COHOMOLOGIES OF THE KLEINIAN 4-GROUP

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Abstract. We calculate explicitly the cohomologies of all G-lattices, where G is the Kleinian 4-group.

Introduction

The calculation of cohomologies of a given group is an important and interesting, but usually a cumbersome problem. So only some cases are known where such calculations were made for a rather wide class of modules. If the group is finite, a special interest is in cohomologies of *lattices*, i.e. G-modules which are finitely generated and torsion free as groups. They are of importance, for example, in the theory of crystallographic groups and of Chernikov groups. Certainly, if we want to know all cohomologies of lattices, one would like to have a classification of G-lattices. In the case of finite p-groups, such classification is only known for cyclic groups of order p and p^2 for a prime p [4, § 34B, § 34C], for the cyclic group of order 8 [11] and for the Kleinian 4-group [8, 9]. In other cases such a classification is *wild*, i.e. includes a description of representations of all finite dimensional algebras. In the first above mentioned case there are only finitely many indecomposable representations, the cohomologies $\hat{H}^n(G,m)$ are periodic of period 2, so the answer can be easily obtained. It was used in [7], where a complete list of Chernikov p-groups with the cyclic top of order p or p^2 was obtained. For the Kleinian group the question becomes much more complicated, since, first, there are infinitely many non-isomorphic indecomposable lattices, and, second, the cohomologies are no more periodic. In this paper we use the description of indecomposable lattices from [9] and some general facts about cohomologies of p-groups and give a complete description of cohomologies of lattices over the Kleinian group.

It is known that for the Kleinian 4-group G every $\mathbb{Z}_2G$-lattice, where $\mathbb{Z}_2$ is the ring of p-adic integers, coincides with the 2-adic completion

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$\hat{M}_2$ of some $\mathbb{Z}G$-lattice M, and $\hat{M}_2 \simeq \hat{N}_2$ if and only if $M \simeq N$ (cf. [8]). As $\hat{H}^n(G, M) = \hat{H}^n(G, \hat{M}_2)$, we only have to consider the $\mathbb{Z}_2G$-lattices.

1. Preliminaries

In this section we establish some results on Auslander–Reiten translate and cohomologies of p-groups.

Let R be a local complete noetherian ring, equidimensional of Krull dimension 1 and without nilpotent elements, R-lat be the category of *R-lattices*, i.e. maximal Cohen–Macaulay R-modules or, equivalently, torsion free R-modules. It is known that R has a *canonical module* $\omega_R \in R$-lat such that the functor $D = \mathrm{Hom}_R(\_, \omega_R)$ is an exact duality on the category R-lat [3, Corollary 3.3.8]. In particular, it induces a bijection on the set R-ind of isomorphism classes of indecomposable R-lattices.

Recall that an *Auslander–Reiten sequence* in R-lat is a non-split exact sequence

$$0 \to M' \xrightarrow{\beta} N \xrightarrow{\alpha} M \to 0, \tag{AR}$$

where M and M' are indecomposable R-lattices and

- for every homomorphism $\xi : L \to M$, where L is an indecomposable R-lattice and ξ is not an isomorphism, there is $\xi' : L \to N$ such that $\xi = \alpha\xi'$;
- for every homomorphism $\eta : M' \to L$, where L is an indecomposable R-lattice and η is not an isomorphism, there is $\eta' : N \to L$ such that $\eta = \eta'\beta$.[1]

In this case M and M' define one another up to an isomorphism. They are denoted: $M' = \tau_R M$ and $M = \tau_R^{-1} M'$. τ_R is called the *Auslander–Reiten translate* for R-lattices. It is known [1] that for every indecomposable R-lattice $M \not\simeq R$ (for every indecomposable R-lattice $M' \not\simeq \omega_R$) there is an Auslander–Reiten sequence (AR), so $\tau_R M$ (respectively, $\tau_R^{-1} M'$) is defined. If R is Gorenstein, i.e. $\omega_R \simeq R$, τ_R induces a bijection on the set R-ind$\setminus\{R\}$.

