm{Hot}(X,Y) \to \mathrm{Hot}(SX,SY)$ *is bijective if* $m < 2n-1$ *and surjective if* $m = 2n-1$.

In particular, the map $\mathrm{Hot}(S^kX,S^kY) \to \mathrm{Hos}(X,Y)$ *is bijective for* $k > m - 2n + 1$ *and surjective for* $k = m - 2n + 1$.

Corollary 1.2. *If X is a polyhedron of dimension at most n, the map* $\mathrm{Hot}(S^kX, S^kY) \to \mathrm{Hos}(X,Y)$ *is bijective for $k > n+1$ and surjective for $k = n+1$. In particular, $\pi_n^S(X) \simeq \pi_{2(n+1)}(S^{n+2}X)$.*

In what follows, when speaking on polyhedra, we always consider them as objects of the stable homotopy category $\mathscr{S}$. In particular, an *isomorphism* always means a *stable isomorphism*. An important feature of the stable homotopy category $\mathscr{S}$ is that all its Hos-groups are finitely generated [7, Corollary X.8.3].

Let CW_n^m be the full subcategory of CW consisting of $(n-1)$-connected polyhedra of dimension at most $n+m$. The suspension functor maps CW_n^m to CW_{n+1}^m. If $n > m+1$ it is an equivalence of categories. If $n = m+1$, it is an *epivalence*, i.e. this functor is full, dense and conservative. In particular, it is one-to-one on the isomorphism classes of objects. Set $\mathsf{CW}^m = \bigcup_{n=1}^\infty \mathsf{CW}_n^m$. We denote by $\mathscr{S}$ the image in Hos of the category CW and by $\mathscr{S}^m$ the image of CW^m in $\mathscr{S}$.

Let $\mathbb{Z}_p = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ p \nmid b \right\}$, where p is a prime integer, $\hat{Z}_p$ be the ring of p-adic integers. We denote by $\mathscr{S}_p$ ($\hat{\mathscr{S}}_p$) the category which has the same objects as $\mathscr{S}$ but the sets of morphisms are $\mathrm{Hos}_p(X,Y) = \mathrm{Hos}(X,Y) \otimes \mathbb{Z}_p$ (respectively, $\widehat{\mathrm{Hos}}_p(X,Y) = \mathrm{Hos}(X,Y) \otimes \hat{Z}_p$). Actually, $\mathrm{Hos}_p(X,Y)$ ($\widehat{\mathrm{Hos}}_p(X,Y)$) coincides with the group of stable maps between the p-localizations (respectively, p-adic completions of X and Y) in the sense of Artin–Mazur–Sullivan [9]. For the sake of convenience, we denote the image in $\mathscr{S}_p$ ($\hat{\mathscr{S}}_p$) of a polyhedron X by X_p (respectively, by $\hat{X}_p$). Since all groups $\mathrm{Hos}(X,Y)$ are finitely generated for $X, Y \in \mathscr{S}$, $\widehat{\mathrm{Hos}}_p(X,Y)$ coincides with the p-adic completion of $\mathrm{Hos}_p(X,Y)$. Therefore, $X_p \simeq Y_p$ in $\mathscr{S}_p$ if and only if $\hat{X}_p \simeq \hat{Y}_p$ in $\hat{\mathscr{S}}_p$.

Definition 1.3. We say that two polyhedra X and Y are *of the same genus* and write $X \sim Y$ if $X_p \simeq Y_p$ for every prime p. We denote by $G(X)$ the *genus of the polyhedron* X, i.e. the full subcategory of $\mathscr{S}$ consisting of all polyhedra which are in the same genus as X, and by $g(X)$ the number of isomorphism classes in $G(X)$. (Further we will see that it is always finite.)

Since all endomorphism rings $\widehat{\mathrm{Hos}}_p(X,X)$ are finitely generated modules over a complete local Noetherian ring $\mathbb{Z}_p$, the *cancellation law* holds in $\hat{\mathscr{S}}_p$ and therefore in $\mathscr{S}_p$: if $\hat{X}_p \vee \hat{Z}_p \simeq \hat{Y}_p \vee \hat{Z}_p$ (or $X_p \vee Z_p \simeq Y_p \vee Z_p$), then $hX_p \simeq \hat{Y}_p$ (respectively, $X_p \simeq Y_p$). So we have:

Proposition 1.4. *If $X \vee Z \simeq Y \vee Z$, then $X \sim Y$.*

Later we will show that the converse is also true. It means that the relation $X \sim Y$ is the same as the relation $X \equiv Z$ from [2, page 90].

