ON NILPOTENT CHERNIKOV p-GROUPS WITH ELEMENTARY TOPS

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Abstract. The description of nilpotent Chernikov p-groups with elementary tops is reduced to the study of tuples of skew-symmetric bilinear forms over the residue field $\mathbb{F}_p$. If $p \neq 2$ and the bottom of the group only consists of 2 quasi-cyclic summands, a complete classification is given. The main tool is the theory of representations of quivers with involution.

Contents

1. Structure theorem

Recall that a *Chernikov p-group* [1, 7] G is an extension of a finite direct sum M of *quasi-cyclic* p-groups, or, the same, the groups of type p^∞, by a finite p-group H. Note that M is the biggest abelian divisible subgroup of G, so both M and H are defined by G up to isomorphism. We call H and M, respectively, the *top* and the *bottom* of G. We denote by $M^{(n)}$ a direct sum of n copies M_i of quasi-cyclic p-groups and fix elements $a_i \in M_i$ of order p. A Chernikov p-group is defined by an action of a finite p-group H on a group $M^{(n)}$ and an element from the second cohomology group $H^2(H, M^{(n)})$ with respect to this action. Such an element is given by a 2-cocycle $\mu : H \times H \to M^{(n)}$, which is defined up to a 2-boundary [5, Chapter 15]. In what follows it is convenient to denote the operations in the groups G, H, M by $+$, so their units are denoted by 0.

It is known [1, Theorem 1.9] that a Chernikov p-group G is nilpotent if and only if the action of H on $M^{(n)}$ is trivial. In this case a cocycle is a map

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$\mu : H \times H \to M^{(n)}$ such that $\mu(y,z) + \mu(x, y+z) = \mu(x+y, z) + \mu(x,y)$ for all $x, y, z \in H$. We can also suppose that μ is *normalized*, i.e. $\mu(0,x) = \mu(x,0) = 0$ for every $x \in H$. A coboundary of a function $\gamma : H \to M^{(n)}$ is the function $\partial\gamma(x,y) = \gamma(x) + \gamma(y) - \gamma(x+y)$.

Let H_m be the elementary abelian p-group with m generators,

$$H_m = \langle\, h_1, h_2, \dots, h_m \mid ph_i = 0,\ h_i + h_j = h_j + h_i \text{ for all } i, j \,\rangle.$$

Let also $M_p^{(n)} = \{a \in M^{(n)} \,|\, pa = 0\} = \langle\, a_1, a_2, \dots, a_n \,\rangle$. We denote by $S(n,m)$ the group of all skew-symmetric maps $\tau : H_m \times H_m \to M_p^{(n)}$, i.e. such bilinear maps that $\tau(x,x) = 0$ for all x (hence $\tau(x,y) = -\tau(y,x)$ for all x, y).

Theorem 1.1. (cf. [9]) *If H_m acts trivially on $M^{(n)}$, then $H^2(H_m, M^{(n)}) \simeq S(n,m)$.*

Proof. Let G be an extension of $M^{(n)}$ by H_m with the trivial action of H_m corresponding to a cocycle μ. Then for every $x \in H_m$ there is a representative $\bar{x} \in G$ such that $\bar{x} + \bar{y} = \overline{x+y} + \mu(x,y)$. Set

$$t(x,y) = [\bar{x}, \bar{y}] = (\overline{x+y} + \mu(x,y)) - (\overline{y+x} + \mu(y,x)) = \mu(x,y) - \mu(y,x),$$

since all values $\mu(x,y)$ are in the center of G. As all commutators are in the center of G as well, we have

$$\begin{aligned}
[\overline{x+y}, \bar{z}] &= (\bar{x} + \bar{y} - \mu(x,y) + \bar{z}) - (\bar{z} + \bar{x} + \bar{y} - \mu(x,y)) \\
&= (\bar{x} + \bar{y} + \bar{z}) - (\bar{z} + \bar{x} + \bar{y}) \\
&= \bar{x} + \bar{y} + \bar{z} - \bar{y} - \bar{x} - \bar{z} \\
&= \bar{x} + \bar{y} + \bar{z} - \bar{y} - \bar{z} + \bar{z} - \bar{x} - \bar{z} \\
&= \bar{x} + [\bar{y}, \bar{z}] + \bar{z} - \bar{x} - \bar{z} \\
&= [\bar{x}, \bar{z}] + [\bar{y}, \bar{z}].
\end{aligned}$$

