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On K_0 of locally finite categories [☆]



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ABSTRACT

We calculate the Grothendieck group $K_0(\mathcal{A})$, where \mathcal{A} is an additive category, locally finite over a Dedekind ring and satisfying some additional conditions. The main examples are categories of modules over finite algebras and the stable homotopy category SW of finite CW-complexes. We show that this group is a direct sum of a free group arising from localizations of the category \mathcal{A} and a group analogous to the groups of ideal classes of maximal orders. As a corollary, we obtain a new simple proof of the Freyd’s theorem describing the group $K_0(SW)$.

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Introduction

In this paper we study Grothendieck groups $K_0(\mathcal{A})$ of additive categories \mathcal{A} which are *locally finite* over a Dedekind ring \mathbf{R} . Among them there are categories of lattices over \mathbf{R} -orders as well as the stable homotopy category \mathbf{SW} of polyhedra (finite CW-complexes). Our main tool is the relation of such categories with the categories of projective modules (Lemma 1.1), which allows to study them “piecewise,” since usually only finitely many objects are involved in the considerations. Perhaps, for the first time this idea was explored in the paper [6]. It replaces the usual technique using abelian or triangulated structure and makes the framework more flexible. So our investigation is quite parallel to the theory of integral representations and we have possibility to avail of the well developed technique and results of this theory. Namely, we localize our categories and define *genera*, like in [4, §31]. We establish “local-global correspondence” (Theorem 2.5) and prove analogues of the known results on genera, such as Jacobinski cancellation (Theorem 2.8) and Roiter addition theorem (Theorem 2.6 (2)). It gives a basis for the calculation of $K_0(\mathcal{A})$ in the next section. Under some, not very restrictive, S-condition we show that in the local case (when \mathbf{R} is a discrete valuation ring) this group is free and almost the same as the group of the adically completed category $\hat{\mathcal{A}}$ (the difference is on the level of their rational envelopes), see Theorem 3.6. Finally, in Section 4, under a bit more restrictive Max-condition, we show that in the global case the group \mathcal{A} splits into a free part $K_0(G\mathcal{A})$, which is of a purely local nature, and an analogue of the group of ideal classes $\bigoplus_S \text{Cl}(S)$, where S runs through special objects called S -objects. They are analogues of maximal orders in the theory of integral representations and of spheres in the stable homotopy theory.

As an application, we calculate the group $K_0(A)$, where A is a hereditary order (Example 4.9), and give a new simple proof of the Freyd’s description of $K_0(\mathbf{SW})$ (Example 4.10). Actually, the results of Section 4 can be considered as a far-reaching generalization of the Freyd’s theorem, which was the original incentive of our investigation.

1. Generalities

All categories and functors that we consider are supposed *preadditive* and *small*. An additive category \mathcal{A} is said to be *fully additive* if every idempotent morphism $e : A \rightarrow A$ in it splits, i.e. there are morphisms $A \xrightarrow{\pi} B \xrightarrow{\iota} A$ such that $\iota\pi = e$ and $\pi\iota = 1_B$. Then $A \simeq B \oplus C$, where C is obtained in the same way from the idempotent $1 - e$. If \mathcal{A} is fully additive and $\mathfrak{S} \subseteq \text{ob } \mathcal{A}$, we denote by $\text{add}(\mathfrak{S})$ the smallest full subcategory of \mathcal{A} containing \mathfrak{S} and closed under (finite) direct sums and direct summands. If $\mathfrak{S} = \{A\}$ consists

of one object, we write $\text{add}(A)$ instead of $\text{add}(\mathfrak{S})$. Obviously, if $\mathfrak{S} = \{A_1, A_2, \dots, A_n\}$ is finite, $\text{add}(\mathfrak{S}) = \text{add}(\bigoplus_{i=1}^n A_i)$. We write $A \in B$ if A is a direct summand of B . It is known that every preadditive category \mathcal{A} can be embedded as a full subcategory into a fully additive category $\tilde{\mathcal{A}}$ such that $\text{add}(\text{ob } \mathcal{A}) = \tilde{\mathcal{A}}$. This category is defined up to a natural equivalence, so we denote it by $\text{add } \mathcal{A}$ (see [7, pp.60-61]). We denote by $\text{End}_{\mathcal{A}} A$ the endomorphism ring $\mathcal{A}(A, A)$ (though we write usual $\text{End}_{\Lambda} M$ instead of $\text{End}_{\Lambda\text{-mod}} M$).

To transform the study of categories to that of rings and modules, we use the following result which is actually a variant of the Yoneda’s lemma [12].

Lemma 1.1. *Let \mathcal{A} be a fully additive category, C be an object of \mathcal{A} and $\Lambda = \text{End}_{\mathcal{A}} C$. The map $A \mapsto \mathcal{A}(C, A)$ induces an equivalence $\text{add } C \xrightarrow{\sim} \text{proj-}\Lambda$, the category of finitely generated projective right Λ -modules.*

Proof. Note that every functor $F : \text{add } C \rightarrow \mathcal{C}$, where \mathcal{C} is a fully additive category, is completely determined (up to isomorphism) by its values on C and on endomorphisms of C . As the functor $\mathcal{A}(C, _)$ maps C to Λ and induces an isomorphism $\text{End}_{\mathcal{A}} C \xrightarrow{\sim} \text{End}_{\Lambda} \Lambda \simeq \Lambda$, it only remains to apply the Yoneda’s lemma. \square

Recall the definition of the Grothendieck group $K_0(\mathcal{A})$.

Definition 1.2. Let \mathcal{A} be a fully additive category. The *Grothendieck group* $K_0(\mathcal{A})$ is a quotient of the free abelian group with the basis $\text{ob } \mathcal{A}$ by the subgroup generated by all elements of the form $A - B - C$, where $A \simeq B \oplus C$. We denote by $[A]$ the image of A in $K_0(\mathcal{A})$.

One easily sees that $[A] = [B]$ if and only if there is an object C such that $A \oplus C \simeq B \oplus C$.¹

We denote by $\text{iso } \mathcal{A}$ the set of isomorphism classes of objects from \mathcal{A} and by $\text{ind } \mathcal{A}$ its subset consisting of the classes of *indecomposable* objects A , i.e. such that there are no decompositions $A \simeq B \oplus C$ with $B \neq 0$ and $C \neq 0$. We say that \mathcal{A} is a *category with decomposition* if every object in it is isomorphic to a direct sum of indecomposable objects and a *Krull–Schmidt category* if, moreover, such a decomposition is unique up to isomorphism and permutation of summands.

A morphism $a \in \mathcal{A}(A, B)$ is said to be *essentially nilpotent* if for every sequence b_1, b_2, b_3, \dots of elements of $\mathcal{A}(B, A)$ there is an integer n such that $ab_1ab_2 \dots ab_n a = 0$. The set of all essentially nilpotent morphisms $A \rightarrow B$ is denoted by $\text{nil}(A, B)$. One easily sees that $\text{nil } \mathcal{A} = \bigcup_{A, B} \text{nil}(A, B)$ is an ideal in \mathcal{A} called the *nilradical* of \mathcal{A} . If \mathcal{A} has one object, hence is identified with a ring \mathbf{A} , $\text{nil } \mathcal{A}$ is the *lower nil radical* (or the *prime radical*) of \mathbf{A} [11]. If $\text{nil } \mathcal{A} = 0$, the category \mathcal{A} is called *semiprime*. It is known [11,

¹ It is just the equivalence denoted by $A \equiv B$ in [8] or [3]. Note that we use the notation \equiv for another equivalence, see Corollary 2.7.

10.30] that if a ring A is left or right noetherian, $\text{nil } A$ is the maximal nilpotent ideal of A and contains all left and right nil-ideals. We denote by \mathcal{A}^0 the quotient $\mathcal{A} / \text{nil } \mathcal{A}$. This category has the same objects, but $\mathcal{A}^0(A, B) = \mathcal{A}(A, B) / \text{nil}(A, B)$. Obviously, it is semiprime. We denote by A^0 the object A considered as an object of \mathcal{A}^0 and by α^0 the class of a morphism α in \mathcal{A}^0 . A morphism α is an isomorphism if and only if so is α^0 , and any idempotent from \mathcal{A}^0 can be lifted to an idempotent in \mathcal{A} . So the following results are evident.

Proposition 1.3.

- (1) \mathcal{A}^0 is fully additive if and only if so is \mathcal{A} .
- (2) $A \in B$ if and only if $A^0 \in B^0$.
- (3) $\text{iso } \mathcal{A}^0 = \text{iso } \mathcal{A}$ and $\text{ind } \mathcal{A}^0 = \text{ind } \mathcal{A}$.
- (4) $K_0(\mathcal{A}^0) = K_0(\mathcal{A})$.

Let \mathbf{R} be a commutative ring. Recall that an \mathbf{R} -category is a category \mathcal{A} such that all sets $\mathcal{A}(A, B)$ are \mathbf{R} -modules and the multiplication of morphisms is bilinear. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between \mathbf{R} -categories is called an \mathbf{R} -functor if all induced maps $\mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ are \mathbf{R} -linear.

Definition 1.4. An \mathbf{R} -category \mathcal{A} is said to be *hom-finite* if all \mathbf{R} -modules $\mathcal{A}(A, B)$ are finitely generated and *finite* if, moreover, there is a finite set of objects \mathfrak{S} such that $\text{add } \mathfrak{S} = \mathcal{A}$. In particular, if A is an \mathbf{R} -algebra, the category $\text{proj-}A$ is finite if and only if A is a finitely generated \mathbf{R} -module. Then we say that A is a *finite \mathbf{R} -algebra*.

If the ring \mathbf{R} is noetherian, a hom-finite \mathbf{R} -category is always a category with decomposition, but not necessarily a Krull–Schmidt category. It is a Krull–Schmidt category if \mathbf{R} is a complete local ring [11, 23.3] (though this condition is not necessary).

If \mathbf{S} is a commutative \mathbf{R} -algebra and \mathcal{A} is an \mathbf{R} -category, we define the \mathbf{S} -category $\mathbf{S} \otimes_{\mathbf{R}} \mathcal{A}$ as the category with the same set of objects and the sets of morphisms $(\mathbf{S} \otimes_{\mathbf{R}} \mathcal{A})(A, B) = \mathbf{S} \otimes_{\mathbf{R}} \mathcal{A}(A, B)$, with the obvious multiplication.

If \mathbf{R} is a domain, we denote by $\text{tors } M$ the *torsion submodule* of an \mathbf{R} -module M , i.e. the set of all periodic elements. If \mathcal{A} is an \mathbf{R} -category and A, B are its objects, we set $\text{tors}(A, B) = \text{tors } \mathcal{A}(A, B)$ and $\text{tors } \mathcal{A} = \bigcup_{A, B} \text{tors}(A, B)$. It is an ideal in \mathcal{A} and the quotient category $\mathcal{A} / \text{tors } \mathcal{A}$ is *torsion free*, i.e. all sets of morphisms in it are torsion free. We call an object A *torsion* if $\text{End}_{\mathcal{A}} A$ is torsion and *torsion reduced* if $\text{tors}(A, A) \subseteq \text{nil}(A, A)$. We denote by \mathcal{A}^t (\mathcal{A}^f) the full subcategory of \mathcal{A} consisting of torsion (respectively, torsion reduced) objects. If $\mathcal{A} = \mathcal{A}^t$ ($\mathcal{A} = \mathcal{A}^f$), we say that \mathcal{A} is *torsion* (respectively, *torsion reduced*). We call an \mathbf{R} -category *torsion free* if all modules $\mathcal{A}(A, B)$ are torsion free. If \mathcal{A} is additive, it is enough to check endomorphism algebras $\text{End}_{\mathcal{A}} A$.