We also consider the category $\mathscr{S}_{\mathbb{Q}}$, having the same objects as $\mathscr{S}$, but with the morphism groups $\mathrm{Hos}_{\mathbb{Q}}(X,Y) = \mathrm{Hos}(X,Y) \otimes \mathbb{Q}$. This category is *semisimple*:

$$\mathrm{Hos}_{\mathbb{Q}}(S^n, S^m) = \begin{cases} \mathbb{Q} & \text{if } m = n, \\ 0 & \text{if } m \neq n \end{cases}$$

and every object in $\mathscr{S}_{\mathbb{Q}}$ is isomorphic to a direct sum of spheres. Namely,

Proposition 1.5. *For any object $X \in \mathscr{S}$, set $r_n(X) = \dim_{\mathbb{Q}} \mathrm{Hos}_{\mathbb{Q}}(S^n, X)$ and $B(X) = \bigvee_n r_n(X) S^n$. Then X is isomorphic to $B(X)$ in the category $\mathscr{S}_{\mathbb{Q}}$.*

Note that $\mathrm{Hos}_{\mathbb{Q}}(S^n, X)$ is a finite dimensional vector space over $\mathbb{Q}$, zero if $n > \dim X$, since the stable homotopy groups $\pi_n^S(S^m)$ are periodic for $n > m$ [10]. For any abelian group A we denote by $\mathrm{tors}(A)$ its *torsion part*, i.e. the subgroup of all torsion elements. Especially, for objects $X, Y \in \mathscr{S}$ we denote $\mathrm{tors}(X,Y) = \mathrm{tors}\,\mathrm{Hos}(X,Y)$.

Proof. We need a definition and an easy lemma. Recall that a *cone sequence* of a map $f : X \to Y$ is the sequence $X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} SX$. Here C_f is the *cone of* f, i.e. the factorspace $CX \cup Y/\approx$, where $CX = X \times [0,1]/X \times 1$ is the cone over X and $\approx$ is the equivalence relation such that all nontrivial equivalences are $(x,0) \approx f(x)$. The map g is the obvious embedding $Y \to C_f$ and h is the surjection $C_f \to C_f/Y \simeq SX$. Any sequence isomorphic (in $\mathscr{S}$) to a cone sequence is called a *cofibration sequence*.

Definition 1.6. Let F be an additive functor from $\mathscr{S}$ to an abelian category $\mathscr{A}$. We call F *exact* if for every cofibration sequence $X \to Y \to Z \to SX$ the induced sequence $FX \to FY \to FZ \to F(SX)$ is exact.

For instance, every representable functor $\mathrm{Hos}(X, \_)$ is exact [2, Theorem 1.25]. The same is true for the representable contravariant functors $\mathrm{Hos}(\_, X)$ (considered as functors $\mathscr{S} \to \mathsf{Ab}^{\mathrm{op}}$) [2, page 6, Property 6].

Lemma 1.7. *Let F, G be exact functors $\mathscr{S} \to \mathscr{A}$ and $\phi : F \to G$ be a morphism of functors such that $\phi(S^k) : F(S^k) \to G(S^k)$ is an isomorphism for each k and if $\phi(X)$ is an isomorphism, so is also $\phi(SX)$. Then ϕ is an isomorphism of functors.*

Proof. We prove that $\phi(X)$ is an isomorphism by induction on $n = \dim X$. The claim is obvious for $n = 1$, since every 1-dimensional polyhedron is isomorphic to a wedge of spheres. Suppose that it holds for polyhedra of dimension n. Let $\dim X = n+1$, $Y = X^n$ be its n-th skeleton. Then there is a cofibration sequence $kS^n \to Y \to X \to kS^{n+1} \to SY$. It gives rise to the commutative diagram with exact rows

$$\begin{tikzcd}
F(kS^n) \arrow[r] \arrow[d, "\phi(kS^n)"'] & FY \arrow[r] \arrow[d, "\phi(Y)"'] & FX \arrow[r] \arrow[d, "\phi(X)"'] & F(kS^{n+1}) \arrow[r] \arrow[d, "\phi(kS^{n+1})"] & F(SY) \arrow[d, "\phi(SY)"] \\
G(kS^n) \arrow[r] & GY \arrow[r] & GX \arrow[r] & G(kS^{n+1}) \arrow[r] & G(SY) .
\end{tikzcd}$$

By the condition on ϕ and the induction supposition, all vertical morphisms except $\phi(X)$ are isomorphisms. Then the 5-Lemma asserts that $\phi(X)$ is an isomorphism too. □

Let now X be a polyhedron, $r_n = r_n(X)$, $B = B(X)$. Choose morphisms $S^n \to X$ such that their images form a basis of $\mathrm{Hos}_{\mathbb{Q}}(S^n, X)$. We get a morphism $B_n = r_n S^n \to X$ which induces an isomorphism $\mathrm{Hos}_{\mathbb{Q}}(S^n, B_n) \to \mathrm{Hos}_{\mathbb{Q}}(S^n, X)$. Altogether, we get a morphism $f : B = B(X) \to X$ which induces isomorphisms $\mathrm{Hos}_{\mathbb{Q}}(S^n, B) \to \mathrm{Hos}_{\mathbb{Q}}(S^n, X)$ for each n. The morphism f gives rise to the morphism of functors $f_* : \mathrm{Hos}_{\mathbb{Q}}(\_, B) \to \mathrm{Hos}_{\mathbb{Q}}(\_, X)$. Since representable functors are exact and the functor $\_ \otimes \mathbb{Q}$ is exact in Ab, these functors are exact too, and f_* satisfies the conditions of Lemma 1.7. Therefore, f_* is an isomorphism of functors. Now the Yoneda Lemma implies that $f : B \to X$ is an isomorphism in $\mathscr{S}_{\mathbb{Q}}$. □

Corollary 1.8. (1) *There are morphisms $\alpha : X \to B(X)$ and $\beta : B(X) \to X$ in $\mathscr{S}$ such that $\alpha\beta = t1_{B(X)}$ and $\beta\alpha = t1_X$ for some integer $t > 0$.*

(2) $\dim_{\mathbb{Q}}(X, S^n) = r_n(X)$.

