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REPRESENTATION TYPE OF ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$

by YURIY DROZD†

(*Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 64 Volodimirska Street, 01033 Kyiv, Ukraine*)

and VOLODYMYR MAZORCHUK‡

(*Department of Mathematics, Uppsala University, PO Box 480, 751 06 Uppsala, Sweden*)

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Abstract

For a semi-simple finite-dimensional complex Lie algebra $\mathfrak{g}$, we classify the representation type of the associative algebras associated with the categories ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ of Harish–Chandra bimodules for $\mathfrak{g}$.

1. Result

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra with a fixed triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, let λ and μ be two dominant and integral (but not necessarily regular) weights, let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$. Denote by χ_λ and χ_μ the central characters of the Verma modules $\Delta(\lambda)$ and $\Delta(\mu)$, respectively. Let further ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ denote the full subcategory of the category of all $U(\mathfrak{g})$-bimodules, which consists of all X satisfying the following conditions [**24**, Kapitel 6]:

(1) X is finitely generated as a bimodule;
(2) X is algebraic, that is, X is a direct sum of finite-dimensional $\mathfrak{g}$-modules with respect to the diagonal action $g \mapsto (g, \sigma(g))$, where σ is the Chevalley involution on $\mathfrak{g}$;
(3) $x(z - \chi_\mu(z)) = 0$, for all $x \in X$ and $z \in Z(\mathfrak{g})$;
(4) for every $x \in X$ and $z \in Z(\mathfrak{g})$, there exists $k \in \mathbb{N}$ such that $(z - \chi_\lambda(z))^k x = 0$.

For regular μ, the category ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ is equivalent to a block of the BGG category $\mathcal{O}$, associated with the triangular decomposition mentioned earlier; see [**7**]. For singular μ, the category ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ is equivalent to a block of the parabolic generalization $\mathcal{O}(\mathfrak{p}, \Lambda)$ of $\mathcal{O}$, studied in [**21**]. Moreover, from [**21**, **29**], it follows that every block of $\mathcal{O}$ and $\mathcal{O}(\mathfrak{p}, \Lambda)$ is equivalent to some ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$. Every ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ is equivalent to the module category of a properly stratified finite-dimensional associative algebra. The regular blocks of ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ can be used to categorify a parabolic Hecke module; see [**26**].

Let $\mathbf{W}$ be the Weyl group of $\mathfrak{g}$ and ρ be the half of the sum of all positive roots of $\mathfrak{g}$. Then, $\mathbf{W}$ acts on $\mathfrak{h}^*$ in the usual way, and we recall the following *dot-action* of $\mathbf{W}$ on $\mathfrak{h}^*$: $w \cdot \nu = w(\nu + \rho) - \rho$.

†E-mail: yuriy@drozd.org, url: http://bearlair.drozd.org/∼yuriy

‡Corresponding author. E-mail: mazor@math.uu.se, url: http://www.math.uu.se/∼mazor

Let $\mathbf{G} \subset \mathbf{W}$ be the stabilizer of λ with respect to the dot-action, and $\mathbf{H} \subset \mathbf{W}$ be the stabilizer of μ with respect to the dot-action. We will say that the triple $(\mathbf{W}, \mathbf{G}, \mathbf{H})$ is associated to ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$. In the present article, we classify the categories ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$ according to their representation type in terms of the associated triples, thus extending the results of [**11**, **22**, **23**]. Let $(\mathbf{W}, \mathbf{G}, \mathbf{H})$ be the triple, associated to ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$, and $(\mathbf{W}, \mathbf{G}', \mathbf{H}')$ be the triple, associated to some ${}^{\infty}_{\lambda'}\mathcal{H}^1_{\mu'}$. Then, from [**6**, Theorem 5.9; **29**, Theorem 11], it follows that ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$ and ${}^{\infty}_{\lambda'}\mathcal{H}^1_{\mu'}$ are equivalent if there exists an automorphism, φ, of the Coxeter system $(\mathbf{W}, S)$, where S is the set of simple reflections associated to our triangular decomposition, such that $\varphi(\mathbf{G}) = \mathbf{G}'$ and $\varphi(\mathbf{H}) = \mathbf{H}'$. The *Coxeter type* of a triple, $(\mathbf{W}, \mathbf{G}, \mathbf{H})$, is the triple, which consists of the Coxeter types of the corresponding components of $(\mathbf{W}, \mathbf{G}, \mathbf{H})$. Note that, in general, the Coxeter type of the triple does not determine the triple in a unique way (for example, one can compare the cases (1e), (2d) and (2e) in the formulation of Theorem 1.1 below). Our main result is the following statement.

THEOREM 1.1 (1) *The category ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$ is of finite type if and only if the Coxeter type of the associated triple is*
- (a) *any and* $\mathbf{W} = \mathbf{G}$*;*
- (b) (A_n, A_{n-1}, A_n), (B_n, B_{n-1}, B_n), (C_n, C_{n-1}, C_n) *or* (G_2, A_1, G_2)*;*
- (c) (A_1, e, e)*;*
- (d) (A_n, A_{n-1}, A_{n-1})*;*
- (e) (A_n, A_{n-1}, A_{n-2}), *where A_{n-2} is obtained from A_n by taking away the first and the last roots;*
- (f) (B_2, A_1, A_1) *or* (C_2, A_1, A_1), *and* $\mathbf{G} = \mathbf{H}$ (*in both cases*)*;*
- (g) (B_n, B_{n-1}, B_{n-1}) *or* (C_n, C_{n-1}, C_{n-1}), *where* $n \geq 3$*;*
- (h) (A_2, A_1, e).

(2) *The category ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$ is tame if and only if the Coxeter type of the associated triple is*
- (a) $(A_3, A_1 \times A_1, A_3)$, (A_2, e, A_2), (B_2, e, B_2), (G_2, e, G_2), (B_3, A_2, B_3), (C_3, A_2, C_3) *or* (D_n, D_{n-1}, D_n), *where* $n \geq 4$*;*
- (b) (B_2, A_1, A_1) *or* (C_2, A_1, A_1), *and* $\mathbf{G} \neq \mathbf{H}$ (*in both cases*)*;*
- (c) $(A_n, A_{n-1}, A_1 \times A_{n-2})$, $n > 2$*;*
- (d) (A_n, A_{n-1}, A_{n-2}), $n > 2$, *where A_{n-2} is included into A_{n-1} and contains either the first or the last root of A_n;*
- (e) (A_n, A_{n-1}, A_{n-2}), $n > 2$, *where A_{n-2} is not included into A_{n-1};*
- (f) (A_3, A_2, e), (B_2, A_1, e), (C_2, A_1, e).

(3) *In all other cases, the category ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$ is wild.*

For regular μ, Theorem 1.1 gives the classification of the representation type of the blocks of the category $\mathcal{O}$ obtained in [**22**] (see [**11**] for a different proof). Formally, we do not use any results from [**22**; **11**], however, the main idea of our proof is similar to the one of [**11**].

In the case $\mathbf{H} = \mathbf{W}$ (that is, μ is most singular), Theorem 1.1 reduces to the classification of the representation type for the algebra $\mathtt{C}(\mathbf{W}, \mathbf{G})$ of $\mathbf{G}$-invariants in the coinvariant algebra associated to $\mathbf{W}$. This result was obtained in [**23**] and, in fact, our argument in the present article is based on it.

The last important ingredient in the proof of Theorem 1.1, the latter being presented in section 3, is the classification of the representation type of all centralizer subalgebras in the Auslander algebra $\mathbb{A}_n$ of $\Bbbk[x]/(x^n)$. This classification is given in section 2. Two series of centralizer subalgebras, namely those considered in Lemmas 2.7 and 2.8, seem to be rather interesting and non-trivial.

The article finishes with an extension of Theorem 1.1 to the case of a semi-simple Lie algebra $\mathfrak{g}$. This is presented in section 4, where one more interesting tame algebra arises.

We would like to finish the introduction with a remark that just recently a first step towards the classification of the representation type of the blocks of Rocha-Caridi's parabolic analogue $\mathcal{O}_S$ of $\mathcal{O}$ was made in [**8**]. The next step would be to complete this classification and then to classify the representation type of the 'mixed' version of $\mathcal{O}_S$ and $\mathcal{O}(\mathfrak{p}, \Lambda)$. As the results of [**8**] and of the present article suggest, this might give some interesting tame algebras in a natural way.

