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Introduction to Singularities and Deformations

Springer

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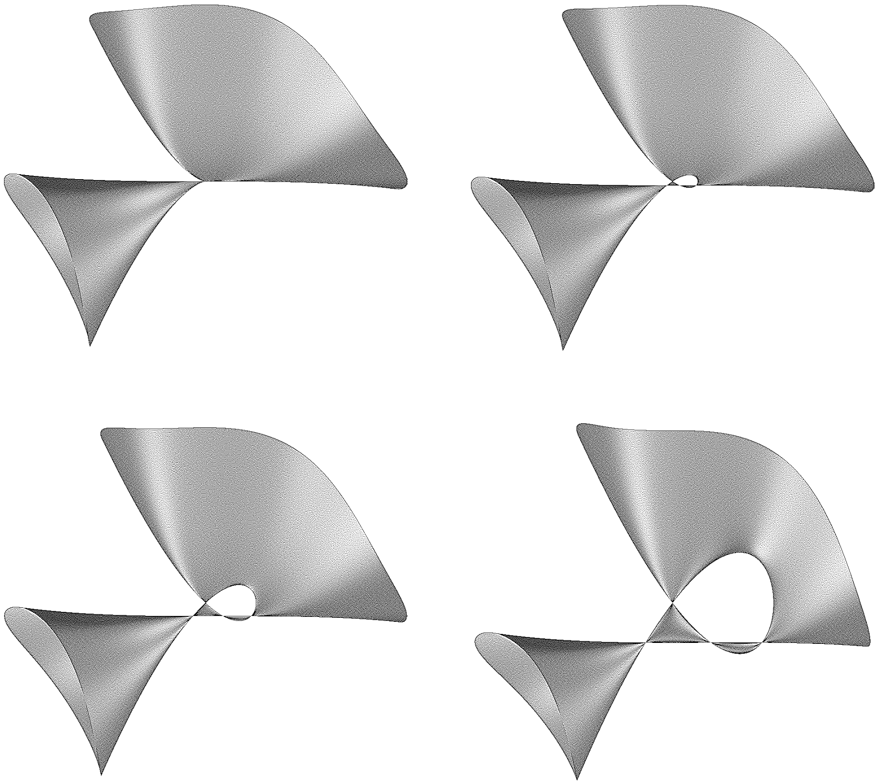
Meiner Mutter Irma und
der Erinnerung meines Vaters Wilhelm
G.-M.G.

Für Carmen, Katrin und Carolin
C.L.

To my parents Isaac and Maya
E.S.



A deformation of a simple surface singularity of type E_7 into four A_1-singularities. The family is defined by the equation

$$F(x,y,z;t) = z^2 - \left(x + \sqrt{\frac{4t^3}{27}}\right) \cdot \left(x^2 - y^2(y+t)\right).$$

The pictures[1] *show the surface obtained for $t=0$, $t=\frac{1}{4}$, $t=\frac{1}{2}$ and $t=1$.*

[1] The pictures were drawn by using the program `surf` which is distributed with SINGULAR [GPS].

Preface

Singularity theory is a field of intensive study in modern mathematics with fascinating relations to algebraic geometry, complex analysis, commutative algebra, representation theory, the theory of Lie groups, topology, dynamical systems, and many more, and with numerous applications in the natural and technical sciences. The specific feature of the present *Introduction to Singularities and Deformations*, separating it from other introductions to singularity theory, is the choice of a material and a unified point of view based on the theory of analytic spaces.

This text has grown up from a preparatory part of our monograph *Singular algebraic curves* (to appear), devoted to the up-to-date theory of equisingular families of algebraic curves and related topics such as local and global deformation theory, the cohomology vanishing theory for ideal sheaves of zero-dimensional schemes associated with singularities, applications and computational aspects. When working at the monograph, we realized that in order to keep the required level of completeness, accuracy, and readability, we have to provide a relevant and exhaustive introduction. Indeed, many needed statements and definitions have been spread through numerous sources, sometimes presented in a too short or incomplete form, and often in a rather different setting. This, finally, has led us to the decision to write a separate volume, presenting a self-contained textbook on the basic singularity theory of analytic spaces, including local deformation theory, and the theory of plane curve singularities.

Having in mind to get the reader ready for understanding the volume *Singular algebraic curves*, we did not restrict the book to that specific purpose. The present book comprises material which can partly be found in other books and partly in research articles, and which for the first time is exposed from a unified point of view, with complete proofs which are new in many cases. We include many examples and exercises which complement and illustrate the general theory. This exposition can serve as a source for special courses in singularity theory and local algebraic and analytic geometry. A special attention is paid to the computational aspect of the theory, illustrated by a number of examples of computing various characteristics via the computer algebra system SINGULAR [GPS][2]. Three appendices, including basic facts from sheaf theory, commutative algebra, and formal deformation theory, make the reading self-contained.

In the first part of the book we develop the relevant techniques, the basic theory of complex spaces and their germs and sheaves on them, including the key ingredients - the Weierstraß preparation theorem and its other forms (division theorem and finiteness theorem), and the finite coherence theorem. Then we pass to the main object of study, isolated hypersurface and plane curve singularities. Isolated hypersurface singularities and especially plane curve singularities form a classical research area which still is in the centre of current research. In many aspects they are simpler than general singularities, but on the other hand they are much richer in ideas, applications, and links to other branches of mathematics. Furthermore, they provide an ideal introduction to the general singularity theory. Particularly, we treat in detail the classical topological and analytic invariants, finite determinacy, resolution of singularities, and classification of simple singularities.

In the second chapter, we systematically present the local deformation theory of complex space germs with an emphasis on the issues of versality, infinitesimal deformations and obstructions. The chapter culminates in the treatment of equisingular deformations of plane curve singularities. This is a new treatment, based on the theory of deformations of the parametrization developed here with a complete treatment of infinitesimal deformations and obstructions for several related functors. We further provide a full disquisition on equinormalizable (δ-constant) deformations and prove that after base change, by normalizing the δ-constant stratum, we obtain the semiuniversal deformation of the parametrization. Equisingularity is first introduced for deformations of the parametrization and it is shown that this is essentially a linear theory and, thus, the corresponding semiuniversal deformation has a smooth base. By proving that the functor of equisingular deformations of the parametrization is isomorphic to the functor of equisingular deformations of the equation, we substantially enhance the original work by J. Wahl [Wah], and, in particular, give a new proof of the smoothness of the μ-constant stratum. Actually, this part of the book is intended for a more advanced reader familiar with the basics of modern algebraic geometry and commutative algebra. A number of illustrating examples and exercises should make the material more accessible and keep the textbook style, suitable for special courses on the subject.