Lemma 1.5. *Let \mathbf{R} be a Dedekind domain, Λ be a finite \mathbf{R} -algebra. There are orthogonal idempotents e_0 and $e_1 = 1 - e_0$ in Λ such that, if we denote $\Lambda_{ij} = e_i \Lambda e_j$, then*

- (1) Λ_{ij} is torsion (hence of finite length) if $(i, j) \neq (1, 1)$.
- (2) $\Lambda_{01} \cup \Lambda_{02} \cup \text{tors } \Lambda_{11} \subseteq \text{nil } \Lambda$.
- (3) $\tilde{\Lambda} = \Lambda_{11} / \text{nil } \Lambda_{11}$ is semiprime and torsion free.

We denote $e_0 = e^t$ and $e_1 = e^f$.

Proof. Let $N = \text{nil } \Lambda$ and $\Lambda^0 = \Lambda / N$. It contains no nilpotent ideals. Hence, every minimal left or right ideal of Λ^0 is generated by an idempotent. As $\text{tors } \Lambda^0$ is an ideal, it is semisimple and generated by an idempotent \bar{e}_0 both as left and as right ideal. Let $\bar{e}_1 = 1 - \bar{e}_0$. Then $\bar{e}_i \Lambda^0 \bar{e}_j = 0$ if $i \neq j$, since it is torsion and $\text{tors } \Lambda^0 = e_0 \Lambda^0 e_0$. Therefore, $\Lambda^0 = \Lambda_0^0 \times \Lambda_1^0$, where Λ_0^0 is a semisimple ring and Λ_1^0 is semiprime and torsion free. We take for $e_i \in \Lambda$ ($i = 0, 1$) a representative of \bar{e}_i . Then Λ_{ij} is torsion for $(i, j) \neq (1, 1)$, $\text{tors } \Lambda_{11} \subseteq N$ and $\tilde{\Lambda} \simeq \Lambda_1^0$, which proves (1)–(3). \square

Applying this lemma to the endomorphism rings of objects of a hom-finite \mathbf{R} -category, we obtain the following results.

Corollary 1.6. *Let \mathbf{R} be a Dedekind domain, \mathcal{A} be a fully additive hom-finite \mathbf{R} -category.*

- (1) *If A is torsion reduced and B is torsion, then $\mathcal{A}(A, B) \cup \mathcal{A}(B, A) \subseteq \text{nil } \mathcal{A}$.*
- (2) *Every object $A \in \mathcal{A}$ is a direct sum $A^t \oplus A^f$, where A^t is torsion and A^f is torsion reduced.*
- (3) *If also $B \simeq B^t \oplus B^f$, where B^t is torsion and B^f is torsion reduced, then $A \simeq B$ if and only if $A^t \simeq B^t$ and $A^f \simeq B^f$.*
- (4) *Any indecomposable object is either torsion or torsion reduced.*
- (5) $K_0(\mathcal{A}) = K_0(\mathcal{A}^t) \oplus K_0(\mathcal{A}^f)$.

Proof. (1) If $a : A \rightarrow B$, the left ideal $\mathcal{A}(B, A)a$ of the ring $\text{End}_{\mathcal{A}} A$ is torsion, hence nilpotent, whence $a \in \text{nil}(A, B)$. The proof for $\mathcal{A}(B, A)$ is analogous.

(2) Let $\Lambda = \text{End}_{\mathcal{A}} A$, e^t and e^f be as in Lemma 1.5. They define a decomposition $A = A^t \oplus A^f$, where $\text{End}_{\mathcal{A}} A^t \simeq e^t \Lambda e^t$ is torsion and $\text{End}_{\mathcal{A}} A^f \simeq e^f \Lambda e^f$ is torsion reduced.

(3) follows from (1), (4) follows from (2), and (5) follows from (2) and (3). \square

Note that if A is torsion, the ring $\text{End}_{\mathcal{A}} A$ is artinian. If A is indecomposable, $\text{End}_{\mathcal{A}} A$ have no non-trivial idempotents, hence is local. Therefore, \mathcal{A}^t is a Krull–Schmidt category [2, I.3.6] and $K_0(\mathcal{A}^t)$ is a free group with a basis $\{[A] \mid A \in \text{ind } \mathcal{A}^t\}$. That is why, when studying Grothendieck groups, we can restrict to the case of torsion reduced categories.

2. Localization and genera

From now on \mathbf{R} denotes a Dedekind domain, \mathbf{Q} its field of fractions, $\max \mathbf{R}$ the set of its maximal ideals, $\hat{\mathbf{R}}_{\mathfrak{p}}$ the completion of the local ring $\mathbf{R}_{\mathfrak{p}}$ in the \mathfrak{p} -adic topology and $\hat{\mathbf{Q}}_{\mathfrak{p}}$ the field of fractions of $\hat{\mathbf{R}}_{\mathfrak{p}}$. \mathcal{A} denotes a fully additive hom-finite \mathbf{R} -category, and we write

- $Q\mathcal{A}$ for $\text{add}(\mathbf{Q} \otimes_{\mathbf{R}} \mathcal{A})$,
- $\mathcal{A}_{\mathfrak{p}}$ for $\text{add}(\mathbf{R}_{\mathfrak{p}} \otimes_{\mathbf{R}} \mathcal{A})$,
- $\hat{\mathcal{A}}_{\mathfrak{p}}$ for $\text{add}(\hat{\mathbf{R}}_{\mathfrak{p}} \otimes_{\mathbf{R}} \mathcal{A})$,
- $Q\hat{\mathcal{A}}_{\mathfrak{p}}$ for $\text{add}(\hat{\mathbf{Q}}_{\mathfrak{p}} \otimes_{\mathbf{R}} \mathcal{A}) \simeq \text{add}(\mathbf{Q} \otimes_{\mathbf{R}} \hat{\mathcal{A}}_{\mathfrak{p}})$.

Usually we denote by $QA, A_{\mathfrak{p}}, \hat{A}_{\mathfrak{p}}, Q\hat{A}_{\mathfrak{p}}$ the object A considered as an object of the corresponding categories. Note that the operation add here is indeed necessary. It often happens, for instance, that $\text{ob } \mathcal{A}_{\mathfrak{p}} \neq \{A_{\mathfrak{p}} \mid A \in \text{ob } \mathcal{A}\}$. Following [7], we identify the objects of $\mathcal{A}_{\mathfrak{p}}$ with the pairs $(A_{\mathfrak{p}}, e)$, where $A \in \text{ob } \mathcal{A}$ and e is an idempotent from $\text{End}_{\mathcal{A}_{\mathfrak{p}}} A_{\mathfrak{p}}$. Then the set of morphisms $\mathcal{A}_{\mathfrak{p}}((A_{\mathfrak{p}}, e), (B_{\mathfrak{p}}, f))$ is identified with $e(\mathcal{A}_{\mathfrak{p}}(A_{\mathfrak{p}}, B_{\mathfrak{p}}))f$. The same is valid for the objects and morphisms of $Q\mathcal{A}$, $\hat{\mathcal{A}}_{\mathfrak{p}}$ and $Q\hat{\mathcal{A}}_{\mathfrak{p}}$.²

Definition 2.1. Let A be an object from \mathcal{A} ,

$$G(A) = \{ B \in \text{ob } \mathcal{A} \mid B_{\mathfrak{p}} \simeq A_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \max \mathbf{R} \}.$$

We call $G(A)$ the *genus* of A . If $G(B) = G(A)$ (or, the same, $B \in G(A)$), we say that A and B are of the same genus. The cardinality of $G(A)$ is denoted by $g(A)$.

We denote by $\mathbf{R}\text{-lat}$ the category of \mathbf{R} -lattices, i.e. finitely generated torsion free \mathbf{R} -modules. Such a lattice M is always considered as a submodule of the finite dimensional \mathbf{Q} -vector space $QM = \mathbf{Q} \otimes_{\mathbf{R}} M$. If Λ is an \mathbf{R} -algebra we denote by $\Lambda\text{-lat}$ the category of Λ -modules which are \mathbf{R} -lattices as \mathbf{R} -modules and call them Λ -lattices. $\Lambda\text{-lat}$ is a hom-finite fully additive torsion free \mathbf{R} -category. If Λ is itself an \mathbf{R} -lattice, it is called an \mathbf{R} -order. Then it is a subalgebra in the finite dimensional \mathbf{Q} -algebra $Q\Lambda = \mathbf{Q} \otimes_{\mathbf{R}} \Lambda$ and they say that Λ is an \mathbf{R} -order in $Q\Lambda$. An *overring* of Λ is an \mathbf{R} -order Λ' such that $\Lambda \subseteq \Lambda' \subseteq Q\Lambda$. If Λ has no proper overrings, it is called a *maximal order*. An overring that is a maximal order is called a *maximal overring*. If the \mathbf{Q} -algebra $Q\Lambda$ is separable or if $Q\Lambda$ is semisimple and \mathbf{R} is an *excellent ring* [13], Λ has maximal overrings and all of them are Morita equivalent [4].

In what follows we use the results on the “local-global correspondence” from the theory of orders and lattices [4]. Actually we need more refined versions, so we formulate them here.

² In the important case, when the Max-condition 4.1 is satisfied, the objects from $Q\mathcal{A}$ are exactly QA with $A \in \text{ob } \mathcal{A}$, though analogous equality can still be wrong for $\mathcal{A}_{\mathfrak{p}}$ and, all the more, for $\hat{\mathcal{A}}_{\mathfrak{p}}$.

Lemma 2.2. *Let M, N be Λ -lattices such that $N \supseteq M \supseteq aN$ for some non-zero $a \in \mathbf{R}$, $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ are all prime ideals of \mathbf{R} containing a . There are Λ -lattices M_1, M_2, \dots, M_r such that*

- $(M_i)_{\mathfrak{p}_i} = M_{\mathfrak{p}_i}$ and $(M_i)_{\mathfrak{q}} = N_{\mathfrak{q}}$ if $\mathfrak{q} \neq \mathfrak{p}_i$;
- $\bigoplus_{i=1}^r M_i \simeq M \oplus (r - 1)N$.

In particular, if M and N are projective, so are all M_i .

Proof. N/M is an \mathbf{R} -module of finite length and $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ are all prime ideals associated with N/M . Hence there are submodules M_1, M_2, \dots, M_r of N such that

- M_i is \mathfrak{p}_i -primary, i.e. $M_i \supseteq \mathfrak{p}_i^{k_i} N$;
- $\bigcap_{i=1}^r M_i = M$;
- $M_i \not\subseteq M'_i = \bigcap_{j \neq i} M_j$.