2. Representation type of the centralizer subalgebras in the Auslander algebra of $\Bbbk[x]/(x^n)$

In the article, we will compose arrows of the quiver algebras from the right to the left. Let $\Bbbk$ be an algebraically closed field. Recall that, according to [**18**], every finite-dimensional associative $\Bbbk$-algebra has either finite tame or wild representation type. In what follows, we will call the latter statement the *Tame and wild theorem*. The algebras, which are not of finite representation type, are said to be of infinite representation type.

Let $A = (A_{\mathrm{ob}}, A_{\mathrm{mor}})$ be a $\Bbbk$-linear category. An A-module, M, is a functor from A to the category of $\Bbbk$-vector spaces. In particular, for $x \in A_{\mathrm{ob}}$ and $\alpha \in A_{\mathrm{mor}}$ we will denote by $M(x)$ and $M(\alpha)$ the images of x and α under M, respectively.

For a positive integer $n > 1$, let $\mathbb{A}_n$ be the algebra given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \cdots \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_{n-1}"] & n \arrow[l, bend left, "b_{n-1}"] \end{tikzcd} \qquad \begin{aligned} a_i b_i &= b_{i+1} a_{i+1}, \quad i = 1, \dots, n-2, \\ a_{n-1} b_{n-1} &= 0. \end{aligned}$$

The algebra $\mathbb{A}_n$ is the Auslander algebra of $\Bbbk[x]/(x^n)$; see, for example, [**14**, Section 7]. For $X \subset \{2, 3, \dots, n\}$, let e_X denote the direct sum of all primitive idempotents of $\mathbb{A}_n$, which corrrespond to the vertexes from $\{1\} \cup X$. Set $\mathbb{A}_n^X = e_X \mathbb{A}_n e_X$. The main result of this section is the following.

THEOREM 2.1 (i) *The algebra $\mathbb{A}_n^X$ has finite representation type if and only if $X \subset \{2, n\}$.*
(ii) *The algebra $\mathbb{A}_n^X$ has tame representation type if and only if either $n > 3$ and $X = \{3\}, \{2, 3\}, \{n-1\}, \{n-1, n\}$, or $n = 4$ and $X = \{2, 3, 4\}$.*
(iii) *The algebra $\mathbb{A}_n^X$ is wild in all other cases.*

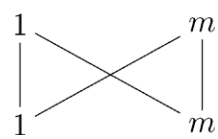
To prove Theorem 2.1, we will need the following lemmas.

LEMMA 2.2 *The algebra $\mathbb{A}_n^{\{m\}}$ has infinite representation type for $m \in \{3, \dots, n-1\}$ and $n \geq 4$.*

Proof. The algebra $\mathtt{A}_n^{\{m\}}$ is given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[loop left, "x"] \arrow[r, bend left, "a"] & m \arrow[l, bend left, "b"] \arrow[loop right, "y"] \end{tikzcd} \qquad \begin{array}{ll} ax = ya, & xb = by, \\ ab = y^{m-1}, & ba = x^{m-1}, \\ y^{n-m+1} = 0, & \end{array} \tag{1}$$

where $x = b_1a_1$, $y = b_ma_m$, $a = a_{m-1}\cdots a_1$, $b = b_1\cdots b_{m-1}$. Modulo the square of the radical $\mathtt{A}_n^{\{m\}}$ gives rise to the following diagram of infinite type.



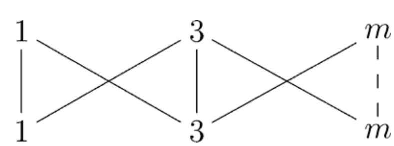
Hence, $\mathtt{A}_n^{\{m\}}$ has infinite representation type as well.

LEMMA 2.3 *The algebra $\mathtt{A}_n^X$ is wild for $X = \{3, m\}$, where $m > 4$.*

Proof. In this case, the algebra $\mathtt{A}_n^X$ is given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[loop left, "x"] \arrow[r, bend left, "a"] & 3 \arrow[l, bend left, "b"] \arrow[loop above, "y"] \arrow[r, bend left, "s"] & m \arrow[l, bend left, "t"] \arrow[loop right, "z"] \end{tikzcd} \qquad \begin{array}{ll} ax = ya, & xb = by, \\ sy = zs, & yt = tz, \\ ab = y^2, & ba = x^2, \\ st = z^{m-3}, & ts = y^{m-3}, \\ z^{n-m+1} = 0, & \end{array}$$

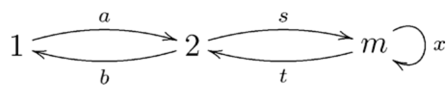
where $x = b_1a_1$, $y = b_3a_3$, $z = b_ma_m$, $a = a_2a_1$, $b = b_1b_2$, $s = a_{m-1}\cdots a_3$, $t = b_3\cdots b_{m-1}$. Note that $z = 0$ if $m = n$. Modulo the square of the radical $\mathtt{A}_n^X$ gives rise to the following diagram:



(where the dashed line disappears in the case $m = n$). With or without the dashed line the diagram is not an extended Dynkin quiver and hence is wild [**13**, **15**]. Hence, $\mathtt{A}_n^X$ is wild as well.

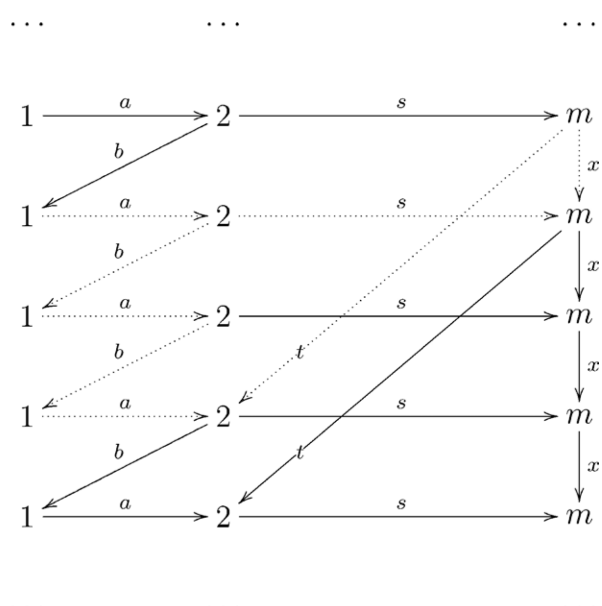
LEMMA 2.4 *The algebra $\mathtt{A}_n^X$ is wild for $X = \{2, n-1\}$ and $n \geq 5$.*

Proof. To make the quivers in the proof below look better we set $m = n - 1$. The algebra $\mathtt{A}_n^X$ is given by the following quiver with relations:



$$sab = xs, \quad abt = tx, \quad st = 0, \quad ts = (ab)^{n-3}, \quad x^2 = 0,$$

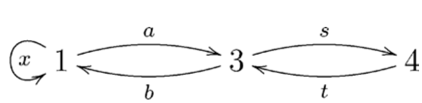
where $a = a_1$, $b = b_1$, $s = a_{n-2} \cdots a_2$, $t = b_2 \cdots b_{n-2}$, $x = b_{n-1}a_{n-1}$. The universal covering of $\mathtt{A}_n^X$ has the wild fragment (a hereditary algebra, whose underlined quiver is not an extended Dynkin diagram, see [**13**, **15**]) indicated by the dotted arrows in the following picture:



Hence, $\mathtt{A}_n^X$ is wild as well.