Cross references to theorems, propositions, etc., within the same chapter are given by, e.g., "Theorem 1.1". References to statements in another chapter are preceded by the chapter number, e.g., "Theorem I.1.1".

[2] See [GrP, DeL] for a thorough introduction to SINGULAR and its applicability to problems in algebraic geometry and singularity theory.

Acknowledgements

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Kaiserslautern - Tel Aviv,
August 2006

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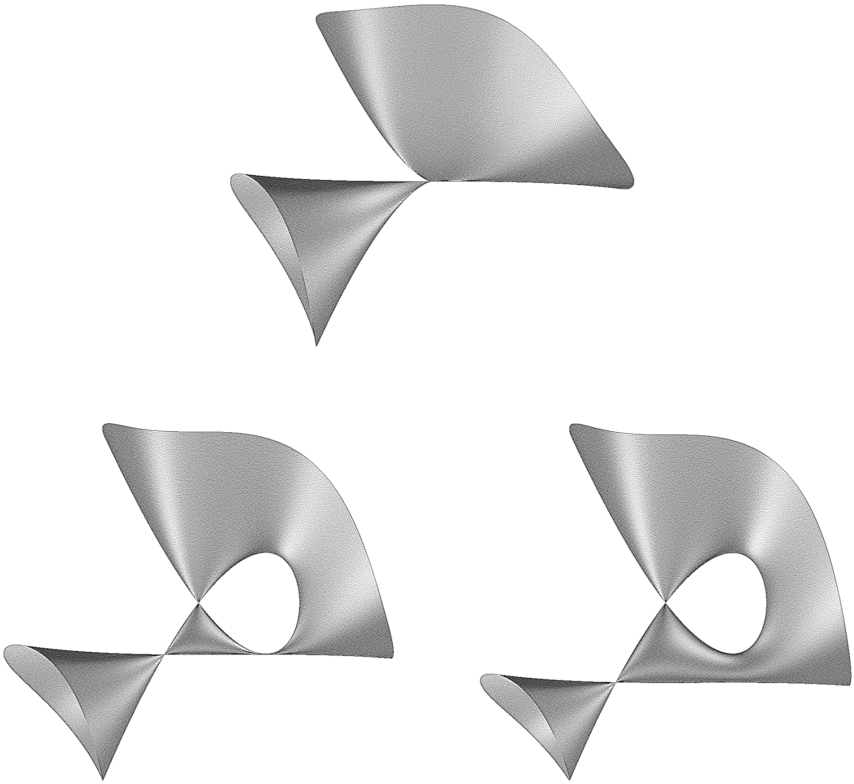
and E. Shustin



(a) (b)

Deformations of a simple surface singularity of type E_7 (a) into two A_1-singularities and one A_3-singularity, resp. (b) into two A_1-singularities, smoothing the A_3-singularity. The corresponding family is defined by

$$F(x,y,z;t) = z^2 - \left(x + \frac{3}{10}\sqrt{t^3}\right) \cdot \left(x^2 - y^2(y+t)\right),$$

resp. by

$$F(x,y,z;t) = z^2 - \left(x + \frac{6}{10}\sqrt{t^3}\right) \cdot \left(x^2 - y^2(y+t)\right).$$

Contents

I

Singularity Theory

> *"The theory of singularities of differentiable maps is a rapidly developing area of contemporary mathematics, being a grandiose generalization of the study of functions at maxima and minima, and having numerous applications in mathematics, the natural sciences and technology (as in the so-called theory of bifurcations and catastrophes)."*
> V.I. Arnol'd, S.M. Guzein-Zade, A.N. Varchenko [AGV].

The above citation describes in a few words the essence of what is called today often "singularity theory". A little bit more precisely, we can say that the subject of this relatively new area of mathematics is the study of systems of finitely many differentiable, or analytic, or algebraic, functions in the neighbourhood of a point where the Jacobian matrix of these functions is not of locally constant rank. The general notion of a "singularity" is, of course, much more comprehensive. Singularities appear in all parts of mathematics, for instance as zeroes of vector fields, or points at infinity, or points of indeterminacy of functions, but always refer to a situation which is not regular, that is, not the usual, or expected, one.

In the first part of this book, we are mainly studying the singularities of systems of complex analytic equations,

$$\begin{aligned} f_1(x_1,\dots,x_n) &= 0\,,\\ \vdots\qquad & \quad\vdots\\ f_m(x_1,\dots,x_n) &= 0\,, \end{aligned} \tag{0.0.1}$$

where the f_i are holomorphic functions in some open set of $\mathbb{C}^n$. More precisely, we investigate geometric properties of the solution set $V = V(f_1,\dots,f_m)$ of a system (0.0.1) in a small neighbourhood of those points, where the analytic set V fails to be a complex manifold. In algebraic terms, this means to study *analytic* $\mathbb{C}$*-algebras*, that is, factor algebras of power series algebras over the field of complex numbers. Both points of view, the geometric one and the algebraic one, contribute to each other. Generally speaking, we can say that geometry provides intuition, while algebra provides rigour.

Of course, the solution set of the system (0.0.1) in a small neighbourhood of some point $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{C}^n$ depends only on the ideal I generated by $f_1, \dots, f_m$ in $\mathbb{C}\{\boldsymbol{x}-\boldsymbol{p}\} = \mathbb{C}\{x_1 - p_1, \dots, x_n - p_n\}$. Even more, if J denotes the ideal generated by $g_1, \dots, g_\ell$ in $\mathbb{C}\{\boldsymbol{x}-\boldsymbol{p}\}$, then the Hilbert-Rückert Nullstellensatz states that $V(f_1, \dots, f_m) = V(g_1, \dots, g_\ell)$ in a small neighbourhood of $\boldsymbol{p}$ iff $\sqrt{I} = \sqrt{J}$. Here, $\sqrt{I} := \big\{ f \in \mathbb{C}\{\boldsymbol{x}-\boldsymbol{p}\} \bigm| f^r \in I \text{ for some } r \geq 0 \big\}$ denotes the *radical* of I.

