AUTOMORPHISMS OF INCIDENCE ALGEBRAS

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Abstract. We study automorphisms of the incidence algebra of a finite quasiordered set $\mathbf{M}$. In particular, we describe explicitly the group of outer automorphisms and give a criterion for any automorphism of this algebra to be a product of an inner one and an automorphism of $\mathbf{M}$, which corrects some results of [3].

Let $\mathbf{M} = (\mathbf{M}, \preccurlyeq)$ be a finite quasiordered set, $m = \#(M)$. Recall that its *incidence algebra* $\mathbf{I} = \mathbf{kM}$ over a filed $\mathbf{k}$ is, by definition, the algebra with a basis $\{e_{ij} \mid i, j \in \mathbf{M},\ i \preccurlyeq j\}$ and the multiplication $e_{ij}e_{kl} = \delta_{jk}e_{il}$, where δ is the Kronecker symbol [4][1]. In particular, $e_i = e_{ii}$ are primitive orthogonal idempotents and $\sum_i e_i$ is the unit of the algebra $\mathbf{I}$. Moreover, $i \preccurlyeq j$ in $\mathbf{M}$ if and only if $e_i\mathbf{I}e_j \neq 0$. We are going to study the automorphism group $\operatorname{Aut}\mathbf{I}$ of this algebra, more precisely, the group of *outer* automorphisms $\operatorname{Out}\mathbf{I} = \operatorname{Aut}\mathbf{I}/\operatorname{Inn}\mathbf{I}$, where $\operatorname{Inn}\mathbf{I}$ is the subgroup of *inner* automorphisms, i.e. those of the form $\sigma_a : x \mapsto a^{-1}xa$ for an invertible element $a \in \mathbf{I}$.

If τ is an automorphism of the quasiordered set $\mathbf{M}$, it induces an automorphism of $\mathbf{I}$ mapping e_{ij} to $e_{\tau i,\tau j}$. We denote the latter by τ too. Thus $\operatorname{Aut}\mathbf{I} \supseteq \operatorname{Aut}\mathbf{M}$. Let $\sim$ be the equivalence relation associated with the quasiorder $\preccurlyeq$, i.e. $i \sim j$ means that $i \preccurlyeq j$ and $j \preccurlyeq i$, and $\tilde{\mathbf{M}} = \{M_1, M_2, \dots, M_s\}$ be the set of equivalence classes with respect to $\sim$. It is obvious that $i \sim j$ if and only if e_i and e_j are conjugate in $\mathbf{I}$. We also write $i \prec j$ if $i \preccurlyeq j$ but $j \not\preccurlyeq i$, and $M_k \leqslant M_l$ if $i \preccurlyeq j$ for some (then any) elements $i \in M_k,\ j \in M_l$. Then $\leqslant$ is an *order* on $\tilde{\mathbf{M}}$.

The subspace $\mathbf{J} = \langle e_{ij} \mid i \prec j\rangle$ is the radical of the algebra $\mathbf{I}$ and $\mathbf{S} = \langle e_{ij} \mid i \sim j\rangle$ is the semisimple part of $\mathbf{I}$, i.e. it is a subalgebra such that $\mathbf{S} \oplus \mathbf{J} = \mathbf{I}$. Moreover, if $\#(M_k) = m_k$, then $\mathbf{S} = \prod_{k=1}^{s} \mathbf{S}_k$, where $\mathbf{S}_k = \langle e_{ij} \mid i, j \in M_k\rangle \simeq \operatorname{Mat}(m_k, \mathbf{k})$. Let $E_k = \sum_{i \in M_k} e_i$ for every equivalence class M_k. Then E_k are central idempotents in $\mathbf{S}$, namely E_k is the unit element of the component $\mathbf{S}_k$. We always suppose that the numeration $M_1, M_2, \dots, M_s$ of the equivalence classes from $\tilde{\mathbf{M}}$ is such that $M_k \leqslant M_l$ implies $k \leqslant l$. Then $\mathbf{I}$ can be considered as the ring of $s \times s$ triangular matrices (a_{kl}), where $a_{kl} \in E_k\mathbf{I}E_l$ $(k, l = 1, \dots, s)$. It easily implies

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[1]Such incidence algebras coincide with *minimal algebras* in the sense of [1, Exercise 3.8].