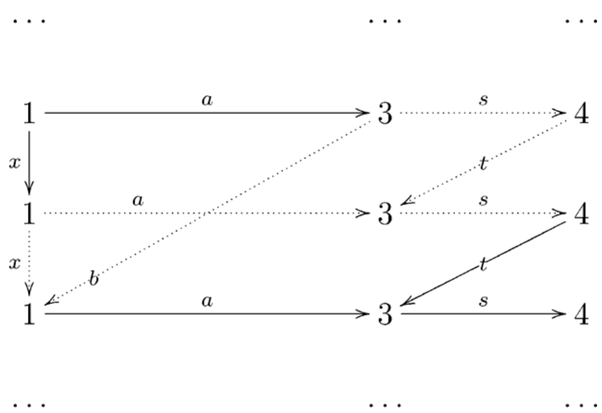
LEMMA 2.5 *The algebra* $\mathtt{A}_5^{\{3,4\}}$ *is wild.*

Proof. The algebra $\mathtt{A}_5^{\{3,4\}}$ is given by the following quiver with relations:



$$ax = tsa, \quad xb = bts, \quad ba = x^2, \quad ab = (ts)^2, \quad (st)^2 = 0,$$

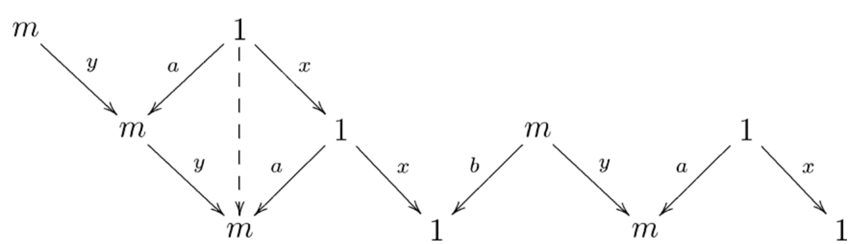
where $a = a_2a_1$, $b = b_1b_2$, $s = a_3$, $t = b_3$, $x = b_1a_1$. The universal covering of $\mathtt{A}_5^{\{3,4\}}$ has the wild fragment (a hereditary algebra, whose underlined quiver is not an extended Dynkin diagram, see [**13**, **15**]) indicated by the dotted arrows in the following picture.



Hence, $\mathtt{A}_5^{\{3,4\}}$ is wild as well.

LEMMA 2.6 *The algebra $\mathtt{A}_n^{\{m\}}$ is wild for $m \in \{4, \dots, n-2\}$ and $n \geq 6$.*

Proof. The algebra $\mathtt{A}_n^{\{m\}}$ is given by (1). We consider its quotient $\mathtt{B}$ given by the additional relations $x^3 = y^3 = ab = ba = 0$ (which is possible because of our restrictions on m and n). Then, the universal covering of $\mathtt{B}$ exists and has the following fragment:



which is wild by [**31**]. This implies that $\mathtt{B}$ and hence $\mathtt{A}_n^{\{m\}}$ is wild.

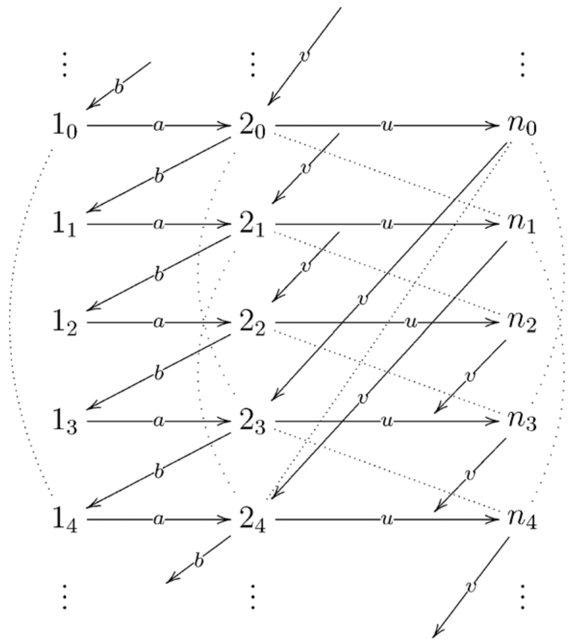
LEMMA 2.7 *The algebra $\mathtt{A}_n^{\{2,n\}}$, $n \geq 2$, is of finite representation type.*

Proof. For $n = 2, 3$, the statement follows from [**14**, section 7]. The algebra $\mathtt{A}_n^{\{2,n\}}$, $n \geq 4$, is given by the following quiver with relations:

$$1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{v}{\overset{u}{\rightleftarrows}} n \quad uv = uab = abv = 0,\ vu = (ab)^{n-2}, \tag{2}$$

where $a = a_1$, $b = b_1$, $u = a_{n-1} \cdots a_2$, $v = b_2 \cdots b_{n-1}$. Note that these relations imply $(ab)^{n-1} = (ba)^n = 0$. The projective $\mathtt{A}_n^{\{2,n\}}$-module $P(1)$ is injective, so we can replace $\mathtt{A}_n^{\{2,n\}}$ by

$\mathtt{A}' = \mathtt{A}_n^{\{2,n\}}/\mathrm{soc}(P(1)) = \mathtt{A}_n^{\{2,n\}}/((ba)^{n-1})$, which has the same indecomposable modules except $P(1)$; see [**19**, Lemma 9.2.2]. So, from now on, we consider the algebra $\mathtt{A}'$, that is, add the relation $(ba)^{n-1} = 0$ to (2). The algebra $\mathtt{A}'$ has a simply connected covering $\widetilde{\mathtt{A}}$, see [**10**], which is the category, given by the following quiver with relations (we show the case $n = 5$, in the general case, the arrow starting at n_k ends at 2_{n-2+k}).



We omit the indices at the arrows a, b, u, v. They satisfy the same relations as in $\mathtt{A}'$, which are shown by the dotted lines. Consider the full subcategory $\mathtt{B}_m$ of $\widetilde{\mathtt{A}}$ with the set of objects $\mathbf{S} = \{1_k, m \le k \le m+n-1;\ 2_k, m \le k \le m+n-2;\ n_m\}$. Let M be an $\widetilde{\mathtt{A}}$-module, N_m be its restriction to $\mathtt{B}_m$, $N_m = \bigoplus_{i=1}^{s} K_i$, where K_i are indecomposable $\mathtt{B}_m$-modules. It is well known that every K_i is completely determined by the subset of objects $\mathbf{S}_i = \{x \mid K_i(x) \neq 0\}$, and if $1_m \in \mathbf{S}_i$, then $1_{m+n-1} \notin \mathbf{S}_i$. Moreover, all $K_i(x)$ with $x \in \mathbf{S}_i$ are one-dimensional and all arrows between these objects correspond to the identity maps. Because $uab = abv = 0$, K_i splits out of the whole module M whenever $\mathbf{S}_i \supseteq \{2_m, 2_{m+n-2}\}$. Suppose that for every integer m, N_m does not contain such direct summands. This implies that $M(vu) = 0$. Therefore, M can be considered as a module over $\overline{\mathtt{A}}$, where $\overline{\mathtt{A}}$ is given by the following quiver:

$$\begin{tikzcd}
\cdots & n' \arrow[d, "v"] & & n' \arrow[d, "v"] & \cdots & n' \arrow[d, "v"] & \cdots \\
\cdots\ 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[d, "u"] & 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[d, "u"] & \cdots \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[d, "u"] & 1\ \cdots \\
\cdots & n & & n & \cdots & n & \cdots
\end{tikzcd}$$

with relations $uv = uab = abv = (ab)^{n-2} = 0$. One easily checks that any indecomposable representation of $\overline{\mathtt{A}}$ is at most of dimension $2n-5$. Hence, $\overline{\mathtt{A}}$ is representation (locally) finite, that is, for every object $x \in \overline{\mathtt{A}}$, there are only finitely many indecomposable representations M with $M(x) \neq 0$. By [**10**], the algebra $\mathtt{A}_n^{\{2,n\}}$ is representation (locally) finite as well, which completes the proof.

LEMMA 2.8 *The algebra* $\mathtt{A}_n^{\{n-1,n\}}$, $n > 3$, *is tame.*

Proof. For $q = n-1$ the algebra $\mathtt{A}_n^{\{q,n\}}$ is given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[loop left, "c"] \arrow[r, bend left, "u"] & q \arrow[l, bend left, "v"] \arrow[r, bend left, "a"] & n \arrow[l, bend left, "b"] \end{tikzcd} \quad c^n = ab = uv = 0, \quad vu = c^{n-2}, \quad cv = vba, \quad uc = bau,$$

where $c = b_1 a_1$, $a = a_q$, $b = b_q$, $u = a_{n-2} \cdots a_1$, $v = b_1 \cdots b_{n-2}$. The projective module $P(1)$ is also injective; hence, using [**19**, Lemma 9.2.2] as it was done in the proof of Lemma 2.7, we can replace $\mathtt{A}$ by $\mathtt{A}' = \mathtt{A}/\mathrm{soc}(P(1)) = \mathtt{A}/(c^q)$. Let M be an $\mathtt{A}'$-module. Choose a basis in $M(1)$ so that the matrix $C = M(c)$ is in the Jordan normal form or, further,

$$M(c) = \bigoplus_{i=1}^{q} J_i \otimes I_{m_i},$$

where J_i is the nilpotent Jordan block of size $i \times i$ and I_{m_i} is the identity matrix of size $m_i \times m_i$ (here, m_i is just the number of Jordan blocks of size i). Thus,

$$J_i \otimes I_m = \begin{pmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & 0 & I_m \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{i \times i}$$