Thus the function $t : H_m \times H_m \to M^{(n)}$ is bilinear. Obviously, it is skew-symmetric. Moreover, $pt(x,y) = t(px, y) = t(0,y) = 0$, so $t(x,y) \in M_p^{(n)}$. We denote this function by $\tau(\mu)$, so defining a map $\tau : Z^2(H_m, M^{(n)}) \to S(n,m)$, where Z^2 denotes the group of cocycles.

If $\mu = \partial\gamma$, it is symmetric: $\mu(x,y) = \mu(y,x)$, hence $\tau(\mu) = 0$. On the contrary, let $\tau(\mu) = 0$. Then the group G is commutative. Therefore, its divisible subgroup $M^{(n)}$ is a direct summand of G [5, Theorem 13.3.1], i.e. $G = M^{(n)} \oplus H_m$, so the class of μ in $H^2(H_m, M^{(n)})$ is zero. It means that μ is a coboundary. Thus $\ker \tau = B^2(H_m, M^{(n)})$, the group of coboundaries.

It remains to prove that τ is surjective. Let $t : H_m \times H_m \to M_p^{(n)}$ be any skew-symmetric function. Set $t_{ij} = t(h_i, h_j)$ and, for any elements $x = \sum_{i=1}^m \alpha_i h_i$, $y = \sum_{j=1}^m \beta_j h_j$, set $\mu(x,y) = \sum_{i<j} \alpha_i \beta_j t_{ij}$. If $z = \sum_{k=1}^m \gamma_k h_k$,

then

$$\mu(y,z)+\mu(x,y+z)=\sum_{i<j}\beta_i\gamma_j t_{ij}+\sum_{i<j}\alpha_i(\beta_i+\gamma_j)t_{ij},$$
$$\mu(x+y,z)+\mu(x,y)=\sum_{i<j}(\alpha_i+\beta_i)\gamma_j t_{ij}+\sum_{i<j}\alpha_i\beta_i t_{ij},$$

so both sums equal $\sum_{i<j}(\alpha_i\beta_j+\alpha_i\gamma_j+\beta_i\gamma_j)t_{ij}$. Hence μ is a cocycle. Moreover,

$$\mu(h_i,h_j)-\mu(h_j,h_i)=\begin{cases} t_{ij} & \text{if } i<j,\\ -t_{ji}=t_{ij} & \text{if } i>j,\end{cases}$$

whence $\tau(\mu)=t$. □

Now we can classify all nilpotent Chernikov p-groups which are extensions of $M^{(n)}$ by H_m up to isomorphism. As we have seen, such a group is generated by the subgroup $M^{(n)}$ and elements $(\bar h_1,\bar h_2,\dots,\bar h_m)$ with the defining relations

$$\begin{aligned}&\bar h_i+a=a+\bar h_i,\\ &p\bar h_i=0,\\ &[\bar h_i,\bar h_j]=t_{ij}\end{aligned}$$

for all $a\in M^{(n)}$ and all $i,j\in\{1,2,\dots,m\}$, where (t_{ij}) is a skew-symmetric $m\times m$ matrix with elements from $M_p^{(n)}$. As $M_p^{(n)}\simeq\mathbb{F}_p^n$, where $\mathbb{F}_p$ is the residue field modulo p, the matrix (t_{ij}) can be considered as an n-tuple $\mathbf{A}=(A_1,A_2,\dots,A_n)$ of $m\times m$ skew-symmetric matrices with elements from $\mathbb{F}_p$. Recall that both $M^{(n)}$ and H_m are uniquely defined by G.

