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Representations of Munn algebras and related semigroups

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ABSTRACT
We establish representation types (finite, tame or wild) of finite dimensional Munn algebras with semisimple bases. As an application, we establish representation types of finite Rees matrix semigroups, in particular, 0-simple semigroups, and their mutually annihilating unions.

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1. Introduction

Munn algebras appeared in the theory of semigroups as semigroup algebras of *completely* 0-*simple* semigroups [1, 6]. They were immediately used for the study of representations of such semigroups. An important input was made by Ponizovskiĭ in the paper [7], where he established the cases when a finite 0-simple semigroup is *representation finite*, i.e., only has finitely many indecomposable representations, over an algebraically closed field $\Bbbk$ whose characteristic does not divide the order of the underlying group of its Rees matrix presentation [1, Th. 3.5]. He also considered the case of semigroups that are unions of mutually annihilating 0-simple semigroups with common zero.

The questions remained what happens if the field is not algebraically closed and when the representation type of such a semigroup is *tame*, i.e., indecomposable representations of each dimension form a finite number of 1-parameter families. In this article we give a complete answer to these questions (also for the fields of characteristics that does non divide the orders of the underlying groups). Of course, in the case of an algebraically closed field our criterion of finiteness coincides with that of Ponizovskiĭ. Actually, we obtain criteria of finiteness and tameness for all Munn algebras with semisimple base, even in a bit more wide context than they are considered in [1]. To prove these results, we establish a relation of modules over Munn algebras with *representations of valued graphs* in the sense of [2] (in the algebraically closed case they are just *representations of quivers* in the sense of [4]). Then we apply the criteria from this paper.

It follows from [3] (and can be easily checked directly) that in all other cases the Munn algebra $\mathbb{M}$ (or the corresponding semigroup) is *representation wild* over the field $\Bbbk$, i.e., for every finitely generated $\Bbbk$-algebra A there is an exact functor A-Mod $\to$ $\mathbb{M}$-Mod mapping non-isomorphic modules to non-isomorphic and indecomposable to indecomposable.

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To the memory of Iosif Solomonovich Ponizovskiĭ.

2. Munn algebras

In this paper *algebra* means an associative algebra over a commutative ring $\Bbbk$. We do not suppose that such an algebra is unital, but always suppose that modules over such algebra are also $\Bbbk$-modules and the multiplication by elements of the algebra is $\Bbbk$-bilinear. We denote by A-Mod and Mod-A, respectively, the categories of left and right A-modules. By A^1 we denote the algebra obtained from an algebra A by the formal attachment of unit. Then the categories of A-modules and unital A^1-modules are equivalent. So A and B are Morita equivalent if and only if so are A^1 and B^1. We consider the elements from A^1 as formal sums $\lambda + a$, where $a \in A$, $\lambda \in \Bbbk$.

Definition 2.1.

(1) Let R be a $\Bbbk$-algebra and $\mu : N \to M$ be a homomorphism of R-modules. Define a multiplication on $\mathrm{Hom}_R(M,N)$ setting $a \cdot b = a\mu b$. The resulting ring is called a *Munn algebra* and denoted by $\mathbb{M}(R,M,N,\mu)$.[1] We say that this Munn algebra *is based* on the algebra R. We denote by $\mathbb{M}^1(R,M,N,\mu)$ the algebra obtained from $\mathbb{M}(R,M,N,\mu)$ by the formal attachment of unit.

(2) A Munn algebra $\mathbb{M}(R,M,N,\mu)$ is said to be *regular* if the homomorphism μ is von Neumann regular, i.e., there is a homomorphism $\theta : M \to N$ such that $\mu\theta\mu = \mu$. For instance, this is the case if R is von Neumann regular, while M and N are finitely generated and projective and $\mu \neq 0$ (it follows from [5, Th. 1.7]).

Remark 2.2. One can see that $\mathbb{M}(R,M,N,\mu)$ has a unit if and only if there are decompositions $M \simeq M_1 \oplus M_2$ and $N \simeq N_1 \oplus N_2$ such that $\mathrm{Hom}_R(M_2,N) = \mathrm{Hom}_R(M,N_2) = 0$ and the map $\bar{\mu} = \mathrm{pr}_1 \circ \mu|_{N_1}$ is an isomorphism $N_1 \xrightarrow{\sim} M_1$. Then the unit $u : M \to N$ coincides with $\bar{\mu}^{-1}$. Actually, in this case $\mathbb{M}(R,M,N,\mu) \simeq \mathbb{M}(R,M_1,N_1,\bar{\mu}) \simeq \mathrm{End}_R M_1$.