Of course, this is analogous to Hilbert's Nullstellensatz for solution sets in $\mathbb{C}^n$ of complex polynomial equations and for ideals in the polynomial ring $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \dots, x_n]$. The Nullstellensatz provides a bridge between algebra and geometry.

The somewhat vague formulation "a sufficiently small neighbourhood of $\boldsymbol{p}$ in V" is made precise by the concept of the *germ* $(V, \boldsymbol{p})$ *of the analytic set* V at $\boldsymbol{p}$. Then the Hilbert-Rückert Nullstellensatz can be reformulated by saying that two analytic functions, defined in some neighbourhood of $\boldsymbol{p}$ in $\mathbb{C}^n$, define the same function on the germ $(V, \boldsymbol{p})$ iff their difference belongs to $\sqrt{I}$. Thus, the algebra of complex analytic functions on the germ $(V, \boldsymbol{p})$ is identified with $\mathbb{C}\{\boldsymbol{x}-\boldsymbol{p}\}/\sqrt{I}$.

However, although I and $\sqrt{I}$ have the same solution set, we loose information when passing from I to $\sqrt{I}$. This is similar to the univariate case, where the sets $V(x)$ and $V(x^k)$ coincide, but where the zero of the polynomial x, respectively x^k, is counted with multiplicity 1, respectively with multiplicity k. The significance of the multiplicity becomes immediately clear if we slightly "deform" x, resp. x^k: while $x - t$ has only one root, $(x-t)^k$ has k different roots for small $t \neq 0$. The notion of a *complex space germ* generalizes the notion of a germ of an analytic set by taking into account these multiplicities. Formally, it is just a pair, consisting of the germ $(V, \boldsymbol{p})$ and the algebra $\mathbb{C}\{\boldsymbol{x}-\boldsymbol{p}\}/I$. As $(V, \boldsymbol{p})$ is determined by I, analytic $\mathbb{C}$-algebras and germs of complex spaces essentially carry the same information (the respective categories are equivalent). One is the algebraic, respectively the geometric, mirror of the other. In this book, the word "singularity" will be used as a synonym for "complex space germ".

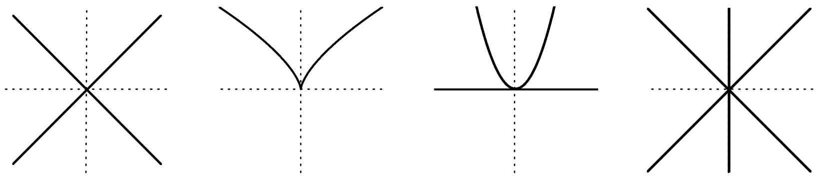
The concept of *coherent analytic sheaves* is used to pass from the local notion of a complex space germ to the global notion of a *complex space*. Indeed, the theory of sheaves is unavoidable in modern algebraic and analytic geometry as a powerful tool for handling questions that involve local solutions and global patching. Coherence of a sheaf can be understood as a local principle of analytic continuation, which allows to pass from properties at a point $\boldsymbol{p}$ to properties in a neighbourhood of $\boldsymbol{p}$.

For easy reference, we give a short account of sheaf theory in Appendix A. It should provide sufficient background on abstract sheaf theory for the unexperienced reader. Anyway, it is better to learn about sheaves via concrete examples such as the sheaf of holomorphic functions, than to start with the rather abstract theory.

Section 1 gives an introduction to the theory of analytic $\mathbb{C}$-algebras (even of analytic K-algebras, where K is any complete real valued field), and to complex spaces and germs of complex spaces. We develop the local Weierstraß theory, which is fundamental to local analytic geometry. The central aim of the first section is then to prove the finite coherence theorem, which states that for a finite morphism $f : X \to Y$ of complex spaces, the direct image $f_*\mathcal{F}$ of a coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-sheaf.

The usefulness of the finite coherence theorem for singularity theory can hardly be overestimated. Once it is proved, it provides a general, uniform and powerful tool to prove theorems which otherwise are hard to obtain, even in special cases. We use it, in particular, to prove the Hilbert-Rückert Nullstellensatz, which provides the link between analytic geometry and algebra indicated above. Moreover, the finite coherence theorem is used to give an easy proof for the (semi)continuity of certain fibre functions.

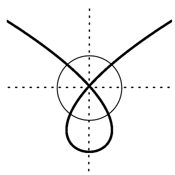
This pays off in Section 2, where we study the solution set of only one equation ($m = 1$ in (0.0.1)). The corresponding singularities, or the defining power series, are called *hypersurface singularities*. Historically, hypersurface singularities given by one equation in two variables, that is, *plane curve singularities*, can be seen as the initial point of singularity theory. For instance, in Newton's work on affine cubic plane curves, the following singularities appear:



$\{x^2 - y^2 = 0\}$ $\{x^2 - y^3 = 0\}$ $\{x^2y - y^2 = 0\}$ $\{x^3 - xy^2 = 0\}$

The pictures only show the set of real solutions. However, in the given cases, they also reflect the main geometric properties of the complex solution set in a small neighbourhood of the origin, such as the number of irreducible components (corresponding to the irreducible factors of the defining polynomial in the power series ring) and the pairwise intersection behaviour (transversal or tangential) of these components.