(See [13, Sec. 8].) As $(\prod_{j \neq i} \mathfrak{p}_j^{k_j})N \subseteq M'_i$ and $\mathfrak{p}_i^{k_i} + \prod_{j \neq i} \mathfrak{p}_j^{k_j} = \mathbf{R}$, it implies that $N = M_i + M'_i$. As M_i is \mathfrak{p}_i -primary, $(M_i)_{\mathfrak{q}} = N_{\mathfrak{q}}$ for $\mathfrak{q} \neq \mathfrak{p}_i$. Moreover, $M_i/M \simeq N/M'_i$ is annihilated by $\prod_{j \neq i} \mathfrak{p}_j^{k_j}$, hence $(M_i)_{\mathfrak{p}_i} = M_{\mathfrak{p}_i}$. Consider the map $\varphi : \bigoplus_{i=1}^r M_i \rightarrow (r - 1)N$ such that $\varphi(u_1, u_2, \dots, u_r) = (u_1 + u_2, u_2 + u_3, \dots, u_{r-1} + u_r)$. For any $v \in N$ there are $u \in M_1$ and $u' \in M'_1$ such that $u + u' = v$, whence $(v, 0, 0, \dots, 0) = \varphi(u, u', -u', u', \dots, (-1)^r u')$. Just in the same way all components of $(r - 1)N$ are in the image, so φ is surjective. Moreover, $\varphi(u_1, u_2, \dots, u_r) = 0$ means that $u_i = -u_{i-1}$ for $1 < i \leq r$, so this row is of the form $(u, -u, u, \dots, (-1)^{r-1}u)$, where $u \in \bigcap_{i=1}^r M_i = M$. Thus we obtain an exact sequence $0 \rightarrow M \rightarrow \bigoplus_{i=1}^r M_i \rightarrow (r - 1)N \rightarrow 0$. One easily sees that its localization $0 \rightarrow M_{\mathfrak{p}} \rightarrow \bigoplus_{i=1}^r (M_i)_{\mathfrak{p}} \rightarrow (r - 1)N_{\mathfrak{p}} \rightarrow 0$ splits for every prime \mathfrak{p} . Therefore, this sequence splits [14, 3.20] and $\bigoplus_{i=1}^r M_i \simeq M \oplus (r - 1)N$. \square

Lemma 2.3. *Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ be different prime ideals of \mathbf{R} , M_i ($1 \leq i \leq r$) be a $\Lambda_{\mathfrak{p}_i}$ -lattice, N be a Λ -lattice and $QM_i = QN$ for all i . There is a Λ -lattice M such that $M_{\mathfrak{p}_i} = M_i$ for all i and $M_{\mathfrak{q}} = N_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$. If the modules M_1, M_2, \dots, M_r, N are projective, so is M .*

Proof. First suppose that $M_i \subseteq N_{\mathfrak{p}_i}$ for all i . Set $M'_i = N \cap M_i$. Then $(M'_i)_{\mathfrak{p}_i} = M_i$ and $(M'_i)_{\mathfrak{q}} = N_{\mathfrak{q}}$ if $\mathfrak{q} \neq \mathfrak{p}_i$, so M'_i is a \mathfrak{p}_i -primary submodule in N . Set $M = \bigcap_{i=1}^r M'_i$. The same arguments as in the preceding proof show that $M_{\mathfrak{p}_i} = M_i$ and $M_{\mathfrak{q}} = N_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$.

In general situation find a non-zero $a \in \mathbf{R}$ such that $aM_i \subseteq N_{\mathfrak{p}_i}$ for all i . Let $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$ be all prime ideals, different from $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$, that contain a . As we have just proved, there is a lattice M^a such that $M^a_{\mathfrak{p}_i} = aM_i$ ($1 \leq i \leq r$), $M^a_{\mathfrak{q}_j} = aN_{\mathfrak{q}_j}$ ($1 \leq j \leq s$) and $M^a_{\mathfrak{q}} = N_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r, \mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s\}$. Then we can set $M = a^{-1}M^a$.

If M_1, M_2, \dots, M_r, N are projective, then $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$ -projective for all prime \mathfrak{p} , thus M is projective [14, 3.23]. \square

Lemma 2.4. *Let M, N be Λ -lattices, $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ be different prime ideals of \mathbf{R} and homomorphisms $\alpha : M \rightarrow N$ and $\beta_i : M_{\mathfrak{p}_i} \rightarrow N_{\mathfrak{p}_i}$ be given such that $Q\beta_i = Q\alpha$ for all i . There is a unique homomorphism $\beta : M \rightarrow N$ such that $\beta_{\mathfrak{p}_i} = \beta_i$ and $\beta_{\mathfrak{q}} = \alpha_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$.*

Proof. Let $A = \{(u, \alpha(u)) \mid u \in M\}$ be the graph of α , B_i be the graph of β_i . They are submodules in $QM \oplus QN$ and $QA = QB_i$. By Lemma 2.3 there is a lattice B such that $B_{\mathfrak{p}_i} = B_i$ and $B_{\mathfrak{q}} = A_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$. As all projections $A \rightarrow QM$ and $B_i \rightarrow QM$ are monomorphisms, so is the projection $B \rightarrow QM$. Therefore, B is the graph of a homomorphism β such that $\beta_{\mathfrak{p}_i} = \beta_i$ and $\beta_{\mathfrak{q}} = \alpha_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$. As $M = \bigcap_{\mathfrak{p} \in \max \mathbf{R}} M_{\mathfrak{p}}$, β is unique. \square

In what follows \mathcal{A} is a hom-finite fully additive \mathbf{R} -category. One easily checks that

$$\begin{aligned} \text{nil}(QA, QB) &= Q \text{nil}(A, B), \\ \text{nil}(A_{\mathfrak{p}}, B_{\mathfrak{p}}) &= \text{nil}(A, B)_{\mathfrak{p}}, \end{aligned}$$

whence

$$(Q\mathcal{A})^0 = Q\mathcal{A}^0 \text{ and } (\mathcal{A}_{\mathfrak{p}})^0 = (\mathcal{A}^0)_{\mathfrak{p}}.$$

If the ring \mathbf{R} is excellent [13], then also

$$\begin{aligned} \text{nil}(\widehat{A}_{\mathfrak{p}}, \widehat{B}_{\mathfrak{p}}) &= \widehat{\text{nil}(A, B)}_{\mathfrak{p}}, \\ \text{nil}(Q\widehat{A}_{\mathfrak{p}}, Q\widehat{B}_{\mathfrak{p}}) &= Q\widehat{\text{nil}(A, B)}_{\mathfrak{p}}, \end{aligned}$$

whence

$$(\widehat{\mathcal{A}}_{\mathfrak{p}})^0 = \widehat{(\mathcal{A}^0)}_{\mathfrak{p}} \text{ and } (Q\widehat{\mathcal{A}}_{\mathfrak{p}})^0 = Q\widehat{(\mathcal{A}^0)}_{\mathfrak{p}}.$$

These equalities for completions are also valid if \mathcal{A} satisfies the Max-condition, see Proposition 4.3 (1).

Following [13], we call an object A \mathfrak{p} -coprimary if it is torsion and $A_{\mathfrak{q}} = 0$ for $\mathfrak{q} \neq \mathfrak{p}$. Note that, if \mathbf{A} is a finite \mathbf{R} -algebra which is torsion as \mathbf{R} -module, $\mathbf{A} \simeq \prod_{i=1}^r \mathbf{A}_{\mathfrak{p}_i}$ for some prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$. If $\mathbf{A} = \text{End}_{\mathcal{A}} A$, where A is a torsion object, it gives a decomposition $A \simeq \bigoplus_{i=1}^r A_i$, where A_i is \mathfrak{p}_i -coprimary. Such decomposition is unique, since $\mathcal{A}(A, B) = 0$ if A is \mathfrak{p} -coprimary and B is \mathfrak{q} -coprimary for $\mathfrak{p} \neq \mathfrak{q}$. It implies that

$$K_0(\mathcal{A}^t) \simeq \bigoplus_{\mathfrak{p} \in \max \mathbf{R}} K_0(\mathcal{A}^{\mathfrak{p}}),$$

where $\mathcal{A}^{\mathfrak{p}}$ is the subcategory of \mathfrak{p} -coprimary objects. So from now on we can suppose that the category \mathcal{A} is torsion reduced. Then $\text{End}_{\mathcal{A}^0} A^0$ is a semiprimary \mathbf{R} -order for every object A and $\mathcal{A}^0(A^0, B^0)$ is a (right) $\text{End}_{\mathcal{A}^0} A^0$ -lattice.

The next theorem provides a background for the theory of genera.

Theorem 2.5. *Let \mathcal{A} be a fully additive hom-finite \mathbf{R} -category, $\mathfrak{n} = \text{nil } \mathcal{A}$.*

(1) *Let A, B be torsion reduced objects and $A \xrightarrow{\alpha} B \xrightarrow{\beta} A$ be such morphisms that $\beta\alpha \equiv a1_A \pmod{\mathfrak{n}}$ and $\alpha\beta \equiv a1_B \pmod{\mathfrak{n}}$ for some non-zero $a \in \mathbf{R}$. If $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ are all prime ideals of \mathbf{R} such that $a \in \mathfrak{p}_i$, there are torsion reduced objects A_1, A_2, \dots, A_r such that*

- (a) $(A_i)_{\mathfrak{p}_i} \simeq A_{\mathfrak{p}_i}$ and $(A_i)_{\mathfrak{q}} \simeq B_{\mathfrak{q}}$ if $\mathfrak{q} \neq \mathfrak{p}_i$;
- (b) $\bigoplus_{i=1}^r A_i \simeq A \oplus (r - 1)B$.

Note that such an element a exists if and only if $QA \simeq QB$.

(2) *Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ be different prime ideals of \mathbf{R} , A_i ($1 \leq i \leq r$) be a torsion reduced object of $\mathcal{A}_{\mathfrak{p}_i}$, B be a torsion reduced object of \mathcal{A} and $QA_i \simeq QB$ for all i . There is a torsion reduced object A such that $A_{\mathfrak{p}_i} \simeq A_i$ ($1 \leq i \leq r$) and $A_{\mathfrak{q}} \simeq B_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$.*

(3) *Suppose that \mathcal{A} is torsion free. Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ be different prime ideals of \mathbf{R} , $\alpha \in \mathcal{A}(A, B)$ and $\beta_i \in \mathcal{A}_{\mathfrak{p}_i}(A_{\mathfrak{p}_i}, B_{\mathfrak{p}_i})$ be such that $Q\beta_i = Q\alpha$ for all i . There is a morphism $\beta : A \rightarrow B$ such that $\beta_{\mathfrak{p}_i} = \beta_i$ and $\beta_{\mathfrak{q}} = \alpha_{\mathfrak{q}}$ if $\mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$.*

Note that the genera of the objects A_i in (1) and of the object A in (2) are uniquely defined.

Proof. (1) Replacing \mathcal{A} by $(\mathcal{A}^f)^0$ we can suppose that \mathcal{A} is torsion reduced and semiprime, hence torsion free. Let $C = A \oplus B$, $\Lambda = \text{End}_{\mathcal{A}} C$, $M = \mathcal{A}(C, A)$, $N = \mathcal{A}(C, B)$. M and N are projective Λ -lattices and the multiplications by α and β give homomorphisms $M \xrightarrow{\alpha} N \xrightarrow{\beta} M$ such that $(\alpha \cdot)(\beta \cdot) = a1_N$, $(\beta \cdot)(\alpha \cdot) = a1_M$. Hence $\alpha \cdot$ and $\beta \cdot$ are monomorphisms, so we can suppose that $\alpha \cdot$ is an embedding $M \subseteq N$ such that $M \supseteq aN$. Now we are in the situation of Lemma 2.2. Therefore, there are Λ -lattices M_i such that

- $(M_i)_{\mathfrak{p}_i} = M_{\mathfrak{p}_i}$ and $(M_i)_{\mathfrak{q}} = N_{\mathfrak{q}}$ if $\mathfrak{q} \neq \mathfrak{p}_i$;
- $\bigoplus_{i=1}^r M_i \simeq M \oplus (r - 1)N$.