Theorem 1.2. (cf. [4]) *Let G and F be two nilpotent Chernikov p-groups with tops H_m and bottoms $M^{(n)}$, t and f be the corresponding skew-symmetric functions $H_m\times H_m\to M_p^{(n)}$. The groups G and F are isomorphic if and only if there are automorphisms σ of $M^{(n)}$ and θ of H_m such that $f(\theta(x),\theta(y))=\sigma(t(x,y))$ for all $x,y\in H_m$.*

Proof. As $M^{(n)}$ is the biggest divisible abelian subgroup of G or F, any isomorphism $\phi:G\to F$ maps $M^{(n)}$ to itself, so defines automorphisms $\sigma=\phi|_{M^{(n)}}$ of $M^{(n)}$ and θ of $H_m=G/M^{(n)}=F/M^{(n)}$. Note that the functions t and f do not depend on the choice of representatives of elements from H in G and F. If $\bar x$ is a preimage of $x\in H_m$ in G, then $\bar x'=\phi(\bar x)$ is a preimage of $\theta(x)$ in F. Therefore, $f(\theta(x),\theta(y))=[\bar x',\bar y']=\phi([\bar x,\bar y])=\sigma(t(x,y))$.

On the other hand, if σ and θ are automorphisms satisfying the condition of the theorem, the map $\phi:G\to F$ such that $\phi(a)=\sigma(a)$ for $a\in M^{(n)}$ and $\phi(\bar x)=\overline{\theta(x)}$ defines an isomorphism between G and F. □

If we identify skew-symmetric functions with n-tuples of skew-symmetric matrices over the field $\mathbb{F}_p$, this theorem can be reformulated as follows. For any n-tuple $\mathbf{A}=(A_1,A_2,\dots,A_n)$ and any invertible matrix $Q=(q_{ij})\in$

$\mathrm{GL}(n,\mathbb{F})$ we set $\mathbf{A}\circ Q = (A'_1, A'_2, \dots, A'_n)$, where $A'_j = \sum_{i=1}^n q_{ij}A_i$. If the matrices A_i are of size $m\times m$ and $P\in \mathrm{GL}(m,\mathbb{F})$, we set

$$P\circ\mathbf{A} = (PA_1P^\top, PA_2P^\top, \dots, PA_nP^\top),$$

where $P^\top$ denotes the transposed matrix. Obviously, these two operations commute. The n-tuples $\mathbf{A}$ and $P\circ\mathbf{A}$ are said to be *congruent*, and the n-tuples $\mathbf{A}$ and $P\circ\mathbf{A}\circ Q$ are called *weakly congruent.*

Corollary 1.3. *Two n-tuples $\mathbf{A}$ and $\mathbf{A}'$ of $m\times m$ skew-symmetric matrices over $\mathbb{F}_p$ define isomorphic nilpotent Chernikov p-groups with tops H_m and bottoms $M^{(n)}$ if and only if they are weakly congruent.*

Proof. The transformation $A\mapsto P\circ\mathbf{A}$ corresponds to an automorphism of $H_m\simeq\mathbb{F}_p^m$ given by the matrix P. On the other hand, automorphisms of $M^{(n)}$ are given by invertible matrices Q from $\mathrm{GL}(n,\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers considered as the endomorphism ring of the group of type p^∞ [7, § 21]. Such an automorphism transforms a sequence of matrices $\mathbf{A}$ to $\mathbf{A}\circ Q$. Moreover, the result only depends on the value of Q modulo p. As every invertible matrix over $\mathbb{F}_p$ can be lifted to an invertible matrix over $\mathbb{Z}_p$, it accomplishes the proof. $\square$

We denote by $G(\mathbf{A})$ the nilpotent Chernikov p-group with the bottom $M^{(n)}$ and elementary top corresponding to an n-tuple of skew-symmetric matrices $\mathbf{A}$.

