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Weight Modules over Generalized Weyl Algebras*

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This work originated in an attempt to comprehend a striking likeness between representations and cohomology theories of some algebras, such as $\mathbf{sl}(2, \mathbf{C})$ and its non-degenerated quantizations, modular $\mathbf{sl}(2)$ and degenerated quantizations of $\mathbf{sl}(2, \mathbf{C})$, Weyl algebra A_1, and others. As a result, the notion of *generalized Weyl algebras* (of which all those mentioned above turned out to be examples) and weight modules have arisen.

Fortunately, almost all the techniques necessary to elaborate the theory are present in the literature. The most important tools can be found in [1, 3]. So we were able to give a nearly complete description of weight modules over generalized Weyl algebras via reduction to a class of linear categories (called *chain and circle categories*) resembling those arising in [1]. We could emphasize that, though categories seem to be more complicated than algebras, as one needs to deal with many objects, this multitude itself provides a certain convenience, and sometimes a decisive one, at least for calculation, but often for comprehension as well.

In Sections 1 and 2 the subjects of the paper—generalized Weyl algebras, chain and circle categories, and weight modules—are introduced and the main correlations established between them. In Sections 3 and 4, a description of weight modules over chain and circle categories is given, while in Section 5 it is translated back to the generalized Weyl algebras.

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1. GW-ALGEBRAS AND WEIGHT MODULES

1.1. Let R be a ring. A category $\mathscr{C}$ is said to be an *R-category* if its morphism sets $\mathscr{C}(i, j)$ for the objects i, j are equipped with R-bimodule structure, the multiplication of its morphisms is R-linear with respect to both left and right R-module structure, and $(ar)b = a(rb)$ for any possible $a, b \in \operatorname{Mor} \mathscr{C}$, $r \in R$. Remark that, even if R is commutative, we do not suppose that $ar = ra$ for $a \in \operatorname{Mor} \mathscr{C}$, $r \in R$ (in the latter case C is said to be *R-linear*). If $\mathscr{C}$ contains only one object (i.e., is simply a ring), we call it an R-ring.

1.2. From now on suppose the ring R is commutative. Let $\mathscr{C}$ be an R-category and M be a $\mathscr{C}$-module, i.e., an additive functor from $\mathscr{C}$ to the category Ab of Abelian groups. Then, for each $i \in \operatorname{Ob} \mathscr{C}$, the group $M(i)$ becomes an R-module if we put $rv = (r1_i)v$. We write, as usual, av instead of $M(a)v$ for $v \in M(i)$, $a: i \to j$ in $\mathscr{C}$, and call elements of $\bigsqcup_{i \in \operatorname{Ob} \mathscr{C}} M(i)$ *the elements of the module* M.

1.3. Denote by $\operatorname{Max} R$ the set of all maximal ideals $\mathbf{m} \subset R$. For $\mathbf{m} \in \operatorname{Max} R$ and an R-module M, put $M_{\mathbf{m}} = \{v \in M \mid \mathbf{m}v = 0\}$. A $\mathscr{C}$-module M is said to be a *weight module* if $M = \sum_{\mathbf{m} \in \operatorname{Max} R} M_{\mathbf{m}}$. Denote by $\mathscr{W}(\mathscr{C})$ the category of all weight $\mathscr{C}$-modules. Put $\operatorname{Supp} M = \{\mathbf{m} \mid M_{\mathbf{m}} \neq 0\}$ and, for each subset $\omega \subseteq \operatorname{Max} R$, denote by $W_\omega(\mathscr{C})$ the full subcategory of $\mathscr{W}(\mathscr{C})$ consisting of all modules M with $\operatorname{Supp} M \subseteq \omega$.

1.4. Suppose we are given an automorphism σ of the ring R and an element $t \in R$. Then define the *generalized Weyl algebra* (*GW-algebra*) $\mathsf{A} = \mathsf{A}(R, \sigma, t)$ as the R-ring generated over R by two elements X, Y subject to the following relations:

- $Xr = \sigma(r)X$ and $rY = Y\sigma(r)$ for any $r \in R$;
- $YX = t$ and $XY = \sigma(t)$.

One can easily check that both as a left and as a right R-module A is free with a basis $\{1, X^n, Y^n \mid n \in \mathbf{N}\}$.

Remark. In most cases (cf. below), R is an algebra over some field K and σ is an algebra automorphism; thus A is really a K-algebra. Nevertheless, we retain the term "GW-algebra" even if it contains no ground field at all.

1.5. The cyclic group $\langle \sigma \rangle$ generated by σ acts on the set $\operatorname{Max} R$. Denote by Ω the corresponding orbit set. It follows from the definition of $\mathsf{A}(R, \sigma, t)$ that $XM_{\mathbf{m}} \subseteq M_{\sigma(\mathbf{m})}$ and $YM_{\mathbf{m}} \subseteq M_{\sigma^{-1}(\mathbf{m})}$ for any A-module M and $\mathbf{m} \in \operatorname{Max} R$. Hence, the following statement is obvious.

PROPOSITION. $\mathscr{W}(\mathsf{A}) = \coprod_{\omega \in \Omega} \mathscr{W}_\omega(\mathsf{A})$, *i.e., any weight* A*-module* M *decomposes into a direct sum*: $M = \bigoplus_{\omega \in \Omega} M_\omega$ *with* $\operatorname{Supp} M_\omega \subseteq \omega$.

1.6. EXAMPLES. (i) Let $R = K[T]$, the polynomial ring over some field K, $t = T$, and $\sigma(T) = qT + 1$ for some $q \in K \setminus \{0\}$. Then A is a quantization of the usual Weyl algebra A_1 over K (if $q = 1$, then $\mathsf{A} = A_1$).

(ii) Let $R = K[H, T]$, the polynomial ring in two variables, $t = T$, $\sigma(H) = H - 1$, and $\sigma(T) = T + H$. Then $\mathsf{A} \simeq U(\mathbf{g})$, the universal enveloping algebra of the 3-dimensional simple split Lie algebra $\mathbf{g}$ over K ($\mathbf{sl}(2, K)$, if char $K \neq 2$).

(iii) Now let $R = K[T, Z, Z^{-1}]$ (polynomials in T and Laurent polynomials in Z), $t = T$, $\sigma(Z) = q^{-1}Z$, and $\sigma(T) = T + (q - q^{-1})^{-1}(Z - Z^{-1})$ for some $q \in K \setminus \{0, \pm 1\}$. Then we obtain as A a quantization $U_q(\mathbf{g})$ of the algebra of the preceding example.