(3) *If $\Lambda = \mathrm{Es}(X)$, then $\Lambda \otimes \mathbb{Q} \simeq \prod_n \mathrm{Mat}(r_n(X), \mathbb{Q})$.*

Definition 1.9. Let $\mathscr{A}$ be a preadditive category. A morphism $f : X \to Y$ in $\mathscr{A}$ is said to be *essentially nilpotent* if for every morphism $g : Y \to X$ the product fg (or, equivalently, gf) is nilpotent. We denote the set of such morphisms by $\mathrm{nil}(X, Y)$ and by $\mathrm{nil}(X)$ if $X = Y$. The class $\mathrm{nil}\,\mathscr{A} = \bigcup_{X,Y} \mathrm{nil}(X, Y)$ of all essentially nilpotent morphisms is called the *nilradical* of the category $\mathscr{A}$.[1] It is an ideal in $\mathscr{A}$, so the factor category $\mathscr{A}^0 = \mathscr{A} / \mathrm{nil}\,\mathscr{A}$ is defined. We call it the *semiprime part* of

[1] If $\mathscr{A}$ only has one object, i.e. is actually a ring, $\mathrm{nil}\,\mathscr{A}$ is its *upper nil radical* in the sense of [8].

$\mathscr{A}$. It contains no essentially nilpotent morphisms and a morphism $f : X \to Y$ is an isomorphism if and only if so is its image in $\mathscr{A}^0$. In particular, two objects X, Y are isomorphic in $\mathscr{A}$ if and only if they are isomorphic in $\mathscr{A}^0$.

In particular, for the category Hos, we write

$$\operatorname{Hos}^0(X,Y) = \operatorname{Hos}(X,Y)/\operatorname{nil}(X,Y),$$
$$\operatorname{Es}^0(X) = \operatorname{Es}(X)/\operatorname{nil}(X).$$

Corollary 1.10. $\operatorname{nil}(X,Y) \subseteq \operatorname{tors}(X,Y)$ *for any objects* $X, Y \in \mathscr{S}$.

Proof. Otherwise $\operatorname{Es}(X \vee Y) \otimes \mathbb{Q}$ contains nilpotent ideals. □

Proposition 1.11. *Let* Λ *be a semiprime ring* (*i.e. a ring without nil-ideals*) *such that* $\operatorname{tors}\Lambda$ *is a finitely generated group. Then* $\Lambda = \operatorname{tors}\Lambda \times \Lambda^{\mathrm{f}}$ *for some torsion free ideal* Λ^{f} (*isomorphic to* $\Lambda/\operatorname{tors}\Lambda$ *as a ring*).

Proof. Obviously, $\operatorname{tors}\Lambda$ is an ideal. Since its additive group is finitely generated, it is of finite length as ${}^{\Lambda}$-module. Therefore, it contains a minimal right ideal I_1. Since I_1 is not nilpotent, it is generated by an idempotent: $I_1 = e_1\Lambda$ where $e_1^2 = e_1$. Hence $\operatorname{tors}\Lambda = I_1 \oplus I'$ for some right ideal I'. The same observation shows that $I' = e_2\Lambda \oplus I''$ and so on. Eventually we get $\operatorname{tors}\Lambda = e\Lambda$ for some idempotent e. Since $(1-e)\Lambda e \subseteq \operatorname{tors}\Lambda$, we get $(1-e)\Lambda e \subseteq e\Lambda$, so $(1-e)\Lambda e = 0$. Then $e\Lambda(1-e)$ is a nilpotent ideal, so $e\Lambda(1-e) = 0$ too, and $\Lambda = e\Lambda \oplus (1-e)\Lambda$, where both summands are two-sided ideals, which implies the statement. □

Corollary 1.12. *Every object* $X \in \mathscr{S}$ *splits uniquely as* $X \simeq X^{\mathrm{t}} \vee X^{\mathrm{f}}$, *so that* $\operatorname{Es}(X^{\mathrm{f}})/\operatorname{nil}(X^{\mathrm{f}})$ *is torsion free, while* $\operatorname{Es}(X^{\mathrm{t}})$ *is torsion. Especially, if* X *is indecomposable in the category* $\mathscr{S}$, *then either* $\operatorname{Es}(X)$ *is torsion or* $\operatorname{Es}(X)/\operatorname{nil}(X)$ *is torsion free.* (The latter condition means that $\operatorname{tors}(X) = \operatorname{nil}(X)$.)