In concrete examples, as above, singularities are given by polynomial equations. However, for a hypersurface singularity given by a polynomial, the irreducible components do not necessarily have polynomial equations, too. Consider, for instance, the plane cubic curve $\{x^2 - y^2(1 + y) = 0\}$:



$\{x^2 - y^2(1 + y) = 0\}$

While $f := x^2 - y^2(1+y)$ is irreducible in the polynomial ring $\mathbb{C}[x,y]$ (and in its localization at $\langle x,y\rangle$), in the power series ring $\mathbb{C}\{x,y\}$ we have a decomposition

$$f = \left(x - y\sqrt{1+y}\right)\left(x + y\sqrt{1+y}\right)$$

into two non-trivial factors $x \pm y\sqrt{1+y} \in \mathbb{C}\{x,y\}$. (Note that $\sqrt{1+y}$ is a unit in $\mathbb{C}\{x,y\}$ but it is not an element of $\mathbb{C}[x,y]$.) As suggested by the picture, this shows that in a small neighbourhood of the origin the curve has two components, intersecting transversally, while in a bigger neighbourhood it is irreducible.

From a geometric point of view, there is no difference between the singularities at the origin of $\{x^2 - y^2 = 0\}$ and of $\{f = 0\}$. Algebraically, this is reflected by the fact that the factor rings $\mathbb{C}\{x,y\}/\langle x^2 - y^2\rangle$ and $\mathbb{C}\{x,y\}/\langle f\rangle$ are isomorphic (via $x \mapsto x$, $y \mapsto y\sqrt{1+y}$). We say that the two singularities have the *same analytic type*, or that the defining equations are *contact equivalent*, if their factor algebras are isomorphic.

Closely related to contact equivalence is the notion of *right equivalence*: two power series f and g are right equivalent if they coincide up to an analytic change of coordinates. In the late 1960's, V.I. Arnol'd started the *classification of hypersurface singularities* with respect to right equivalence. His work culminated, among others, in impressive lists of normal forms of singularities [AGV, II.16]. The singularities in these lists turned out to be of great improtance in other parts of mathematics and physics.

Most prominent is the list of *simple*, or *Kleinian*, or *ADE-singularities*, which have appeared in surprisingly diverse areas of mathematics. The above examples of plane curve singularities belong to this list: the corresponding classes are named A_1, A_2, A_3 and D_4. The letters A, D result from their relation to the simple Lie groups of type A, D. The indices $1,\dots,4$ refer to an important invariant of hypersurface singularities, the *Milnor number*, which for simple singularities coincides with another important invariant, the *Tjurina number*.

These invariants are introduced and studied in Section 2.1. We show, as an application of the finite coherence theorem, that they behave semicontinuously under deformation. Section 2.2 shows also that each isolated hypersurface singularity f has a polynomial normal form. They are actually determined (up to right as well as up to contact equivalence) by the Taylor series expansion up to a sufficiently high order. The remaining part of Section 2 is devoted to the (analytic) classification of singularities. In particular, in Section 2.4, we give a full proof for the classification of simple singularities as given by Arnol'd.

We actually do this for right and for contact equivalence. While the theory with respect to right equivalence is well-developed, even in textbooks, this is not the case for contact equivalence (which is needed in the second volume). It appears that Section 2 provides the first systematic treatment with full proofs for contact equivalence.

In Section 3, we focus on plane curve singularities, a particular case of hypersurface singularities, which is a classical object of study, but still in the centre of current research. Plane curve singularities admit a much more deep and complete description than general hypersurface singularities.

The aim of Section 3 is to present the two most powerful technical tools — the parametrization of local branches (irreducible components of germs of analytic curves) and the embedded resolution of singularities by a sequence of blowing ups — and then to give the complete topological classification of plane curve singularities. We also present a detailed treatment of various topological and analytic invariants.

The existence of analytic parametrizations is naturally linked with the algebraic closeness of the field of complex convergent Puiseux series $\bigcup_{m\geq 1}\mathbb{C}\{x^{1/m}\}$, and it can be proved by *Newton's constructive method.* Solving a polynomial equation in two variables with respect to one of them, Newton introduced what is nowadays called a *Newton diagram*. Newton's algorithm is a beautiful example of a combinatoric-geometric idea, solving an algebraic-analytic problem.

An immediate application of parametrizations is realized in the study of the intersection multiplicity of two plane curve germs, introduced as the total order of one curve on the parametrizations of the local branches of the other curve. This way of introducing the intersection multiplicity is quite convenient in computations as well as in deriving the main properties of the intersection multiplicity.

One of the most important geometric characterizations of plane curve singularities is based on the embedded resolution (desingularization) via subsequent blowing ups. Induction on the number of blowing ups to resolve the singularity serves as a universal technical tool for proving various properties and for computing numerical characteristics of plane curve singularities.

Our next goal is the topological classification of plane curve singularities. In contrast to analytic or contact equivalence, the topological one does not come from an algebraic group action. Another important distinction is that the topological classification is discrete, that is, it has no moduli, whereas the contact and right equivalences have. We give two descriptions of the topological type of a plane curve singularity: one via the characteristic exponents of the Puiseux parametrizations of the local branches and their mutual intersection multiplicities, and another one via the sequence of infinitely near points in the minimal embedded resolution and their multiplicities. Both descriptions are used to express the main topological numerical invariants, the Milnor number (the maximal number of critical points in a small deformation of the defining holomorphic function), the δ-invariant (the maximal number of critical points lying on the deformed curve in a small deformation of the curve germ), the κ-invariant (the number of ramification points of a generic projection onto a line of a generic deformation), and the relations between them.

General Notations and Conventions

We set $\mathbb{N} := \{n \in \mathbb{Z} \mid n \geq 0\}$, the set of non-negative integers.

(A) *Rings and Modules.* We assume the reader to be familiar with the basic facts from ideal and module theory. For more advanced topics, we refer to Appendix B and the literature given there.

All rings A are assumed to be commutative with unit 1, all modules M are unitary, that is, the multiplication by 1 is the identity map. If S is a subset of A (resp. of M), we denote by

$$\langle S\rangle := \langle S\rangle_A := \Big\{\sum_{\text{finite}} a_i f_i \mid a_i \in A,\ f_i \in S\Big\}$$

the ideal in A (resp. the submodule of M) generated by S.

We say that M is a *finite A-module* or *finite over A* if M is generated as A-module by a finite set. If $\varphi : A \to B$ is a ring map, $I \subset A$ an ideal, and M a B-module, then M is via $am := \varphi(a)m$ an A-module and IM denotes the submodule $\varphi(I)M$.