As M and N are projective, so are M_i . By Lemma 1.1, there are objects A_i such that $M_i = \mathcal{A}(C, A_i)$. Then A_i satisfy conditions (a) and (b).

(2) follows in the same way from Lemma 2.3 if we choose an object \tilde{A} such that $A_i \in \tilde{A}_{\mathfrak{p}_i}$ for all i , set $C = \tilde{A} \oplus B$ and apply the functor $\mathcal{A}(C, _)$.

(3) is deduced in the same way from Lemma 2.4, setting $C = A \oplus B$. \square

We transfer to a hom-finite \mathbf{R} -category \mathcal{A} some results on genera of lattices over orders.

Theorem 2.6.

- (1) If $A_{\mathfrak{p}} \in B_{\mathfrak{p}}$ for all prime \mathfrak{p} , there is an object $A' \in G(A)$ such that $A' \in B$.
- (2) (Roiter addition theorem) Let $QA \in \text{add } QB$, $A' \in G(A)$. There is an object $B' \in G(B)$ such that $A \oplus B \simeq A' \oplus B'$.
- (3) If $A' \in G(A)$, then $A' \in A \oplus A$.
- (4) Let $A', A'' \in G(A)$. There is $A''' \in G(A)$ such that $A' \oplus A'' \simeq A \oplus A'''$.

Proof. In view of Proposition 1.3, we may suppose that \mathcal{A} is semiprime. If A is torsion, all claims are trivial. So we may suppose that \mathcal{A} is torsion free. We use Lemma 1.1 for $C = A \oplus B$. Set $\Lambda = \text{End}_{\mathcal{A}} C$, $M = \mathcal{A}(C, A)$ and $N = \mathcal{A}(C, B)$. Then Λ is a semiprime order and M, N are projective Λ -lattices.

(1) $M_{\mathfrak{p}} \in N_{\mathfrak{p}}$ for all \mathfrak{p} , hence there is a lattice $M' \in G(M)$ such that $M' \in N$ [4, 31.12]. This lattice is projective, as so is M [14, 3.23], hence there is an object A' such that $\mathcal{A}(C, A') \simeq M'$. Then $A' \in G(A)$ and $A' \in B$.

(2) In this situation $QC \in \text{add } QB$, hence $QA \in \text{add } QN$, so N is a faithful Λ -module. Set $M' = \mathcal{A}(C, A')$, then $M' \in G(M)$. By Roiter addition theorem [14, 31.28], there is a lattice $N' \in G(N)$ such that $M \oplus N \simeq M' \oplus N'$. Again, N' is projective, hence $N' \simeq \mathcal{A}(C, B')$, where $B' \in G(B)$ and $A \oplus B \simeq A' \oplus B'$.

(3) and (4) follow from (2) (in (3) set $A = B$). \square

Corollary 2.7. Define an equivalence relation \equiv on the set $G(A)$ such that $A' \equiv A''$ means $A' \oplus A \simeq A'' \oplus A$. Denote by $\text{Cl}(A)$ the set of equivalence classes under this relation and by $c(A')$ the class of A' in $\text{Cl}(A)$. Define an algebraic operation $+$ on $\text{Cl}(A)$ setting $c(A') + c(A'') = c(A''')$ if $A' \oplus A'' \simeq A \oplus A'''$. Then $(\text{Cl}(A), +)$ is an abelian group and $c(A)$ is its neutral element.

Note that this group does not depend on the choice of an object A in the genus (see Remark 2.10 below). We call it the *class group* of the object A or of the genus $G(A)$ and denote its cardinality by $cl(A)$. If $\mathcal{A} = \Lambda\text{-lat}$, it is the group of β -classes of the genus $G(A)$ from [15] (see also [14, 35.5] in the case of maximal orders).

Now we prove the *cancellation theorem* for genera.

Theorem 2.8 (Jacobinski cancellation theorem). Let A be such an object that $Q\text{End}_{\mathcal{A}} A / \text{nil } Q\text{End}_{\mathcal{A}} A \simeq \prod_{i=1}^m \text{Mat}(n_i, \mathbf{D}_i)$, where \mathbf{D}_i are skewfields, and for every i either the

skewfield D_i is commutative or $n_i > 1$ (or both). If $A', A'' \in G(A)$, then $A' \equiv A''$ if and only if $A' \simeq A''$.

Proof. As before, we may suppose that \mathcal{A} is torsion reduced and semiprimary. Then $\Lambda = \text{End}_{\mathcal{A}} A$ is a semiprimary \mathbf{R} -order, $M' = \mathcal{A}(A, A')$ and $M'' = \mathcal{A}(A, A'')$ are projective Λ -lattices, $Q\Lambda \simeq Q\text{End}_{\mathcal{A}} A$, $M', M'' \in G(\Lambda)$ and, if $A' \equiv A''$, also $M' \equiv M''$. By the Jacobinski cancellation theorem [5, 51.24], if $M' \equiv M''$, then $M' \simeq M''$, hence $A' \simeq A''$. \square

Corollary 2.9. Let A, A' and B be such objects that $QA \in \text{add } QB$ and $A' \in G(A)$.

- (1) If $A'' \in G(A)$ and $A' \equiv A''$, then $A' \oplus B \simeq A'' \oplus B$.
- (2) If $A \oplus B \simeq A' \oplus B'$, where $B' \equiv B$ in $G(B)$, then $A \oplus B \simeq A' \oplus B$.
- (3) If $A \oplus B \simeq A' \oplus B$, where $B \in \text{add } A$, then $A \equiv A'$ in $G(A)$.

Proof. (1) If $A' \oplus A \simeq A'' \oplus A$, then $(A' \oplus B) \oplus (A \oplus B) \simeq (A'' \oplus B) \oplus (A \oplus B)$. As $QA \in \text{add } QB$, the object $A \oplus B$ satisfies the conditions of Theorem 2.8 (namely, all $n_i > 1$), whence $A' \oplus B \simeq A'' \oplus B$.

(2) Let $B \oplus B \simeq B' \oplus B$. Then $(A \oplus B) \oplus (A \oplus B) \simeq (A' \oplus B) \oplus (A \oplus B)$. Again the object $A \oplus B$ satisfies the conditions of Theorem 2.8, whence $A \oplus B \simeq A' \oplus B$.

(3) immediately reduces to the case $B = nA$, where it is proved by an easy induction. \square

Remark 2.10. Item (1) of this corollary immediately implies that the relation \equiv on the genus $G(A)$ does not depend on the choice of the object A in this genus: just take for B any object from $G(A)$.

2.1. Arithmetic case

We call a Dedekind ring \mathbf{R} *arithmetic* if its field of fractions \mathbf{Q} is a global field, i.e. either a field of algebraic numbers or a field of algebraic functions of one variable over a finite field. Then the preceding results can be precised.

Theorem 2.11.

- (1) $g(A) < \infty$ for every object A .
- (2) If $QA \in \text{add } QB$, then $g(A \oplus B) \leq cl(B)$.

Proof. (1) is obviously reduced to the semiprime and torsion free case, where it follows from the Jordan–Zassenhaus theorem [14, 26.4].

(2) By Theorem 2.6(1), for every $C \in G(A \oplus B)$ there is $A' \in G(A)$ such that $C \simeq A' \oplus B'$. Then $B' \in G(B)$. By item (2) of the same theorem, there is $B'' \in G(A)$

such that $C \simeq A \oplus B''$. Moreover, if $B' \equiv B''$ in $G(B)$, i.e. $B' \oplus B \simeq B'' \oplus B$, then $(A \oplus B') \oplus (A \oplus B) \simeq (A \oplus B'') \oplus (A \oplus B)$, whence $A \oplus B' \simeq A \oplus B''$ by Theorem 2.8. \square

If v is a valuation of the field \mathbf{Q} , we denote by $\hat{\mathbf{Q}}_v$ the completion of \mathbf{Q} with respect to v . We say that v is *infinite with respect to \mathbf{R}* if it is not equivalent to the \mathfrak{p} -adic valuation for any $\mathfrak{p} \in \max \mathbf{R}$. Let \mathbf{D} be a finite dimensional division algebra over the field \mathbf{Q} . We say that \mathbf{D} *satisfies the Eichler condition* if for some valuation v of the field \mathbf{Q} that is infinite with respect to \mathbf{R} the $\hat{\mathbf{Q}}_v$ -algebra $\hat{\mathbf{Q}}_v \otimes_{\mathbf{Q}} \mathbf{D}$ is not a product of skewfields. Note that if \mathbf{R} is the ring of algebraic integers from \mathbf{Q} , the only exceptions are when the center \mathbf{K} of \mathbf{D} is a totally real field, $\dim_{\mathbf{K}} \mathbf{D} = 4$ and $\hat{\mathbf{D}}_w$ is the skewfield of quaternions for every infinite valuation w of the field \mathbf{K} .

Theorem 2.12. *Jacobinski cancellation theorem 2.8 remains valid if the condition $n_i > 1$ holds true for those \mathbf{D}_i that do not satisfy the Eichler condition.*

Proof. It is just the situation when the Jacobinski cancellation theorem is valid in the arithmetic case [5, 51.24]. So the same proof can be applied. \square

3. K_0 , local case

In this section \mathbf{R} is a discrete valuation ring with the maximal ideal \mathfrak{m} , $\hat{\mathbf{R}}$ is its \mathfrak{m} -adic completion and $\hat{\mathbf{Q}}$ is the field of fractions of $\hat{\mathbf{R}}$. We have a diagram of categories and functors, commutative up to isomorphism,

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\hat{}} & \hat{\mathcal{A}} \\
 \mathcal{Q} \downarrow & & \downarrow \mathcal{Q} \\
 \mathcal{Q}\mathcal{A} & \xrightarrow{\hat{}} & \mathcal{Q}\hat{\mathcal{A}}
 \end{array} \tag{3.1}$$

The K_0 -groups form an analogous diagram

$$\begin{array}{ccc}
 K_0(\mathcal{A}) & \longrightarrow & K_0(\hat{\mathcal{A}}) \\
 \downarrow & & \downarrow \\
 K_0(\mathcal{Q}\mathcal{A}) & \longrightarrow & K_0(\mathcal{Q}\hat{\mathcal{A}})
 \end{array} \tag{3.2}$$

The categories $\mathcal{Q}\mathcal{A}, \hat{\mathcal{A}}$ and $\mathcal{Q}\hat{\mathcal{A}}$ are Krull–Schmidt categories, so their K_0 -groups are free and their bases consist of classes of indecomposable objects. On the other hand, the category \mathcal{A} need not be a Krull–Schmidt category (see Example 3.8 below). Nevertheless, as $A \simeq B$ if and only if $\hat{A} \simeq \hat{B}$ and cancellation holds true in $\hat{\mathcal{A}}$, it also holds true in \mathcal{A} .

The category \mathcal{A} can be reconstructed from the other components of the diagram (3.1).