Proposition 1. $\operatorname{Inn}\mathbf{I}\cap\operatorname{Aut}\mathbf{M}=\operatorname{Inn}\mathbf{M}$, *where*

$$\operatorname{Inn}\mathbf{M}=\{\tau\in\operatorname{Aut}\mathbf{M}\mid\tau(M_k)=M_k\text{ for all }k=1\dots,s\}.$$

In particular, if $\preccurlyeq$ *is an* order, *then* $\operatorname{Inn}\mathbf{I}\cap\operatorname{Aut}\mathbf{M}=\{\mathrm{id}\}$, *thus* $\operatorname{Aut}\mathbf{I}$ *contains the semidirect product* $\operatorname{Inn}\mathbf{I}\rtimes\operatorname{Aut}\mathbf{M}$.

We set $\operatorname{Out}\mathbf{M}=\operatorname{Aut}\mathbf{M}/\operatorname{Inn}\mathbf{M}$.

Proposition 2. $\operatorname{Out}\mathbf{M}$ *is isomorphic to the subgroup of the group* $\operatorname{Aut}\tilde{\mathbf{M}}$ *consisting of all automorphisms* τ *such that* $\#(\tau M_k)=\#(M_k)$ *for all* $k=1,\dots,s$. *Especially, if* $\preccurlyeq$ *is an* order, $\tilde{\mathbf{M}}=\mathbf{M}$ *and* $\operatorname{Out}\mathbf{M}=\operatorname{Aut}\mathbf{M}$. (Further we always identify $\operatorname{Out}\mathbf{M}$ with this subgroup.)

Proof. Every automorphism σ of $\mathbf{M}$ induces an automorphism $\bar\sigma$ of $\tilde{\mathbf{M}}$. Moreover, $\#(\bar\sigma M_k)=\#(\sigma(M_k))=\#(M_k)$ for all k and $\bar\sigma=\mathrm{id}$ if and only if $\sigma\in\operatorname{Inn}\mathbf{M}$. On the other hand, let $\tau\in\operatorname{Aut}\tilde{\mathbf{M}}$ and $\#(\tau M_k)=\#(M_k)$ for all k. We fix a bijection $\beta_k:M_k\to\tau M_k$ for every k and define $\sigma\in\operatorname{Aut}\mathbf{M}$ setting $\sigma i=\beta_k i$ if $i\in M_k$. It is evident that σ is indeed an automorphism of $\mathbf{M}$ and $\bar\sigma=\tau$. □

Recall that the ordered set $\tilde{\mathbf{M}}$ defines a simplicial complex, which we also denote by $\tilde{\mathbf{M}}$. Its n-simplicies are sequences $(x_0,x_1,\dots,x_n)$, where $x_i\in\tilde{\mathbf{M}}$ and $x_0\prec x_1\prec\dots\prec x_n$. Thus, the groups of *cochains, cocycles, coboundaries* and *cohomologies* of this complex with the values in any abelian group G are defined, which we denote, respectively, by $\mathrm{C}^n(\tilde{\mathbf{M}},G)$, $\mathrm{Z}^n(\tilde{\mathbf{M}},G)$, $\mathrm{B}^n(\tilde{\mathbf{M}},G)$ and $\mathrm{H}^n(\tilde{\mathbf{M}},G)$.

Proposition 3. *Let* $\operatorname{Sid}\mathbf{I}=\{\psi\in\operatorname{Aut}\mathbf{I}\mid\psi|_{\mathbf{S}}=\mathrm{id}\}$. *There is a natural isomorphism* $\operatorname{Sid}\mathbf{I}\simeq\mathrm{Z}^1(\tilde{\mathbf{M}},\mathbf{k}^\times)$, *which maps* $\operatorname{Sid}\mathbf{I}\cap\operatorname{Inn}\mathbf{I}$ *onto* $\mathrm{B}^1(\tilde{\mathbf{M}},\mathbf{k}^\times)$.

Proof. If $\psi\in\operatorname{Sid}\mathbf{I}$, then $\psi(e_i)=e_i$ for all i. Since $e_i\mathbf{I}e_j=\langle e_{ij}\rangle$ if $i\preccurlyeq j$, it implies that $\psi(e_{ij})=\gamma(i,j)e_{ij}$ for some $\gamma(i,j)\in\mathbf{k}^\times$. Moreover, if $i\sim i'$ and $j\sim j'$, then $e_{i'j'}=e_{i'i}e_{ij}e_{jj'}$, where both $e_{i'i}$ and $e_{jj'}$ belong to $\mathbf{S}$, wherefrom $\gamma(i,j)=\gamma(i',j')$. Hence ψ can be identified with the cochain $\gamma\in\mathrm{C}^1(\tilde{\mathbf{M}},\mathbf{k}^\times)$. If $i\prec j\prec k$, then $e_{ij}e_{jk}=e_{ik}$, wherefrom $\gamma(i,j)\gamma(j,k)=\gamma(i,k)$, which means that γ is actually a cocycle. On the other hand, every cocycle $\gamma\in\mathrm{Z}^1(\tilde{\mathbf{M}},\mathbf{k}^\times)$ induces an automorphism $\psi\in\operatorname{Sid}\mathbf{M}$ such that $\psi(e_{ij})=\gamma(k,l)e_{ij}$ for $i\in M_k,\ j\in M_l,\ k\prec l$, and $\psi(e_{ij})=e_{ij}$ for $i\sim j$.