If K is a field, $K[\varepsilon]$ denotes the two-dimensional K-algebra with $\varepsilon^2 = 0$, that is, $K[\varepsilon] \cong K[x]/\langle x^2\rangle$. If A is a local ring, $\mathfrak{m}_A$ or $\mathfrak{m}$ denotes its maximal ideal.

(B) *Power Series and Polynomials.* If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we use the standard notations $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ to denote monomials, and

$$f = \sum_{|\boldsymbol{\alpha}|=0}^{\infty} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n}^{\infty} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n}^{\infty} c_{\alpha_1\cdots\alpha_n} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}\,,$$

$c_{\boldsymbol{\alpha}} \in A$, $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$, to denote *formal power series* over a ring A. If $c_{\boldsymbol{\alpha}} \neq 0$ then $c_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$ is called a (non-zero) *term* of the power series, and $c_{\boldsymbol{\alpha}}$ is called the *coefficient* of the term. The monomial $\boldsymbol{x}^{\mathbf{0}}$, $\mathbf{0} = (0, \dots, 0)$, is identified with $1 \in A$ and $c_{\mathbf{0}} =: f(\mathbf{0})$ is called the *constant term of f*. We write $f = 0$ iff $c_{\boldsymbol{\alpha}} = 0$ for all $\boldsymbol{\alpha}$. For f a non-zero power series, we introduce the *support* of f,

$$\operatorname{supp}(f) := \big\{\boldsymbol{\alpha} \in \mathbb{N}^n \bigm| c_{\boldsymbol{\alpha}} \neq 0\big\}\,,$$

and the *order* (also called the *multiplicity* or *subdegree*) of f,

$$\operatorname{ord}(f) := \operatorname{ord}_{\boldsymbol{x}}(f) := \operatorname{mt}(f) := \min\big\{|\boldsymbol{\alpha}| \bigm| \boldsymbol{\alpha} \in \operatorname{supp}(f)\big\}\,.$$

We set $\operatorname{supp}(0) = \emptyset$ and $\operatorname{ord}(0) = \infty$. Note that f is a *polynomial* (with coefficients in A) iff $\operatorname{supp}(f)$ is finite. Then the *degree* of f is defined as

$$\deg(f) := \deg_{\boldsymbol{x}}(f) := \begin{cases} \max\big\{|\boldsymbol{\alpha}| \bigm| \boldsymbol{\alpha} \in \operatorname{supp}(f)\big\} & \text{if } f \neq 0\,,\\ -\infty & \text{if } f = 0\,.\end{cases}$$

A polynomial f is called *homogeneous* if all (non-zero) terms have the same degree $|\boldsymbol{\alpha}| = \deg(f)$.

Polynomials in one variable are called *univariate*, those in several variables are called *multivariate*. For a univariate polynomial f, there is a unique term of highest degree, called the *leading term* of f. If the leading term has coefficient 1, we say that f is *monic*.

The usual addition and multiplication of power series $f = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} c_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$, $g = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} d_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$,

$$f + g = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} (c_{\boldsymbol{\alpha}} + d_{\boldsymbol{\alpha}})\boldsymbol{x}^{\boldsymbol{\alpha}}\,, \quad f \cdot g = \sum_{\nu=0}^{\infty} \sum_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|=\nu} (c_{\boldsymbol{\alpha}} d_{\boldsymbol{\beta}})\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}},$$

make the set of (formal) power series with coefficients in A a commutative ring with 1. We denote this ring by $A[[\boldsymbol{x}]] = A[[x_1, \dots, x_n]]$. As the A-module structure on $A[[\boldsymbol{x}]]$ is compatible with the ring structure, $A[[\boldsymbol{x}]]$ is an A-algebra. The polynomial ring $A[\boldsymbol{x}]$ is a subalgebra of $A[[\boldsymbol{x}]]$.

(C) *Spaces.* We denote by $\{\mathrm{pt}\}$ the topological space consisting of one point. As a complex space (see Section 1.3), we assume that $\{\mathrm{pt}\}$ carries the reduced structure (with local ring $\mathbb{C}$). T_ε denotes the complex space $(\{\mathrm{pt}\}, \mathbb{C}[\varepsilon])$ with $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/\langle t^2\rangle$, which is also referred to as a *fat point of dimension* 2. If X is a complex space and x a point in X, then $\mathfrak{m}_{X,x}$ or $\mathfrak{m}_x$ denotes the maximal ideal of the analytic local ring $\mathcal{O}_{X,x}$.

If X and S are complex spaces (or complex space germs), then X is called a *space (germ) over* S if a morphism $X \to S$ is given. A *morphism* $X \to Y$ *of spaces (space germs) over* S, or an S*-morphism*, is a morphism $X \to Y$ which commutes with the given morphisms $X \to S$, $Y \to S$. We denote by $\mathrm{Mor}_S(X, Y)$ the set of S-morphisms from X to Y. If $S = \{\mathrm{pt}\}$, we get morphisms of complex spaces (or of space germs), and we just write $\mathrm{Mor}(X, Y)$ instead of $\mathrm{Mor}_{\{\mathrm{pt}\}}(X, Y)$.

(D) *Categories and Functors.* We use the language of categories and functors mainly in order to give short and precise definitions and statements. If $\mathscr{C}$ is a category, then $C \in \mathscr{C}$ means that C is an object of $\mathscr{C}$. The set of morphisms in $\mathscr{C}$ from C to D is denoted by $\mathrm{Mor}_{\mathscr{C}}(C, D)$ or just by $\mathrm{Mor}(C, D)$. For the basic notations in category theory we refer to [GeM, Chapter 2].

The category of sets is denoted by $\mathcal{S}ets$. To take care of the usual logical difficulties, all sets are assumed to be in a fixed universe. Further, we denote by $\mathscr{A}_K$ the category of analytic K-algebras and by $\mathscr{A}_A$ the category of analytic A-algebras, where A is an analytic K-algebra (see Section 1.2).

1 Basic Properties of Complex Spaces and Germs

In the first half of this section, we develop the local Weierstraß theory and introduce the basic notions of complex spaces and germs, together with the notions of singular and regular points.