We call X^{t} the *torsion part* of X and X^{f} its *torsion reduced* part. If $X = X^{\mathrm{t}}$, we call X *torsion*; if $X = X^{\mathrm{f}}$, we call it *torsion reduced.*

Proof. Let $\Lambda = \operatorname{Es}(X)$, $\bar\Lambda = \operatorname{Es}^0(X)$. According to Proposition 1.11, there are central idempotents e_1, e_2 such that $\bar\Lambda = e_1\bar\Lambda \oplus e_2\bar\Lambda$, where $e_1\bar\Lambda$ is torsion and $e_2\bar\Lambda$ is torsion free. Idempotents modulo $\operatorname{nil}(X)$ can be lifted to idempotents in Λ, so $\Lambda = e_1^*\Lambda \oplus e_2^*\Lambda$ where $e_i = e_i^* \bmod \operatorname{nil}(X)$. It gives rise to a decomposition $X = X_1 \vee X_2$ of X so that $\operatorname{Es}(X_i) \simeq e_i^*\Lambda e_i^*$. Since $\operatorname{nil}(e\Lambda e) = e(\operatorname{nil}\Lambda)e$, we get the statement if we set $X^{\mathrm{t}} = X_1$ and $X^{\mathrm{f}} = X_2$. □

Proposition 1.13. (1) *For any polyhedra* X, Y, $X \sim Y$ *if and only if* $X^{\mathrm{f}} \sim Y^{\mathrm{f}}$ *and* $X^{\mathrm{t}} \simeq Y^{\mathrm{t}}$.

(2) *If* $X_1 \vee Y_1 \simeq X_2 \vee Y_2$*, where* X_i *are torsion reduced and* Y_i *are torsion, then* $X_1 \simeq X_2$ *and* $Y_1 \simeq Y_2$*.*
(3) $g(X) = g(X^{\mathrm{t}})$*.*

Proof. (1) Obviously, every morphism $f : X_p \to Y_p$ maps $X^{\mathrm{t}}_p \to Y^{\mathrm{t}}_p$ and if f is an isomorphism, so is its restriction on X^{t}_p. But $\mathrm{Hos}(Z, X^{\mathrm{t}})$ is always torsion, hence, isomorphic to $\bigoplus_p \mathrm{Hos}_p(Z, X^{\mathrm{t}})$. Therefore, if $X^{\mathrm{t}}_p \simeq Y^{\mathrm{t}}_p$ for all p, also $X^{\mathrm{t}} \simeq Y^{\mathrm{t}}$. Thus if $X \sim Y$, then $X^{\mathrm{t}} \simeq Y^{\mathrm{t}}$. Since cancellation holds in $\mathscr{S}_p$, also $X^{\mathrm{f}} \sim Y^{\mathrm{f}}$. The converse is evident.

(2) Proposition 1.11 implies that $\mathrm{Hos}(X_i, Y_j)$ and $\mathrm{Hos}(Y_j, X_i)$ belong to nil(Hos). Hence, any morphism from $\mathrm{Hos}^0(X_i \vee Y_i, X_j \vee Y_j)$ is given by a diagonal matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha : X_i \to X_j$ and $\beta : Y_i \to Y_j$. It is invertible if and only if so are α and β. Since a morphism from Hos is invertible if and only if its image in Hos^0 is invertible, it proves the statement.

(3) follows immediately from (1) and (2). □

2. $G(X)$ and $G(\Lambda)$

We are going to establish relations between genera of polyhedra and of modules. First recall a fact from general nonsense. For an object X of a fully additive category $\mathscr{A}$ we denote by $\operatorname{add} X$ the full subcategory of $\mathscr{A}$ consisting of all objects that are isomorphic to direct summands of direct multiples kX of the object X.

Proposition 2.1. *Let* $\Lambda = \operatorname{End} X$ *be the ring of endomorphism of an object* X *from a fully additive category* $\mathscr{A}$*. Then the functor* $X^* : Y \mapsto \mathscr{A}(X, Y)$ *establishes an equivalence* $\operatorname{add} X \simeq \operatorname{add} \Lambda$*, where* Λ *is considered as an object of the category of right* Λ*-modules.*

Note that $\operatorname{add} \Lambda$ is actually the category of finitely generated projective right Λ-modules.