(here, $i \times i$ means i boxes times i boxes, each of size m_i). Choose bases in $M(q)$ and $M(n)$ such that the matrices $A = M(a)$ and $B = M(b)$ are of the form

$$A = \begin{pmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the vertical (horizontal) stripes of A are of the same size as the horizontal (respectively, vertical) stripes of B, and I is the identity matrix; we do not specify these sizes here. Set $r = nq/2$; it is the number of the horizontal and vertical stripes in C. Then $M(u)$ and $M(v)$ can be considered as block matrices: $M(u) = U = (U_k^{ij})_{5\times r}$ and $M(v) = V = (V_{ij}^k)_{r\times 5}$, where $k = 1, \ldots, 5$ correspond to the kth horizontal stripe of B; $i = 1, \ldots, q$, $j = 1, \ldots, i$, and the stripe (ij) corresponds to the jth horizontal stripe of the matrix $J_i \otimes I_{m_i}$ in the decomposition of C. The conditions $uc = bau$ and $cv = vba$ imply that for $i > 1$, the only non-zero blocks U_k^{ij} and V_{ij}^k can be

$$U_k^{ii} \text{ and } U_1^{i,i-1} = U_5^{ii},$$
$$V_{i1}^k \text{ and } V_{i2}^5 = V_{i1}^1.$$

Moreover, we also have $U_5^{11} = V_{11}^1 = 0$. Changing bases in the spaces $M(x)$, $x = 1, q, n$, so that the matrices A, B and C remain of the same form, we can replace U and V, respectively, by $T^{-1}US$ and $S^{-1}VT$, where S, T are invertible matrices of the appropriate sizes such that $SA = AS$ and $TU = UQ$, $QV = VT$ for an invertible matrix Q. We also consider S and T as block matrices: $S = (S_{st}^{ij})_{r\times r}$ and $T = (T_l^k)_{5\times 5}$ with respect to the division of A, B, C. Then, the conditions mentioned earlier can be rewritten as follows:

(1) S_{st}^{ij} can only be non-zero if $i - j < s - t$ or $i - j = s - t$, $s \le i$;
(2) $S_{st}^{ij} = S_{st'}^{ij'}$ if $t - j = t' - j'$;
(3) T is block triangular: $T_l^k = 0$, if $k < l$, and $T_1^1 = T_5^5$;
(4) all diagonal blocks S_{ij}^{ij} and T_k^k are invertible.

Especially for the vertical stripes U^{ii} and for the horizontal stripes U_k of the matrix U, the following transformations are allowed.

(1) Replace U^{ii} by $U^{ii}Z$.
(2) Replace U_k by ZU_k, where $k = 2, 3, 4$.
(3) Replace U_1 and U_5 respectively by ZU_1 and ZU_5.
(4) Replace U^{ii} by $U^{ii} + U^{jj}Z$, where $j < i$.
(5) Replace U_k by $U_k + U_lZ$, where $k < l$.

Here, Z denotes an arbitrary matrix of the appropriate size; moreover, in the cases (1)–(3), it must be invertible. One can easily see that, using these transformations, one can subdivide all blocks U_k^{ii} into subblocks so that each stripe contains at most one non-zero block, which is an identity matrix. Note that the sizes of the horizontal substripes of U_1 and U_5 must be the same. Let Λ^{ii} and Λ_k be, respectively, the sets of the vertical and the horizontal stripes of these subdivisions. Note that all stripes U^{ij} must be subdivided, respectively, to the subdivision of U^{ii}, and recall that $U_1^{i,i-1} = U_5^{ii}$. Especially there is a one-to-one correspondence $\lambda \mapsto \lambda'$ between Λ_5 and Λ_1.

We make the respective subdivision of the blocks of the matrix V, in addition. The condition $UV = 0$ implies that, whenever the λth vertical stripe of U is non-zero ($\lambda \in \Lambda^{ii}$), the λth horizontal stripe of V is zero. The conditions $VU = C^q$ can be rewritten as

$$V_{ij}U^{st} = \begin{cases} I & \text{if } (i, j, s, t) = (q, 1, q, q), \\ 0 & \text{otherwise.} \end{cases}$$

This implies that there are no zero vertical stripes in the new subdivision of $U^{q,q}$. Moreover, if $\lambda \in \Lambda^{ii}$, $\mu \in \Lambda_k$ and the block V_μ^λ is non-zero, then the μth vertical stripe of U is zero if $i \ne q$; if $i = q$, this stripe contains exactly one non-zero block, namely, $U_\lambda^\mu = I$. We denote by $\overline{\Lambda}^{ii}$ and $\overline{\Lambda}_k$ the set of those stripes from Λ^{ii} and Λ_k that are not completely defined by these rules. Let $\lambda \in \Lambda_5$, λ' be the corresponding element of Λ_1. If the blocks U_λ^μ and $U_{\lambda'}^{\mu'}$ are both non-zero, write $\mu \sim \mu'$. Note that there is at most one element μ' such that this holds, and $\mu' \neq \mu$.

One can verify that the sets $\overline{\Lambda}^{ii}$ and $\overline{\Lambda}_k$ can be linearly ordered, so that applying the transformations of the types 1–5 above, we can replace a stripe V^λ by $V^\lambda + V^{\lambda'}Z$ with $\lambda' < \lambda$ and a stripe V_μ by $V_\mu + ZV_{\mu'}$, where $\lambda' < \lambda$, $\mu' < \mu$ for any matrix Z (of the appropriate size). We can also replace V^λ by $V^\lambda Z$, where Z is invertible, and simultaneously replace V_μ and $V_{\mu'}$, where $\mu' \sim \mu$, by ZV_μ and $ZV_{\mu'}$ (if μ' does not exist, just replace V_μ by ZV_μ) with invertible Z. Therefore, we obtain a special sort of the matrix problems considered in [**9**], which is known to be tame. Hence, the algebra $\mathtt{A}_n^{\{q,n\}}$ is tame as well.

Proof of Theorem 2.1. Lemma 2.7 and Lemma 2.2 imply Theorem 2.1(i). The statement of Theorem 2.1(iii) follows from Theorem 2.1(i) and (ii) using the Tame and wild theorem. Hence, we have to prove Theorem 2.1(ii) only.

It is known, see, for example, [**14**], that $\mathtt{A}_n$ has finite representation type for $n \leq 3$, is tame for $n = 4$ and is wild for all other n. This, in particular, proves Theorem 2.1(ii) for $n \leq 4$.

If $n \geq 6$, then from Lemma 2.6, it follows that if $\mathtt{A}_n^X$ is tame, then $X \subset \{2, 3, n-1, n\}$. From Theorem 2.1(i), we know that $X \not\subset \{2, n\}$. From Lemma 2.3, it follows that $\{3, n-1\} \not\subset X$ and $\{3, n\} \not\subset X$. From Lemma 2.4, it follows that $\{2, n-1\} \not\subset X$. This leaves the cases $X = \{n-1, n\}$, $\{n-1\}$, $\{2, 3\}$ and $\{3\}$. In the first two cases, $\mathtt{A}_n^X$ is tame by Lemma 2.8. The algebra $\mathtt{A}_n^{\{2,3\}}$, $n \geq 3$, is given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[r, bend left, "s"] & 3 \arrow[l, bend left, "t"] \end{tikzcd} \qquad \begin{aligned} &ab = ts, \\ &(st)^{n-2} = 0, \end{aligned}$$

where $a = a_1$, $b = b_1$, $s = a_2$, $t = b_2$. For $n \geq 5$, this algebra is tame as a quotient of the classical tame problem from [**27**]. Hence, $\mathtt{A}_n^{\{3\}}$ is tame as well.

For $n = 5$, Lemma 2.5 implies that $\mathtt{A}_n^X$ is wild if $X \supset \{3, 4\}$, Lemma 2.3 implies that $\mathtt{A}_n^X$ is wild if $X \supset \{3, 5\}$ and Lemma 2.4 implies that $\mathtt{A}_n^X$ is wild if $X \supset \{2, 4\}$. We have already shown that the algebra $\mathtt{A}_5^{\{2,3\}}$ is tame, and hence, $\mathtt{A}_5^{\{3\}}$ is tame as well. Finally, that the algebras $\mathtt{A}_5^{\{4,5\}}$ and $\mathtt{A}_5^{\{4\}}$ are tame follows from Lemma 2.8. This completes the proof.