The Weierstraß techniques are then exploited for a proof of the finite coherence theorem, the main result of this section. We apply the finite coherence theorem to prove the Hilbert-Rückert Nullstellensatz and to show the semicontinuity of the fibre dimension of a coherent sheaf under a finite morphism of complex spaces. We study in some detail flat morphisms which are at the core of deformation theory. Flat morphisms impose several strong continuity properties on the fibres, in particular, for finite morphisms. These continuity properties will be of outmost importance in the study of invariants in families of complex spaces and germs.

Finally, we apply the theory of differential forms to give a characterition for singular points of complex spaces, respectively of morphisms of complex spaces. In particular, we show that in both cases the set of singular points is an analytic set.

1.1 Weierstraß Preparation and Finiteness Theorem

The Weierstraß preparation theorem is a cornerstone of local analytic algebra and, hence, of singularity theory. Its idea and purpose is to "prepare" a power series such that it becomes a *polynomial* in one variable with power series in the remaining variables as coefficients.

More or less equivalent to the Weierstraß preparation theorem is the Weierstraß division theorem which is the generalization of division with remainder for univariate polynomials. An equivalent, modern and invariant, way to formulate the Weierstraß division theorem is to express it as a finiteness theorem for morphisms of analytic algebras.

The preparation theorem, the division theorem and the finiteness theorem have numerous applications. They are used, in particular, to prove the Hilbert basis theorem and the Noether normalization theorem for power series rings.

Although we are mainly interested in complex analytic geometry, we eventually like to apply the results to questions about real varieties. Since the Weierstraß preparation theorem, as well as the division theorem and the finiteness theorem, can be proven without any extra cost for any complete real valued field, we formulate it in this generality.

Thus, throughout this section, let K denote a complete real valued field with real valuation $|\ | : K \to \mathbb{R}_{\geq 0}$ (see (A) on page 18). Examples are $\mathbb{C}$ and $\mathbb{R}$ with the usual absolute value, or any field with the trivial valuation.

For each $\boldsymbol{\varepsilon} \in (\mathbb{R}_{>0})^n$, we define a map

$$\|\ \|_{\boldsymbol{\varepsilon}} : K[[x_1, \ldots, x_n]] \to \mathbb{R}_{>0} \cup \{\infty\}$$

by setting

$$\|f\|_{\boldsymbol{\varepsilon}} := \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} |c_{\boldsymbol{\alpha}}| \cdot \boldsymbol{\varepsilon}^{\boldsymbol{\alpha}} \in \mathbb{R}_{>0} \cup \{\infty\}\,.$$

Note that $\|\ \ \|_{\boldsymbol{\varepsilon}}$ is a norm on the set of all power series f with $\|f\|_{\boldsymbol{\varepsilon}} < \infty$.

Definition 1.1. (1) A formal power series $f = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} c_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$ is called *convergent* iff there exists a real vector $\boldsymbol{\varepsilon} \in (\mathbb{R}_{>0})^n$ such that $\|f\|_{\boldsymbol{\varepsilon}} < \infty$.

$K\langle \boldsymbol{x}\rangle = K\langle x_1,\dots,x_n\rangle$ denotes the subring of all *convergent power series* in $K[[x_1,\dots,x_n]]$ (see also Exercise 1.1.3). For $K=\mathbb{C}$, $\mathbb{R}$ with the valuation given by the usual absolute value, we write $\mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1,\dots,x_n\}$, respectively $\mathbb{R}\{\boldsymbol{x}\}$, for the ring of convergent power series.

(2) A K-algebra A is called *analytic* if it is isomorphic (as K-algebra) to $K\langle x_1,\dots,x_n\rangle/I$ for some $n\geq 0$ and some ideal $I\subset K\langle \boldsymbol{x}\rangle$. A *morphism φ of analytic K-algebras* is, by definition, a morphism of K-algebras[1]. The *category of analytic K-algebras* is denoted by $\mathscr{A}_K$.

Remark 1.1.1. (1) $K[[\boldsymbol{x}]] = K\langle \boldsymbol{x}\rangle$ iff the valuation on K is trivial.

(2) $K\langle \boldsymbol{x}\rangle$ is a local ring, with maximal ideal

$$\mathfrak{m} = \mathfrak{m}_{K\langle \boldsymbol{x}\rangle} = \langle x_1,\dots,x_n\rangle = \big\{f\in K\langle \boldsymbol{x}\rangle \;\big|\; f(0)=0\big\}\,.$$

It follows that any analytic K-algebra is local with maximal ideal being the image of $\langle x_1,\dots,x_n\rangle$. In particular, the units in $K\langle \boldsymbol{x}\rangle/I$ are precisely the residue classes of power series with non-zero constant term.

(3) $K\langle \boldsymbol{x}\rangle$ is an *integral domain*, that is, it has no zerodivisors. To see this, note that the product of the lowest terms of two non-zero power series does not vanish. It follows that $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$.

(4) Any morphism $\varphi : A\to B$ of analytic K-algebras is automatically *local* (that is, it maps the maximal ideal of A to the maximal ideal of B).

Indeed, let $x\in\mathfrak{m}_A$, $\varphi(x) = y+c$ with $c\in K$, $y\in\mathfrak{m}_B$, and suppose that $c\neq 0$. Clearly, $x-c$ is a unit in A, hence $\varphi(x-c) = y$ is a unit, too, a contradiction.

(5) Any morphism $\varphi : K\langle x_1,\dots,x_n\rangle \to K\langle y_1,\dots,y_m\rangle$ is uniquely determined by the images $\varphi(x_i) =: f_i$, $i=1,\dots,n$. Indeed, φ is given by substituting the variables $x_1,\dots,x_n$ by power series $f_1,\dots,f_n$, and these power series necessarily satisfy $f_i\in\mathfrak{m}_{K\langle \boldsymbol{y}\rangle}$. Conversely, any collection of power series $f_1,\dots,f_n\in\mathfrak{m}_{K\langle \boldsymbol{y}\rangle}$ defines a unique morphism by mapping $g\in K\langle \boldsymbol{x}\rangle$ to

$$\varphi(g) = \varphi\Big(\sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\boldsymbol{x}^{\boldsymbol{\nu}}\Big) := \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\varphi(x_1)^{\nu_1}\cdot\ldots\cdot\varphi(x_n)^{\nu_n} = g(f_1,\dots,f_n)$$

(Exercise 1.1.4). We use the notation $g|_{(x_1,\dots,x_n)=(f_1,\dots,f_n)} := g(f_1,\dots,f_n)$.