Theorem 3.1. *The category \mathcal{A} is equivalent to the pull-back (or recollement) $Q\mathcal{A} \times_{Q\hat{\mathcal{A}}} \hat{\mathcal{A}}$ [9, VI.1] of the categories $Q\mathcal{A}$ and $\hat{\mathcal{A}}$ over $Q\hat{\mathcal{A}}$.³*

Recall that the category $Q\mathcal{A} \times_{Q\hat{\mathcal{A}}} \hat{\mathcal{A}}$ consists of triples (V, \hat{A}, σ) , where $V \in Q\mathcal{A}$, $\hat{A} \in \hat{\mathcal{A}}$ and $\sigma : \hat{V} \xrightarrow{\sim} Q\hat{A}$, and a morphism $(V, \hat{A}, \sigma) \rightarrow (V', \hat{A}', \sigma')$ is a pair $\beta : V \rightarrow V'$, $\alpha : \hat{A} \rightarrow \hat{A}'$ such that $(Q\alpha)\sigma = \sigma'\beta$.

Proof. We define a functor $F : \mathcal{A} \rightarrow Q\mathcal{A} \times_{Q\hat{\mathcal{A}}} \hat{\mathcal{A}}$ setting $F(A) = (QA, \hat{A}, \sigma)$, where $\sigma : \widehat{QA} \xrightarrow{\sim} Q\hat{A}$ comes from the identity morphism of A , and $F(\alpha) = (Q\alpha, \hat{\alpha})$. Note that the diagram

$$\begin{array}{ccc} M & \longrightarrow & \hat{M} \\ \downarrow & & \downarrow \\ QM & \longrightarrow & Q\hat{M} \end{array}$$

is cartesian for every finitely generated \mathbf{R} -module M . It implies that F is fully faithful. So it remains to show that it is dense.

An object from $Q\mathcal{A}$ is a pair $B_1 = (A_1, e_1)$ where $A_1 \in \text{ob } \mathcal{A}$ and e_1 is an idempotent in $\text{End}_{Q\mathcal{A}} A_1$. An object from $\hat{\mathcal{A}}$ is a pair $B_2 = (A_2, e_2)$ where $A_2 \in \text{ob } \mathcal{A}$ and e_2 is an idempotent in $\text{End}_{\mathcal{A}} A_2$. Setting $B = A_1 \oplus A_2$, we can replace both A_1 and A_2 by B , so suppose that $B_1 = (B, e_1)$ and $B_2 = (B, e_2)$. An isomorphism $\hat{B}_1 \xrightarrow{\sim} QB_2$ is then given by an automorphism σ of $Q\hat{B}$ such that $\sigma e_1 \sigma^{-1} = e_2$. Let $\Lambda = \text{End}_{\mathcal{A}} B$. As $(\hat{\Lambda})^\times (Q\Lambda)^\times = (Q\hat{\Lambda})^\times$, there are automorphisms σ_1 of QB and σ_2 of \hat{B} such that $\sigma = \sigma_2 \sigma_1$, whence $\sigma_1 e_1 \sigma_1^{-1} = \sigma_2^{-1} e_2 \sigma_2$. It implies that there is an idempotent $e \in \Lambda$ such that its image in QA is $\sigma_1 e_1 \sigma_1^{-1}$ and its image in \hat{A} is $\sigma_2^{-1} e_2 \sigma_2$. This idempotent arises from a direct summand $A \in B$ such that $B_1 \simeq QA$, $B_2 \simeq \hat{A}$ and $(B_1, B_2, \sigma) \simeq F(A)$, which accomplishes the proof. \square

Corollary 3.2.

- (1) *The diagram (3.2) is cartesian.*
- (2) *The group $K_0(\mathcal{A})$ is free.*

Proof. (1) follows immediately from Theorem 3.1. As the groups $K_0(Q\mathcal{A})$ and $K_0(\hat{\mathcal{A}})$ are free, it implies (2). \square

There is one important case, when the generators of the group $K_0(\mathcal{A})$ can be explicitly calculated. Fortunately, most examples that occur in applications are of this sort. (As a rule, they even satisfy much more restrictive *Max-condition*, see Definition 4.1.)

³ Actually, it means that the diagram (3.1) is *cartesian* as a diagram of categories and functors.

Definition 3.3 (*S-condition*). We say that a hom-finite \mathbf{R} -category \mathcal{A} satisfies the *S-condition* if for every indecomposable object $U \in \text{ind } Q\hat{\mathcal{A}}$ there is an object $S \in \text{ob } \hat{\mathcal{A}}$ such that $U \simeq QS$. Obviously, then S is also indecomposable.

If this condition is satisfied, we fix an object $S(U)$ such that $QS(U) \simeq U$ for every $U \in \text{ind } Q\hat{\mathcal{A}}$ and set $S(V) = \bigoplus_{i=1}^m S(U_i)$ if $V \simeq \bigoplus_{i=1}^m U_i$, where $U_i \in \text{ind } Q\hat{\mathcal{A}}$.

We set $S(\hat{\mathcal{A}}) = \{S(U) \mid U \in \text{ind } Q\hat{\mathcal{A}}\}$ and $\mathbb{A}(\hat{\mathcal{A}}) = \text{ind } \hat{\mathcal{A}} \setminus S(\hat{\mathcal{A}})$.

In the rest of this section we suppose that \mathcal{A} satisfies the S-condition and we use the notations of Definition 3.3. Fix some more notations and terms. Note that $Q\mathcal{A}^0$ is a semisimple category, i.e. $Q\mathcal{A}^0(W, W') = 0$ if $W, W' \in \text{ind } Q\mathcal{A}$ and $W \not\cong W'$. It implies that $Q\mathcal{A}(W, W') \in \text{nil } Q\mathcal{A}$, whence also $Q\hat{\mathcal{A}}(\hat{W}, \hat{W}') \in \text{nil } Q\hat{\mathcal{A}}$. Therefore, \hat{W} and \hat{W}' have no isomorphic direct summands.

Definition 3.4. Let $U \in \text{ind } Q\hat{\mathcal{A}}, V \in \text{ob } \hat{\mathcal{A}}, W \in \text{ind } Q\mathcal{A}$.

- (1) The multiplicities $\mu(U, V)$ are defined from the decomposition $V \simeq \bigoplus_{U \in \text{ind } Q\hat{\mathcal{A}}} \mu(U, V)U$.
- (2) We set $\mu(U) = \mu(U, \hat{W})$ if $\mu(U, \hat{W}) \neq 0$. Note that there is exactly one W with this property.
- (3) If $\mu(U) \mid \mu(U, V)$ for all $U \in \text{ind } Q\mathcal{A}$, we define S_V as an object from \mathcal{A} such that $\widehat{S}_V \simeq S(V)$ (it exists by Theorem 3.1). Note that if this condition is not satisfied there is no object V_a in $Q\mathcal{A}$ such that $V \simeq \widehat{V}_a$.
- (4) We set $S(W) = S_{\widehat{W}}$, denote by $S(\mathcal{A})$ the set $\{S(W) \mid W \in \text{ind } Q\mathcal{A}\}$ and call these objects *S-objects*.
- (5) Let $V \in \text{ob } Q\hat{\mathcal{A}}, U \in \text{ind } Q\hat{\mathcal{A}}$. We denote by $\delta(U, V)$ the smallest non-negative integer such that $\mu(U) \mid \mu(U, V) + \delta(U, V)$.
- (6) For an object $C \in \mathbb{A}(\hat{\mathcal{A}})$ we set $\tilde{C} = C \oplus \left(\bigoplus_{U \in \text{ind } Q\hat{\mathcal{A}}} \delta(U, QC)S(U) \right)$.
- (7) We call an object $A \in \text{ob } \mathbf{A}$ *atomic* if $\hat{A} \simeq \tilde{C}$ for some $C \in \mathbb{A}(\hat{\mathcal{A}})$. Note that such object A exists and is unique up to isomorphism for every $C \in \mathbb{A}(\hat{\mathcal{A}})$ by Theorem 3.1. In this case we call C *the core* of A and denote it by $\text{co}(A)$. We denote by $\mathbb{A}(\mathcal{A})$ the set of isomorphism classes of atomic objects.

The following properties immediately follow from definitions.

Proposition 3.5.

- (1) Any atomic object or S-object is indecomposable.
- (2) Two atomic objects are isomorphic if and only if their cores are isomorphic.
- (3) S-objects S and S' are isomorphic if and only if $QS \simeq QS'$.
- (4) Neither atomic object is isomorphic to an S-object.

Theorem 3.6. Let \mathcal{A} satisfy S-condition.

- (1) For every indecomposable object $B \notin \mathcal{S}(\mathcal{A})$ there are atomic objects A_1, A_2, \dots, A_m and S -objects S_1, S_2, \dots, S_n such that $\bigoplus_{i=1}^m A_i \simeq B \oplus (\bigoplus_{j=1}^n S_j)$. These objects are uniquely defined, up to isomorphism and permutation.
- (2) $\{[A] \mid A \in \mathbb{A}(\mathcal{A}) \cup \mathcal{S}(\mathcal{A})\}$ is a basis of $K_0(\mathcal{A})$.

Note that the subgroup generated by $\mathbb{A}(\mathcal{A})$ is isomorphic to the subgroup of $K_0(\hat{\mathcal{A}})$ with the basis $\mathbb{A}(\hat{\mathcal{A}})$ and the subgroup generated by $\mathcal{S}(\mathcal{A})$ is isomorphic to $K_0(Q\mathcal{A})$.

Proof. (1) Let $\hat{B} \simeq C \oplus B'$, where $C = (\bigoplus_{i=1}^m C_i)$ with $C_i \in \mathbb{A}(\hat{\mathcal{A}})$ and B' has no direct summands from $\mathbb{A}(\hat{\mathcal{A}})$. Then $B' \simeq \bigoplus_{U \in \text{ind } Q\mathcal{A}} \mu(U, QB')S(U)$ and

$$\mu(U, Q\hat{B}) = \sum_{i=1}^m (\mu(U, QC_i) + \delta(U, QC_i)) - (\sum_{i=1}^m \delta(U, QC_i) - \mu(U, QB')).$$

Moreover, $\mu(U, QB') = \delta(U, QC)$. Indeed, $\mu(U, QB') \geq \delta(U, QC)$, since $\mu(U) \mid \mu(U, QC) + \mu(U, QB')$, and

$$Q\hat{B} \simeq (QC \oplus (\bigoplus_U \delta(U, QC)U)) \oplus (\bigoplus_U (\mu(U, QB') - \delta(U, QC))U),$$

whence, by Theorem 3.1, $B \simeq B_1 \oplus B_2$, where

$$\hat{B}_1 \simeq C \oplus (\bigoplus_U \delta(U, QC)S(U))$$

and

$$\hat{B}_2 \simeq \bigoplus_U (\mu(U, QB') - \delta(U, QC))S(U).$$

As B is indecomposable, $B_2 = 0$, so $\mu(U, qB') = \delta(U, QC)$.

Let $A_i = \hat{C}_i$, $A = \bigoplus_{i=1}^m A_i$. Then $Q\hat{A} \simeq QC \oplus (\bigoplus_U \delta(U, QC_i)U)$. Obviously, $\delta(U, QC) \leq \sum_{i=1}^m \delta(U, QC_i)$ and $\mu(U) \mid \sum_{i=1}^m \delta(U, QC_i) - \mu(U, QB')$. Therefore, there is an object $V \in \text{ob } Q\mathcal{A}$ such that $\hat{V} \simeq \bigoplus_U (\sum_{i=1}^m \delta(U, QC_i) - \mu(U, QB'))U$. Let $S = S_V$. Then $Q(B \oplus S) \simeq QA$ and $\hat{B} \oplus \hat{S} \simeq \hat{A}$. By Theorem 3.1, $B \oplus S \simeq A$. If $V \simeq \bigoplus_{j=1}^n W_j$ with $W_j \in \text{ind } Q\mathcal{A}$, then $S \simeq \bigoplus_{j=1}^n S_j$, where $S_j = \mathcal{S}(W_j)$ are S -objects. Note that the cores $\text{co}(A_i)$ (hence A_i) are uniquely determined: they are direct summands of \hat{B} that are not S -objects. As $Q\mathcal{A}$ is a Krull–Schmidt category, the objects W_j are uniquely determined. Hence S -objects S_j are also uniquely determined.