2. CHAIN AND CIRCLE CATEGORIES

2.1. Let L be a local commutative ring, $\mathbf{m}$ its only maximal ideal, and $K = L/\mathbf{m}$ its residue field. Suppose we are given a family $\{t_i \mid i \in \mathbf{Z}\}$ of elements of L. Then we define the *chain category* $\mathscr{C}(L, t_i)$ as the L-category with the object set $\mathbf{Z}$, generated (over L) by the set of morphisms $\{X_i, Y_i \mid i \in \mathbf{Z}\}$, where X_i: $i \to i + 1$ and Y_i: $i + 1 \to i$, subject to the relations:

- $X_i r = rX_i$ and $Y_i r = rY_i$ for each $r \in L$ and each $i \in \mathbf{Z}$;
- $Y_i X_i = t_i 1_i$ and $X_i Y_i = t_i 1_{i+1}$ for each $i \in \mathbf{Z}$.

2.2. PROPOSITION. *Let $\mathsf{A} = \mathsf{A}(R, \sigma, t)$ be a GW-algebra, $\omega \in \Omega$ an infinite orbit* (*cf.* 1.5), *and $\mathbf{m} \in \omega$. Then $\mathscr{W}_\omega(\mathsf{A}) \simeq \mathscr{W}(\mathscr{C})$ for the chain category $\mathscr{C} = \mathscr{C}(R_{\mathbf{m}}, \sigma^{-i}(t))$.*

Proof. Given a weight module $M \in \mathscr{W}(\mathsf{A})$, put $N(i) = M_{\sigma^i(\mathbf{m})}$. As σ^i induces an isomorphism $R_{\mathbf{m}} \xrightarrow{\sim} R_{\sigma^i(\mathbf{m})}$, we can consider $N(i)$ as an $R_{\mathbf{m}}$-module. For $v \in N(i)$, put $X_i v = Xv$ and $Y_{i-1} v = Yv$. Then $X_i v \in N(i + 1)$ and $Y_{i-1} v \in N(i - 1)$ (cf. 1.5). Moreover, if $r \in R_{\mathbf{m}}$, then $X_i rv = X\sigma(r)v = \sigma^{i+1}(r)Xv = rX_i v$. Analogously, $Y_i rv = rY_i v$. Further, $Y_i X_i v = YXv = \sigma^{-i}(t)v$ and $X_i Y_i v = \sigma^{-i}(t)v$. Hence, N is a $\mathscr{C}$-module and we obtain a functor F: $\mathscr{W}_\omega(\mathsf{A}) \to \mathscr{W}(\mathscr{C})$.

Conversely, if $N \in \mathscr{W}(\mathscr{C})$, put $M = \oplus_i N(i)$ and supply M with the A-module structure putting $rv = \sigma^{-i}(r)v$, $Xv = X_i v$, and $Yv = Y_{i-1}v$ for $v \in N(i)$ and $r \in R$. It gives us a functor F': $\mathscr{W}(\mathscr{C}) \to \mathscr{W}_\omega(\mathsf{A})$ obviously inverse to F.

2.3. Now suppose we are given a natural $p \in \mathbf{N}$ (not necessarily prime, perhaps even $p = 1$), an automorphism τ of L, and a family $\{t_i \mid i \in \mathbf{Z}_p\}$ of elements of L. Then we define the *circle category* $\mathscr{C}_p(L, t_i, \tau)$ as the L-category with the object set $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, generated by the set of morphisms $\{X_i, Y_i \mid i \in \mathbf{Z}_p\}$, where $X_i\colon i \to i+1$ and $Y_i\colon i+1 \to i$, subject to the relations:

- $X_i r = rX_i$, $Y_i r = rY_i$, and $X_i Y_i = t_i 1_{i+1}$ for each $r \in L$ and each $i \in \mathbf{Z}_p$ except for $p-1$;
- $X_{p-1} r = \tau(r) X_{p-1}$, $rY_{p-1} = Y_{p-1}\tau(r)$ for each $r \in L$ and $X_{p-1}Y_{p-1} = \tau(t_{p-1})1_0$;
- $Y_i X_i = t_i 1_i$ for each $i \in \mathbf{Z}_p$.

2.4. PROPOSITION. *Let* $\mathsf{A} = \mathsf{A}(R, \sigma, t)$ *be a GW-algebra,* $\omega \in \Omega$ *an orbit with p elements, and* $\mathbf{m} \in \omega$. *Then* $\mathscr{W}_\omega(\mathsf{A}) \simeq \mathscr{W}(\mathscr{C})$ *for the circle category* $\mathscr{C} = \mathscr{C}_p(R_m, \sigma^{-i}(t), \sigma^p)$.

The proof is analogous to that of 2.2 and hence is omitted.

3. WEIGHT MODULES OVER CHAIN CATEGORIES

3.1. To describe weight modules, we need some simple and known results. Recall that a full subcategory $\mathscr{S} \subseteq \mathscr{C}$ is called a *skeleton* of $\mathscr{C}$ provided the objects of $\mathscr{S}$ are pairwise non-isomorphic and any object of $\mathscr{C}$ is isomorphic to some object of $\mathscr{S}$. It is evident that in this case the categories of $\mathscr{C}$- and of $\mathscr{S}$-modules are equivalent.

3.2. LEMMA (cf., e.g., [2]). (i) *Let M be a simple* $\mathscr{C}$*-module and* $M(i) \neq 0$ *for some* $i \in \operatorname{Ob}\mathscr{C}$. *Then* $M(i)$ *is a simple* $\mathscr{C}(i, i)$*-module.*

(ii) *Conversely, for any simple* $\mathscr{C}(i, i)$*-module N there exists a unique (up to isomorphism) simple* $\mathscr{C}$*-module M such that* $M(i) \simeq N$ *as* $\mathscr{C}(i, i)$*-modules.*

3.3. From now on in this section $\mathscr{C}$ denotes a chain category $\mathscr{C}(L, t_i)$ (cf. 2.1).

Call $i \in \mathbf{Z}$ a *break* (for $\mathscr{C}$) if $t_i \in \mathbf{m}$. Denote by $\mathbf{B}$ the set of all breaks. As $\mathbf{B} \subseteq \mathbf{Z}$, it inherits the natural order. The following statement is evident.