Let in the Auslander–Reiten sequence (AR) $N = \bigoplus_{i=1}^m N_i$, where N_i are indecomposable, $\alpha_i : N_i \to M$ and $\beta_i : M' \to N_i$ be the components of α and β with respect to this decomposition. The *Auslander–Reiten quiver* of the category R-lat is a quiver whose vertices are the isomorphism classes of indecomposable R-lattices and arrows are given by the following rules:

[1] It is known (see, for instance, [2]) that each of these two conditions implies the other.

- If $M \not\simeq R$, the arrows ending in M are just $\alpha_i : N_i \to M$.
- If $M' \not\simeq \omega_R$, the arrows starting in M are just $\beta_i : M' \to N_i$.
- If $\operatorname{rad} R = \bigoplus_{i=1}^k Q_i$, where Q_i are indecomposable, the arrows ending in R are just the embeddings $\iota_i : Q_i \to R$.
- The arrows starting in ω_R are just $D\iota_i : DQ_i \to \omega_R$.

We denote by ΩM the *syzygy* of M, i.e. the kernel of an epimorphism $\varphi : P \to M$, where P is projective and $\operatorname{Ker} \varphi \subseteq \operatorname{rad} P$. Again, if R is Gorenstein, Ω induces a bijection on the set R-ind$\setminus\{R\}$. So in this case $\Omega^{-1}M$ is well defined for any indecomposable lattice $M \not\simeq R$.

Proposition 1.1. *If R is Gorenstein, $\tau_R M \simeq \Omega M$ for every indecomposable R-lattice $M \not\simeq R$.*

Proof. For an indecomposable R-lattice $M \not\simeq R$, consider a minimal projective presentation $P_1 \xrightarrow{\psi} P_0 \xrightarrow{\varphi} M \to 0$, i.e. an exact sequence, where P_0, P_1 are projective and both $\operatorname{Ker} \varphi \subseteq \operatorname{rad} P_0$ and $\operatorname{Ker} \psi \subseteq \operatorname{rad} P_1$. Then $\operatorname{Ker} \varphi = \Omega M$. Let $N = \operatorname{Ker} \psi$. Set $\operatorname{Tr} M = \operatorname{Coker} D\psi = DN$. Then there is an Auslander–Reiten sequence (AR), where $M' = D\Omega \operatorname{Tr} M$ [1]. Since the exact sequence $DP_0 \xrightarrow{D\psi} DP_1 \to DN \to 0$ is a minimal projective resolution of DN, we have that $\Omega \operatorname{Tr} M = \operatorname{Im} D\psi$ and $D\Omega \operatorname{Tr} M = D(\operatorname{Im} D\psi) = \operatorname{Im} \psi = \operatorname{Ker} \varphi = \Omega M$. □

Recall that if R is Gorenstein and not regular there is a unique ring $A \supset R$ such that $A/R \simeq R/\operatorname{rad} R$, $A \in R$-lat (hence A-lat is a full subcategory of R-lat) and every indecomposable R-lattice $M \not\simeq R$ is an A-lattice. A is called the *minimal overring* of R (see [5]). By duality, $DA \simeq \operatorname{rad} R$.

Proposition 1.2. *Let the ring R be Gorenstein and non-regular, A be the minimal overring of R.*

1. *If $0 \to M' \xrightarrow{\beta} N \xrightarrow{\alpha} M \to 0$ is an Auslander–Reiten sequence in A-lat, it is also an Auslander–Reiten sequence in R-lat.*
2. *Let M be an indecomposable A-lattice. If $M \not\simeq A$, then $\tau_R M = \tau_A M$, and if $M \not\simeq \omega_A$, then $\tau_R^{-1} M = \tau_A^{-1} M$.*