(2) By (1), $\{[A] \mid A \in \mathbb{A}(\mathcal{A}) \cup \mathcal{S}(\mathcal{A})\}$ is a set of generators of $K_0(\mathcal{A})$. So we only have to show that if $(\bigoplus_{i=1}^m A_i) \oplus (\bigoplus_{j=1}^n S_j) \simeq (\bigoplus_{i=1}^k A'_i) \oplus (\bigoplus_{j=1}^l S'_j)$, where A_i, A'_i are atomic and S_j, S'_j are S -objects, then $m = k, n = l$ and the objects A_i and A'_i , as well as the objects S_j and S'_j , are isomorphic up to a permutation. Taking cores, we obtain that $\bigoplus_{i=1}^m \text{co}(A_i) \simeq \bigoplus_{i=1}^k \text{co}(A'_i)$, whence $m = k$ and $\text{co}(A_i) \simeq \text{co}(A'_{\sigma i})$, hence $A_i \simeq A'_{\sigma i}$ for some permutation σ . As the cancellation holds true in the category \mathcal{A} , also $\bigoplus_{j=1}^n S_j \simeq \bigoplus_{j=1}^l S'_j$, whence $\bigoplus_{j=1}^n QS_j \simeq \bigoplus_{j=1}^l QS'_j$. As QS_j, QS'_j are indecomposable

and $Q\mathcal{A}$ is a Krull–Schmidt category, $n = l$ and $QS_j \simeq QS'_{\tau_j}$, hence $S_j \simeq S'_{\tau_j}$ for some permutation τ . \square

Corollary 3.7. *Suppose that \mathcal{A} satisfies the S-condition and \hat{V} is indecomposable for every $V \in \text{ind } Q\mathcal{A}$. Then $K_0(\mathcal{A})$ is a Krull–Schmidt category.*

Note that this condition is not necessary for \mathcal{A} to be a Krull–Schmidt category.

Proof. It follows from Theorem 3.6, since in this case any indecomposable object from \mathcal{A} is either atomic or an S-object. \square

Example 3.8. We present here an example when \mathcal{A} is not a Krull–Schmidt category.⁴ Let \mathbf{K} be an extension of the field \mathbf{Q} such that $\dim_{\mathbf{Q}} \mathbf{K} = 3$ and, if \mathbf{S} is the integral closure of \mathbf{R} in \mathbf{K} , $\mathfrak{p}\mathbf{S}$ is a product of 3 different prime ideals of \mathbf{S} : $\mathfrak{p}\mathbf{S} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$. For instance, $\mathbf{R} = \mathbb{Z}_7 = \{a \in \mathbb{Q} \mid a = m/n, \text{ where } m, n \in \mathbb{Z} \text{ and } 7 \nmid n\}$ and $\mathbf{K} = \mathbb{Q}(\sqrt[3]{6})$. Then $\mathbf{S}/\mathfrak{p}_i \simeq \mathbf{R}/\mathfrak{p}$. Fix isomorphisms $\varphi_i : \mathbf{S}/\mathfrak{p}_i \xrightarrow{\sim} \mathbf{R}/\mathfrak{p}$ and set $\Lambda = \{a \in \mathbf{S} \mid \varphi_1(a) = \varphi_2(a) = \varphi_3(a)\}$, $N_{ij} = \{a \in \mathbf{S} \mid \varphi_i(a) = \varphi_j(a)\}$. Let $\mathcal{A} = \Lambda\text{-lat}$. The ring $\hat{\Lambda}$ is local, $\hat{\mathbf{S}} = \mathbf{S}_1 \times \mathbf{S}_2 \times \mathbf{S}_3$, where \mathbf{S}_i is the \mathfrak{p}_i -adic completion of \mathbf{S} , and $\hat{N}_{ij} \simeq \hat{\Lambda}_{ij} \oplus \mathbf{S}_k$, where $\hat{\Lambda}_{ij}$ is the projection of $\hat{\Lambda}$ onto $\mathbf{S}_i \times \mathbf{S}_j$ and $k \notin \{i, j\}$. We set $\mathbb{S}(\hat{\mathcal{A}}) = \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\}$ (actually, this choice is unique). The $\hat{\Lambda}$ -lattices $\hat{\Lambda}_{ij}$ are indecomposable. Hence the Λ -lattices N_{ij} are atomic. According to Theorem 3.1, there is a Λ -lattice M such that $\hat{M} \simeq \hat{\Lambda}_{12} \oplus \hat{\Lambda}_{13} \oplus \hat{\Lambda}_{23}$ and it is indecomposable. Then $M \oplus \mathbf{S} \simeq N_{12} \oplus N_{13} \oplus N_{23}$. (It is the decomposition from Theorem 3.6(1).) Thus we have decompositions of the same module into direct sums both of 2 and of 3 indecomposables and all of them are non-isomorphic.

In this example $\hat{\Lambda}$ is a *triad of discrete valuation rings* [1, p. 23]. So it follows from the calculations there that $\mathbb{A}(\mathcal{A}) = \{N_{12}, N_{13}, N_{23}, \Lambda, \Lambda^*\}$, where $\Lambda^* = \{a \in \mathbf{S} \mid \varphi_1(a) = \varphi_2(a) + \varphi_3(a)\}$. Therefore, $K_0(\Lambda\text{-lat}) = \mathbb{Z}^6$.

4. K_0 , global case

In this section we suppose that \mathbf{R} is any Dedekind domain and the hom-finite \mathbf{R} -category \mathcal{A} satisfies the following condition.

Definition 4.1 (Max-condition). We say that a hom-finite \mathbf{R} -category \mathcal{A} satisfies the *Max-condition* if for every indecomposable object $W \in \text{ob } Q\mathcal{A}$ there is an object $S \in \text{ob } \mathcal{A}$ such that $W \simeq QS$ and $\Delta(W) = \text{End}_{\mathcal{A}^0} S^0$ is a maximal order in the skewfield $Q\mathcal{A}^0(W^0, W^0)$.

If this condition is satisfied, we fix, for every $W \in \text{ind } Q\mathcal{A}$, an object $\mathbb{S}(W)$ such that $\mathbb{S}(W) \simeq W$ and $\text{End}_{\mathcal{A}^0} \mathbb{S}(W)^0$ is a maximal order, and set $\mathbb{S}(V) = \bigoplus_{i=1}^m \mathbb{S}(W_i)$

⁴ Perhaps, it is the simplest example. For other examples see [4, §35] and [10], where the case of group rings $\mathbb{Z}_{(p)}G$ is studied.

if $V \simeq \bigoplus_{i=1}^m W_i$, where $W_i \in \text{ind } Q\mathcal{A}$. We set $\mathbb{S}(\mathcal{A}) = \{S(W) \mid W \in \text{ind } Q\mathcal{A}\}$. We denote by $S_{\mathfrak{p}}(W)$ the image of $S(W)$ in $\mathcal{A}_{\mathfrak{p}}$ and $\mathbb{S}(\mathcal{A}_{\mathfrak{p}}) = \{S_{\mathfrak{p}}(W) \mid W \in \text{ind } Q\mathcal{A}\}$. Since $\text{End}_{\mathcal{A}_{\mathfrak{p}}^0} S_{\mathfrak{p}}^0 = (\text{End}_{\mathcal{A}^0} S^0)_{\mathfrak{p}}$ is a maximal $\mathbf{R}_{\mathfrak{p}}$ -order if $S = S(W)$, $\mathcal{A}_{\mathfrak{p}}$ satisfies the Max-condition as well.

Remark 4.2. If $\text{End}_{\mathcal{A}^0} S^0$ is a maximal order and $S = \bigoplus_{i=1}^k S_i$, where S_i are indecomposable, then QS_i^0 are simple $Q\mathcal{A}^0$ -objects and all $\text{End}_{\mathcal{A}^0} S_i^0$ are also maximal. It follows from [14, 21.2]. Therefore, the Max-condition is satisfied if for every $A \in \text{ob } \mathcal{A}^f$ there is an object S such that $QS \simeq QA$ and $\text{End}_{\mathcal{A}^0} S^0$ is a maximal order.

For instance, Max-condition is satisfied if \mathbf{R} is excellent and $\mathcal{A} = \Lambda\text{-mod}$, where Λ is a finite \mathbf{R} -algebra. It also is satisfied if $\mathcal{A} = \text{SW}$, the stable homotopy category of polyhedra. In this case all indecomposable objects in $Q\text{SW}$ are QS^n , where S^n is an n -dimensional sphere, and $\text{End}_{\text{SW}} S^n = \mathbb{Z}$.

Proposition 4.3. *Suppose that \mathcal{A} satisfies the Max-condition, $\mathfrak{p} \in \text{max } \mathbf{R}$.*

- (1) $\text{nil } \hat{\mathcal{A}} = \hat{\mathbf{R}}_{\mathfrak{p}} \otimes_{\mathbf{R}} \text{nil } \mathcal{A}$ and $\text{nil } Q\hat{\mathcal{A}} = \hat{\mathbf{Q}}_{\mathfrak{p}} \otimes_{\mathbf{R}} \text{nil } \mathcal{A}$. Therefore, $(\hat{\mathcal{A}}_{\mathfrak{p}})^0 = (\widehat{\mathcal{A}^0})_{\mathfrak{p}}$ and $(Q\hat{\mathcal{A}}_{\mathfrak{p}})^0 = Q(\widehat{\mathcal{A}^0})_{\mathfrak{p}}$.
- (2) The category $\hat{\mathcal{A}}_{\mathfrak{p}}$ satisfies the Max-condition and the category $\mathcal{A}_{\mathfrak{p}}$ satisfies the S -condition.
- (3) For every object $B \in \mathcal{A}_{\mathfrak{p}}$ there is an object $A \in \mathcal{A}$ such that $A_{\mathfrak{p}} \simeq B$.

Proof. (1) If Δ is a maximal order, its center \mathbf{C} is integrally closed, hence is a Dedekind domain. Let \mathbf{K} be its field of fractions, $\mathbf{D} = Q\Delta$ and $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_k$ be all primes of \mathbf{C} containing \mathfrak{p} . Then $\hat{\Delta}_{\mathfrak{p}} \simeq \prod_{i=1}^k \hat{\Delta}_{\mathfrak{p}_i}$ and $\hat{\mathbf{K}}_{\mathfrak{p}} \simeq \prod_{i=1}^k \hat{\mathbf{K}}_{\mathfrak{p}_i}$ [13, 24.C]. Since \mathbf{D} is central over \mathbf{K} , all $\hat{\mathbf{D}}_{\mathfrak{p}_i}$ are central simple algebras, so $\hat{\mathbf{D}}_{\mathfrak{p}}$ is semisimple. It implies that $\hat{\mathbf{Q}}_{\mathfrak{p}} \otimes_{\mathbf{R}} \mathcal{A}^0$ is semisimple, whence $\text{nil } Q\hat{\mathcal{A}} = \hat{\mathbf{Q}}_{\mathfrak{p}} \otimes_{\mathbf{R}} \text{nil } \mathcal{A}$ and $\text{nil } \hat{\mathcal{A}} = \hat{\mathbf{R}}_{\mathfrak{p}} \otimes_{\mathbf{R}} \text{nil } \mathcal{A}$.