PROPOSITION. *A skeleton* $\mathscr{S}$ *of the chain category* $\mathscr{C}$ *can be chosen as follows:*

1. *If* $\mathbf{B} = \emptyset$, *then* $\operatorname{Ob}\mathscr{S} = \{0\}$.
2. *If* $\mathbf{B} \neq \emptyset$ *and has no maximal element, then* $\operatorname{Ob}\mathscr{S} = \mathbf{B}$.
3. *If* $\mathbf{B} \neq \emptyset$ *and m is its maximal element, then* $\operatorname{Ob}\mathscr{S} = \mathbf{B} \cup \{m+1\}$.

3.4. Surely, the objects of $\mathscr{S}$ can be numbered by integers belonging to some interval $I \subseteq \mathbf{Z}$ (perhaps left or right or two-sided unbounded). Denote by $\bar{j}$ the object of $\mathscr{S}$ corresponding to $j \in I$.

It follows from the definition (2.1) that $\mathbf{m}\mathscr{C}$ is a two-sided ideal in $\mathscr{C}$. Hence, we can form the factor $\bar{\mathscr{S}} = \mathscr{S}/\mathbf{m}\mathscr{S}$ and, of course, weight $\mathscr{S}$-modules are just $\bar{\mathscr{S}}$-modules. But the same definition implies the following description of $\bar{\mathscr{S}}$.

3.5. PROPOSITION. *If* $\mathbf{B} = \emptyset$, *then* $\bar{\mathscr{S}}(0, 0) = K$ *for the only object* 0 *in* $\bar{\mathscr{S}}$. *Otherwise,* $\bar{\mathscr{S}}$ *is the K-category generated by the set of morphisms*

$$\left\{\bar{X}_j, \bar{Y}_j \mid J \in I, \textit{non-maximal}\right\}, \qquad \textit{where } \bar{X}_j\colon \bar{j} \to \overline{j+1}, \bar{Y}_j\colon \overline{j+1} \to \bar{j},$$

commuting with the elements of K and subject to the relations

$$\bar{X}_j\bar{Y}_j = 0 \qquad \textit{and} \qquad \bar{Y}_j\bar{X}_j = 0.$$

In particular, $\bar{\mathscr{S}}(\bar{j}, \bar{j}) = K$ *for any* $j \in I$.

3.6. COROLLARY. *Isomorphism classes of simple weight* $\mathscr{C}$*-modules are in* $1-1$ *correspondence with the elements of* I. *Namely, for each* $j \in I$, *the corresponding simple* $\mathscr{C}$*-module* M_j *is defined as follows*:

1. *If* $\bar{j}$ *is a break, then* $M_j(i) = K$ *for* $\overline{j-1} < i \le \bar{j}$ (*we put* $\overline{j-1} = -\infty$ *if* $j-1 \notin I$, *i.e.*, j *is the minimal break*), $M_j(i) = 0$ *otherwise*; $M_j(Y_i) = 1$ *and* $M_j(X_i) = \bar{t}_i$ (*the multiplication by* t_i *in* K) *for all* $\overline{j-1} < i < \bar{j}$ (*otherwise these mappings are automatically zeros*).

2. *If* $\bar{j} = m+1$ *for the maximal break* m, *then* $M_j(i) = K$ *for* $i > m$, $M_j(i) = 0$ *otherwise*; $M_j(Y_i) = 1$ *and* $M_j(X_i) = \bar{t}_i$ *for all* $i > m$.

3. *Last, if* $\mathbf{B} = \emptyset$, *then* $M_0(i) = K$, $M_0(Y_i) = 1$, *and* $M_0(X_i) = \bar{t}_i$ *for all* $i \in \mathbf{Z}$.

Here, as before, $K = L/\mathbf{m}$.

3.7. In the case considered one can also determine all $\bar{\mathscr{S}}$-modules, hence, all weight $\mathscr{C}$-modules. Namely, let $J \subseteq I$ be a non-empty subinterval of I and $J' \subseteq J$ be any subset not containing the maximal element of J (if the latter exists). Then define an $\bar{\mathscr{S}}$-module $N = N(J, J')$ by the rules:

- $N(\bar{j}) = K$ if $j \in J$ and $N(\bar{j}) = 0$ otherwise;
- $N(\bar{X}_j) = 1$ if $j \in J'$ and $N(\bar{X}_j) = 0$ otherwise;
- $N(\bar{Y}_j) = 1$ if $j \notin J'$ and $N(\bar{Y}_j) = 0$ otherwise.

Now an easy exercise in linear algebra leads to the following result.

3.8. THEOREM. *All* $\bar{\mathscr{S}}$*-modules* $N(J, J')$ *are indecomposable and any* $\bar{\mathscr{S}}$*-module decomposes uniquely into a direct sum of modules isomorphic to some of the* $N(J, J')$.

3.9. COROLLARY. *For each pair* $J' \subseteq J$ *as in* 3.7, *define a* $\mathscr{C}$-*module* $M = M(J, J')$ *as follows. Denote by* l *the maximal break preceding* $\bar{J}$ (*or* $l = -\infty$ *if such breaks do not exist*) *and by* m *the maximal element of* J *provided it is a break or* $m = \infty$ *otherwise. Put*:

- $M(i) = K$ *for* $l < i \leq m$ *and* $M(i) = 0$ *otherwise*;
- *if* $l < i < m$ *and* i *is not a break, then* $M(Y_i) = 1$ *and* $M(X_i) = \bar{t}_i$;
- *if* $i \in \bar{J}$ *and is a break, then*

$$M(X_i) = 1 \qquad \text{if } i \in \bar{J}' \text{ and } M(X_i) = 0 \text{ otherwise;}$$

$$M(Y_i) = 1 \qquad \text{if } i \notin \bar{J}' \cup \{m\} \text{ and } M(Y_i) = 0 \text{ otherwise.}$$

(*In other cases* $M(X_i) = 0$ *and* $M(Y_i) = 0$ *automatically*).

Then all $M(J, J')$ *are indecomposable weight* $\mathscr{C}$-*modules and any weight* $\mathscr{C}$-*module decomposes uniquely into a direct sum of modules isomorphic to some of the* $M(J, J')$.

4. WEIGHT MODULES OVER CIRCLE CATEGORIES

4.1. In this section $\mathscr{C}$ denotes a circle category $\mathscr{C}_p(L, t_i, \tau)$ (cf. 2.3). Again call $i \in \mathbf{Z}_p$ a *break* (for $\mathscr{C}$) if $t_i \in \mathbf{m}$ and denote by $\mathbf{B}$ the set of all breaks.