Proposition 2.3. *Let $\mathbb{M}(R,M,N,\mu)$ be a regular Munn algebra. There are isomorphisms $M \simeq L \oplus M'$ and $N \simeq L \oplus N'$ such that with respect to these decompositions $\mu = \left(\begin{smallmatrix} 1_L & 0 \\ 0 & 0 \end{smallmatrix}\right)$.*

Proof. Let $\theta : M \to N$ be such that $\mu\theta\mu = \mu$. Then $\mu\theta : M \to M$ and $\theta\mu : N \to N$ are idempotents. Therefore, $M = M_1 \oplus M_2$, where $M_1 = \mathrm{Im}\,\mu\theta$, $M_2 = \mathrm{Ker}\,\mu\theta$ and $N = N_1 \oplus N_2$, where $N_1 = \mathrm{Im}\,\theta\mu$, $N_2 = \mathrm{Ker}\,\theta\mu$. One easily sees that $\mathrm{Ker}\,\mu = \mathrm{Ker}\,\theta\mu$ and $\mathrm{Im}\,\mu = \mathrm{Im}\,\mu\theta$, so $\bar{\mu} = \mathrm{pr}_1 \circ \mu|_{N_1}$ is an isomorphism and $\bar{\mu}^{-1} = \mathrm{pr}_1 \circ \theta|_{M_1}$, while $\mu|_{N_2} = 0$ and $\mathrm{pr}_2 \circ \mu = 0$, hence $\mu = \left(\begin{smallmatrix} \bar{\mu} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with respect to these decompositions. Obviously, it implies the claim. □

Definition 2.4. We write $\mathbb{M}(R,L,M,N)$ instead of $\mathbb{M}(R,L \oplus M, L \oplus N,\mu)$, where $\mu = \left(\begin{smallmatrix} 1_L & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and call such a Munn algebra *normal*. Thus every regular Munn algebra is isomorphic to a normal one. As above, we denote by $\mathbb{M}^1(R,L,M,N)$ the algebra obtained from $\mathbb{M}(R,L,M,N)$ by the formal attachment of unit.

Lemma 2.5. *Let A and C be two rings, P be a right C-module, M be a right A-module and N be a right A - left C-bimodule. Define the natural map $\phi : P \otimes_C \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M, P \otimes_C N)$ mapping $p \otimes f$ to the homomorphism $x \mapsto p \otimes f(x)$. If P is projective and either P or M is finitely generated, ϕ is an isomorphism.*

The proof is obvious. □

[1] This definition is a bit more general than that from [1, 6], where only the case of free modules is considered.

Lemma 2.6. *Let A be a unital ring, $1 = e_1 + e_2$, where e_1, e_2 are orthogonal idempotents. We denote $A_i = e_iA$, $A_{ij} = e_iAe_j \simeq \mathrm{Hom}_A(A_j, A_i)$ and identify A with the ring of matrices*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{2.1}$$

Let P be a progenerator of the category Mod-A_{11}*. Then $P^\sharp = (P \otimes_{A_{11}} A_1) \oplus A_2$ is a progenerator of the category* Mod-A*, hence A-*Mod $\simeq$ *B-*Mod*, where $B = \mathrm{End}_A P^\sharp$. The ring B can be identified with the ring of matrices*

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{2.2}$$

where $B_{11} = \mathrm{End}_{A_{11}} P$, $B_{12} = P \otimes_{A_{11}} A_{12}$, $B_{21} = \mathrm{Hom}_{A_{11}}(P, A_{21})$, $B_{22} = A_{22}$.

Proof. For some m there is an epimorphism of A_{11}-modules $P^m \twoheadrightarrow A_{11}$, which induces an epimorphism $(P \otimes_{A_{11}} A_1)^m \twoheadrightarrow A_1$. Hence, A is a direct summand of $(P \otimes_{A_{11}} A_1)^m \oplus A_2$ and $P^\sharp$ is a progenerator of A-Mod. Using Lemma 2.5, we obtain:

$$\begin{aligned}
&\mathrm{Hom}_A(P \otimes_{A_{11}} A_1, P \otimes_{A_{11}} A_1) \simeq \\
&\simeq \mathrm{Hom}_{A_{11}}(P, \mathrm{Hom}_A(A_1, P \otimes_{A_{11}} A_1) \simeq \\
&\simeq \mathrm{Hom}_{A_{11}}(P, P \otimes_{A_{11}} A_{11}) \simeq \mathrm{End}_{A_{11}} P; \\
&\mathrm{Hom}_A(A_2, P \otimes_{A_{11}} A_1) \simeq P \otimes_{A_{11}} A_{12}; \\
&\mathrm{Hom}_A(P \otimes_{A_{11}} A_1, A_2) \simeq \mathrm{Hom}_{A_{11}}(P, A_{21}).
\end{aligned}$$

It gives the presentation (2.2) for $\mathrm{End}_A P^\sharp$. □

Theorem 2.7. *Let $\mathbb{M} = \mathbb{M}(R, L, M, N)$ be a normal Munn algebra, $C = \mathrm{End}_R L$ and P be a progenerator of the category* Mod-C*. Then $\mathbb{M}$ is Morita equivalent to the normal Munn algebra $\mathbb{M}(R, P \otimes_C L, M, N)$.*