Suppose that $\psi\in\operatorname{Sid}\mathbf{I}\cap\operatorname{Inn}\mathbf{I}$, $\psi=\sigma_a$, where a is presented by a matrix (a_{kl}) as above. Since $\sigma_a(b)=b$, i.e. $ab=ba$ for all $b\in\mathbf{S}$, one easily sees that $a_{kl}=0$ for $k\neq l$ and $a_{kk}=\lambda(k)E_k$. Therefore, $\psi(e_{ij})=\lambda(k)^{-1}\lambda(l)$ if $i\in M_k,\ j\in M_l$, i.e. the corresponding cocycle is actually a coboundary. On the contrary, if $\gamma\in\mathrm{Z}^1(\tilde{\mathbf{M}},\mathbf{k}^\times)$ is a coboundary, $\gamma(k,l)=\lambda(k)^{-1}\lambda(l)$ for some $\lambda\in\mathrm{C}^0(\tilde{\mathbf{M}},\mathbf{k}^\times)$, the corresponding automorphism coincides with σ_a, where $a=\sum_{k=1}^s\lambda(k)E_k$. □

Obviously, the group $\operatorname{Aut}\mathbf{M}$ acts on $\mathrm{C}^n(\tilde{\mathbf{M}},G)$ for any G and the groups of cocycles and coboundaries are stable under this action. Thus $\operatorname{Aut}\mathbf{M}$ acts on $\mathrm{H}^n(\tilde{\mathbf{M}},G)$.

Theorem 4. *The group* $\operatorname{Out}\mathbf{I}$ *is isomorphic to the semidirect product* $\mathrm{H}^1(\tilde{\mathbf{M}},\mathbf{k}^\times)\rtimes\operatorname{Out}\mathbf{M}$.

Proof. Proposition 3 implies that $\mathrm{H}=\mathrm{H}^1(\tilde{\mathbf{M}},\mathbf{k}^\times)$ embeds into $\operatorname{Out}\mathbf{I}$. Obviously, $\operatorname{Out}\mathbf{M}$ normalizes H and $\mathrm{H}\cap\operatorname{Out}\mathbf{M}=\{\mathrm{id}\}$. Let $\phi\in\operatorname{Aut}\mathbf{I}$, $f_i=\phi(e_i)$. Then f_i are primitive idempotents and $\sum_i f_i=1$, hence there is an inner automorphism σ such that $\sigma(f_i)=e_{\tau i}$ for some permutation τ of $\mathbf{M}$ [1, Theorem 3.4.1]. Since $i\preccurlyeq j$ if and only if $e_i\mathbf{I}e_j\neq 0$, τ is an automorphism of $\mathbf{M}$ such that $\tau^{-1}\sigma\phi(e_i)=e_i$ for all $i\in\mathbf{M}$. Therefore, the automorphism $\phi'=\tau^{-1}\sigma\phi$ maps $\mathbf{S}_k$ to $\mathbf{S}_k$ for all k. Since all algebras $\mathbf{S}_k$ are central simple, there is an inner automorphism σ' such that $\sigma'\phi'$ is identical on $\mathbf{S}$, i.e. $\sigma'\phi'\in\operatorname{Sid}\mathbf{I}$ [1, Theorem 4.4.3]. Thus the image of ϕ in $\operatorname{Out}\mathbf{I}$ belongs to $\mathrm{H}\cdot\operatorname{Out}\tilde{\mathbf{M}}$, and $\operatorname{Out}\mathbf{I}=\mathrm{H}\rtimes\operatorname{Out}\mathbf{M}$. □