PROPOSITION. *A skeleton* $\mathscr{S}$ *of the circle category* $\mathscr{C}$ *can be chosen as follows*:

1. *If* $\mathbf{B} = \varnothing$, *then* $Ob\,\mathscr{S} = \{0\}$.
2. *If* $\mathbf{B} \neq \varnothing$, *then* $Ob\,\mathscr{S} = \mathbf{B}$.

4.2. Of course, there is no natural order on $\mathbf{Z}_p$. Instead we can define a "*circular order*": $i < j < k$ means that $0 < j - i < k - i < p$ (in $\mathbf{Z}$). If $\mathscr{S}$ contains m objects, put them in 1–1 correspondence with $\mathbf{Z}_m$ in such way that $i < j < k$ in $\mathbf{Z}_m$ implies $\bar{i} < \bar{j} < \bar{k}$ in $\mathbf{Z}_p$ (surely, it means nothing if $m = 1$). Here, again, $\bar{j}$ denotes the element of Ob$\,\mathscr{S}$ corresponding to $j \in \mathbf{Z}_m$.

4.3. The definition 2.3 again implies that $\mathbf{m}\mathscr{C}$ is an ideal in $\mathscr{C}$, so we can form the factor $\bar{\mathscr{S}} = \mathscr{S}/\mathbf{m}\mathscr{S}$ and give the following description.

PROPOSITION. (i) *If* $\mathbf{B} = \varnothing$, *then* $\bar{\mathscr{S}}(0, 0) \simeq \mathsf{P} = K[x, x^{-1}, \tau]$, *the skew Laurent polynomial ring over* K *with the automorphism* $\bar{\tau}$ *induced by* τ.

(ii) *If* $\mathbf{B} \neq \emptyset$, *then* $\bar{\mathscr{S}}$ *is the* K*-category generated by the set of morphisms* $\{\bar{X}_j\colon \bar{j} \to \overline{j+1}$ *and* $\bar{Y}_j\colon \overline{j+1} \to \bar{j} \mid j \in \mathbf{Z}_m\}$ *subject to the relations*:

- $\bar{X}_j$ *and* $\bar{Y}_j$ *commute with the elements of* K *if* $j \neq m-1$;
- $\bar{X}_{m-1} a = \bar{\tau}(a)$ *and* $a\bar{Y}_{m-1} = \bar{Y}_{m-1}\bar{\tau}(a)$ *for all* $a \in K$;
- $\bar{X}_j \bar{Y}_j = 0$ *and* $\bar{Y}_j\, \bar{X}_j = 0$ *for all* $j \in \mathbf{Z}_m$.

In particular,

$$\bar{\mathscr{S}}\big(\bar{j}, \bar{j}\big) \simeq \Delta = K\langle x, y \mid xy = yx = 0,\ xa = \bar{\tau}(a)x \quad \text{and} \quad ay = y\bar{\tau}(a) \ \text{for all } a \in K\rangle.$$

4.4. PROPOSITION. (i) *All simple* P*-modules are of the form* $\mathsf{P}/f\mathsf{P}$ *for irreducible elements* $f \in \mathsf{P}$.

(ii) *All simple* Δ*-modules are the following*:

- $N_0 = \Delta/(x, y)$;
- $N_{1,f} = \Delta/(f(x), y)$, *where* $f \neq x$ *is an irreducible element of the skew polynomial ring* $\mathsf{P}_0 = K[x, \bar{\tau}]$;
- $N_{2,f} = \Delta/(x, f(y))$, *where* $f \neq y$ *is an irreducible element of the skew polynomial ring* $\mathsf{P}_0^\circ = K[y, \bar{\tau}^{-1}]$.

Proof. The assertion (i) is well known, as P is a principal ideal ring [4]. To prove (ii), we need to remark that both xN and yN are submodules in any Δ-module N. Hence, if N is simple, each of them coincides either with N or with 0. But $xN = N$ implies $yN = 0$ and vice versa. Thus, either $xN = yN = 0$ (i.e., $N \simeq N_0$), or $xN = N$, $yN = 0$, or $xN = 0$, $yN = N$. Obviously, $\Delta/y\Delta \simeq \mathsf{P}_0$, $\Delta/x\Delta \simeq \mathsf{P}_0^\circ$, and both of them are principal ideal rings, which implies the assertion (ii).

4.5. Remark that the rings P_0 and P_0° are anti-isomorphic; hence, there is a natural 1–1 correspondence between their irreducible elements. Namely, to a polynomial $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d$ of P_0 there corresponds a polynomial $f^\circ = a_1 + ya_2 + \cdots + y^{d-1}a_d + y^d \in \mathsf{P}_0^\circ$. Using Lemma 3.2, we are able to reconstruct all simple weight $\mathscr{C}$-modules.

If $\mathbf{B} = \emptyset$, take an irreducible element $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d$ ($a_1 \neq 0$) in P. The corresponding simple module can be defined as follows:

- $M_f(i) = K^d$ for all $i \in \mathbf{Z}_p$;
- $M_f(Y_i) = 1$ and $M_f(X_i) = \bar{t}_i$ for all $i \neq p-1$;
- $M_f(X_{p-1})v = \bar{\tau}(F_f \overline{t_{p-1}} v)$, where

$$F_f = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & 0 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_d \end{pmatrix}$$

- $M_f(Y_{p-1})v = F_f^{-1}\bar{\tau}^{-1}(v)$.

If $|\mathbf{B}| = m > 0$, define first the simple modules M_j for $j \in \mathbf{Z}_m$:

- $M_j(i) = K$ if $\overline{j-1} < i \leq \bar{j}$ and $M_j(i) = 0$ otherwise;
- $M_j(Y_i) = 1$ and $M_j(X_i) = \bar{t}_i$ if $\overline{j-1} < i < \bar{j}$ and $i \neq p-1$;
- $M_j(Y_{p-1}) = \bar{\tau}^{-1}$ and $M_j(X_{p-1}) = \bar{\tau}\overline{t_{p-1}}$ if $\overline{j-1} < p-1 < j$.