Proof. Let $A = \mathbb{M}^1(R, L, M, N)$. Consider the idempotents $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = 1 - e_1$. The presentation (2.1) of the algebra A is of the form

$$\begin{pmatrix} C & \mathrm{Hom}_R(M, L) \\ \mathrm{Hom}_R(L, N) & \Bbbk + \mathrm{Hom}_R(M, N) \end{pmatrix} \tag{2.3}$$

By Lemma 2.6, A is Morita equivalent to the algebra B of the matrices of the form (2.2), where, due to Lemma 2.5,

$$\begin{aligned}
B_{11} &= \mathrm{Hom}_C(P, P) \simeq \mathrm{Hom}_C\big(P, P \otimes_C \mathrm{Hom}_R(L, L)\big) \\
&\simeq \mathrm{Hom}_C(P, \mathrm{Hom}_R(L, P \otimes_C L) \simeq \mathrm{Hom}_R(P \otimes_C L, P \otimes_C L); \\
B_{12} &= P \otimes_C \mathrm{Hom}_R(M, L) \simeq \mathrm{Hom}_R(M, P \otimes_C L); \\
B_{21} &= \mathrm{Hom}_C(P, \mathrm{Hom}_R(L, N)) \simeq \mathrm{Hom}_R(P \otimes_C L, N); \\
B_{22} &= \Bbbk + \mathrm{Hom}_R(M, N).
\end{aligned}$$

But it is just the matrix presentation of $\mathbb{M}^1(R, P \otimes_C L, M, N)$. □

The following fact is evident.

Proposition 2.8. *$\prod_{k=1}^s \mathbb{M}(R_k, M_k, N_k, \mu_k) \simeq \mathbb{M}(R, M, N, \mu)$, where $R = \prod_{k=1}^s R_k$, $M = \bigoplus_{k=1}^s M_k$, $N = \bigoplus_{k=1}^s N_k$ and $\mu|_{N_k} = \mu_k$.*

Remark 2.9. Note that $\prod_{k=1}^s \mathbb{M}^1(R_k, M_k, N_k, \mu_k) \not\simeq \mathbb{M}^1(R, M, N, \mu)$.

Let now R be a semisimple ring. Then $R = \prod_{k=1}^{s} R_k$, where $R_k = \mathrm{Mat}(d_k, F_k)$ for some integers d_k and some skewfields F_k. So any Munn algebra based on R is a product of Munn algebras based on the simple algebras R_k. All of them are regular, so can be supposed normal.

Proposition 2.10. *Let $R = \mathrm{Mat}(d, F)$, where F is a skewfield, U be the simple R-module, $L = U^r$, $M = U^m$, $N = U^n$. The algebra $\mathbb{M}(R, L, M, N)$, up to isomorphism, only depends on r, m, n and does not depend on d. In particular, it is isomorphic to $\mathbb{M}(F, F^r, F^m, F^l)$.*

We denote the algebra $\mathbb{M}(F, F^r, F^m, F^n)$ by $\mathbb{M}(F, r, m, n)$.[2]

Proof. Indeed, $\mathrm{Hom}_R(U^k, U^l) \simeq \mathrm{Mat}(l \times k, F)$ does not depend on d and with respect to such isomorphisms $\mathbb{M}(R, L, M, N) = \mathrm{Mat}((r+n) \times (r+m), F)$ with the multiplication $a \cdot b = a\mu b$, where $\mu = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ (of size $(r+m) \times (r+n)$) and I is the $r \times r$ unit matrix. □

Theorem 2.11. *Let $\mathbb{M} = \prod_{k=1}^{s} \mathbb{M}(F_k, r_k, m_k, n_k)$, where F_k are skewfields. Then $\mathbb{M}$ is Morita equivalent to $\prod_{k=1}^{s} \mathbb{M}(F_k, 1, m_k, n_k)$.*

Proof. Let $R = \prod_{k=1}^{s} F_k$, $L_k = F_k^{r_k}$ and $L = \prod_{k=1}^{s} L_k$. Then $C_k = \mathrm{End}_R L_k \simeq \mathrm{Mat}(r_k \times r_k, F)$. Let P_k be the simple right C_k-module. It is a progenerator of the category Mod-C_k and $P_k \otimes_{C_k} L_k \simeq F_k$. Now apply Theorem 2.7. □

We denote the algebra $M(F, 1, m, n)$ by $\mathbb{M}(F, m, n)$. It is the algebra of $(n+1) \times (m+1)$ matrices over F with the multiplication $a \cdot b = a\mu b$, where μ is the $(m+1) \times (n+1)$ matrix with 1 at the $(1,1)$-place and 0 elsewhere.