Let $\mathrm{H}_n(\tilde{\mathbf{M}})=\mathrm{H}_n(\tilde{\mathbf{M}},\mathbb{Z})$. Since $\tilde{\mathbf{M}}$ is a finite simplicial complex, this group is finitely generated and can be effectively (and easily) calculated. Recall that $\mathrm{H}^1(\tilde{\mathbf{M}},G)\simeq\operatorname{Hom}(\mathrm{H}_1(\tilde{\mathbf{M}}),G)$ for any abelian group G [2, Corollary XII.4.6]. Moreover, if $\tilde{\mathbf{M}}$ is connected, $\mathrm{H}_1(\tilde{\mathbf{M}},\mathbb{Z})$ is isomorphic to $\pi(\tilde{\mathbf{M}})/\pi'(\tilde{\mathbf{M}})$, where $\pi(\tilde{\mathbf{M}})$ is the fundamental group of $\tilde{\mathbf{M}}$ and $\pi'(\tilde{\mathbf{M}})$ is its commutant [2, Theorem VIII.7.1]; therefore, $\mathrm{H}^1(\tilde{\mathbf{M}},G)\simeq\operatorname{Hom}(\pi(\tilde{\mathbf{M}}),G)$. Note that $\tilde{\mathbf{M}}$ is connected if and only if $\mathbf{M}$ does not split as $\mathbf{M}=X\cup Y$ so that $i\not\preccurlyeq j$ and $j\not\preccurlyeq i$ for any $i\in X$, $j\in Y$.

In what follows we write H for $\mathrm{H}_1(\tilde{\mathbf{M}})$. For any abelian group G we denote by $\Pi(G)$ the set of all prime numbers p such that G contains an element of order p.

Corollary 5. *The following conditions are equivalent:*

(1) $\operatorname{Aut}\mathbf{I}=\operatorname{Inn}\mathbf{I}\cdot\operatorname{Aut}\mathbf{M}$.

(2) *Either* $\#(\mathbf{k})=2$ *or the group* H *is finite and* $\Pi(\mathrm{H})\cap\Pi(\mathbf{k}^\times)=\emptyset$.

Note that if $\preccurlyeq$ is an *order* and the condition (2) holds, then $\operatorname{Aut}\mathbf{M}=\operatorname{Inn}\mathbf{M}\rtimes\operatorname{Aut}\mathbf{M}$.

Remark. This corollary shows that Theorem 1 of [3] is not correct, since it implies, in particular, that $\operatorname{Aut}\mathbf{I}=\operatorname{Inn}\mathbf{I}\cdot\operatorname{Aut}\mathbf{M}$ for any finite ordered set $\mathbf{M}$. An easy example, when $\operatorname{Aut}\mathbf{I}\neq\operatorname{Inn}\mathbf{I}\cdot\operatorname{Aut}\mathbf{M}$, is the ordered set $\mathbf{M}=\{1,2,3,4\mid 1<3,\ 1<4,\ 2<3,\ 2<4\}$, since $\mathrm{H}_1(\tilde{\mathbf{M}})=\mathbb{Z}$ in this case. If we add one more element 5 with $1<5$, we get a counterexample with $\operatorname{Aut}\mathbf{M}=\{\mathrm{id}\}$ and $\operatorname{Aut}\mathbf{I}\neq\operatorname{Inn}\mathbf{I}$ (namely, $\operatorname{Aut}\mathbf{I}/\operatorname{Inn}\mathbf{I}\simeq\mathbf{k}^\times$).

Corollary 6. *Suppose that* $\mathbf{k}$ *is algebraically closed.*

(1) *If* $\operatorname{char}\mathbf{k}=0$, *then* $\operatorname{Aut}\mathbf{I}=\operatorname{Inn}\mathbf{I}\cdot\operatorname{Aut}\mathbf{M}$ *if and only if* $\mathrm{H}=0$.

(2) *If* $\operatorname{char}\mathbf{k} = p > 0$*, then* $\operatorname{Aut}\mathbf{I} = \operatorname{Inn}\mathbf{I}\cdot\operatorname{Aut}\mathbf{M}$ *if and only if* H *is a finite* p*-group.*

Remark. Actually, Corollary 6 remains valid for any field $\mathbf{k}$ such that for every prime $q \neq \operatorname{char}\mathbf{k}$ the cyclotomic polynomial $(x^q - 1)/(x - 1)$ has a root in $\mathbf{k}$.

Corollary 7. *Suppose that* $\tilde{\mathbf{M}}$ *is simply connected, i.e. it is connected and its fundamental group is trivial. Then* $\operatorname{Aut}\mathbf{I} = \operatorname{Inn}\mathbf{I}\cdot\operatorname{Aut}\mathbf{M}$.

For instance, it holds if the Hasse diagram of $\tilde{\mathbf{M}}$ is a tree.

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