Now let $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d \neq x$ be an irreducible element of P_0. Define the modules $M_{1,f}$ and $M_{2,f}$ in the following way:

- $M_{1,f}(i) = M_{2,f}(i) = K^d$ for all i;
- $M_{1,f}(Y_i) = 0$ and $M_{2,f}(X_i) = 0$ for all i;
- $M_{1,f}(X_i) = \bar{t}_i$ and $M_{2,f}(Y_i) = 1$ for $i \neq p-1$;
- $M_{1,f}(X_{p-1})$ is the semi-linear mapping with automorphism $\bar{\tau}$ defined by the matrix F_f (cf. above);
- $M_{2,f}(Y_{p-1})$ is the semi-linear mapping with automorphism $\bar{\tau}^{-1}$ defined by the matrix F_f^{-1}.

4.6. Recall that two elements f, g of a ring Λ are said to be *similar* if $\Lambda/f\Lambda \simeq \Lambda/g\Lambda$.

COROLLARY. (i) *If $\mathbf{B} = \emptyset$, then, for each irreducible polynomial $f \in \mathsf{P}$, the module M_f is a simple weight $\mathscr{C}$-module, each simple weight $\mathscr{C}$-module is isomorphic to one of them, and $M_f \simeq M_g$ if and only if f and g are similar in P.*

(ii) *If $\mathbf{B} \neq \emptyset$, then all modules M_j, $M_{1,f}$, and $M_{2,f}$, for irreducible $f \in \mathsf{P}_0$ $(f \neq x)$, are simple weight $\mathscr{C}$-modules, each simple $\mathscr{C}$-module is isomorphic to one of them, and the only isomorphisms between these modules are $M_{1,f} \simeq M_{1,g}$ and $M_{2,f} \simeq M_{2,g}$ if f and g are similar in P_0.*

4.7. To classify non-simple $\mathscr{C}$-modules, we need to suppose them to be *finite-dimensional* (i.e., all $M(i)$ to be finite-dimensional vector spaces over K): otherwise we must at least classify all linear (or semi-linear) mappings in infinite-dimensional vector spaces—the problem, which seems hopeless.

The case $\mathbf{B} = \varnothing$ is rather simple: we must determine finite-dimensional P-modules. But P is a principal ideal domain; thus the answer is well known [4]: the only indecomposable modules are $N_f = \mathsf{P}/f\mathsf{P}$ with f indecomposable in P (it means, by definition, that $\mathsf{P}/f\mathsf{P}$ is indecomposable) and $N_f \simeq N_g$ if and only if f and g are similar in P.

4.8. If $\mathbf{B} \neq \varnothing$, the morphisms $\overline{X}_j$, $\overline{Y}_j$ satisfy the relations $\overline{X}_j\overline{Y}_j = 0$ and $\overline{Y}_j\overline{X}_j = 0$. Thus the problem almost coincides with that of classification of pairs of mutually annihilating linear mappings solved in [3]. We need only take into account the graduation, different from that in [3], and the fact that our mappings are not linear but semi-linear. However, we encounter no significant trouble and, accurately following the quoted work, we arrive at a description of finite-dimensional $\overline{\mathscr{S}}$-modules.

Denote by $\mathbf{D}$ the free monoid generated by two letters x, y. Let $|w|$ be the length of the word $w \in \mathbf{D}$. For any such word $w = z_1 z_2 \cdots z_n$, where $z_k \in \{x, y\}$, and any $j \in \mathbf{Z}_m$, define the $\overline{\mathscr{S}}$-module $N = N_{j,w}$ (*module of the first kind* in the terminology of [3]) as follows.

Consider $n+1$ symbols $e_1, e_2, \dots, e_n$. As a K-basis of $N(\bar{l})$ take the set of all e_k with $j + k = l$ (in $\mathbf{Z}_m$). If there are no such indices, then $N(\bar{l}) = 0$. Define the action of $\overline{X}_l$ and $\overline{Y}_l$ on this basis by the rules

$$\overline{X}_l e_k = \begin{cases} e_{k+1} & \text{if } z_{k+1} = x \\ 0 & \text{otherwise;} \end{cases}$$

$$\overline{Y}_{l-1} e_k = \begin{cases} e_{k-1} & \text{if } z_k = y \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\overline{X}_{j+n} e_n = 0$ and $\overline{Y}_{j-1} e_0 = 0$. For instance, if $w = \varepsilon$ (the *empty word*, $|\varepsilon| = 0$), then $N_{j,\varepsilon}$ is the simple $\overline{\mathscr{S}}$-module corresponding to j.

Of course, defining $\overline{X}_{m-1}$ and $\overline{Y}_{m-1}$ on other elements, we must take into account the fact that they are not linear but semi-linear mappings with automorphisms $\bar{\tau}$ and $\bar{\tau}^{-1}$, respectively (the next paragraph, as well as 5.5 and 5.6, is concerned with the same remark).

4.9. Now call a word w an *m-word* if its length n is a multiple of m and *non-periodic* if it is not a power of another m-word. Let $\mathsf{P}_0 = K[x, \bar{\tau}]$ be the skew polynomial ring with automorphism $\bar{\tau}$ and $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d \neq x^d$ be an indecomposable polynomial in P_0. Put $a_r^\circ = \tau^{r-1}(a_r)$ $(r = 1, \dots, d)$. For any non-periodic m-word w and any $j \in \mathbf{Z}_m$ define an $\overline{\mathscr{S}}$-module $N = N_{w,f}$ (*of the second kind* [3]) as follows.

Consider dn elements e_{ks} $(k = 1, \dots, n,\ s = 1, \dots, d)$. Take as a basis of $N(\bar{l})$ the set of all e_{ks} with $k \equiv l \pmod m$. Define the action of $\overline{X}_l$ and

$\bar{Y}_{l-1}$ on this basis by the rules

$$\bar{X}_l e_{ks} = \begin{cases} e_{k+1,s} & \text{if } k \neq n \text{ and } z_{k+1} = x \\ e_{1,s+1} & \text{if } k = n,\ z_1 = x, \text{ and } s \neq d \\ -\sum_{r=1}^{d} a_r e_{1r} & \text{if } k = n,\ z_1 = x, \text{ and } s = d \\ 0 & \text{otherwise;} \end{cases}$$

$$\bar{Y}_{l-1} e_{ks} = \begin{cases} e_{k-1,s} & \text{if } k \neq 1 \text{ and } z_k = y \\ e_{n,s+1} & \text{if } k = 1,\ z_1 = y, \text{ and } s \neq d \\ -\sum_{r=1}^{d} a_r^{\circ} e_{nr} & \text{if } k = 1,\ z_1 = y, \text{ and } s = d \\ 0 & \text{otherwise.} \end{cases}$$