3. Representations

In this section we consider representations of finite dimensional regular Munn algebras over a field $\Bbbk$ with a semisimple base. According to Theorem 2.11, such an algebra is Morita equivalent to a direct product $\mathbb{M} = \prod_{k=1}^{s} \mathbb{M}_k$, where $\mathbb{M}_k = \mathbb{M}(F_k, m_k, n_k)$ and F_k are skewfields. If $m_k = n_k = 0$, $\mathbb{M}(F_k, m_k, n_k) = F_k$ and is a direct factor of $\mathbb{M}^1$. So we can and will suppose that there are no such components in $\mathbb{M}$. The algebra $\mathbb{M}_k$ contains an idempotent e_k which is the $(n_k+1) \times (m_k+1)$ matrix with 1 at the $(1,1)$-place and 0 elsewhere. Let $e_0 = 1 - \sum_{k=1}^{s} e_k$. Then, if $k \neq 0$, $e_k \mathbb{M}^1 e_k = F_k$, $e_0 \mathbb{M}^1 e_k = F_k^{n_k}$, $e_k \mathbb{M}^1 e_0 = F_k^{m_k}$, $e_0 \mathbb{M}^1 e_0 = \Bbbk + \bigoplus_{k=1}^{s} M_k$, where $M_k \simeq \mathrm{Mat}(n_k \times m_k, F_k)$, and $e_k \mathbb{M}^1 e_l = 0$ if $0 \neq k \neq l \neq 0$. Choose an F_k-basis $\{a_{k1}, a_{k2}, \dots, a_{km_k}\}$ in each space $e_k \mathbb{M}^1 e_0$ and an F_k-basis $\{b_{k1}, b_{k2}, \dots, b_{kn_k}\}$ in each space $e_0 \mathbb{M}^1 e_k$. Then $a_{ki} b_{lj} = 0$ for all k, l, i, j, $b_{ki} a_{lj} = 0$ if $k \neq l$ and $\{b_{ki} a_{kj}\}$ is a basis of M_k. For every $\mathbb{M}^1$-module V set $V_k = e_k V$ $(0 \leq k \leq s)$. It is a vector space over F_k. The multiplication by a_{ki} gives rise to a $\Bbbk$-linear map $\alpha_{ki} : V_0 \to V_k$ and the multiplication by b_{kj} gives rise to a $\Bbbk$-linear map $\beta_{ki} : V_k \to V_0$. Since $\mathrm{Hom}_{\Bbbk}(V_0, V_k) \simeq \mathrm{Hom}_{F_k}(F_k \otimes_{\Bbbk} V_0, V_k))$ and $\mathrm{Hom}_{\Bbbk}(V_k, V_0) \simeq \mathrm{Hom}_{F_k}(V_k, \mathrm{Hom}_{\Bbbk}(F_k, V_0))$, both α and β can be considered as matrices over F_k of appropriate sizes. So V is defined by the set of maps (or of matrices) $\{\alpha_{ki}, \beta_{lj}\}$ such that $\alpha_{ki}\beta_{lj} = 0$ for all k, l, i, j. We present it by the diagram

$$V : \ \{V_k\} \overset{0}{\circlearrowleft} \underset{\{\beta_{kj}\}}{\overset{\{\alpha_{ki}\}}{\rightleftarrows}} V_0,$$

[2] Ponizovskiĭ [7] denotes this algebra by $\mathfrak{A}(E_{m+r,n+r,r}, F)$.

A homomorphism $\phi : V \to V'$ is given by a set of F_k-linear maps $\phi_k : V_k \to V'_k$ $(0 \le k \le s)$, where $F_0 = \Bbbk$, such that $\phi_k\alpha_{ki} = \alpha'_{ki}\phi_0$ and $\phi_0\beta_{kj} = \beta'_{kj}\phi_k$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \{V_k\} \overset{0}{\circlearrowleft} & \underset{\{\beta_{kj}\}}{\overset{\{\alpha_{ki}\}}{\rightleftarrows}} & V_0 \\ {\scriptstyle \{\phi_k\}}\downarrow & & \downarrow{\scriptstyle \phi_0} \\ \{V'_k\} \overset{0}{\circlearrowleft} & \underset{\{\beta'_{kj}\}}{\overset{\{\alpha'_{ki}\}}{\rightleftarrows}} & V'_0 \end{array} \tag{3.1}$$

ϕ is an isomorphism if and only if so are all ϕ_k.