3. Proof of Theorem 1.1

We briefly recall the structure of ${}^\infty_{\ \lambda}\mathcal{H}^1_\mu$. We refer the reader to [**6**, **21**, **25**, **29**] for details. By [**6**, Theorem 5.9], the category ${}^\infty_{\ \lambda}\mathcal{H}^1_0$ is equivalent to the block $\mathcal{O}_\lambda$ of the BGG category $\mathcal{O}$ [**7**]. Let $\mathtt{O}(\mathbf{W}, \mathbf{G})$ denote the basic associative algebra, whose module category is equivalent to $\mathcal{O}_\lambda$. The simple modules in $\mathcal{O}_\lambda$ are in natural bijection with the cosets $\mathbf{W}/\mathbf{G}$ (under this bijection, the coset $\mathbf{G}$ corresponds to the dominant highest weight). For $w \in \mathbf{W}$, let $L(w)$ denote the corresponding simple module in $\mathcal{O}_\lambda$, $P(w)$ be the projective cover of $L(w)$, $\Delta(w)$ be the corresponding Verma module and $I(w)$ be the injective envelope of $L(w)$. Then, [**29**] implies that for the longest element $w_0 \in \mathbf{W}$, one has $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0)) \cong \mathtt{C}(\mathbf{W}, \mathbf{G})$ (recall that this is the subalgebra of $\mathbf{G}$-invariants in the coinvariant algebra associated to $\mathbf{W}$). The left multiplication in $\mathbf{W}$ induces an action of $\mathbf{H}$ on the set $\mathbf{W} \cdot \lambda$. Let $P(\lambda, \mathbf{H})$ denote the direct sum of indecomposable projective modules that correspond to the longest elements in all orbits of this action. The category ${}^\infty_{\ \lambda}\mathcal{H}^1_\mu$ is equivalent, by [**25**], to the module category over $\mathtt{B}(\mathbf{G}, \mathbf{H}) = \mathrm{End}_{\mathcal{O}_\lambda}(P(\lambda, \mathbf{H}))$. From [**29**], it follows that $\mathtt{B}(\mathbf{G}, \mathbf{H})$ depends on $\mathbf{G}$ rather than on λ.

We start with Theorem 1.1(1), that is, with the case of finite representation type.

Note that $P(w_0)$ is always a direct summand of $P(\lambda, \mathbf{H})$. Hence, $\mathtt{C}(\mathbf{W}, \mathbf{G})$ is a centralizer subalgebra of $\mathtt{B}(\mathbf{G}, \mathbf{H})$. In particular, for ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$ to be of finite representation type, $\mathtt{C}(\mathbf{W}, \mathbf{G})$ must be of finite representation type as well. According to [**23**, Theorem 7.2], $\mathtt{C}(\mathbf{W}, \mathbf{G})$ is of finite representation type in the following cases:

(I) $\mathbf{W} = \mathbf{G}$;
(II) $\mathbf{W}$ is of type A_n, and $\mathbf{G}$ is of type A_{n-1};
(III) $\mathbf{W}$ is of type B_n, and $\mathbf{G}$ is of type B_{n-1};
(IV) $\mathbf{W}$ is of type C_n, and $\mathbf{G}$ is of type C_{n-1};
(V) $\mathbf{W}$ is of type G_2, and $\mathbf{G}$ is of type A_1.

Moreover, in all these cases, $\mathtt{C}(\mathbf{W}, \mathbf{G}) \cong \mathbb{C}[x]/(x^r)$, where $r = [\mathbf{W} : \mathbf{G}]$. The last observation and [**21**, Theorem 1] imply that in all the above cases, the category $\mathcal{O}_\lambda$ is equivalent to $\mathtt{A}_r$-mod. In particular, the algebra $\mathtt{B}(\mathbf{G}, \mathbf{H})$ is isomorphic to $\mathtt{A}_r^X$ for appropriate X, and in the notation of section 2, the algebra $\mathtt{C}(\mathbf{W}, \mathbf{G})$ is the centralizer subalgebra, which corresponds to the vertex 1.

The case (I) gives Theorem 1.1(1a). In the cases (II)–(V), it follows from Theorem 2.1(i) that we have the following possibilities for $\mathtt{B}(\mathbf{G}, \mathbf{H})$:

$\mathtt{B}(\mathbf{G}, \mathbf{H})$ *has one simple module*. This implies $\mathbf{W} = \mathbf{H}$ and gives Theorem 1.1(b);

$\mathtt{B}(\mathbf{G}, \mathbf{H})$ *has two simple modules*. These simple modules correspond either to the dominant and the anti-dominant weights in $\mathcal{O}_\lambda$ or to the anti-dominant weight and its neighbour. By a direct calculation, we get the following: the case $r = 2$ gives Theorem 1.1(1c), and the case $r > 2$ gives Theorem 1.1(1d);

$\mathtt{B}(\mathbf{G}, \mathbf{H})$ *has three simple modules*. These simple modules correspond to the following weights in $\mathcal{O}_\lambda$: the anti-dominant one, its neighbour and the dominant one. By a direct calculation, we get the following: the case $r = 3$ gives Theorem 1.1(1h), and the case $r > 3$ gives Theorem 1.1(1e), (1f) and (1g). This proves Theorem 1.1(1).

Let us now proceed with the tame case, that is, with Theorem 1.1(2). If $\mathtt{C}(\mathbf{W}, \mathbf{G})$ is of finite representation type, that is, in the cases (I)–(V), Theorem 2.1(ii) give us the following possibilities for $\mathtt{B}(\mathbf{G}, \mathbf{H})$:

$\mathtt{B}(\mathbf{G}, \mathbf{H})$ *has two simple modules*. These simple modules correspond to the following weights in $\mathcal{O}_\lambda$: either the anti-dominant one and the neighbour of its neighbour or the anti-dominant one and the neighbour of the dominant one. By a direct calculation, we get that these cases lead to Theorem 1.1(2b) and (2c);

$\mathtt{B}(\mathbf{G}, \mathbf{H})$ *has three simple modules*. These simple modules correspond to the following weights in $\mathcal{O}_\lambda$: either the anti-dominant one, its neighbour and the neighbour of its neighbour or the anti-dominant, its neighbour and the dominant one. By a direct calculation, we get that these cases lead to Theorem 1.1(2d) and (2e);

$\mathtt{B}(\mathbf{G}, \mathbf{H})$ *has four simple modules*. In this case, $r = 4$ and a direct calculation gives Theorem 1.1(2f).

The rest (that is, Theorem 1.1(2a)) should correspond to the case when $\mathtt{C}(\mathbf{W}, \mathbf{G})$ is tame. According to [**23**, Theorem 7.2], $\mathtt{C}(\mathbf{W}, \mathbf{G})$ is tame in the following cases:

(VI) $\mathbf{W}$ has rank 2 and $\mathbf{G} = \{e\}$;
(VII) $\mathbf{W}$ is of type A_3 and $\mathbf{G}$ is of type $A_1 \times A_1$;
(VIII) $\mathbf{W}$ is of type B_3 and $\mathbf{G}$ is of type A_2;

(IX) $\mathbf{W}$ is of type C_3 and $\mathbf{G}$ is of type A_2;
(X) $\mathbf{W}$ is of type D_n and $\mathbf{G}$ is of type D_{n-1}.

For $\mathbf{W} = \mathbf{H}$, the cases (VI)–(X) give exactly Theorem 1.1(2a). Let us now show that the rest is wild.

If $\mathbf{W} \neq \mathbf{H}$, then ${}^{\infty}_{\lambda}\mathcal{H}^1_{\mu}$ has at least two non-isomorphic indecomposable projective modules, one of which is $P(w_0)$ and the other one is some $P(w)$. We first consider the cases (VII)–(X). In all these cases, the restriction of the Bruhat order to $\mathbf{W}/\mathbf{G}$ gives the following poset.

$$\begin{tikzcd}
& & & & u_1 \arrow[dr, dash] & & & & \\
w_0 \arrow[r, dash] & w_1 \arrow[r, dash] & \cdots \arrow[r, dash] & w_s \arrow[ur, dash] \arrow[dr, dash] & & v_s \arrow[r, dash] & \cdots \arrow[r, dash] & v_1 \arrow[r, dash] & v_0. \\
& & & & u_2 \arrow[ur, dash] & & & &
\end{tikzcd} \qquad (3)$$

From [**23**, Theorem 7.3], it follows that in all these cases, the algebra $\mathtt{C}(\mathbf{W}, \mathbf{G})$ has two generators.