[1] A map $\varphi : A\to B$ of K-algebras is called a *morphism*, if $\varphi(x+y) = \varphi(x)+\varphi(y)$, $\varphi(x\cdot y) = \varphi(x)\cdot\varphi(y)$ for all $x,y\in A$ and $\varphi(c) = c$ for all $c\in K$.

Many constructions for (convergent) power series are inductive, in each step producing new summands contributing to the final result. To get a well-defined (formal) limit for such an inductive process, the sequence of intermediate results has to be convergent with respect to the $\mathfrak{m}$-adic topology:

Definition 1.2. A sequence $(f_n)_{n\in\mathbb{N}} \subset K\langle \boldsymbol{x}\rangle$ is called *formally convergent*, or *convergent in the* $\mathfrak{m}$*-adic topology*, to $f \in K\langle \boldsymbol{x}\rangle$ if for each $k \in \mathbb{N}$ there exists a number N such that $f_n - f \in \mathfrak{m}^k$ for all $n \geq N$.

It is called a *Cauchy sequence* if for each $k \in \mathbb{N}$ there exists a number N such that $f_n - f_m \in \mathfrak{m}^k$ for all $m, n \geq N$.

Note that $K[[\boldsymbol{x}]]$ is complete with respect to the $\mathfrak{m}$-adic topology, that is, each Cauchy sequence in $K\langle \boldsymbol{x}\rangle$ is formally convergent to a formal power series. The limit series is uniquely determined as $\bigcap_{i\geq 0} \mathfrak{m}^i_{K\langle \boldsymbol{x}\rangle} = 0$. To show that it is a convergent power series requires then extra work.

Lemma 1.3. *Let A be an analytic algebra and M a finite A-module. Then*

$$\bigcap_{i\geq 0} \mathfrak{m}_A^i M = 0\,.$$

Proof. Let $A = K\langle \boldsymbol{x}\rangle/I$. Then $\mathfrak{m}_A^i = (\mathfrak{m}^i_{K\langle \boldsymbol{x}\rangle} + I)/I$, and $\bigcap_{i\geq 0} \mathfrak{m}_A^i = 0$ as $\bigcap_{i\geq 0} \mathfrak{m}^i_{K\langle \boldsymbol{x}\rangle} = 0$.

If M is a finite A-module, generated by $m_1, \dots, m_p \in M$, the map $\varphi : A^p \to M$ sending the canonical generators $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ to $m_1, \dots, m_p$ is an epimorphism inducing an epimorphism

$$0 = \Bigl(\bigcap_{i\geq 0} \mathfrak{m}_A^i\Bigr) \cdot A^p \cong \bigcap_{i\geq 0} \mathfrak{m}_A^i A^p \longrightarrow \bigcap_{i\geq 0} \mathfrak{m}_A^i M\,.$$

$\square$

Definition 1.4. $f \in K\langle x_1, \dots, x_n\rangle$ is called x_n*-general of order* b iff

$$f(0, \dots, 0, x_n) = c \cdot x_n^b + (\text{terms in } x_n \text{ of higher degree})\,,\ \ c \in K \setminus \{0\}\,.$$

Of course, not every power series is x_n-general of finite order, even after a permutation of the variables: consider, for instance, $f = x_1 x_2$. However, x_n-generality can always be achieved after some (simple) coordinate change (see also Exercise 1.1.6):

Lemma 1.5. *Let $f \in K\langle \boldsymbol{x}\rangle \setminus \{0\}$. Then there is an automorphism φ of $K\langle \boldsymbol{x}\rangle$, given by $x_i \mapsto x_i + x_n^{\nu_i}$, $\nu_i \geq 1$, for $i = 1, \dots, n-1$, and $x_n \mapsto x_n$, such that $\varphi(f)$ is of finite x_n-order.*

Proof. By Exercise 1.1.5 there exist $\boldsymbol{x}^{\boldsymbol{\alpha}^{(1)}},\dots,\boldsymbol{x}^{\boldsymbol{\alpha}^{(m)}}$ being a system of generators for the monomial ideal of $K[\boldsymbol{x}]$ spanned by $\{\boldsymbol{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha}\in \operatorname{supp}(f)\}$. That is, $\boldsymbol{\alpha}^{(1)},\dots,\boldsymbol{\alpha}^{(m)}\in \operatorname{supp}(f)$ and, for each $\boldsymbol{\alpha}\in\operatorname{supp}(f)$, there is some i such that $\alpha_j^{(i)}\le\alpha_j$ for each $j=1,\dots,n$.

Choose now $\boldsymbol{\nu}=(\nu_1,\dots,\nu_{n-1})\in(\mathbb{Z}_{>0})^{n-1}$ such that $\langle\boldsymbol{\nu},\boldsymbol{\alpha}^{(i)}\rangle\neq\langle\boldsymbol{\nu},\boldsymbol{\alpha}^{(j)}\rangle$ for $i\neq j$, where

$$\langle\boldsymbol{\nu},\boldsymbol{\alpha}\rangle \;:=\; \alpha_n+\sum_{j=1}^{n-1}\nu_j\alpha_j\,.$$

This means, in fact, that $\boldsymbol{\nu}$ has to avoid finitely many affine hyperplanes in $\mathbb{R}^{n-1}$, defined by $\langle\boldsymbol{\nu},\boldsymbol{\alpha}^{(i)}-\boldsymbol{\alpha}^{(j)}\rangle=0$, which is clearly possible.