2. Relation with representations of quivers

Theorem 1.2 and Corollary 1.3 reduce the classification of nilpotent Chernikov p-groups with top H_m and bottom $M^{(n)}$ up to isomorphism to a problem of linear algebra, namely, to the classification of n-tuples of skew-symmetric bilinear forms over the residue field $\mathbb{F}_p$. If $p\neq 2$, this problem is closely related with the study of representations of the so called *generalized Kronecker quiver*

$$K_n = \begin{tikzcd} 1 \arrow[r, bend left=40, "a_1"] \arrow[r, bend left=15, "a_2"'] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "a_n"'] & 2 \end{tikzcd} .$$

Recall this relation [8]. A *representation* R of K_n over a field $\Bbbk$ consists of two finite dimensional vector spaces $R(1)$ and $R(2)$ and n linear maps $R(a_i): R(1)\to R(2)$ $(1\le i\le n)$. A *morphism* f from a representation R to a representation R' is a pair of linear maps $f(k): R(k)\to R'(k)$ $(k=1,2)$ such that $f(2)R(a_i) = R'(a_i)f(1)$ for all $1\le i\le n$. We define an involution * on the quiver K_n setting $1^*=2$, $2^*=1$ and $a_i^* = -a_i$ for all $1\le i\le n$. If R is a representation of K_n, we define the dual representation R^* setting $R^*(k) = R(k^*)^*$, where V^* denotes the dual vector space to V, and $R^*(a_i) = -R(a_i)^*$, where $L^*: W^*\to V^*$ denotes the dual linear map to $L: V\to W$. A representation R is said to be *self-dual* if $R^* = R$. Then $R(a_i): R(1)\to R(1)^*$ is identified with a bilinear form on $R(1)$ and, if $\operatorname{char}\Bbbk \neq 2$, this form is skew-symmetric, since $R(a_i)^* = -R(a_i)$. One can check (cf. [8]) that a representation R is isomorphic to a self-dual one if and only if there is a *self-dual isomorphism* $f : R \to R^*$, i.e. such an isomorphism that $f(2) = f(1)^*$. We usually identify a representation R with the n-tuple of matrices describing the linear maps $R(a_i)$.

Let R be an indecomposable representation of K_n which is not isomorphic to a self-dual one. Then $= R \oplus R^*$ is isomorphic to a self-dual representation R^+, which cannot be decomposed into a direct sum of non-zero self-dual representations. Namely, R^+ is given by the n-tuple of skew-symmetric matrices

$$R^+(a_i) = \begin{pmatrix} 0 & R(a_i) \\ -R(a_i)^\top & 0 \end{pmatrix}.$$

If $\operatorname{char}\Bbbk \neq 2$, every self-dual representation decomposes into a direct sum of indecomposable self-dual representations and representations of the form R^+, where R is an indecomposable representation which is not isomorphic to any self-dual one. Moreover, the direct summands of the form R^+ are defined uniquely up to permutation, isomorphisms of the corresponding indecomposable representations R and replacing R by R^* [8, Theorem 1].

Obviously, if $n = 1$, there are no indecomposable self-dual representations. In the next section we will see that the same holds for $n = 2$. On the contrary, if $n = 3$, the representation R such that $R(1) = R(2) = \Bbbk^3$ and

$$R(a_1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ R(a_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ R(a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is indecomposable and self-dual.