Proof. (1) Let L be an indecomposable R-lattice, $\xi : L \to M$ be not an isomorphism. If $L \not\simeq R$, it is an A-lattice, hence there is a homomorphism $\xi' : L \to N$ such that $\xi = \alpha\xi'$. If $L = R$, such a homomorphism exists since R is projective. Let now $\eta : M' \to L$ be not an isomorphism. Again, if $L \not\simeq R$, there is $\eta' : N \to L$ such that $\eta = \eta'\beta$. If $L = R$, $\operatorname{Im} \beta \in \operatorname{rad} R$ and $\operatorname{rad} R \in A$-lat, which implies again the existence of η'.

(2) is an immediate consequence of (1). □

From now on $R = \mathbb{Z}_p G$, where G is a finite commutative p-group. It is local, Gorenstein, non-regular and $R/\operatorname{rad} R \simeq \mathbb{F}_p$ (with the trivial action). Let A be the minimal overring of R.

Proposition 1.3. $\tau_R A \simeq \omega_A$.

Proof. Otherwise $\tau_R A \simeq M$ for some $M \not\simeq \omega_A$. Hence $A = \tau_R^{-1} M = \tau_A^{-1} M$, which is impossible, since A is a projective A-module. □

Proposition 1.4. *Let $G = \prod_{i=1}^{s} G_i$, where G_i are cyclic groups.*

$$\hat{H}^n(G, A) \simeq \hat{H}^n(G, \mathbb{F}_p) \simeq \begin{cases} \binom{n+s-1}{s-1} \mathbb{F}_p & \text{if } n \geq 0, \\ \binom{s-n-2}{s-1} \mathbb{F}_p & \text{if } n < 0. \end{cases}$$

Proof. The first isomorphism follows from the exact sequence $0 \to R \to A \to \mathbb{F}_p \to 0$, since $\hat{H}^n(G, R) = 0$. The cohomologies of $\mathbb{F}_p$ can be easily calculated using the free resolution of $\mathbb{Z}$ described in [6]. Namely, the latter has a free R-module of rank $\binom{n+s-1}{s-1}$ as the n-th component and the image of the differential d_n is in the radical, which implies the result. □

Proposition 1.5. *Let $M \in \mathrm{CM}(R)$, $M \not\simeq R$. Then*

$$\hat{H}^n(G, M) \simeq \hat{H}^{n+1}(G, \tau_R M) \simeq \hat{H}^{n-1}(G, \tau_R^{-1} M)$$

Proof. It follows immediately from Proposition 1.1. □

Now G is the Kleinian 4-group: $G = \langle a, b \mid a^2 = b^2 = 1,\, ab = ba \rangle$. Then $\mathbb{Z}_2 G$ has 4 irreducible lattices L_{uv}, where $u, v \in \{+, -\}$. Namely, $L_{uv} = \mathbb{Z}_2$ as $\mathbb{Z}_2$-module, a acts as $u1$ and b acts as $v1$.

Proposition 1.6.

$$\hat{H}^n(G, L_{++}) = \begin{cases} (|n|/2 + 1)\mathbb{F}_2 & \text{if } n \neq 0 \text{ is even}, \\ [|n|/2]\mathbb{F}_2 & \text{if } n \text{ is odd}, \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n = 0, \end{cases}$$

and if $(u, v) \neq (+, +)$, then

$$\hat{H}^n(G, L_{uv}) = ([(|n| + 1)/2])\mathbb{F}_2$$

Proof. It is a partial case of [6, Theorem 4.3 & Corollary 4.2].[2] □

[2] Note that there is an obvious misprint in [6, Theorem 4.3]: in the formula (4.4) there must be $|n| - 2$ instead of $|n| - 1$.