Proof. The Yoneda Lemma claims that $B \mapsto \mathscr{A}(\_, Y)$ is an equivalence between $\operatorname{add} X$ and the category of representable functors $\operatorname{add} X \to \mathsf{Ab}$. But since every object from $\operatorname{add} X$ is a direct summand of kX, any additive functor from $\operatorname{add} X$ is completely defined (up to isomorphism) by its value at X. It implies the statement. □

We will use this result in case when $\mathscr{A} = \mathscr{S}$, so $\Lambda = \mathrm{Es}(X)$ is the ring of stable endomorphisms of X. Especially, we will use it to study genera. Recall that two Λ-modules M, N *are in the same genus* [3] if $M_p \simeq N_p$ for all prime p, where $M_p = M \otimes \mathbb{Z}_p$. Then we write $N \sim M$. Just as above, we denote by $G(M)$ the category of all Λ-modules N

such that $N \sim M$ and by $g(M)$ the number of isomorphism classes in $G(M)$. Let $\bar{\Lambda} = \Lambda/\operatorname{tors}\Lambda$. It is an order in the semisimple algebra $\prod_i \operatorname{Mat}(r_i(X), \mathbb{Q})$ by Corollary 1.8 (3). Obviously, $g(\Lambda) = g(\bar{\Lambda})$. Therefore, $g(\Lambda) < \infty$ by Jordan–Zassenhaus Theorem [3, Theorem 24.1].

Proposition 2.2. *If $Y \sim X$, then $Y \in \operatorname{add} X$.*

Proof. Note that $\operatorname{Hos}_p(X,Y)/p\operatorname{Hos}_p(X,Y) \simeq \operatorname{Hos}(X,Y)/p\operatorname{Hos}(X,Y)$ and an endomorphism of X_p is invertible if and only if it is invertible modulo p. Thus if $X_p \simeq Y_p$, there are morphisms $f_p : Y \to X$ and $g_p : X \to Y$ such that $g_p f_p$ does not belong to any maximal ideal $M \subset \operatorname{Es}(Y)$ such that $M \supseteq p\operatorname{Es}(Y)$. Since any maximal ideal of $\operatorname{Es}(Y)$ contains some $p\operatorname{Es}(Y)$, the set $\{ g_p f_p \}$ generates the unit ideal. It implies that there are morphisms $f_i : Y \to X$ and $g_i : X \to Y$ $(i = 1, 2, \dots, k)$ such that $\sum_{i=1}^k g_i f_i = 1$. Let $f : Y \to kX$ be the morphism with the components $f_1, f_2, \dots, f_k$ and $g : kX \to Y$ be the morphism with the components $g_1, g_2, \dots, g_k$. Then $gf = 1_Y$, which means that Y is a direct summand of kX. □

Corollary 2.3. *$G(X) \simeq G(\Lambda)$, in particular, $g(X) = g(\Lambda) < \infty$.*

Now we can apply the known facts from the theory of integral representations to genera of polyhedra.

Theorem 2.4. *Let X, Z be two polyhedra such that $Z \in \operatorname{add} X$ (for instance, $Z \sim kX$ for some k). If $X \vee Z \simeq Y \vee Z$, then $X \simeq Y$.*

Proof. Proposition 1.13 implies that we can suppose X torsion reduced. Then so are also Y and Z. We consider the functor X^* of Proposition 2.1. Set $\Lambda = \operatorname{Es}(X) = X^*(X)$, $M = X^*(Y)$, $N = X^*(Z)$. Then $\Lambda \oplus N \simeq M \oplus N$ and $N \in \operatorname{add}\Lambda$. We have to show that $M \simeq \Lambda$. Set also $R = \operatorname{nil}\Lambda$, $\bar{\Lambda} = \Lambda/R$, $\bar{M} = M/MR$, $\bar{N} = N/NR$. Note that $\bar{\Lambda}$ is torsion free as an abelian group. Since M is a projective Λ-module, it is enough to prove that $\bar{M} \simeq \bar{\Lambda}$. By Corollary 1.8 (3), $\mathbb{Q} \otimes \bar{\Lambda}$ is a product of full matrix rings over $\mathbb{Q}$, so it satisfies the *Eichler condition* in the sense of [4, § 51A]. Therefore we can apply the Jacobinski cancellation theorem [4, Theorem 51.24]: if $A \oplus k\bar{\Lambda} \simeq B \oplus k\bar{\Lambda}$ for some locally free $\bar{\Lambda}$-modules A, B, then $A \simeq B$. Since $\bar{M} \oplus \bar{N} \simeq \bar{\Lambda} \oplus \bar{N}$, where $\bar{N} \in \operatorname{add}\bar{\Lambda}$, i.e. $\bar{N} \oplus N' \simeq k\bar{\Lambda}$ for some N' and k, and $\bar{M}_p \simeq \bar{\Lambda}_p$ for all p, it implies that $\bar{M} \simeq \bar{\Lambda}$, wherefrom $M \simeq \Lambda$ and $Y \simeq X$. □

Recall that we have set $r_n(X) = \dim_{\mathbb{Q}} \operatorname{Hos}_{\mathbb{Q}}(S^n, X)$. Set

$$B_0(X) = \bigvee_{r_n(X) \neq 0} S^n.$$

In other words, $B_0(X)$ is the wedge of all spheres S^n such that the stable homotopy group $\pi_n^S(X)$ is not torsion, or, the same, is infinite.