Actually, a classification of representations of the quiver K_n for $n > 2$ is a so-called *wild problem*. It means that it contains the classification of representations of every finitely generated algebra over the field $\Bbbk$ (see [2] for precise definitions). The same is true for representations which are not isomorphic to self-dual. Namely, let $n = 3$, $R(1) = \Bbbk^d$, $R(2) = \Bbbk^{2d}$,

$$R(a_1) = \begin{pmatrix} I_d \\ 0 \end{pmatrix},\ R(a_2) = \begin{pmatrix} 0 \\ I_d \end{pmatrix},\ R(a_3) = \begin{pmatrix} X \\ Y \end{pmatrix},$$

where I_d is the unit $d \times d$ matrix, X, Y are arbitrary square $d \times d$ matrices. Obviously, R is not self-dual. One can easily check that two such representations given by the pairs (X, Y) and (X', Y') are isomorphic if and only if the pairs (X, Y) and (X', Y') are conjugate, i.e. $X' = SXS^{-1}$, $Y' = SYS^{-1}$ for some invertible matrix S. The problem of classification of pairs of square matrices up to conjugacy is a "standard" wild problem [2]. Thus one cannot hope to get a more or less comprehensible classification of triples of skew-symmetric forms. This is even more so for n-tuples with $n > 3$. In the next section we will see that for $n = 2$ the problem is "*tame*", hence there is a quite clear description of the corresponding groups.

Remark 2.1. If $\operatorname{char} \Bbbk = 2$, the definition of a skew-symmetric bilinear form cannot be "linearised", since the condition $B(x,x) = 0$ is no more the consequence of the condition $B(x,y) = -B(y,x)$. Hence, we cannot identify n-tuples of skew-symmetric forms with self-dual representations of the quiver K_n. Moreover, the results of [8] are also valid only if $\operatorname{char} \Bbbk \neq 2$. Thus, to study Chernikov 2-groups, we have to use quite different methods.

3. Case $n = 2$

If $n = 1$, G is described by one skew-symmetric matrix A. This matrix is congruent to a direct sum of k matrices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and l matrices (0), where $m = 2k + l$. It gives a simple description.

Proposition 3.1. *A nilpotent Chernikov p-group G with elementary top and quasi-cyclic bottom M decomposes as $G_k \times H_l$, where G_k is generated by M and $2k$ elements $\bar{h}_1, \bar{h}_2, \dots, \bar{h}_{2k}$ which are of order p, commute with all elements from M and their commutators $[\bar{h}_i, \bar{h}_j]$ for $i < j$ are given by the rule*

$$[\bar{h}_i, \bar{h}_j] = \begin{cases} a_1 & \text{if } j = k + i, \\ 0 & \text{otherwise,} \end{cases}$$

where a_1 is a fixed element of order p from the group M.

Now we consider the case $n = 2$.

Following the preceding consideration, we classify the pairs of skew-symmetric bilinear forms over a field $\Bbbk$ with $\operatorname{char} \Bbbk \neq 2$. Equivalently, we classify the self-dual representations of the Kronecker quiver K_2 with the involution $1^* = 2$, $2^* = 1$, $a_i^* = -a_i$. Recall [3, Chapter XII] that indecomposable representations of K_2 ("matrix pencils") are given by the following pairs of matrices;

$$\begin{aligned}
R_f: &\quad R_f(a_1) = I_d,\ R_f(a_2) = F(f), \\
R_{\infty,d}: &\quad R_{\infty,d}(a_1) = F(x^d),\ R_{\infty,d}(a_2) = I_d, \\
R_{-,d}: &\quad R_{-,d}(a_1) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \\
&\quad R_{-,d}(a_2) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \\
R_{+,d}: &\quad R_{+,d}(a_i) = R_{-,d}(a_i)^\top.
\end{aligned} \tag{3.1}$$

Here $f = f(x)$ is a polynomial of degree d from $\Bbbk[x]$ which is a power of a unital irreducible polynomial and $F(f)$ is the Frobenius matrix with the characteristic polynomial $f(x)$. The size of the matrices in $R_{-,d}$ is $(d-1) \times d$; respectively, the size of the matrices in $R_{+,d}$ is $d \times (d-1)$.