Set $V_+ = \sum_{l,j} \operatorname{Im} \beta_{lj} \subseteq V_0$, $V_- = V_0/V_+$. Then $\alpha_{ki}(V_+) = 0$. Hence α_{ki} can be considered as a map $V_- \to V_k$ and we obtain a diagram

$$\tilde{V} : \ \{V_k\} \begin{array}{c} \xleftarrow{\{\alpha_{ki}\}} V_- \\ \xrightarrow[\{\beta_{kj}\}]{} V_+ \end{array}$$

with the condition $\sum_{k,j} \operatorname{Im} \beta_{kj} = V_+$. Such diagram can be considered as a representation of the *realization* $(\mathfrak{M}, \Omega)$ *of the valued graph* (Γ, d) in the sense of [2]. Namely the vertices of the graph Γ are $\{+, -, 1, 2, \dots, s\}$, $d_k = \dim_{\Bbbk} F_k$, $d_{k+} = (m_k, m_k d_k)$, $d_{-k} = (n_k d_k, n_k)$ and $d_{ij} = 0$ otherwise. The *orientation* Ω of the edge $\{k, +\}$ is $k \to +$ and that of the edge $\{-, k\}$ is $- \to k$. The *modulation* $\mathfrak{M}$ of Γ is given by the algebras F_k and $F_\pm = \Bbbk$, F_k-F_--bimodules ${}_kM_- = m_k F_k$ and F_+-F_k-bimodules ${}_+M_k = n_k F_k$. Thus a representation of this realization is indeed given by a set of F_k-vector spaces V_k, F_0-vector spaces $V_\pm$ and a set of linear maps $\tilde{\alpha}_k : n_k V_- \to V_k$ and $\tilde{\beta}_l : m_l V_l \to V_+$. There components are just α_{ki} and β_{lj}.

Theorem 3.1. *Let* $\operatorname{Rep}^+(\mathfrak{M}, \Omega)$ *be the full subcategory of the category of representations of* $(\mathfrak{M}, \Omega)$ *such that* $\sum_{l=1}^{s} \operatorname{Im} \tilde{\beta}_l = V_+$ *and* $\bigcap_{k,i} \operatorname{Ker} \tilde{\alpha}_k = 0$. *Let also* $\mathbb{M}$-Mod^+ *be the full subcategory of* $\mathbb{M}$-Mod *consisting of such modules* V *that* $\sum_{l,j} \operatorname{Im} \beta_{lj} = \bigcap_i \operatorname{Ker} \alpha_{ki}$. *Denote by* $\mathcal{I}$ *the ideal of the category* $\mathbb{M}$-Mod^+ *consisting of all morphisms* $\phi : V \to V'$ *such that* $\phi_k = 0$ *for* $k \neq 0$, $\phi_0(V_+) = 0$ *and* $\operatorname{Im} \phi_0 \subseteq V'_+$. *Then* $\mathbb{M}$-$\mathrm{Mod}^+/\mathcal{I} \simeq \operatorname{Rep}^+(\mathfrak{M}, \Omega)$ *and* $\mathcal{I}^2 = 0$.

Proof. We have already constructed, for any $\mathbb{M}$-module V, the representation $\tilde{V}$. By definition, $\tilde{V} \in \operatorname{Rep}^+(\mathfrak{M}, \Omega)$. Given a homomorphism $\phi = \{\phi_k\} : V \to V'$ as in (3.1), we obtain linear maps $\phi_+ : V_+ \to V'_+$ and $\phi_- : V_- \to V'_-$ such that together with the maps ϕ_k they give a morphism $\tilde{\phi} : \tilde{V} \to \tilde{V}'$. Obviously, $\tilde{\phi} = 0$ if and only if $\phi \in \mathcal{I}$. Thus we obtain a functor $\Phi : \mathbb{M}$-$\mathrm{Mod}^+/\mathcal{I} \to \operatorname{Rep}^+(\mathfrak{M}, \Omega)$. Obviously $\mathcal{I}^2 = 0$.

Let $W = (W_k, W_+, W_-, \alpha_k, \beta_k \mid 1 \le k \le s)$ be a representation from $\operatorname{Rep}^+(\mathfrak{M}, \Omega)$. Set $\tilde{W}_0 = W_+ \oplus W_-$, take for $\tilde{\alpha}_{ki} : W_0 \to W_k$ the maps that are 0 on W_+ and coincide with the components of α_k on W_-, and take for $\tilde{\beta}_{lj} : W_l \to \tilde{W}_0$ the components of $\beta_l : W_l \to W_+$. It defines an $\mathbb{M}$-module $\tilde{W} \in \mathbb{M}$-Mod^+. If $\psi : W \to W'$ is a morphism of representations, set $\tilde{\psi}_0(w) = \psi_+(w_+) + \psi_-(w_-)$ if $w = w_+ + w_-$, where $w_\pm \in W_\pm$. It gives a homomorphism $\tilde{\psi} : \tilde{W} \to \tilde{W}'$. Taking its class modulo $\mathcal{I}$, we obtain a functor $\Psi : \operatorname{Rep}^+(\mathfrak{M}, \Omega) \to \mathbb{M}$-$\mathrm{Mod}^+/\mathcal{I}$. One easily verifies that this functor is quasi-inverse to Φ. □