Theorem 2.5. *$X \sim Y$ if and only if $X \vee B_0(X) \simeq Y \oplus B_0(Y)$.*

Proof. We have already seen (Proposition 1.4) that $X \vee B_0(X) \simeq Y \oplus B_0(Y)$ implies $X \sim Y$. To show the converse, we can suppose X and Y torsion reduced. Let $B_0 = B_0(X)$, $\tilde{X} = X \vee B_0$. Then $X, Y, B_0 \in \operatorname{add} \tilde{X}$. Set $\Lambda = \operatorname{Es}(\tilde{X})$, and apply the functor $\tilde{X}^*$ of Proposition 2.1. Denote $M = \tilde{X}^*(X)$, $N = \tilde{X}^*(Y)$, $L = \tilde{X}^*(B_0)$; $R = \operatorname{nil}\Lambda$, $\bar{\Lambda} = \Lambda/R$, $\bar{M} = M/MR$, $\bar{N} = N/NR$, $\bar{L} = L/LR$. Since $X \simeq B(X)$ in the category $\mathscr{S}_{\mathbb{Q}}$, $\mathbb{Q} \otimes L$ is a faithful $\mathbb{Q} \otimes \Lambda$-module. Hence $\bar{L}$ is a faithful $\bar{\Lambda}$-module. Now we can use a Roiter's theorem on genera [3, Theorem 31.28]. It claims that there is a $\bar{\Lambda}$-module L' such that $L' \sim \bar{L}$ and $\bar{M} \oplus \bar{L} \simeq \bar{N} \oplus L'$. Note now that $\operatorname{End}_{\bar{\Lambda}}(\bar{L}) \simeq \Lambda_0/\operatorname{nil}\Lambda_0$, where $\Lambda_0 = \operatorname{Es}(B_0)$, thus $\operatorname{End}_{\bar{\Lambda}}(\bar{L}) \simeq \mathbb{Z}^m$, where $m = \#\{\, n \in \mathbb{Z} \mid r_n(X) \neq 0 \,\}$. Therefore, $g(\bar{L}) = 1$ and $L' \simeq \bar{L}$, that is $\bar{M} \oplus \bar{L} \simeq \bar{N} \oplus \bar{L}$, so $M \oplus L \simeq N \oplus L$ and $X \vee B_0 \simeq Y \vee B_0$. $\square$

Finally, we propose the following conjecture, which is, in some sense, an analogue of another Roiter's theorem on genera [3, Theorem 31.34] (boundedness of $g(M)$ for all modules M over an order in a semisimple algebra).

Conjecture 1. *For every positive integer n there is an integer c_n such that $g(X) \leq c_n$ for every polyhedron X of dimension n.*

3. Calculations and examples

For calculation of $g(X)$ for concrete polyhedra X the following facts are useful.

Proposition 3.1 ([5]). *Let Λ be an order in a semisimple $\mathbb{Q}$-algebra, Γ be a maximal order containing Λ and $\Lambda \supseteq m\Gamma$ for some integer $m > 1$. Then $g(\Lambda) = g(\Gamma)g(\Lambda, \Gamma)$, where $g(\Lambda, \Gamma)$ is the number of double cosets*

$$\Gamma^\times \backslash \prod_{p|m} \Gamma_p^\times / \prod_{p|m} \Lambda_p^\times .$$

Proposition 3.2. *Let Λ be an order in a semisimple $\mathbb{Q}$-algebra, Γ be a maximal order containing Λ and $\Lambda \supseteq m\Gamma$ for some integer $m > 1$. Then $g(\Lambda, \Gamma)$ equals the number of cosets*

$$\operatorname{Im}\gamma \backslash (\Gamma/m\Gamma)^\times / (\Lambda/m\Gamma)^\times ,$$

where γ is the natural map $\Gamma^\times \to (\Gamma/m\Gamma)^\times$.

Proof. It is evident, since $\Lambda_p^\times \supseteq 1+m\Gamma_p$ and $\Gamma_p^\times/(1+m\Gamma_p) \simeq (\Gamma_p/m\Gamma_p)^\times$. $\square$

Corollary 3.3. *Let Λ be an order in $\prod_{i=1}^k \mathrm{Mat}(r_i,\mathbb{Q})$ such that $\Gamma \supseteq \Lambda \supseteq m\Gamma$ for some maximal order Γ and some integer $m > 1$. Then $g(\Lambda) = 1$ if $m = 2$ and $g(\Lambda) \leq (\varphi(m)/2)^k$ if $m > 2$, where $\varphi(m)$ is the Euler function.*

In particular, $g(\Lambda) = 1$ if $m \in \{2, 3, 4, 6\}$.