Actually, $R_{+,d} = (R_{-,d})^*$, $R_f^* \simeq R_f$ and $R_{\infty,d}^* \simeq R_{\infty,d}$. Nevertheless, there are no self-dual indecomposable representations.

Proposition 3.2. *Neither of indecomposable representations from the preceding list is isomorphic to a self-dual one.*

Proof. It is evident for $R_{\pm,d}$. The representation R_f^* is given by the pair of matrices $(-I_d, -F(f)^\top)$. If it were isomorphic to a self-dual one, there would be an invertible $d \times d$ matrix P such that $PI_d = -I_dP^*$ and $PF(f) = -F(f)^\top P^*$. Hence P is skew-symmetric, and $PF(f) = F(f)^\top P$. One easily checks that it is impossible. The same holds for $R_{\infty,d}$. $\square$

Combining this result with those from [8], we get a complete classification of pairs of skew-symmetric bilinear forms. We denote by $\mathfrak{A}$ the set of all pairs R^+, where $R \in \{R_f, R_{\infty,d}, R_{-,d}\}$, and by $\mathfrak{F}$ the set of functions $\kappa : \mathfrak{A} \to \mathbb{Z}_{\geq 0}$ such that $\kappa(\mathbf{A}) = 0$ for almost all $\mathbf{A}$. For any function $\kappa \in \mathfrak{F}$ we set $\mathfrak{A}^\kappa = \bigoplus_{\mathbf{A}\in\mathfrak{A}} \mathbf{A}^{\kappa(\mathbf{A})}$.

Theorem 3.3. *Let* $\operatorname{char}\Bbbk \neq 2$. *Any pair of skew-symmetric bilinear forms over the field* $\Bbbk$ *is congruent to a direct sum* $\mathfrak{A}^\kappa$ *for a uniquely defined function* $\kappa \in \mathfrak{F}$.

To obtain a classification of Chernikov p-groups with elementary tops and the bottom $M^{(2)}$, we also have to answer the question:

Given two functions with finite supports $\kappa, \kappa' : \mathfrak{A} \to \mathbb{Z}_{\geq 0}$, *when are the pairs* $\mathfrak{A}^\kappa$ *and* $\mathfrak{A}^{\kappa'}$ *weakly congruent*?

Evidently, $(\mathbf{A}_1 \oplus \mathbf{A}_2) \circ Q = (\mathbf{A}_1 \circ Q) \oplus (\mathbf{A}_2 \circ Q)$, so the pairs $\mathbf{A}$ and $\mathbf{A} \circ Q$ are indecomposable simultaneously. For every pair $\mathbf{A} \in \mathfrak{A}$ we denote by $\mathbf{A} * Q$ the unique pair from $\mathfrak{A}$ which is congruent to $\mathbf{A} \circ Q$. The map $\mathbf{A} \mapsto \mathbf{A} * Q$ defines an action of the group $\mathfrak{g} = \mathrm{GL}(2, \Bbbk)$ on the set $\mathfrak{A}$, hence on the set $\mathfrak{F}$ of functions $\kappa : \mathfrak{A} \to \mathbb{Z}_{\geq 0}$: $(Q * \kappa)(\mathbf{A}) = \kappa(A * Q)$. Theorem 3.3 implies the following result.

Corollary 3.4. *The pairs* $\mathfrak{A}^\kappa$ *and* $\mathfrak{A}^{\kappa'}$ *are weakly congruent if and only if the functions* κ *and* κ' *belong to the same orbit of the group* $\mathfrak{g}$.