Remark 3.2. Since $\mathcal{I}^2 = 0$, the isomorphism classes of objects in $\mathbb{M}$-Mod^+ are the same as in $\mathbb{M}$-$\mathrm{Mod}^+/\mathcal{I}$. The only indecomposable representations not belonging to $\operatorname{Rep}^+(\mathfrak{M}, \Omega)$ are two *trivial representations* such that $V_+ = \Bbbk$ (or $V_- = \Bbbk$) and $V_k = 0$ for $k \neq +$ (respectively, for $k \neq -$). The only indecomposable $\mathbb{M}$-module not belonging to $\mathbb{M}$-Mod^+ is the 1-dimensional vector space with zero multiplication by the elements of $\mathbb{M}$. Therefore, the *representation type* of the algebra $\mathbb{M}$ (finite, tame or wild) is the same as that of the realization $(\mathfrak{M}, \Omega)$ of the valued graph Γ.

It is proved in [2] that the representation type of $(\mathfrak{M}, \Omega)$ actually only depends on the valued graph itself. Namely, it is representation finite if and only if all its connected components are *Dynkin graphs* and representation tame if and only if all of them are Dynkin or *Euclidean* (*extended Dynkin*) graphs and at least one Euclidean graph occurs. For the list of these graphs see [2, p. 3]. In all other cases it is representation wild.

Taking into account the construction of the valued graph Γ from the algebra $\mathbb{M}$, we can establish the representation type of any finite dimensional Munn algebra with a semisimple base. Actually it only depends on the set of triples $\{(d_k, m_k, n_k)\}$, where $d_k = \dim_{\Bbbk} F_k$. We use the following notations:

$$\begin{aligned}&\mathfrak{T}(d_1, \dots, d_r \mid d_{r+1}, \dots, d_s) = \\ &= \{(d_1, 1, 0), \dots, (d_r, 1, 0), (d_{r+1}, 0, 1), \dots, (d_s, 0, 1)\},\end{aligned}$$

and, for $\mathfrak{T} = \mathfrak{T}(d_1, \dots, d_r \mid d_{r+1}, \dots, d_s)$,

$$\begin{aligned}S^-(\mathfrak{T}) &= \sum\nolimits_{k=1}^{r} d_k,\\ S^+(\mathfrak{T}) &= \sum\nolimits_{k=r+1}^{s} d_k,\\ S(\mathfrak{T}) &= S^-(\mathfrak{T}) + S^+(\mathfrak{T}).\end{aligned}$$

Certainly, maybe $r = 0$ or $r = s$.

Theorem 3.3. *Let* $\mathbb{M} = \prod_{k=1}^{s} \mathbb{M}(F_k, m_k, n_k)$, $\mathfrak{T} = \{(d_k, m_k, n_k) \mid (m_k, n_k) \neq (0, 0)\}$, *where* $d_k = \dim_{\Bbbk} F_k$.

(1)[3] *$\mathbb{M}$ is representation finite if and only if $\mathfrak{T} = \mathfrak{T}_0 \cup \mathfrak{T}_1$, where $\mathfrak{T}_0 = \mathfrak{T}(d_1, \dots, d_r \mid d_{r+1}, \dots, d_s)$ for some d_k and*

  (a) *either $\mathfrak{T}_1 = \emptyset$ and* $\max\{S^-(\mathfrak{T}_0), S^+(\mathfrak{T}_0)\} \le 3$
  (b) *or $\mathfrak{T}_1 = \{(1, 1, 1)\}$, $S(\mathfrak{T}_0) \le 3$ and* $\max\{S^-(\mathfrak{T}_0), S^+(\mathfrak{T}_0)\} \le 2$.

(2) *$\mathbb{M}$ is representation tame if and only if $\mathfrak{T} = \mathfrak{T}_0 \cup \mathfrak{T}_1$, where $\mathfrak{T}_0 = \mathfrak{T}(d_1, \dots, d_r \mid d_{r+1}, \dots, d_s)$ for some d_k and*

  (a) *either $\mathfrak{T}_0 = \emptyset$ and $\mathfrak{T}$ is one of the sets*

$$\{(1, 1, 1), (1, 1, 1)\},\ \{(2, 1, 1)\},\ \{(1, 2, 0)\},\ \{(1, 0, 2)\},$$

  (b) *or $\mathfrak{T}_1 = \emptyset$ and* $\max\{S^-(\mathfrak{T}_0), S^+(\mathfrak{T}_0)\} = 4$,
  (c) *or $\mathfrak{T}_1 = \{(1, 1, 1)\}$ and $S^-(\mathfrak{T}_0) = S^+(\mathfrak{T}_0) = 2$.*

(3) *In all other cases $\mathbb{M}$ is representation wild.*

Proof.