Proof. We can and will identify Γ with $\prod_{i=1}^k \mathrm{Mat}(r_i,\mathbb{Z})$. Then $g(\Gamma) = 1$,

$$(\Gamma/m\Gamma)^\times \simeq \prod_{i=1}^k \mathrm{GL}(r_i,\mathbb{Z}/m),$$

and

$$\Gamma^\times = \prod_{i=1}^k \mathrm{GL}(r_i,\mathbb{Z}).$$

The group $\mathrm{GL}(r_i,\mathbb{Z})$ contains all elementary matrices, hence its image in $\mathrm{GL}(r_i,\mathbb{Z}/m)$ contains the subgroup generated by elementary matrices, which is $\mathrm{SL}(r_i,\mathbb{Z}/m)$. The quotient group $\mathrm{GL}(r_i,\mathbb{Z}/m)/\mathrm{SL}(r_i,\mathbb{Z}/m)$ is isomorphic to $(\mathbb{Z}/m)^\times$, the isomorphism being induced by the determinate. Moreover, $\pm 1 = \det\gamma$ for a diagonal matrix from $\mathrm{Mat}(r_i,\mathbb{Z})$. Therefore,

$$g(\Lambda) \leq \prod_{i=1}^k (\mathbb{Z}_m^\times : \{\pm 1\}) = \begin{cases} 1 & \text{if } m = 2, \\ (\varphi(m)/2)^k & \text{if } m > 2. \end{cases}$$

$\square$

Applied to polyhedra, it gives the following result.

Theorem 3.4. *Let X be a polyhedron, $B = \bigvee_{i=1}^k r_i S^{n_i}$ with different $n_1, n_2, \dots, n_k$. Suppose that there are maps $X \xrightarrow{\beta} B \xrightarrow{\alpha} X$ such that $\alpha\beta \equiv m1_X \bmod \mathrm{tors}(X)$ and $\beta\alpha \equiv m1_B \bmod \mathrm{tors}(B)$ for some integer $m > 1$. Then $g(X) = 1$ if $m = 2$ and $g(X) \leq (\varphi(m)/2)^k$ if $m > 2$.*

Proof. Denote $\Gamma = \mathrm{Es}^0(B)$, $\Lambda = \mathrm{Es}^0(X)$. Then $g(X) = g(\Lambda)$, Γ and Λ are orders in the semisimple $\mathbb{Q}$-algebra $\mathsf{A} = \prod_{i=1}^k \mathrm{Mat}(r_i,\mathbb{Q})$, and $\Gamma \simeq \prod_{i=1}^k \mathrm{Mat}(r_i,\mathbb{Z})$ is a maximal order with $g(\Gamma) = 1$. Define a map $\phi : \Gamma \to \Lambda$ induced by the map $\mathrm{Es}(B) \to \mathrm{Es}(\Lambda)$, $f \mapsto \alpha f \beta$. It is injective, additive and $\phi(f)\phi(g) = m\phi(fg)$. Thus the assertion of the theorem follows from Corollary 3.3 and the following lemma.

Lemma 3.5. *Let Λ be an order in a semisimple $\mathbb{Q}$-algebra A, Γ be a maximal order in A and $\phi : \Gamma \to \Lambda$ be an injective homomorphism of additive groups such that $m\phi(ab) = \phi(a)\phi(b)$ for every $a, b \in \Gamma$ and some integer $m > 1$. Then there is a maximal order Γ' in A such that $\Gamma' \supseteq \Lambda \supset m\Gamma'$.*

Proof. We extend ϕ to a homomorphism of $\mathbb{Q}$-modules $\tilde{\phi} : \mathsf{A} \to \mathsf{A}$. Then $m\tilde{\phi}(ab) = \tilde{\phi}(a)\tilde{\phi}(b)$ for any $a, b \in \mathsf{A}$, so $\psi = \frac{\tilde{\phi}}{m}$ is an automorphism of the algebra A and $\psi(\Gamma) = \Gamma_1$ is a maximal order in A too. Moreover, $m\Gamma_1 = \phi(\Gamma) \subseteq \Lambda$. Consider the Λ-Γ_1-bimodule $M = m\Lambda\Gamma_1 \subset \mathsf{A}$ and set $\Gamma' = \{\, a \in \mathsf{A} \mid aM \subseteq M \,\} \simeq \operatorname{End}_{\Gamma_1} M$. Obviously, $\Gamma' \supseteq \Lambda$ and $m\Gamma' \subseteq \Gamma' M = M \subset \Lambda$. Since M is right torsion free Γ_1-module and Γ_1 is maximal, Γ' is also maximal [3, Theorem 26.25]. $\square$

Theorem 3.4 implies that Conjecture 1 above follows from the following.

Conjecture 2. *For every integer d there is an integer $m > 0$ such that for every polyhedron X of dimension d there are maps $X \xrightarrow{\beta} B \xrightarrow{\alpha} X$, where B is a wedge of spheres, such that $\alpha\beta \equiv m1_X \bmod \operatorname{tors}(X)$ and $\beta\alpha \equiv m1_B \bmod \operatorname{tors}(B)$.*

In the following examples we use definitions and calculations from [6, Section 3]. In particular, we denote by a the a-th multiple of a generator of the group $\pi_n(S^n) \simeq \mathbb{Z}$ and by η the nonzero element of $\pi_n^S(S^{n-1}) \simeq \mathbb{Z}/2$. We also denote by $\mathbb{Z} \times_m \mathbb{Z}$ the subring of $\mathbb{Z} \times \mathbb{Z}$ consisting of all pairs (α, β) with $\alpha \equiv \beta \pmod{m}$. Note that $\mathbb{Z} \times_m \mathbb{Z} \supseteq m(\mathbb{Z} \times \mathbb{Z})$, so $g(\mathbb{Z} \times_m \mathbb{Z})$ equals the number of double cosets

$$\{\pm 1\} \times \{\pm 1\} \backslash \mathbb{Z}_m^\times \times \mathbb{Z}_m^\times / \mathbb{Z}_m^\times$$

under the diagonal embedding of $\mathbb{Z}_m^\times$ into $\mathbb{Z}_m^\times \times \mathbb{Z}_m^\times$. It easily gives $g(\mathbb{Z} \times_m \mathbb{Z}) = \varphi(m)/2$.