We consider the centralizer subalgebra $\mathtt{D}(w) = \mathrm{End}_{\mathcal{O}_\lambda}(P(w_0) \oplus P(w))$, and let $Q(w)$ denote the quotient of $\mathtt{D}(w)$ modulo the square of the radical. Recall that the algebra $\mathtt{O}(\mathbf{W}, \mathbf{G})$ is Koszul, see [**4**], and hence, the category $\mathcal{O}_\lambda$ is positively (Koszul) graded, see also [**30**]. Hence, $\mathtt{D}(w)$ is positively graded as well. We are going to show that $\mathtt{D}(w)$ is always wild. We start with the following statement.

LEMMA 3.1 *Let* $w \in \{w_1, \dots, w_s, u_1, u_2, v_0, \dots, v_s\}$. *Then*

$$[P(v_0) : L(w)] = \begin{cases} 1, & w \in \{u_1, u_2, v_0, \dots, v_s, w_0\}, \\ 2, & w \in \{w_1, w_3, \dots, w_s\}, \end{cases}$$

where $[P(v_0) : L(w)]$ *denotes the composition multiplicity.*

Proof. By [**4**], the category $\mathcal{O}_\lambda$ is Koszul dual to the regular block of the corresponding parabolic category of Rocha-Caridi; see [**28**]. Hence, the multiplicity question for $\mathcal{O}_\lambda$ reduces, via the Koszul duality, to the computation of the extensions in the parabolic case. The latter are given by Kazhdan–Lusztig polynomials, and for the algebras of type (VIII)–(X), these multiplicities are computed in [**20**, section 14]. The statement of our lemma follows directly from [**20**, section 14].

As $L(w_0)$ is a simple Verma module, it occurs exactly one time in the composition series of $\Delta(w)$, which gives rise to a morphism, $\alpha\colon P(w_0) \to P(w)$. This morphism has the minimal possible degree (with respect to our positive grading) and hence does not belong to the square of the radical. Further, the unique (now by the BGG reciprocity) occurrence of $\Delta(w)$ in the Verma flag of $P(w_0)$ gives a morphism, $\beta : P(w) \to P(w_0)$, which does not belong to the square of the radical either because it again has the minimal possible degree. Now, we will have to consider several cases.

Case A Assume that $w \in \{v_0, v_1, \dots, v_s\}$. The quiver of $Q(w)$ contains the arrows, corresponding to α and β. Moreover, $Q(w)$ also contains two loops at the point w_0, which correspond to the generators of $\mathtt{C}(\mathbf{W},\mathbf{G})$. Passing, if necessary, to a quotient of $Q(w)$, we obtain the following configuration.

$$\begin{tikzcd}
w_0 \arrow[d, dash, bend right] \arrow[d, dash, bend left] \arrow[dr, dash] & w \arrow[dl, dash] \\
w_0 & w
\end{tikzcd} \quad . \tag{4}$$

Because the underlined diagram is not an extended Dynkin diagram, the configuration is wild; see [**13**, **15**]. This implies that $\mathtt{D}(w)$, and hence, ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ is wild in this case.

Case B Consider now the case $w = u_1$ (the case $w = u_2$ is analogous). Lemma 3.1 implies that in this case, the multiplicity of $L(w)$ in $\Delta(v_0)$ is 1. Hence, from [**2**, Proposition 2.12], it follows that $P(w)$ has simple socle $L(w_0)$, in particular, and $P(w)$ is a submodule of $P(w_0) = I(w_0)$. Injectivity of $P(w_0)$ thus gives a surjection from $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0)) \cong \mathtt{C}(\mathbf{W},\mathbf{G})$ to $\mathrm{End}_{\mathcal{O}_\lambda}(P(w))$. Note that, by [**29**], $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0))$ is the centre of $\mathtt{O}(\mathbf{W},\mathbf{G})$ and hence is central in $\mathtt{B}(\mathbf{G},\mathbf{H})$. We still have the elements α and β as mentioned earlier, which do not belong to the square of the radical. Further, using the embedding $P(w) \hookrightarrow P(w_0)$ one also obtains that α generates $\mathrm{Hom}_{\mathcal{O}_\lambda}(P(w_0), P(w))$ as a $\mathtt{C}(\mathbf{W},\mathbf{G})$-module and β generates $\mathrm{Hom}_{\mathcal{O}_\lambda}(P(w), P(w_0))$ as a $\mathtt{C}(\mathbf{W},\mathbf{G})$-module.

With this notation, $\mathtt{D}(w)$ has the following quiver.

$$\begin{tikzcd}
w_0 \arrow[loop left, "x"] \arrow[r, bend left, "\alpha"] & w \arrow[l, bend left, "\beta"] \arrow[loop right, "y"]
\end{tikzcd} .$$

Note that α is surjective as a homomorphism from $\mathrm{End}_{\mathtt{D}(w)}(P(w_0))$ to $\mathrm{End}_{\mathtt{D}(w)}(P(w))$ because $P(w)$ has simple socle. This and the fact that $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0))$ is central implies the relations $\alpha x = y\alpha$ and $\beta y = x\beta$. Using [**23**, 7.12–7.16], one also easily gets the following additional relations: $y^{s+2} = 0$, $\alpha\beta = cy^{s+1}$ for some $0 \neq c \in \mathbb{C}$, $x\beta\alpha = \beta\alpha x = 0$ and $(\beta\alpha)^2 = x^{2s+3}$. Thus the universal covering of $\mathtt{D}(w)$ has the following fragment (shown for $s = 1$):

$$\begin{tikzcd}
\mathbf{w_0} \arrow[r, "\alpha"] \arrow[d, "x"'] \arrow[dr, dashed, dash] & w \arrow[d, "y"] \arrow[ddl, "\beta"] \\
w_0 \arrow[r, "\alpha"] \arrow[d, "x"'] & w \\
w_0 &
\end{tikzcd} \tag{5}$$

(here, the dashed line indicates the commutativity of the corresponding square). Evaluating the Tits form of this fragment at the point $(1, 2, 2, 2, 2)$, where 1 is placed in the bold vertex, we obtain $-1 < 0$ implying that the fragment (5) is wild; see, for example, [**12**, **17**]. Hence, $\mathtt{D}(w)$ is wild as well.

Case C Assume now that $w = w_i$, $i = 2, \ldots, s-1$. Hence, by Lemma 3.1, the multiplicity of $L(w)$ in $P(v_0)$ is 2. We will need the following lemma.

LEMMA 3.2 *Let A be a basic associative algebra, let e be an idempotent of A and f be a primitive direct summand of e. Assume that there exist two non-isomorphic A-modules M and N satisfying the following properties:*

(1) *both M and N have simple top and simple socles isomorphic to the simple A-module $L^A(f)$, corresponding to f;*
(2) $e\,\mathrm{rad}(M)/\mathrm{soc}(M) = e\,\mathrm{rad}(N)/\mathrm{soc}(N) = 0$.

Then, $\dim \mathrm{Ext}^1_{eAe}(L^{eAe}(f), L^{eAe}(f)) > 1$.

Proof. Recall from [**1**; **2**, section 5] that eAe-mod is equivalent to the full subcategory $\mathcal{M}$ of A-mod, consisting of all Ae-approximations of modules from A-mod. Let M' and N' be the Ae-approximations of M and N, respectively. Both M' and N' are indecomposable because M and N are indecomposable by (1). Then, the eAe-modules eM' and eN' are indecomposable as well, and, because of (1) and (2), both eM' and eN' have length 2 with both composition subquotients isomorphic to the simple eAe-module $L^{eAe}(f)$.

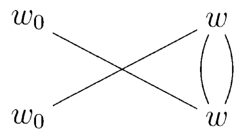
Assume that $eM' \cong eN'$. Then, by [**1**;**2**, section 5], any eAe-isomorphism between eM' and eN' induces an A-isomorphism between M' and N'. From (1), we also have that the canonical maps $N \to N'$ and $M \to M'$ are injective, that is, we have

$$N \hookrightarrow N' \cong M' \hookleftarrow M.$$

From (1), the definition of the Ae-approximation, and the fact that f is a direct summand of e, it follows that the image of N in N' coincides with the trace of the projective module Af in N'. Analogously, the image of M in M' coincides with the trace of the projective module Af in M'. This implies $M \cong N$, a contradiction. The statement follows.