(1a) In this case the graph Γ is a disjoint union of 2 graphs of the types A_2, A_3, D_4, B_2, or B_3.
(1b) In this case Γ is of one of the types $A_3, A_4, A_5, D_5, D_6, B_4$, or B_5.
  In other cases Γ is not a disjoint union of Dynkin graphs.
  From now on we only list the cases when $\mathbb{M}$ is not representation finite.
(2a) In these cases Γ is, respectively, of type $\tilde{A}_3$, or $\tilde{B}_2$, or $\tilde{A}_{12}$.
(2b) In this case Γ is a disjoint union of two graphs, where either both are of types $\tilde{D}_4, \tilde{BD}_3, \tilde{B}_2, \tilde{A}_{11}$, or $\tilde{G}_2$ or one is of one of these types while the other is of a type cited in case (1a).

[3] If the field $\Bbbk$ is algebraically closed, hence all $d_k = 1$, this result coincides with that of Ponizovskiĭ [7, n° 5].

(2c) In this case Γ is of type $\tilde{D}_6, \tilde{BD}_5$, or $\tilde{B}_4$.
(3) In all other cases the graph Γ is not a disjoint union of Dynkin and Euclidean graphs.

□

4. Semigroups

We apply the obtained result to representations of finite *Rees matrix semigroups*. Recall [1, Section 3.1] that such semigroup $\mathcal{M}(G,p,q,\mu)$ is given by a finite group G and a matrix μ of size $p\times q$ with coefficients from the group G. The elements of $\mathcal{M}(G,p,q,\mu)$ are $q\times p$ matrices with coefficients from $G^0 = G\sqcup\{0\}$ containing at most one non-zero element and the multiplication is defined by the rule $a \cdot b = a\mu b$. If the sandwich matrix μ is *regular*, i.e., every column and every row of μ contains a non-zero element, the semigroup $\mathcal{M}(G,p,q,\mu)$ is 0-*simple* (hence *completely* 0-*simple*) and every finite 0-simple semigroup is isomorphic to a Rees matrix semigroup with a regular sandwich matrix [1, Th. 3.5]. We always suppose that the matrix μ is non-zero; otherwise $\mathcal{M}(G,p,q,\mu)$ is just a semigroup with zero multiplication.

Let $\Bbbk$ be a field, $R = \Bbbk G$ and $\mathcal{M} = \mathcal{M}(G,p,q,\mu)$. Obviously, $\Bbbk\mathcal{M} = \mathbb{M}(R,R^p,R^q,\mu)$, where μ is considered as an element of $\mathrm{Mat}(p\times q,R)$ and is identified with an R-homomorphism $R^q \to R^p$. We suppose that $\operatorname{char}\Bbbk \nmid \#(G)$. Then R is semisimple. Namely, let $U_1,U_2,\dots,U_s$ be all irreducible representations of G over $\Bbbk$, $F_k = \mathrm{End}_G\, U_k$, $d_k = \dim_{\Bbbk} F_k$ and $u_k = \dim_{\Bbbk} U_k$. Set $c_k = \frac{u_k}{d_k}$. Then $R \simeq \prod_{k=1}^s R_k$, where $R_k = \mathrm{Mat}(c_k\times c_k,F_k)$, and $\mathrm{Mat}(p\times q,R_k) = \mathrm{Mat}(pc_k\times qc_k,F_k)$. Denote by μ_k the projection of μ onto $\mathrm{Mat}(pc_k\times qc_k,F_k)$ and set $r_k = \mathrm{rk}\,\mu_k$. As $\mu\neq 0$, also all $\mu_k\neq 0$ and the Munn algebra $\Bbbk\mathcal{M}$ is regular. Then $\Bbbk\mathcal{M} \simeq \prod_{k=1}^s \mathbb{M}(F_k,r_k,m_k,n_k)$, where $m_k = pc_k - r_k$ and $n_k = qc_k - r_k$. Theorem 2.11 now implies the following result.

Corollary 4.1. *$\Bbbk\mathcal{M}$ is Morita equivalent to $\prod_{k=1}^s \mathbb{M}(F_k,m_k,n_k)$.*

Remark 4.2. Note that $c_k \mid m_k - n_k$ and $\frac{m_k-n_k}{c_k} = p-q$ does not depend on k. In particular, if $m_k = n_k$, or $m_k > n_k$, or $m_k < n_k$ for some k, the same holds for all k.

From Corollary 4.1 and Theorem 3.3, taking into account Remark 4.2, we obtain a classification of representation types of Rees matrix semigroups, in particular, of 0-simple semigroups. In the next theorem we use the just introduced notations.

Theorem 4.3. *Let $\mathcal{M} = \mathcal{M}(G,p,q,\mu)$ be a finite Rees matrix semigroup, $\Bbbk$ be a field such that $\operatorname{char}\Bbbk \nmid \#(G)$. Set $\mathfrak{T}(\mathcal{M}) = \{(d_k,m_k,n_k) \mid (m_k,n_k)\neq(0,0)\}$.*

(1)[4] $\mathcal{M}$ is representation finite over the field $\Bbbk$ if and only if

(a) either $\mathfrak{T} = \{(1,1,1)\}$
(b) or $\#(G)\le 3$ and $\mathfrak{T}$ contains either only triples $(d_k,1,0)$ or only triples $(d_k,0,1)$.