(2) If Δ is a maximal order in a skewfield \mathbf{D} and $\hat{\mathbf{D}}_{\mathfrak{p}} = \prod_{i=1}^m \text{Mat}(n_i, \mathbf{D}_i)$, where \mathbf{D}_i are skewfields, $\hat{\Delta}_{\mathfrak{p}}$ also splits as $\hat{\Delta}_{\mathfrak{p}} \simeq \prod_{i=1}^m \text{Mat}(n_i, \Delta_i)$, where Δ_i is a maximal order in \mathbf{D}_i [14, 18.8]. Then $\hat{\mathbf{D}}_{\mathfrak{p}} \simeq \bigoplus_{i=1}^m n_i \mathbf{D}_i^{n_i}$ as $\hat{\mathbf{D}}_{\mathfrak{p}}$ -module, where all summands are simple modules with the endomorphism rings \mathbf{D}_i . Respectively, $\hat{\Delta} \simeq \bigoplus_{i=1}^m n_i \Delta_i^{n_i}$, where the summands have endomorphism rings Δ_i . Suppose that $\mathbf{D} = \text{End}_{Q\mathcal{A}^0} W^0$ and $\Delta = \text{End}_{\mathcal{A}^0} S^0$, where $S = S(W)$. By Lemma 1.1 and Proposition 1.3, \hat{W} and \hat{S} split in the same way, namely, every indecomposable summand of $\hat{W}_{\mathfrak{p}}$ is of the form QA , where A is an indecomposable summand of $\hat{S}_{\mathfrak{p}}$, and the endomorphism rings of QA^0 and A^0 are, respectively, \mathbf{D}_i and Δ_i for some i .

(3) Obviously, we can suppose \mathcal{A} semiprime and torsion free. The object B arises from an idempotent $e \in \text{End}_{\mathcal{A}_{\mathfrak{p}}} B'_{\mathfrak{p}} \subseteq \text{End}_{Q\mathcal{A}} QB'$, where $B' \in \text{ob } \mathcal{A}$. Let $S = S(QB), C = S \oplus B', \Lambda = \text{End}_{\mathcal{A}}(C), M = \mathcal{A}(C, S)$ and $L = \mathcal{A}_{\mathfrak{p}}(C_{\mathfrak{p}}, B)$. Then $QM \simeq QL$. Thus we can suppose that $L \subseteq M_{\mathfrak{p}}$ and consider the Λ -lattice $N = L \cap M$. Note that $N_{\mathfrak{p}} = L$ and

$N_q = M_q$ if $q \neq p$, so all localizations of N are projective over the localizations of Λ and N is projective over Λ . Therefore $N \simeq \mathcal{A}(C, A)$ for some A and $A_p \simeq B$. \square

Following the proof of item (2) above, we fix, for every $U_p \in \text{ind } Q\hat{\mathcal{A}}_p$, an object $W \in \text{ind } Q\mathcal{A}$ such that U_p is a direct summand of \hat{W}_p and denote by $S(U_p)$ an indecomposable direct summand of $\widehat{S(W)}_p$ (it is unique up to an isomorphism). We set $S(\hat{\mathcal{A}}_p) = \{S(U_p) \mid U_p \in \text{ind } Q\hat{\mathcal{A}}_p\}$ and $\mathbb{A}(\hat{\mathcal{A}}_p) = \text{ind } \hat{\mathcal{A}}_p \setminus S(\hat{\mathcal{A}}_p)$. We call the objects from $S(\mathcal{A}), S(\mathcal{A}_p)$ and $S(\hat{\mathcal{A}}_p)$ the *S-objects* of the corresponding categories. As we have the notion of S-objects in $\hat{\mathcal{A}}_p$, we define the set $\mathbb{A}(\mathcal{A}_p)$ of the *atomic objects* in \mathcal{A}_p as in the preceding section.

Proposition 4.3 (3) and Theorem 2.5 (2) imply that, if the Max-condition is satisfied, a genus $G(A)$ of objects from \mathcal{A} is defined by the object $V = QA$ from $Q\mathcal{A}$, the (finite) set \mathbf{P} of prime ideals p such that $A_p \not\cong S(V)_p$ and the localizations A_p for $p \in \mathbf{P}$. If $\mathbf{P} = \emptyset$, $A = S(V)$. Moreover, these data can be prescribed arbitrary, with the only restriction that $QA_p \simeq V$ for all $p \in \mathbf{P}$.

Definition 4.4. An object $A \in \text{ob } \mathcal{A}$ is said to be *p-atomic* if $A_p \in \mathbb{A}(\mathcal{A}_p)$ and $A_q \simeq S(QA)_q$ for $q \neq p$. We denote by $\mathbb{A}(p, \mathcal{A})$ the set of *p-atomic* objects, set $\mathbb{A}(\mathcal{A}) = \bigcup_{p \in \max \mathbf{R}} \mathbb{A}(p, \mathcal{A})$ and call the objects from $\mathbb{A}(\mathcal{A})$ *atomic*.

By the remark above, every atomic object from $\mathbb{A}(\mathcal{A}_p)$ is the *p*-localization of a *p-atomic* object, so there is a one-to-one correspondence between $\mathbb{A}(\mathcal{A}, p)$ and $\mathcal{A}(\mathcal{A}_p)$ (or, the same, $\mathbb{A}(\hat{\mathcal{A}}_p)$). Obviously, atomic objects are indecomposable.

We denote by $G\mathcal{A}$ the set of genera of \mathcal{A} and define the group $K_0(G\mathcal{A})$ as the quotient of the free group with the basis $G\mathcal{A}$ by the subgroup generated by the elements of the form $G(A \oplus B) - G(A) - G(B)$. We denote by $[G(A)]$ the class of $G(A)$ in $K_0(G\mathcal{A})$. There is a commutative diagram of groups

$$\begin{array}{ccccc}
 & & & \wedge & \\
 & & & \nearrow & \\
 & & K_0(\mathcal{A}_p) & \xrightarrow{\quad} & K_0(\hat{\mathcal{A}}_p) \\
 & & \downarrow Q & & \downarrow Q \\
 K_0(\mathcal{A}) & \xrightarrow{G} & K_0(G\mathcal{A}) & \begin{array}{l} \nearrow (\cdot)_p \\ \searrow Q \end{array} & \\
 & & K_0(Q\mathcal{A}) & \xrightarrow{\wedge_p} & K_0(Q\hat{\mathcal{A}}_p)
 \end{array} \tag{4.1}$$

The arrow \xrightarrow{G} is surjective by definition. The Max-condition ensures that the arrows \xrightarrow{Q} are surjective too.

Theorem 4.5. *If \mathcal{A} satisfies the Max-condition, the group $K_0(G\mathcal{A})$ is a free abelian group with a basis $\mathfrak{S} = \{ [G(A)] \mid A \in \mathbb{A}(\mathcal{A}) \cup S(\mathcal{A}) \}$.*

Note that the subgroup generated by $\{ [G(A)] \mid A \in S(\mathcal{A}) \}$ is isomorphic to $K_0(Q\mathcal{A})$ and the subgroup generated by $\{ [G(A)] \mid A \in \mathbb{A}(p, \mathcal{A}) \cup S(\mathcal{A}) \}$ is isomorphic to $K_0(\mathcal{A}_p)$.

Proof. Let \mathbf{A} be the subgroup of $K_0(G\mathcal{A})$ generated by the set \mathfrak{S} . Suppose first that $A \in \text{ind } \mathcal{A}$ is such that $A_{\mathfrak{q}} \simeq S(QA)$ for all prime \mathfrak{q} except a unique \mathfrak{p} . By Theorem 2.6 (1), $A_{\mathfrak{p}}$ is also indecomposable. By Theorem 3.6, either $A_{\mathfrak{p}} \simeq S(QA)_{\mathfrak{p}}$ and $G(A) = G(S(QA))$ or $A_{\mathfrak{p}} \oplus S_{\mathfrak{p}} \simeq \bigoplus_{i=1}^m (A_i)_{\mathfrak{p}}$, where S is a direct sum of S -objects, and all A_i are \mathfrak{p} -atomic. In the last case $G(A \oplus S) = G(\bigoplus_{i=1}^m A_i)$, so $[G(A)] + [G(S)] = \sum_{i=1}^m [G(A_i)]$ and $[A] \in \mathbf{A}$.

If A is arbitrary, let $S = S(QA)$. As $QS \simeq QA$, there are morphisms $A \xrightarrow{\alpha} S \xrightarrow{\beta} A$ such that $\beta\alpha = a1_A$ and $\alpha\beta = a1_S$ for some non-zero $a \in \mathbf{R}$. By Theorem 2.5 (1), there are objects $A_i (1 \leq i \leq r)$ such that for each i there is at most one prime \mathfrak{p}_i such that $(A_i)_{\mathfrak{p}_i} \not\cong S_{\mathfrak{p}_i}$ and $A \oplus (r-1)S \simeq \bigoplus_{i=1}^r A_i$. By the preceding consideration, $[G(A_i)] \in \mathbf{A}$, therefore also $[G(A)] \in \mathbf{A}$. Thus $\mathbf{A} = K_0(G\mathcal{A})$, i.e. \mathfrak{S} generates $K_0(G\mathcal{A})$.

Suppose now that $\sum_{i=1}^n [G(A_i)] = \sum_{j=1}^m [G(B_j)]$, where all $[G(A_i)] \in \mathfrak{S}$. Let A_1, A_2, \dots, A_k and B_1, B_2, \dots, B_l be all objects from this list that belong to $\mathbb{A}(\mathfrak{p}, \mathcal{A})$. Then in the group $K_0(\mathcal{A}_{\mathfrak{p}})$ all classes $[(A_i)_{\mathfrak{p}}]$ with $i > k$ and all classes $[(B_j)_{\mathfrak{p}}]$ with $j > l$ belong to the subgroup generated by $S(\mathcal{A}_{\mathfrak{p}})$. By Theorem 3.6 (2), $k = l$ and $(A_i)_{\mathfrak{p}} = (B_{\sigma i})_{\mathfrak{p}}$ for some permutation σ , whence $G(A_i) = G(B_{\sigma i})$. As it is valid for all primes \mathfrak{p} , it remains the case when all summands are from $S(\mathcal{A})$, which is evident. \square

Corollary 4.6. $K_0(\mathcal{A}) \simeq \text{Ker } G \oplus K_0(G\mathcal{A})$, where G is the homomorphism from the diagram (4.1).

Now we have to calculate $\text{Ker } G$. We use the relation \equiv and the groups $\text{Cl}(A)$ defined in Corollary 2.7.

Theorem 4.7. If \mathcal{A} satisfies the Max-condition, $\text{Ker } G \simeq \bigoplus_{S \in \mathfrak{S}(\mathcal{A})} \text{Cl}(S)$.

Recall that $\text{Cl}(S) \simeq \text{Cl}(\Delta)$, where $\Delta = \text{End}_{\mathcal{A}^0} S^0$ is a maximal order in a skewfield. If Δ is commutative (that is, a Dedekind domain), $\text{Cl } \Delta$ is just the group of ideal classes of Δ . In the arithmetic case, when \mathbf{Q} is a global field, all groups $\text{Cl}(S)$ are finite. Thus, if $\text{ind } Q\mathcal{A}$ is finite (for instance, $\mathcal{A} = \Lambda\text{-mod}$ for a semiprime \mathbf{R} -order Λ), $\text{Ker } G$ is finite.