As we are not in the multiplicity-free case, from the Kazhdan–Lusztig theorem, it follows that the quiver of $\mathrm{O}(\mathbf{W}, \mathbf{G})$ contains more arrows than is indicated on the diagram (3). Namely, from the results of [**20**, section 14], we have $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(v_{i-1})) \neq 0$. Note that $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(w_{i+1})) \neq 0$ also follows from the Kazhdan–Lusztig theorem, because w_i and w_{i+1} are neighbours (it follows from [**20**, section 14] as well). Let now $u \in \{v_{i-1}, w_{i+1}\}$. Then, we can fix a non-zero element from $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(u))$. This means that $L(u)$ occurs in degree 1 in the projective module $P(w)$. The module $P(w)$ has a Verma flag, and the above occurrence of $L(u)$ gives rise to an occurrence of $\Delta(u)$ as a subquotient of $P(w)$. Because $L(u)$ is in degree 1 and $\mathcal{O}_\lambda$ is positively graded, we can factor all the Verma subquotients of $P(w)$ except $\Delta(w)$ and $\Delta(u)$ obtaining a non-split extension, $N(u)$ say, of $\Delta(u)$ by $\Delta(w)$. By duality, we have $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(u), L(w)) \neq 0$ as well, and as $w < u$, the module $L(w)$ occurs in degree 2 in the module $N(u)$. This occurrence gives rise to a map from N to the injective module $I(w)$. Let $N'(u)$ denote the image of this map. By construction, the module $N'(u)$ is an indecomposable module of Loewy length 3 with simple top and simple socle isomorphic to $L(w)$. Moreover, $\mathrm{rad}(N'(u))/\mathrm{soc}(N'(u))$ (the latter is considered as an object of $\mathcal{O}_\lambda$) does not contain $L(w)$ as a subquotient because of the quasi-hereditary vanishing $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(w)) = 0$. Because $w \neq w_0, w_1$, all occurrences of $L(w_0)$ in $P(w)$ are in degrees at least 2. Hence, $\mathrm{rad}(N'(u))/\mathrm{soc}(N'(u))$ does not contain $L(w_0)$ as a subquotient either. Finally, we observe that $\mathrm{rad}(N'(v_{i-1}))/\mathrm{soc}(N'(v_{i-1}))$ contains $L(v_{i-1})$ as a subquotient, Whereas $\mathrm{rad}(N'(w_{i+1}))/\mathrm{soc}(N'(w_{i+1}))$ does not contain $L(v_{i-1})$ as a subquotient. This implies that $N'(v_{i-1}) \not\cong N'(w_{i+1})$. Hence, applying Lemma 3.2, we obtain that the quiver of $Q(w)$ contains at least two loops at the point w. This quiver also contains the elements

α and β described earlier. Factoring, if necessary, the extra arrows out, $Q(w)$ thus gives rise to the following configuration.



(6)

Because this is not an extended Dynkin quiver, this configuration is wild, see [**13**, **15**]. Hence, $\mathtt{D}(w)$, and thus, ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ is wild in this case.

Case D Let $w = w_s$. In this case, from [**20**, section 14], we have $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(v_{s-1})) \neq 0$. We also have $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(u_i)) \neq 0$, $i = 1, 2$, as w_s and u_i are neighbours. Hence, the module $P(w)$ contains exactly three copies of $L(w)$ in degree 2: each lying in the top of the radical of some of the Verma modules $\Delta(x)$, $x = u_1, u_2, v_{s-1}$, occurring in degree 1 in the Verma filtration of $P(w)$. Note that $L(w)$ does not occur in degree 1 (see case C). Further, $L(w_0)$ occurs at most one time in degree 1 (this happens if $s = 1$, in which case the occurrence in degree 1 corresponds to the socle of $\Delta(w)$). In any case, because we have three occurrences of $L(w)$ in degree 2, at most one occurrence of $L(w_0)$ in degree 1, and since $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(w_0)) \cong \mathbb{C}$ in the case $s = 1$, mapping the degree-2 occurrences to $I(w)$, we obtain at least two non-isomorphic modules, N_1 and N_2, which have simple top and socle isomorphic to $L(w)$ and no other occurrences of $L(w)$ and $L(w_0)$. Taking into account α and β, from Lemma 3.2, it now follows that some quotient of $Q(w)$ gives rise to the wild configuration (6). Hence, $\mathtt{D}(w)$, and thus, ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ is wild in this case as well.

Case E Finally, let $w = w_1$ and $s > 1$. In this case, both α and β have degree 1. From [**20**, section 14], we have $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), L(v_0)) \neq 0$, which gives two occurrences of $L(w)$ in degree 2 of the module $P(w)$. One of them comes from the subquotient $\Delta(v_0)$ in the Verma flag of $P(w)$. But, v_0 is dominant, and hence, $\Delta(v_0)$ is in fact a submodule. Denote by γ the endomorphism of $P(w)$ of degree 2, which corresponds to this occurrence of $L(w)$ in $\Delta(v_0)$. Because $(\beta\alpha)^2 \neq 0$ by [**23**, 7.12–7.16], it follows that the image of $\alpha\beta$ contains some $L(w_0)$ in degree 3. However, $\Delta(v_0)$ does not contain any $L(w_0)$ in degree 2 (note that $\Delta(v_0)$ itself starts in degree 1 in $P(w)$). Hence, $\alpha\beta$ and γ are linearly independent and thus γ does not belong to the square of the radical. Now, we claim that $\gamma^2 = \gamma\alpha\beta = \alpha\beta\gamma = 0$. The first and the second equalities, that is, $\gamma^2 = \gamma\alpha\beta = 0$, follow from the easy observation that $\Delta(v_0)$ does not have any $L(w)$ in degree $3 = 1 + 2$. The last one, that is, $\alpha\beta\gamma = 0$, follows from the fact that the degree-1 copy of $\Delta(v_0)$ belongs to the kernel of β as $P(w_0)$ does not have any $L(v_0)$ in degree 2. Now, $P(w)$ has two copies of $L(w)$ in the degree $2s$, which correspond to the subquotients $\Delta(u_1)$ and $\Delta(u_2)$ in the Verma flag of $P(w)$. Hence, there should exist an endomorphism of $P(w)$ of degree $2s$, which is linearly independent with $\alpha\beta$. Because $\gamma^2 = \gamma\alpha\beta = \alpha\beta\gamma = 0$, it follows that this new endomorphism does not belong to the square of the radical of $Q(w)$. Taking into account α and β, from Lemma 3.2, it now follows that some quotient of $Q(w)$ gives rise to the wild configuration (6). Hence, $\mathtt{D}(w)$, and thus, ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ is wild in this case as well.

This completes the cases (VII)–(X).

Finally, we consider the case (VI). Let t_1 and t_2 be the simple reflections in $\mathbf{W}$, and θ_{t_1} and θ_{t_2} be translation functors through the t_1 and t_2 walls, respectively. If $\mathbf{H} \neq \mathbf{W}$, then ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ necessarily contains an indecomposable projective module, which corresponds to some w such that $\mathfrak{l}(w_0) - \mathfrak{l}(w) = 2$.

The modules $\theta_{t_1} L(w_0)$ and $\theta_{t_2} L(w_0)$ are indecomposable and have the following Loewy filtrations:

$$\theta_{t_1} L(w_0): \begin{array}{c} L(w_0) \\ L(t_1{}'w_0) \\ L(w_0) \end{array} \qquad \theta_{t_2} L(w_0): \begin{array}{c} L(w_0) \\ L(t_2{}'w_0) \\ L(w_0) \end{array}$$

for some t_1', t_2' such that $\{t_1, t_2\} = \{t_1', t_2'\}$ (the exact values of t_1' and t_2' depend on the type of $\mathbf{W}$). In particular, $\theta_{t_1} L(w_0) \not\cong \theta_{t_2} L(w_0)$, both have simple top and simple socle isomorphic to $L(w_0)$, and both do not contain any subquotient isomorphic to $L(w)$ because $\mathfrak{l}(w_0) - \mathfrak{l}(w) = 2$. Hence, from Lemma 3.2 it follows that the quotient of the corresponding $\mathtt{D}(w)$ modulo the square of the radical gives rise to the wild configuration (4). Hence, $\mathtt{D}(w)$ is wild in this case. This proves Theorem 1.1(2).