2. Cohomologies

In this section $R = \mathbb{Z}_2 G$, where G is the Kleinian 4-group, A is the minimal overring of R. Recall the structure of the Auslander–Reiten quiver of A-lat. It follows from [10] and [9] that this quiver consists of the *preprojective-preinjective* component and *tubes*. The preprojective-preinjective component has the form

$$\begin{tikzcd}[column sep=small]
& L^2_{++} \arrow[ddr] & & L^1_{++} \arrow[ddr] & & L_{++} \arrow[ddr] & & L^{-1}_{++} \arrow[ddr] & & L^{-2}_{++} \arrow[ddr] & & L^{-3}_{++} \arrow[ddr] & \\
& L^2_{+-} \arrow[dr] & & L^1_{+-} \arrow[dr] & & L_{+-} \arrow[dr] & & L^{-1}_{+-} \arrow[dr] & & L^{-2}_{+-} \arrow[dr] & & L^{-3}_{+-} \arrow[dr] & \\
\cdots \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & A^2 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & A^1 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & A \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & A^{-1} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & A^{-2} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \cdots \\
& L^2_{-+} \arrow[ur] & & L^1_{-+} \arrow[ur] & & L_{-+} \arrow[ur] & & L^{-1}_{-+} \arrow[ur] & & L^{-2}_{-+} \arrow[ur] & & L^{-3}_{-+} \arrow[ur] & \\
& L^2_{--} \arrow[uur] & & L^1_{--} \arrow[uur] & & L_{--} \arrow[uur] & & L^{-1}_{--} \arrow[uur] & & L^{-2}_{--} \arrow[uur] & & L^{-3}_{--} \arrow[uur] &
\end{tikzcd}$$

Here M^k denotes $\tau_R^k M$. In particular, $A^1 = \omega_A$. Propositions 1.5 and 1.6 imply the following values of cohomologies of these lattices.

Theorem 2.1.

$$\hat{H}^n(G, A^k) = \begin{cases} (n-k+1)\mathbb{F}_2 & \text{if } n \geq k, \\ (k-n)\mathbb{F}_2 & \text{if } n < k; \end{cases}$$

$$\hat{H}^n(G, L^k_{++}) = \begin{cases} (|n-k|/2+1)\mathbb{F}_2 & \text{if } n-k \neq 0 \text{ is even}, \\ [|n-k|/2]\mathbb{F}_2 & \text{if } n-k \text{ is odd}, \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n = k; \end{cases}$$

and if $(u, v) \neq (+, +)$, *then*

$$\hat{H}^n(G, L^k_{uv}) = [(|n-k|+1)/2]\mathbb{F}_2. \qquad \square$$

To calculate the cohomologies of the lattices that belong to tubes, we need the following considerations. Let $\tilde{A}$ be the integral closure of A in $A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. For any A-lattice M the lattice $\tilde{M} = \tilde{A}M$ is a direct sum $\bigoplus_{u,v \in \{+,-\}} M_{uv}$, where $M_{uv} \simeq (r_{uv}M)L_{uv}$ for some integers $r_{uv}M$. Note that every $\mathbb{Z}_2$-submodule of M_{uv} is actually an $\tilde{A}$-submodule.

Lemma 2.2. *If $M \not\simeq L_{++}$ is an indecomposable R-lattice, then* $\hat{H}^0(G, M) = (r_{++}M)\mathbb{F}_2$

Proof. Recall that $\hat{H}^0(G, M) = M^G/trM$, where M^G is the set of invariants: $M^G = \{\, m \in M \mid gm = m \text{ for all } g \in G \,\}$, and $tr = \sum_{g \in G} g$.