Example 3.6. (1) If $M^{n+1}(a)$ is a *Moore atom*, i.e. is defined by the cofibration sequence

$$S^n \xrightarrow{a} S^n \to M^{n+1}(a) \to S^{n+1},$$

its endomorphism ring is torsion, therefore $g(M^n) = 1$.

(2) The same holds for the Chang atom $C^n(2^r\eta 2^s)$ defined by the cofibration sequence

$$S^{n-1} \vee S^n \xrightarrow{\begin{pmatrix} 2^r & \eta \\ 0 & 2^s \end{pmatrix}} S^{n-1} \vee S^n \to C^{n+1}(2^r\eta 2^s) \to S^n \vee S^{n+1},$$

which also has torsion endomorphism ring.

(3) For Chang atoms $C^{n+1}(2^r\eta)$ and $C^{n+1}(\eta 2^s)$ defined, respectively, by the cofibration sequences

$$S^{n-1} \vee S^n \xrightarrow{\begin{pmatrix} 2^r & \eta \end{pmatrix}} S^{n-1} \to C^{n+1}(2^r\eta) \to S^n \vee S^{n+1}$$

and

$$S^n \xrightarrow{\begin{pmatrix} \eta \\ 2^s \end{pmatrix}} S^{n-1} \vee S^n \to C^{n+1}(\eta 2^s) \to S^{n+1}$$

the calculations of [6, Section 3] show that

$$\mathrm{Es}^0(C^{n+1}(2^r\eta)) \simeq \mathrm{Es}^0(C^{n+1}(\eta 2^s)) \simeq \mathbb{Z},$$

wherefrom

$$g(C^{n+1}(2^r\eta)) = g(C^{n+1}(\eta 2^s)) = 1.$$

(4) Let $C^{n+1}(\eta)$ be the Chang atom defined by the cofibration sequence

$$S^n \xrightarrow{\eta} S^{n-1} \to C^{n+1}(\eta) \to S^{n+1}.$$

One easily verifies that $\mathrm{Es}^0(C^{n+1}(\eta)) \simeq \mathbb{Z} \times_2 \mathbb{Z}$, wherefrom $g(C^{n+1}(\eta)) = 1$. The same holds for the *double Chang atom* $C(\eta^2)$ from [6, Section 5] defined by the cofibration sequence

$$S^n \xrightarrow{\eta^2} S^{n-2} \to C^{n+1}(\eta) \to S^{n+1},$$

where η^2 is the nonzero element of $\pi_n^S(S^{n-2}) \simeq \mathbb{Z}/2$.

(5) Finally, we consider the atom $A^{n+1}(v)$ defined by the cofibration sequence

$$S^n \xrightarrow{v\nu} S^{n-3} \to C^{n+1}(\eta) \to A^{n+1}(V) \to S^{n+1},$$

where $0 < v \le 12$ and ν is a generator of the group $\pi_n^S(S_{n-3}) \simeq \mathbb{Z}/24$. It follows from [6, Theorem 2.4] that $\mathrm{Es}^0(A^{n+1}(v))$ is isomorphic to the ring of pairs (α, β), where $\alpha, \beta \in \mathbb{Z}$ and $\alpha v\nu = \beta v\nu$, that is $\alpha \equiv \beta \pmod{m}$, where $m = 24/d$ and $d = \gcd(v, 24)$. It is the ring $\mathbb{Z} \times_m \mathbb{Z}$. Therefore

$$g(A(v)) = \begin{cases} 4 & \text{if } d = 1, \\ 2 & \text{if } d = 2 \text{ or } d = 3, \\ 1 & \text{if } d > 3. \end{cases}$$

Actually, all atoms $A^{n+1}(v)$ with fixed $\gcd(v,24)$ are in the same genus. One easily verifies that $B_0(A^{n+1}(v)) = S^{n-3} \vee S^{n+1}$, so

$$A^{n+1}(v) \vee S^{n-3} \vee S^{n+1} \simeq A^{n+1}(v') \vee S^{n-3} \vee S^{n+1}$$

if $\gcd(v,24) = \gcd(v',24)$. Indeed, one can even check that in this case already $A^{n+1}(v) \vee S^{n-3} \simeq A^{n+1}(v') \vee S^{n-3}$, as well as $A^{n+1}(v) \vee S^{n+1} \simeq A^{n+1}(v) \vee S^{n+1}$.

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