4.10. THEOREM. (i) *If $\mathbf{B} \neq \varnothing$, then the modules $N_{j,w}$ and $N_{w,f}$ built in* 4.8 *and* 4.9 *are indecomposable $\bar{\mathscr{S}}$-modules and any indecomposable finite-dimensional $\bar{\mathscr{S}}$-module is isomorphic to one of them.*

(ii) *The only isomorphisms between these modules are $N_{w,f} \simeq N_{w(k),g}$, where $k \equiv 0 \pmod m$, f and g are similar in P_0, and $w(k) = z_{k+1} \cdots z_n z_1 \cdots z_k$, i.e., is a cyclic permutation of w.*

4.11. We can reconstruct weight $\mathscr{C}$-modules M_f, $M_{j,w}$, and $M_{w,f}$ corresponding to the $\bar{\mathscr{S}}$-modules N_f, $N_{j,w}$, and $N_{w,f}$, respectively. Namely, M_f is defined as in 4.5. If $N = N_{j,w}$ or $N_{w,f}$, as defined in 4.8 and 4.9, then the corresponding M is defined by the rules

- $M(i) = N(\bar{l})$ if $\overline{l-1} < i \leq \bar{l}$;
- $X_{\bar{l}}$ and $Y_{\bar{l}}$ acts as $\bar{X}_l$ and $\bar{Y}_l$;
- if i is not a break and $i \neq p-1$, then $M(X_i) = \bar{t}_i$ and $M(Y_i) = 1$;
- last, if $p-1$ is not a break, then $M(X_{p-1}) = \bar{\tau}\overline{t_{p-1}}$ and $M(Y_{p-1}) = \bar{\tau}^{-1}$.

In particular, the simple modules M_j, $M_{1,f}$, and $M_{2,f}$ (cf. 4.5) coincide, respectively, with $M_{j,\varepsilon}$, $M_{x^m,f}$, and $M_{y^m,f}$.

4.12. COROLLARY. (i) *If $\mathbf{B} = \varnothing$, any indecomposable weight $\mathscr{C}$-module is isomorphic to M_f for some indecomposable element $f \in \mathsf{P}$ and $M_f \simeq M_g$ if and only if f and g are similar in P.*

(ii) *If $|\mathbf{B}| = m > 0$, any indecomposable weight $\mathscr{C}$-module is isomorphic either to $M_{j,w}$ for some $j \in \mathbf{Z}_m$ and $w \in \mathbf{D}$ or to $M_{w,f}$ for some non-periodic m-word $w \in \mathbf{D}$ and an indecomposable element $f \in \mathsf{P}_0$, $f \neq x^d$. The only isomorphisms between these modules are $M_{w,f} \simeq M_{w(k),g}$ with f and g similar in P_0 and $k \equiv 0 \pmod m$.*

4.13. *Remark.* Corollaries 4.6 and 4.12 "almost" solve the problem of describing weight $\mathscr{C}$-modules. We still have the problem of classifying indecomposable elements in P and P_0 up to similarity, which may be highly non-trivial. However, even the "usual" problem of describing irreducible polynomials over an arbitrary field is far from being solved, though no one doubts that the Frobenius normal form is a "good classification" of linear mappings in finite-dimensional vector spaces.

5. BACK TO GW-ALGEBRAS

5.1. Now return to the definitions and notations of 1.4 and 1.5. So A denotes a GW-algebra $\mathsf{A}(R, \sigma, t)$. We are able to restore the weight A-modules according to Propositions 2.2 and 2.4 and the description of the weight modules over chain and circle categories.

For $\mathbf{m} \in \operatorname{Max} R$, put $K_{\mathbf{m}} = K/\mathbf{m}$ and $t_{\mathbf{m}} = t + \mathbf{m} \in K_{\mathbf{m}}$. Call $\mathbf{m}$ *a break* (for A) if $t_{\mathbf{m}} = 0$. Let $\mathbf{B}$ be the set of all breaks. If $\omega \in \Omega = \operatorname{Max} R/\langle \sigma \rangle$, denote $\mathbf{B}_\omega = \mathbf{B} \cap \omega$.

For each finite orbit ω of p elements, fix a maximal ideal $\mathbf{m}(\omega) \in \omega$ supposing that it is a break (provided $\mathbf{B}_\omega \neq \varnothing$). Put $K_\omega = K_{\mathbf{m}(\omega)}$, $\mathsf{P}_\omega = K_\omega[x, x^{-1}, \tau_\omega]$ and $\mathsf{P}_{0\omega} = K_\omega[x, \tau_\omega]$, where τ_ω denotes the automorphism of K_ω induced by σ^p (note that $\sigma^p(\mathbf{m}) = \mathbf{m}$ for $\mathbf{m} \in \omega$).

The maximal ideals $\mathbf{m} \subset R$ belonging to finite $\langle \sigma \rangle$-orbits will be called *periodic*. Really, it means that $\sigma^p(\mathbf{m}) = \mathbf{m}$ for some $p > 0$.

5.2. Now construct the full set of representatives of isomorphism classes of indecomposable weight $\mathscr{C}$-modules M with $\operatorname{Supp} M \subseteq \omega$.

Suppose first that $|\omega| = \infty$ and $\mathbf{B}_\omega = \varnothing$. Let $V(\omega) = \bigoplus_{\mathbf{m} \in \omega} K_{\mathbf{m}}$. Supply $V(\omega)$ with A-module structure putting $Xv = \sigma(t_{\mathbf{m}} v)$ and $Yv = \sigma^{-1}(v)$ for $v \in K_{\mathbf{m}}$.

5.3. Suppose $|\omega| = p < \infty$ and $\mathbf{B}_\omega = \varnothing$. For any indecomposable polynomial $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d \in \mathsf{P}_\omega$ (with $a_1 \neq 0$), let $V(\omega, f) = \bigoplus_{\mathbf{m} \in \omega} K_{\mathbf{m}}^d$. Supply it with A-module structure putting, for $v \in K_{\mathbf{m}}^d$:

- $Xv = \sigma(t_{\mathbf{m}} v)$ if $\mathbf{m} \neq \mathbf{m}(\omega)$;
- $Xv = \sigma(F_f t_{\mathbf{m}} v)$ if $\mathbf{m} = \mathbf{m}(\omega)$;
- $Yv = \sigma^{-1}(v)$ if $\sigma^{-1}(\mathbf{m}) \neq (\mathbf{m})(\omega)$;
- $Yv = F_f^{-1}\sigma^{-1}(v)$ if $\sigma^{-1}(\mathbf{m}) = \mathbf{m}(\omega)$,

where the matrix F_f was defined in 4.5.