(2) $\mathcal{M}$ is representation tame over the field $\Bbbk$ if and only if

(a) either $\mathfrak{T}(\mathcal{M}) = \{(1,1,1),(1,1,1)\}$, or $\mathfrak{T}(\mathcal{M}) = \{(2,1,1)\}$,
(b) or $\#(G) = 4$ and $\mathfrak{T}(\mathcal{M})$ contains either only triples $(d_k,1,0)$ or only triples $(d_k,0,1)$,
(c) $G = \{1\}$ and $\mathfrak{T}(\mathcal{M}) = \{(1,2,0)\}$ or $\mathfrak{T}(\mathcal{M}) = \{(1,0,2)\}$.

(3) In all other cases $\mathcal{M}$ is representation wild over the field $\Bbbk$.

Note that in cases (1a) and (2a) $p = q$, while in cases (1b) and (2b) the group G is commutative.

[4] If the field $\Bbbk$ is algebraically closed, hence all $d_k = 1$, this result was proved by Ponizovskiĭ [7].

Remark 4.4. According to Proposition 2.10, the algebra $\Bbbk\mathcal{M}(G, p, q, \mu)$ only depends on the ranks r_k. Elementary transformations of the matrix μ do not change these ranks. Obviously, using them one can obtain a matrix μ' such that there is a non-zero element in every row and in every column. Therefore, $\Bbbk\mathcal{M}(G, p, q, \mu) \simeq \Bbbk\mathcal{M}(G, p, q, \mu')$ and $\mathcal{M}(G, p, q, \mu')$ is a 0-simple semigroup [1, Thm.3.3]. Thus, for every Rees matrix semigroup with a non-zero sandwich matrix there is a 0-simple semigroup with the same representation theory.

If a finite semigroup $\mathcal{S} = \bigvee_{i=1}^{t} \mathcal{M}_i$ is a union of pairwise annihilating Rees matrix semigroups $\mathcal{M}_i$ with common 0, its semigroup algebra $\Bbbk\mathcal{S}$ is a direct product of semigroup algebras $\Bbbk\mathcal{M}_i$ and all of them are Munn algebras. So we obtain the following result.

Theorem 4.5. *Let $\mathcal{S} = \bigvee_{i=1}^{t} \mathcal{M}_i$, where $\mathcal{M}_i = \mathcal{M}(G_i, m_i, n_i, \mu_i)$ are finite Rees matrix semigroups, $\Bbbk$ be a field such that* $\operatorname{char} \Bbbk \nmid \#(G_i)$ *for all i. Denote*

$$T_> = \sum_{m_i > n_i} \#(G_i),$$
$$T_< = \sum_{m_i < n_i} \#(G_i),$$
$$\mathfrak{T}_0 = \bigcup_{m_i \neq n_i} \mathfrak{T}(\mathcal{M}_i),$$
$$\mathfrak{T}_1 = \bigcup_{m_i = n_i} \mathfrak{T}(\mathcal{M}_i)$$

(1)[5] *$\mathcal{S}$ is representation finite over the field $\Bbbk$ if and only if*

- *(a) either $\mathfrak{T}_1 = \emptyset$, $\max\{T_>, T_<\} \le 3$ and all triples from $\mathfrak{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$*
- *(b) or $\mathfrak{T}_1 = \{(1, 1, 1)\}$, $T_> + T_< \le 3$, $\max\{T_>, T_<\} \le 2$ and all triples from $\mathfrak{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$.*

(2) $\mathcal{S}$ is representation tame over the field $\Bbbk$ if and only if

- *(a) $\mathfrak{T}_0 = \emptyset$, $\max\{T_>, T_<\} = 4$ and all triples from $\mathfrak{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$,*
- *(b) or $\mathfrak{T}_1 = \{(1, 1, 1)\}$, $T_> = T_< = 2$ and all triples from $\mathfrak{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$,*
- *(c) or $\mathfrak{T}_0 = \emptyset$ and either $\mathfrak{T}_1 = \{(1, 1, 1), (1, 1, 1)\}$ or $\mathfrak{T}_1 = \{(2, 1, 1)\}$,*
- *(d) or $\mathfrak{T}_1 = \emptyset$ and $\mathfrak{T}_0 = \{(1, 2, 0)\}$ or $\mathfrak{T}_0 = \{(1, 0, 2)\}$.*

In the last case there is a unique index i such that $m_i \neq n_i$ and the corresponding group $G_i = \{1\}$.

(3) In all other cases $\mathcal{S}$ is representation wild over the field $\Bbbk$.

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[5] If the field $\Bbbk$ is algebraically closed, this result easily follows from that of Ponizovskiĭ [7, n°5] and Remark 4.2.

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