Obviously, $R^+ \circ Q = (R \circ Q)^+$ for every representation R of the quiver K_2. Thus we have to know when $R \circ Q \simeq R'$ for indecomposable representations from the list (3.1). As $R_{-,d}$ is a unique (up to isomorphism) indecomposable representation R such that $\dim R(1) = d - 1$, $\dim R(2) = d$, we only have to consider the representations from the set $\{R_f, R_{\infty,d}\}$. From [3, Chapter XII, § 3] it follows that a pair $R = (R_1, R_2)$ from this set is completely defined by its *homogeneous characteristic polynomial* $\chi_R(x_1, x_2) = \det(x_1R_1 - x_2R_2)$. Actually, $\chi_{R_f} = x_2^d f(x_1/x_2)$, where $d = \deg f$, and $\chi_{R_{\infty,d}} = x_2^d$. The group $\mathfrak{g}$ naturally acts on the ring $\Bbbk[x_1, x_2]$: $Q \circ f = f(q_{11}x_1 + q_{12}x_2, q_{21}x_1 + q_{22}x_2)$, where $Q = (q_{ij})$, and

$$\begin{aligned}\chi_{R\circ Q} &= \det\big((q_{11}R_1 + q_{21}R_2)x + (q_{12}R_1 + q_{22}R_2)\big) = \\ &= \det\big((q_{11}x + q_{12})R_1 + (q_{21}x + q_{22})R_2\big) = Q\circ\chi_R.\end{aligned}$$

We say that an irreducible homogeneous polynomial $g \in \Bbbk[x_1, x_2]$ is *unital* if either $g = x_2$ or its leading coefficient with respect to x_1 equals 1. Let $\mathbb{P} = \mathbb{P}(\Bbbk)$ be the set of unital homogeneous irreducible polynomials from $\Bbbk[x_1, x_2]$ and $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}(\Bbbk) = \mathbb{P} \cup \{\varepsilon\}$. Note that $\mathbb{P}$ actually coincides with the set of the closed points of the projective line $\mathbb{P}^1_\Bbbk = \operatorname{Proj}\Bbbk[x_1, x_2]$ [6]. For $g \in \mathbb{P}$ and $Q \in \mathfrak{g}$, let $Q * g$ be the unique polynomial $g' \in \mathbb{P}$ such that $Q \circ g = \lambda g'$ for some non-zero $\lambda \in \Bbbk$. (It is the natural action of $\mathfrak{g}$ on $\mathbb{P}^1_\Bbbk$.) We also set $Q * \varepsilon = \varepsilon$ for any Q. It defines an action of $\mathfrak{g}$ on $\tilde{\mathbb{P}}$. Denote by $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(\Bbbk)$ the set of all functions $\rho : \tilde{\mathbb{P}} \times \mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that $\rho(g, d) = 0$ for almost all pairs (g, d). Define the actions of the group $\mathfrak{g}$ on $\tilde{\mathfrak{F}}$ setting $(\rho * Q)(g, d) = \rho(Q * g, d)$. For every pair $(g, d) \in \tilde{\mathfrak{F}}$ we define a pair of skew-symmetric forms $R(g, d)$:

$$R(g,d) = \begin{cases} R^+_{-,d} & \text{if } g = \varepsilon, \\ R^+_{\infty,d} & \text{if } g = x_2, \\ R^+_{g(x,1)^d} & \text{otherwise.} \end{cases}$$

Let $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}(\Bbbk) = \{R(g, d) \mid (g, d) \in \tilde{\mathbb{P}} \times \mathbb{N}\}$. For every function $\rho \in \tilde{\mathfrak{F}}$ we set $\tilde{\mathfrak{A}}^\rho = \bigoplus_{(g,d)\in\tilde{\mathbb{P}}\times\mathbb{N}} R(g, d)^{\rho(g,d)}$. The preceding considerations imply the following theorem.

Theorem 3.5. *Let* $\operatorname{char}\Bbbk \neq 2$.