5.4. Suppose that $|\omega| = \infty$ and $\mathbf{B}_\omega \neq \varnothing$. Consider the natural order on ω: $\mathbf{m} < \sigma(\mathbf{m})$. If $\mathbf{B}_\omega$ possesses a maximal element $\mathbf{m}$, put $\mathbf{B}'_\omega = \mathbf{B} \cup \{\sigma(\mathbf{m})\}$, otherwise $\mathbf{B}'_\omega = \mathbf{B}_\omega$. Let $J \subseteq \mathbf{B}'_\omega$ be an interval (i.e., $\mathbf{m}' < \mathbf{m} < \mathbf{m}''$ in $\mathbf{B}_\omega$ and $\mathbf{m}', \mathbf{m}'' \in J$ implies $\mathbf{m} \in J$) and $J' \subseteq J$ be any subset not containing the maximal element of J if the latter exists. Denote by $\mathbf{m}_0$ the maximal break in ω preceding all elements of J or $-\infty$ if it does not exist; by $\mathbf{m}_1$ the maximal element of J if it exists and is a break or $+\infty$ otherwise. For each $\mathbf{m} \in \omega$, put $V_{\mathbf{m}} = K_{\mathbf{m}}$ if $\mathbf{m}_0 < \mathbf{m} \leq \mathbf{m}_1$ and $V_{\mathbf{m}} = 0$ otherwise. Supply $V(\omega, J, J') = \bigoplus_{\mathbf{m} \in \omega} V_{\mathbf{m}}$ with A-module structure putting, for $v \in V_{\mathbf{m}}$,

$$Xv = \begin{cases} \sigma(t_{\mathbf{m}} v) & \text{if } \mathbf{m} \text{ is not a break} \\ \sigma(v) & \text{if } \mathbf{m} \in J' \\ 0 & \text{otherwise;} \end{cases}$$

$$Yv = \begin{cases} 0 & \text{if } \sigma^{-1}(\mathbf{m}) \in J' \cup \{\mathbf{m}_1\} \\ \sigma^{-1}(v) & \text{otherwise.} \end{cases}$$

5.5. Suppose $|\omega| = p < \infty$ and $|\mathbf{B}_{\omega}| = m > 0$. Consider the natural "circular order" on ω: $\mathbf{m} < \mathbf{m}' < \mathbf{m}''$ means that $\mathbf{m}' = \sigma^{i}(\mathbf{m})$ and $\mathbf{m}'' = \sigma^{k}(\mathbf{m})$ for some $0 < i < k < p$. Define the 1–1 correspondence $\mathbf{Z}_m \to \mathbf{B}_{\omega}$, $j \to \mathbf{m}_j$ in such a way that $i < j < k$ in $\mathbf{Z}_m$ implies $\mathbf{m}_i < \mathbf{m}_j < \mathbf{m}_k$ in ω and $\mathbf{m}_0 = \mathbf{m}(\omega)$ (cf. 5.1). Denote by $j(\mathbf{m})$ the only $j \in \mathbf{Z}_m$ such that $\mathbf{m}_{j-1} < \mathbf{m} \leq \mathbf{m}_j$.

For each $j \in \mathbf{Z}_m$ and each word $w = z_1 z_2 \cdots z_n \in \mathbf{D}$ (cf. 4.8), consider $n+1$ symbols $e_1, e_2, \ldots, e_n$. If $\mathbf{m} \in \omega$, denote by $V_{\mathbf{m}}$ the vector space over $K_{\mathbf{m}}$ with a basis consisting of all pairs $(\mathbf{m}, e_k)$ such that $j + k = j(\mathbf{m})$ (in $\mathbf{Z}_m$). Put $V(\omega, j, w) = \bigoplus_{\mathbf{m} \in \omega} V_{\mathbf{m}}$ and supply it with A-module structure by the rules

$$X(\mathbf{m}, e_k) = \begin{cases} t_{\mathbf{m}}(\sigma(\mathbf{m}), e_k) & \text{if } \mathbf{m} \text{ is not a break} \\ (\sigma(\mathbf{m}), e_{k+1}) & \text{if } \mathbf{m} \in \mathbf{B} \text{ and } z_{k+1} = x \\ 0 & \text{otherwise;} \end{cases}$$

$$Y(\mathbf{m}, e_k) = \begin{cases} (\sigma^{-1}(\mathbf{m}), e_k) & \text{if } \mathbf{m} \text{ is not a break} \\ (\sigma^{-1}(\mathbf{m}), e_{k-1}) & \text{if } \mathbf{m} \in \mathbf{B} \text{ and } z_k = y \\ 0 & \text{otherwise} \end{cases}$$

(taking into account, of course, that X and Y are semi-linear).

5.6. Now let $w \in \mathbf{D}$ be a non-periodic m-word (cf. 4.9) and $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d \neq x^d$ be an indecomposable polynomial from $\mathsf{P}_{0\omega}$. Consider dn symbols e_{ks} $(k = 1, \ldots, n,\ s = 1, \ldots, d)$. If $\mathbf{m} \in \omega$, denote by $V_{\mathbf{m}}$ the vector space over K_m with a basis consisting of all pairs $(\mathbf{m}, e_{ks})$ such that $k \equiv j \pmod{m}$. Put $V(\omega, w, f) = \bigoplus_{\mathbf{m} \in \omega} V_{\mathbf{m}}$ and supply it with