To complete the proof we just note that Theorem 1.1(3) follows from Theorem 1.1(1) and Theorem 1.1(2) using the Tame and wild theorem.

4. The case of a semi-simple algebra $\mathfrak{g}$

Theorem 1.1 is formulated for a simple algebra $\mathfrak{g}$. However, in the case of a semi-simple algebra, the result is almost the same. In a standard way, it reduces to the description of the representation types of the tensor products of algebras, described in Theorem 1.1.

THEOREM 4.1 *Let $k > 1$ be a positive integer, and $\mathtt{X}_i$, $i = 1, \ldots, k$, be basic algebras associated to non-semi-simple categories from the list of Theorem* 1.1. *Then, the algebra $\mathtt{X}_1 \otimes \cdots \otimes \mathtt{X}_k$ is never of finite representation type, and it is of tame representation type only in the following two cases*:

(1) $k = 2$, *and both $\mathtt{X}_1$ and $\mathtt{X}_2$ have Coxeter type (A_1, e, A_1);*
(2) $k = 2$, *one of $\mathtt{X}_1$ and $\mathtt{X}_2$ has Coxeter type (A_1, e, A_1) and the other one has Coxeter type (A_1, e, e).*

Proof. The algebra in (1) is isomorphic to $\mathbb{C}[x, y]/(x^2, y^2)$ and hence is tame with well-known representations. Let us thus consider the algebra $\mathtt{X}$ of the case (2). This algebra is given by the following quiver with relations

$$\begin{tikzcd} 1 \arrow[loop left, "x"] \arrow[r, bend left, "u"] & 2 \arrow[l, bend left, "v"] \arrow[loop right, "y"] \end{tikzcd} \qquad x^2 = y^2 = uv = 0, \; ux = yu, \; xv = vy. \tag{7}$$

LEMMA 4.2 *The algebra of* (7) *is tame.*

Proof. This algebra is tame by [**5**], however, because this paper is not easily available and does not contain a complete argument, we prove the tameness of $\mathtt{X}$. Consider the subalgebra $\mathtt{X}' \subset \mathtt{X}$ generated by x, y, u. Its indecomposable representations are as follows.

$$\begin{tikzcd}
e_8 \arrow[d] & e_9 \arrow[d] \arrow[dr] & f_{10} \arrow[d] & e_{10} \arrow[d] \arrow[dr] & & e_{11} \arrow[r] \arrow[d] & f_8 \arrow[d] \\
e_1 & e_2 & f_3 & e_3 & f_6 & e_4 \arrow[r] & f_1
\end{tikzcd} \tag{8}$$

$$\begin{tikzcd}
e_5 & e_6 \arrow[dr] & f_9 \arrow[d] & e_7 \arrow[r] & f_5 & f_{11} \arrow[d] & f_7 \\
 & & f_2 & & & f_4 &
\end{tikzcd} .$$

Here, the elements e_i form a basis of the space corresponding to the vertex 1, the elements f_j form a basis of the space corresponding to the vertex 2, the vertical arrows show the action of x and y and the arrows going from left to right show the action of u. Let M be an $\mathtt{X}$-module. Decompose it as an $\mathtt{X}'$-module. Then, the matrix V describing the action of v divides into the blocks $V_{ij},\ i, j = 1, 2, \dots, 11$, corresponding to the basic elements e_i and f_j from above. Moreover, as $uv = 0$, the blocks V_{ij} can only be non-zero if $i \in \{1, 2, 3, 5, 8\}$; as $xv = vy$, $V_{ij} = 0$ if $i > 4,\ j < 5$ or $i > 7,\ j < 8$ and $V_{ij} = V_{i+7,j+7}$ for $i, j \in \{1, 2, 3, 4\}$. If M' is another $\mathtt{X}$-module, $V' = (V'_{ij})$ is the corresponding block matrix, a homomorphism $M \to M'$ is given by a pair of matrices S, T, where $S : M(1) \to M(1),\ T : M(2) \to M(2)$. Divide them into blocks corresponding to the division of V: $S = (S_{ij}),\ T = (T_{ij}),\ i, j = 1, 2, \dots, 11$. One can easily check that such block matrices define a homomorphism $M \to M'$ if and only if the following conditions hold.

(1) S and T are block triangular, that is, $S_{ij} = 0$ and $T_{ij} = 0$ if $i > j$;
(2) $S_{ij} = S_{i+7,j+7}$ and $T_{ij} = T_{i+7,j+7}$ for $i, j \in \{1, 2, 3, 4\}$;
(3) $S_{ii} = T_{jj}$, if in the list (8), there is an arrow $e_i \to f_j$;
(4) $S_{ij} = T_{kl}$, if in the list (8), there are arrows $e_i \to f_k$ and $e_j \to f_l$;
(5) $S_{ij} = 0$, if $(i, j) \in \{(4, 5), (4, 6), (6, 8), (7, 8), (7, 9)\}$;
(6) $T_{ij} = 0$, if $(i, j) \in \{(3, 5), (4, 5), (4, 6), (7, 8)\}$.

Certainly, S, T define an isomorphism if and only if all diagonal blocks are invertible. In particular, we can replace the part $V_1 = (V_{11}\ V_{12}\ V_{13}\ V_{14})$ by $S_1^{-1} V_1 T_1$, where S_1 is any invertible matrix and $T_1 = (T_{ij}),\ i, j \in \{1, 2, 3, 4\}$ is any invertible block triangular matrix. Therefore, we can suppose that V_1 is of the form

$$\left(\begin{array}{cc|cc|cc|cc}
0 & I^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I^{(2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I^{(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(4)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),$$

where the vertical lines show the division of V_1 into blocks, $I^{(k)}$ denote identity matrices (of arbitrary sizes). Denote the parts of the blocks V_{1j} to the right of $I^{(k)}$ by $V_{1k,j}$ and those to the right of the zero part of V_1 by V_{5j}. Using automorphisms, we can make zero all $V_{11,j}$ and $V_{12,j}$, as well as the blocks $V_{13,j}$ and $V_{14,j}$, zero for $j > 6$. Note that $V_{1j} = V_{8,j+7}$, and we can also make zero all parts of the blocks $V_{1,j+7}$ over the parts $I^{(j)}$ of the blocks $V_{8,j+7}$. Subdivide the blocks of S and T corresponding to this subdivision of V_1. Note that as $S_{22} = T_{99} = T_{33}$, we must also subdivide the blocks S_{2j} into $S_{20,j}$ and $S_{21,j}$ respective to the zero and non-zero parts of V_{13}. Then, the extra conditions for the new blocks are

$$S_{21,20} = 0 \quad \text{and} \quad S_{1k,1l} = 0 \quad \text{if } k > l.$$

Therefore, we get a matrix problem considered in [**9**]. It is described by the semichain

$$\begin{tikzcd} f_5 \arrow[r] & f_6 \arrow[r] \arrow[dr] & f_7 \arrow[dr] & & & \\ & & f_8 \arrow[r] & f_9 \arrow[r] & f_{10} \arrow[r] & f_{11} \end{tikzcd} \tag{9}$$

for the columns, the chain

$$e_5 \to e_3 \to e_{21} \to e_{20} \to e_{15} \to e_{14} \to e_{13}$$

for the rows, and the unique equivalence $e_3 \sim f_6$. This matrix problem is tame; hence, the algebra $\mathtt{X}$ is tame as well.

If $k > 2$, then each of $\mathtt{X}_1$, $\mathtt{X}_2$ and $\mathtt{X}_3$ has at least one projective module with non-trivial endomorphism ring and thus $\mathtt{X}_1 \otimes \mathtt{X}_2 \otimes \mathtt{X}_3$ contains a centralizer subalgebra, which surjects onto $\mathbb{C}[x, y, z]/(x, y, z)^2$. The latter algebra is wild by [**16**], and hence $\mathtt{X}$ is wild.

If $k = 2$, but neither of the conditions (1), (2) is satisfied, then one of the algebras $\mathtt{X}_1$ and $\mathtt{X}_2$ has a projective module, whose endomorphism algebra surjects onto $\mathbb{C}[x]/(x^3)$, and the other one has a projective module, whose endomorphism algebra surjects onto $\mathbb{C}[y]/(y^3)$. Hence, there is a centralizer subalgebra in $\mathtt{X}$, which surjects onto $\mathbb{C}[x, y]/(x^3, y^2)$, the latter being wild by [**16**]. This shows that $\mathtt{X}_1 \otimes \mathtt{X}_2$ is wild as well and completes the proof.

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