Note that $\operatorname{rad} A = 2\tilde{A}$, hence $2\tilde{M} = \operatorname{rad} M \subset M \subseteq \tilde{M}$. Therefore, $M^G \supseteq 2M_{++}$. Suppose that $m \in M^G \setminus 2M_{++}$. Then $\pi(m) = m' \in M_{++} \setminus 2M_{++}$, where $\pi : M \to M_{++}$ is the projection. Hence $M_{++} = \mathbb{Z}_2 m' \oplus N$ for some submodule $N \subset M_{++}$. Let $\theta : M \to \mathbb{Z}_2 m'$ be the composition of π and the projection $M_{++} \to \mathbb{Z}_2 m'$. There is a homomorphism $\eta : \mathbb{Z}_2 m' \to M$ which maps m' to m. Then $\theta\eta$ is identity on $\mathbb{Z}_2 m'$, so $M \simeq \mathbb{Z}_2 m \oplus \operatorname{Ker}\theta$. As M is indecomposable, $M = \mathbb{Z}_2 m \simeq L_{++}$, which is impossible.

Therefore, $M^G = 2M_{++}$. On the other hand, as $\operatorname{Im}\pi = \tilde{A}\operatorname{Im}\pi$, it coincides with the image of the projection $\tilde{M} \to M_{++}$, which is M_{++}. Hence π is a surjection and its restriction onto trM is also a surjection $trM \to trM_{++} = 4M_{++}$. As $trM \subseteq M^G = 2M_{++}$, it implies that $trM = 4M_{++}$ and $\hat{H}^0(G, M) = 2M_{++}/4M_{++} \simeq (r_{++}M)\mathbb{F}_2$. $\square$

Now recall the structure of tubes [9]. *Homogeneous tubes* are parametrized by irreducible unital polynomials $f \in \mathbb{F}_2[x]$, $f \notin \{x, x-1\}$. The tube $\mathcal{T}^f$ is of the form

$$\begin{tikzcd} T_1^f \arrow[r, shift left] & T_2^f \arrow[l, shift left] \arrow[r, shift left] & T_3^f \arrow[l, shift left] \arrow[r, shift left] & T_4^f \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \end{tikzcd}$$

where $r_{++}(T_k^f) = kd$ for $d = \deg f$ and $\tau_R T_k^f = T_k^f$.

There are also 3 *special tubes* $\mathcal{T}^j$ $(j \in \{2,3,4\})$. They are of the form

$$\begin{tikzcd} T_1^{j1} \arrow[r] & T_2^{j1} \arrow[r] \arrow[dl] & T_3^{j1} \arrow[r] \arrow[dl] & T_4^{j1} \arrow[r] \arrow[dl] & \cdots \arrow[dl] \\ T_1^{j2} \arrow[r] & T_2^{j2} \arrow[r] \arrow[ul] & T_3^{j2} \arrow[r] \arrow[ul] & T_4^{j2} \arrow[r] \arrow[ul] & \cdots \arrow[ul] \end{tikzcd}$$

where $r_{++}T_{2k}^{j1} = r_{++}T_{2k}^{j2} = k$, $r_{++}T_{2k-1}^{j1} = k$, $r_{++}T_{2k-1}^{j2} = k-1$ and $\tau_R T_k^{j1} = T_k^{j2}$, $\tau_R T_k^{j2} = T_k^{j1}$.

Using Lemma 2.2 and Proposition 1.5, we obtain the following result that accomplishes the calculation of cohomologies of lattices over the Kleinian group.

Theorem 2.3.

$$\hat{H}^n(G, T_k^f) = kd\,\mathbb{F}_2, \text{ where } d = \deg f,$$

and for every $j \in \{2,3,4\}$

$$\hat{H}^n(G,T^{ji}_{2k}) = k\,\mathbb{F}_2 \text{ for both } i=1,2,$$
$$\hat{H}^n(G,T^{j1}_{2k-1}) = \begin{cases} k\,\mathbb{F}_2 & \text{if } |n| \text{ is even,} \\ (k-1)\mathbb{F}_2 & \text{if } |n| \text{ is odd,} \end{cases}$$
$$\hat{H}^n(G,T^{j2}_{2k-1}) = \begin{cases} k\,\mathbb{F}_2 & \text{if } |n| \text{ is odd,} \\ (k-1)\mathbb{F}_2 & \text{if } |n| \text{ is even.} \end{cases}$$

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