(1) *Every pair of skew-symmetric bilinear forms over the field* $\Bbbk$ *is weakly congruent to* $\tilde{\mathfrak{A}}^\rho$ *for some function* $\rho \in \tilde{\mathfrak{F}}(\Bbbk)$.
(2) *The pairs* $\tilde{\mathfrak{A}}^\rho$ *and* $\tilde{\mathfrak{A}}^{\rho'}$ *are weakly congruent if and only if the functions* ρ *and* ρ' *belong to the same orbit of the group* $\mathfrak{g} = \mathrm{GL}(2, \Bbbk)$.

From Theorem 3.5 and Corollary 1.3 we immediately obtain a classification of nilpotent Chernikov p-groups with elementary tops and the bottom $M^{(2)}$. Namely, for every function $\rho \in \tilde{\mathfrak{F}}(\mathbb{F}_p)$ set $G(\rho) = G(\tilde{\mathfrak{A}}^\rho)$.

Theorem 3.6. *Let* $\mathfrak{R}$ *be a set of representatives of orbits of the group* $\mathfrak{g} = \mathrm{GL}(2, \mathbb{F}_p)$ *acting on the set of functions* $\tilde{\mathfrak{F}}(\mathbb{F}_p)$. *Then every nilpotent Chernikov p-group with elementary top and the bottom* $M^{(2)}$ *is isomorphic to the group* $G(\rho)$ *for a uniquely defined function* $\rho \in \mathfrak{R}$.

One can easily describe these groups in terms of generators and relations. Note that all of them are of the form $G(\mathbf{A})$, where $\mathbf{A} = \bigoplus_{k=1}^s \mathbf{A}_k$ and all $\mathbf{A}_k$ belong to the set $\{R^+_{-,d}, R^+_{\infty,d}, R^+_f\}$. Therefore $G(\mathbf{A})$ is generated by the subgroup $M^{(2)}$ and elements $\bar{h}_{ki}$, where $1 \leq k \leq s$, $1 \leq i \leq d_k$, $d_k = 2\deg f$ if $\mathbf{A}_k = R^+_f$, $d_k = 2d$ if $\mathbf{A}_k = R^+_{\infty,d}$, and $d_k = 2d-1$ if $\mathbf{A} = R^+_{-,d}$. All elements $\bar{h}_{ki}$ are of order p, commute with the elements from $M^{(2)}$, $[\bar{h}_{ki}, \bar{h}_{lj}] = 0$ if

$k \neq l$ and the values of the commutators $[\bar{h}_{ki}, \bar{h}_{kj}]$ for $i < j$, according to the type of $\mathbf{A}_k$, are given in Table 1. In this table a_1 and a_2 denote some fixed generators of the subgroup $M_p^{(2)}$.

TABLE 1.

$\mathbf{A}_k$	i, j	$[\bar{h}_{ki}, \bar{h}_{kj}]$
$R^+_{-,d}$	$j = d + i$	a_1
	$j = d + i - 1$	a_2
	otherwise	0
$R^+_{\infty,d}$	$j = d + i$	$a_2,$
	$j = d + i - 1$	$a_1,$
	otherwise	0
R^+_f	$j = d + i < 2d$	a_1
	$j = d + i - 1$	a_2
	$i < d,\ j = 2d$	$-\lambda_{d-i+1} a_2$
	$i = d,\ j = 2d$	$a_1 - \lambda_1 a_2$
	otherwise	0

where $f(x) = x^d + \lambda_1 x^{d-1} + \cdots + \lambda_d$

Corollary 3.7. *Let $G = G(\mathbf{A})$.*

1. *G has a finite direct factor if and only if $\mathbf{A} \simeq (R_{-,1})^k \oplus \mathbf{A}'$; then $G \simeq H_k \times G(\mathbf{A}')$.*
2. *Suppose that G has no finite direct factors. It is decomposable if and only if $\mathbf{A} \simeq (R_x^+)^k \oplus (R^+_{\infty,1})^l$; then $G = G_k \times G_l$.*

(See Proposition 3.1 for the definition of G_k.)

Proof is evident. □

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