A-module structure by the rules

$$X(\mathbf{m}, e_{ks}) = \begin{cases} t_m(\sigma(\mathbf{m}), e_{ks}) & \text{if } \mathbf{m} \text{ is not a break} \\ (\sigma(\mathbf{m}), e_{k+1,s}) & \text{if } \mathbf{m} \in \mathbf{B},\ k \neq n \text{ and } z_{k+1} = x \\ (\sigma(\mathbf{m}), e_{1,s+1}) & \text{if } \mathbf{m} \in \mathbf{B},\ k = n,\ z_1 = x,\ s \neq d \\ -\sum_{r=1}^{d-1} a_r(\sigma(\mathbf{m}), e_{1r}) & \text{if } \mathbf{m} \in \mathbf{B},\ k = n,\ z_1 = x,\ s = d \\ 0 & \text{otherwise;} \end{cases}$$

$$Y(\mathbf{m}, e_{ks})$$

$$= \begin{cases} (\sigma^{-1}(\mathbf{m}), e_{ks}) & \text{if } \sigma^{-1}\mathbf{m} \text{ is not a break} \\ (\sigma^{-1}(\mathbf{m}), e_{k-1,s}) & \text{if } \sigma^{-1}(\mathbf{m}) \in \mathbf{B},\ k \neq 1 \text{ and } z_k = y \\ (\sigma^{-1}(\mathbf{m}), e_{n,s+1}) & \text{if } \sigma^{-1}(\mathbf{m}) \in \mathbf{B},\ k = 1,\ z_1 = y,\ s \neq d \\ -\sum_{r=1}^{d-1} a_r^\circ (\sigma^{-1}(\mathbf{m}), e_{nr}) & \text{if } \sigma^{-1}(\mathbf{m}) \in \mathbf{B},\ k = 1,\ z_1 = y,\ s = d \\ 0 & \text{otherwise} \end{cases}$$

(cf. 4.9 for the definition of a_r°).

5.7. THEOREM. (i) *The* A*-modules* $V(\omega)$, $V(\omega, f)$, $V(\omega, J, J')$, $V(\omega, j, w)$, *and* $V(\omega, w, f)$ *defined, respectively, in* 5.2–5.6 *are indecomposable weight* A*-modules*.

(ii) *Every weight* A*-module* M *such that all* $M(\mathbf{m})$ *with periodic* $\mathbf{m}$ *are finite-dimensional decomposes uniquely into a direct sum of modules isomorphic to those listed in* (i).

(iii) *The only isomorphisms between the listed modules are*:

- $V(\omega, f) \simeq V(\omega, g)$ *if* f *and* g *are similar in* P_ω;
- $V(\omega, w, f) \simeq V(\omega, w(k), g)$ *if* f *and* g *are similar in* $\mathsf{P}_{0\omega}$ *and* $k \equiv 0 \pmod{|\mathbf{B}_\omega|}$.

5.8. THEOREM. *The weight* A*-modules* $V(\omega)$, $V(\omega, f)$ *for irreducible* $f \in \mathsf{P}_\omega$, $V(\omega, j, 0)$, $V(\omega, j, \varepsilon)$, *and* $V(\omega, w, f)$ *for* f *irreducible in* $\mathsf{P}_{0\omega}$ *and* $w = x^p$ *or* y^p, *where* $p = |\omega|$, *are simple and each simple weight* A*-module is isomorphic to one from this list*.

5.9. The construction of indecomposable weight modules implies directly the description of their supports. Here is their list:

- If $V = V(\omega)$, $V(\omega, f)$, or $V(\omega, w, f)$, then Supp $V = \omega$.
- Let $\mathbf{m}_1$ be the maximal element of J provided it exists and is a break or $+\infty$ otherwise; $\mathbf{m}_0$ be the maximal break preceding J provided it exists or $-\infty$ otherwise. Then Supp $V(J, J') = \{\mathbf{m} \in \omega \mid \mathbf{m}_0 < \mathbf{m} \leq \mathbf{m}_1\}$.
- Supp $V(\omega, j, w) = \omega$ if $l = |w| \geq |\mathbf{B}_\omega| - 1$, otherwise Supp $V(\omega, j, w) = \{\sigma^k(\mathbf{m}(j)) \mid 0 \geq k \geq l\}$.

5.10. Call a weight A-module M *R-finite* if it has finite length as an R-module. It means that $M(\mathbf{m}) = 0$ for all but a finite number of maximal ideals $\mathbf{m}$ and $\dim_{K_\mathbf{m}} M(\mathbf{m}) < \infty$ for each $\mathbf{m}$. Remark that if R is a finitely generated algebra over some field K, it means that the module M is finite-dimensional over K.

COROLLARY. (i) *A has R-finite weight modules if and only if there exists such orbit ω that either $|\omega| < \infty$ or $|\mathbf{B}_\omega| \geq 2$.*

(ii) *All indecomposable R-finite weight A-modules are isomorphic to either $V(\omega, f)$, $V(\omega, j, w)$, $V(\omega, w, f)$ for some finite orbit ω or to $V(\omega, J, J')$, where J has both a minimal and a maximal element and the latter is a break.*

(iii) *All R-finite weight A-modules are semi-simple if and only if all orbits are infinite and have at most two breaks.*

5.11. *Remark.* For the algebras $A = U(\mathbf{g})$ or $U_q(\mathbf{g})$ from Example 1.6(ii) or (iii) our notion of "weight modules" seems rather unusual. As a rule, in these cases an A-module M is said to be a weight module if it possesses an eigenbasis for the only operator H or, respectively, Z. We demand also the same for $T = XY$, which one can replace by the central element $C = H^2 + H + 2T$ (if char $K \neq 2$) or, respectively, $C = (q-1)(q-q^{-1}) + qZ + Z^{-1}$ (the "Casimir operator").

This, of course, does not imply the description of simple modules but has a great influence on indecomposable ones. Namely, under this weaker condition, the classification of indecomposable modules with finite-dimensional eigenspaces for H or Z can be given if orbits are infinite, i.e., if char $K = 0$ or, respectively, q is not a root of unity. This is due essentially to the fact that there can be no more than two breaks in an orbit. But whenever orbits become finite, there is no chance, in some sense, to give a good classification. For details in the case of $U(\mathbf{g})$, cf. [1]; for $U_q(\mathbf{g})$ the calculations are similar.

REFERENCES

1. Yu. A. Drozd, On representations of Lie algebra $\mathbf{sl}(2)$, *Visnik Kiev. Univ. Ser. Mat. Mekh.* **25** (1983), 70–77.
2. Yu. A. Drozd, S. A. Ovsienko, and V. M. Futorny, Harish-Chandra subalgebras and Gelfand–Zetlin modules, *in* "Finite Dimensional Algebras and Related Topics," Kluwer Academic, Dordrecht, 1994.
3. I. M. Gelfand and V. A. Ponomarev, Indecomposable representations of the Lorenz group, *Uspekhi Mat. Nauk* **23**, No. 2 (1968), 3–40. [*Russian Math. Surveys* **23**, No. 2 (1968), 1–58]
4. N. Jacobson, "The Theory of Rings," Amer. Math. Soc., Providence, RI, 1943.