Proof. First we prove a lemma.

Lemma 4.8. Suppose that $\text{End}_{\mathcal{A}^0} A^0$ is a maximal order, $A_1, A_2 \in G(A)$. If $[A_1] = [A_2]$, then $A_1 \equiv A_2$ in $G(A)$.

Proof. By Proposition 1.3, we can suppose that \mathcal{A} is semiprime. Let $[A_1] = [A_2]$, i.e. $A_1 \oplus B \simeq A_2 \oplus B$ for some object B . Let $\Lambda = \text{End}_{\mathcal{A}}(A \oplus B)$, $M = \mathcal{A}(A \oplus B, A)$, $M_i = \mathcal{A}(A \oplus B, A_i) (i = 1, 2)$ and $N = \mathcal{A}(A \oplus B, B)$. Then Λ is a semiprime order, M, M_1, M_2 and N are right Λ -lattices and $M_1 \oplus N \simeq M_2 \oplus N$. Note that $\text{End}_{\Lambda} M_i \simeq \text{End}_{\Lambda} M \simeq \Delta$, hence $\Gamma = \text{End}_{\Delta} M \simeq \text{End}_{\Delta} M_i$ is a maximal order, which is an overorder of $\Lambda / \text{ann}_{\Lambda} M$. If $\text{ann}_{\Lambda} M \neq 0$, then $Q\Lambda = \mathbf{A}_1 \times \mathbf{A}_2$ such that $\mathbf{A}_2 = Q \text{ann}_{\Lambda} M$. If B_1 is the projection of N onto $(Q\Lambda)\mathbf{A}_1$, then $M_1 \oplus B_1 \simeq M_2 \oplus B_1$. Therefore, we can suppose that M is faithful

and Γ is an overorder of Λ . Now $M_1 \oplus N \simeq M_2 \oplus N$ implies $M_1 \oplus N\Gamma \simeq M_2 \oplus N\Gamma$, where all summands are Γ -lattices. Since Γ is maximal, all Γ -lattices L with fixed QL are in the same genus [4, 31.2], hence all Γ -lattices belong to add M (it follows from Theorem 2.6). By Corollary 2.9 (3), $M_1 \oplus N\Gamma \simeq M_2 \oplus N\Gamma$ implies $M_1 \equiv M_2$. Returning, by Lemma 1.1, to \mathcal{A} , we obtain that $A_1 \equiv A_2$. \square

Fix now a set \mathfrak{A} of representatives of indecomposable genera of the category \mathcal{A} . For convenience, we suppose that $\mathfrak{A} \supseteq \mathfrak{S} = \{S(W) \mid W \in \text{ind } Q\mathcal{A}\}$. One easily verifies that $\text{Ker } G = \{[A'] - [A] \mid A' \in G(A)\}$. Let $S = S(QA)$. By Theorem 2.6 (2), there is an object $S' \in G(S)$ such that $A' \oplus S \simeq A \oplus S'$, whence $[A'] - [A] = [S'] - [S]$. Therefore, $\text{Ker } G = \{[S'] - [S] \mid S \in \mathfrak{S}(\mathcal{A}), S' \in G(S)\}$. By Lemma 4.8, $[S'] - [S] = [S''] - [S]$ if and only if $S' \equiv S''$. Hence, for any fixed S , mapping $[S'] - [S]$ to $c(S')$, we obtain an isomorphism of the subgroup $\mathbf{k}(S)$ generated by the differences $[S'] - [S]$ with the group $\text{Cl}(S)$. By definition, if $S_1, S_2 \in \mathfrak{S}(\mathcal{A})$ and $S_1 \not\equiv S_2$, then $\mathcal{A}(S_1, S_2) \in \text{nil } \mathcal{A}$. It easily implies that $\mathbf{k}(S_1) \cap \sum_{S_2 \not\equiv S_1} \mathbf{k}(S_2) = 0$, which accomplishes the proof. \square

Example 4.9. Let $\mathcal{A} = \Lambda\text{-lat}$, where Λ is a hereditary \mathbf{R} -order. As every Λ -lattice is projective, $K_0(\mathcal{A})$ coincides with $K_0(\Lambda)$, the Grothendieck group of projective Λ -modules. Since Λ decomposes just as $Q\Lambda$ and the center of each component of Λ is a Dedekind ring [14, 10.8, 10.9], we may suppose that $Q\Lambda$ is a central simple Q -algebra. Then $\mathfrak{S}(\mathcal{A})$ consists of a unique lattice S . For every prime \mathfrak{p} , $Q\hat{\Lambda}_{\mathfrak{p}} \simeq \text{Mat}(n_{\mathfrak{p}}, \mathbf{F}_{\mathfrak{p}})$, where $\mathbf{F}_{\mathfrak{p}}$ is a skewfield. Let $\Gamma_{\mathfrak{p}}$ be the (unique) maximal $\hat{\mathbf{R}}_{\mathfrak{p}}$ -order in $\mathbf{F}_{\mathfrak{p}}$ [14, 12.8], $\mathfrak{m}_{\mathfrak{p}}$ be its maximal ideal. There is an integer $m_{\mathfrak{p}} \leq n_{\mathfrak{p}}$ such that $\hat{\Lambda}_{\mathfrak{p}}$ is Morita equivalent to the ring $\mathbf{H}(m_{\mathfrak{p}}, \Gamma_{\mathfrak{p}})$ of $m_{\mathfrak{p}} \times m_{\mathfrak{p}}$ matrices of the form

$$\begin{pmatrix} \Gamma_{\mathfrak{p}} & \Gamma_{\mathfrak{p}} & \Gamma_{\mathfrak{p}} & \dots & \Gamma_{\mathfrak{p}} \\ \mathfrak{m}_{\mathfrak{p}} & \Gamma_{\mathfrak{p}} & \Gamma_{\mathfrak{p}} & \dots & \Gamma_{\mathfrak{p}} \\ \mathfrak{m}_{\mathfrak{p}} & \mathfrak{m}_{\mathfrak{p}} & \Gamma_{\mathfrak{p}} & \dots & \Gamma_{\mathfrak{p}} \\ \dots & \dots & \dots & \dots & \dots \\ \mathfrak{m}_{\mathfrak{p}} & \mathfrak{m}_{\mathfrak{p}} & \mathfrak{m}_{\mathfrak{p}} & \dots & \Gamma_{\mathfrak{p}} \end{pmatrix}$$

(see [14, 39.14]). As $\Lambda_{\mathfrak{p}}$ is maximal for almost all \mathfrak{p} , almost all $m_{\mathfrak{p}} = 1$. This ring, hence also $\hat{\Lambda}_{\mathfrak{p}}$, has $m_{\mathfrak{p}}$ indecomposable lattices and one of them must be chosen as $S(U)$, so $\mathbb{A}(\hat{\mathcal{A}}_{\mathfrak{p}})$ consists of $m_{\mathfrak{p}} - 1$ lattices, as well as $\mathbb{A}(\mathcal{A}_{\mathfrak{p}})$. If $Q\Lambda \simeq \text{Mat}(n, \mathbf{D})$, where \mathbf{D} is a skewfield, then $\text{End}_{\Lambda} S = \Delta$ is a maximal order in \mathbf{D} . Let $m = 1 + \sum_{\mathfrak{p}} (m_{\mathfrak{p}} - 1)$. Theorems 4.5 and 4.7 imply that $K_0(\Lambda) \simeq \text{Cl}(\Delta) \oplus \mathbb{Z}^m$.

Example 4.10. Consider the *stable homotopy category of polyhedra* SW . Its objects are *pointed polyhedra*, that is finite CW-complexes with a fixed point, and the sets of morphisms are *stable homotopy classes* of continuous maps, $\text{Hos}(A, B) = \varinjlim \text{Hot}(S^n A, S^n B)$, where $\text{Hot}(A, B)$ is the set of homotopy classes of continuous maps preserving fixed points and $S^n A$ is the n -fold suspension of A . It is known [3] that SW is a fully additive and locally finite \mathbb{Z} -category. The bouquet (one-point union) $A \vee B$ plays the role of direct sum in this category. For a polyhedron A set

$$\begin{aligned}
 r_n(A) &= \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} \text{Hos}(S^n, A), \text{ where } S^n \text{ is the } n\text{-dimensional sphere,} \\
 B(A) &= \bigvee_n r_n(A) S^n, \\
 B_0(A) &= \bigvee_{r_n(A) > 0} S^n.
 \end{aligned}$$

Note that $\text{Hos}(S^n, A)$ is torsion if $n > \dim A$, so $B(A)$ and $B_0(A)$ are finite bouquets of spheres. Moreover, $QA \simeq QB(A)$ in the category QSW [6, Prop. 1.5], so we can take $\{S^n \mid n \in \mathbb{N}\}$ for $\mathbb{S}(\mathcal{A})$. Then $B(A) = \mathbb{S}(QA)$. It also implies that the map $QA \mapsto Q\hat{A}_p$ gives a bijection $\text{iso } QSW \xrightarrow{\sim} \text{iso } Q\widehat{SW}_p$. Therefore, every object from \widehat{SW}_p is of the form \hat{A}_p for an object from SW . In particular, $\mathbb{A}(\widehat{SW}_p) = \{\hat{A}_p \mid A \in \mathbb{A}(SW, p)\}$. Thus the p -atomic polyhedra are just indecomposable p -primary polyhedra in the sense of [8] or [3]. Recall that a polyhedron A is said to be p -primary if $A_p \not\simeq B(A)_p$, but $A_q \simeq B(A)_q$ for any prime $q \neq p$. As $g(\mathbb{Z}) = 1$, Theorems 4.5 and 4.7 imply the well-known theorem of Freyd [8] (see also [3, Th. 4.44]).

Theorem 4.11. *$K_0(SW)$ is a free abelian group with a basis consisting of isomorphism classes of spheres and genera of indecomposable p -primary polyhedra for all prime p .*

As $QB(A) \in \text{add } QB_0(A)$ and $g(B_0(A)) = 1$, Theorem 2.6 (2) implies the following result proved in [6, Th. 2.5], which is a strengthened variant of [8, Th. 1.3].

Theorem 4.12. *$G(A) = G(A')$ if and only if $A \oplus B_0(A) \simeq A' \oplus B_0(A)$.*

Note also that, using Theorem 2.5 we obtain the known example of non-uniqueness of decomposition in the category SW . Namely, let $A(n)$ denote the cone of the map $n\nu : S^6 \rightarrow S^3$, where ν is the generator of the groups $\pi_6(S^3) \simeq \mathbb{Z}/24\mathbb{Z}$. One can easily check that there are morphisms $A(1) \xrightarrow{\alpha} S^3 \vee S^7 \xrightarrow{\beta} A(1)$ such that $\alpha\beta = 24 \cdot 1_{S^3 \vee S^7}$ and $\beta\alpha = 24 \cdot 1_{A(1)}$. Then Theorem 2.5 (1) implies that $A(1) \vee S^3 \vee S^7 \simeq A(3) \vee A(8)$ (the polyhedra in the right part of this equality are, respectively, 2-primary and 3-primary). It is even a homotopic equivalence of spaces, since we are in the stable range. All polyhedra in these decompositions are indecomposable. Unlike Example 3.8, this one is of “global” nature. It is essential, since Corollary 3.7 implies that all localizations SW_p are Krull–Schmidt categories.

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