Finally, define $\varphi(x_j):=x_j+x_n^{\nu_j}$; by Remark 1.1.1 (5), this defines a unique morphism $\varphi: K\langle\boldsymbol{x}\rangle\to K\langle\boldsymbol{x}\rangle$. For any monomial $\boldsymbol{x}^{\boldsymbol{\beta}}$ we have

$$\varphi(\boldsymbol{x}^{\boldsymbol{\beta}})\big|_{\boldsymbol{x}'=0}=x_n^{\langle\boldsymbol{\nu},\boldsymbol{\beta}\rangle}\,.$$

On the other hand, since the $\langle\boldsymbol{\nu},\boldsymbol{\alpha}^{(i)}\rangle$ are pairwise different, there is a unique $i_0\in\{1,\dots,m\}$ such that $b=\langle\boldsymbol{\nu},\boldsymbol{\alpha}^{(i_0)}\rangle$ is minimal among the $\langle\boldsymbol{\nu},\boldsymbol{\alpha}^{(i)}\rangle$. Thus, $\varphi(f)\big|_{\boldsymbol{x}'=0}=c_{\boldsymbol{\alpha}^{(i_0)}}\cdot x_n^b+$ higher order terms in x_n. □

Together with Lemma 1.5, the Weierstraß preparation theorem says now that each $f\in K[[x_1,\dots,x_n]]$ is, up to a change of coordinates and up to multiplication by a unit, a polynomial in x_n (with coefficients in $K[[x_1,\dots,x_{n-1}]]$):

Theorem 1.6 (Weierstraß preparation theorem – WPT).
Let $f\in K\langle\boldsymbol{x}\rangle=K\langle x_1,\dots,x_n\rangle$ be x_n-general of order b. Then there exists a unit $u\in K\langle\boldsymbol{x}\rangle$ and $a_1,\dots,a_b\in K\langle\boldsymbol{x}'\rangle=K\langle x_1,\dots,x_{n-1}\rangle$ such that

$$f\;=\;u\cdot\left(x_n^b+a_1x_n^{b-1}+\ldots+a_b\right). \tag{1.1.1}$$

Moreover, $u,a_1,\dots,a_b$ are uniquely determined.

Supplement: *If $f\in K\langle\boldsymbol{x}'\rangle[x_n]$ is a monic polynomial in x_n of degree b then $u\in K\langle\boldsymbol{x}'\rangle[x_n]$.*

Note that, in particular, $a_1(0)=\ldots=a_b(0)=0$, that is, $a_i\in\mathfrak{m}_{K\langle\boldsymbol{x}'\rangle}$.

Definition 1.7. A monic polynomial $x_n^b+a_1x_n^{b-1}+\ldots+a_b\in K\langle\boldsymbol{x}'\rangle[x_n]$ with $a_i\in\mathfrak{m}_{K\langle\boldsymbol{x}'\rangle}$ for all i is called a *Weierstraß polynomial* (in x_n, of degree b).

In some sense, the preparation turns f upside down, as the x_n-order (the lowest degree in x_n) of f becomes the x_n-degree (the highest degree in x_n) of the Weierstraß polynomial. This indicates that the unit u and the a_i must be horribly complicated.

Example 1.7.1. $f=xy+y^2+y^4$ is y-general of order 2. We have

$$f=\left(1+x^2-xy+y^2-2x^4+x^3y-\ldots\right)\cdot\left(y^2+y(x-x^3+3x^5+\ldots)\right),$$

which is correct up to degree 5.

The importance of the Weierstraß preparation theorem comes from the fact that, in inductive arguments with respect to the number of variables, only finitely many coefficients a_i have to be considered. In particular, we can find a common range of convergence for all $a_i \in K\langle \boldsymbol{x}' \rangle$.

We deduce the Weierstraß preparation theorem from the Weierstraß division theorem, which itself follows from the Weierstraß finiteness theorem.

Theorem 1.8 (Weierstraß division theorem – WDT).
Let $f \in K\langle \boldsymbol{x} \rangle$ be x_n-general of order b, and let $g \in K\langle \boldsymbol{x} \rangle$ be an arbitrary power series. Then there exist unique $h \in K\langle \boldsymbol{x} \rangle$, $r \in K\langle \boldsymbol{x}' \rangle[x_n]$ such that

$$g = h \cdot f + r, \qquad \deg_{x_n}(r) \le b-1\,. \tag{1.1.2}$$

In other words, as $K\langle \boldsymbol{x}' \rangle$-modules,

$$K\langle \boldsymbol{x} \rangle \cong K\langle \boldsymbol{x} \rangle \cdot f \oplus K\langle \boldsymbol{x}' \rangle \cdot x_n^{b-1} \oplus K\langle \boldsymbol{x}' \rangle \cdot x_n^{b-2} \oplus \cdots \oplus K\langle \boldsymbol{x}' \rangle\,.$$

In particular, $K\langle \boldsymbol{x} \rangle / \langle f \rangle$ is a free $K\langle \boldsymbol{x}' \rangle$-module with basis $1, x_n, \dots, x_n^{b-1}$.

Supplement: *If $f, g \in K\langle \boldsymbol{x}' \rangle[x_n]$, with f a monic polynomial of degree b in x_n then also $h \in K\langle \boldsymbol{x}' \rangle[x_n]$ and, hence, as $K\langle \boldsymbol{x}' \rangle$-modules,*

$$K\langle \boldsymbol{x}' \rangle[x_n] \cong K\langle \boldsymbol{x}' \rangle \cdot f \oplus K\langle \boldsymbol{x}' \rangle \cdot x_n^{b-1} \oplus \cdots \oplus K\langle \boldsymbol{x}' \rangle\,.$$

The division theorem reminds very much to the Euclidean division with remainder in the polynomial ring in one variable over a field K. Indeed, the Weierstraß division theorem says that every g is divisible by f with remainder r (provided f has finite x_n-order) such that the x_n-degree of r is strictly smaller than the x_n-order of f. If f is monic, then we can apply Euclidean division with remainder by f in $K\langle \boldsymbol{x}' \rangle[x_n]$. The uniqueness statement of the Weierstraß division theorem shows that the results of Euclidean and Weierstraß division coincide. This proves the supplement.

Corollary 1.9. *Let $g, g_1, \dots, g_m \in K\langle \boldsymbol{x} \rangle = K\langle \boldsymbol{x}', x_n \rangle$, and let $a \in \mathfrak{m}_{K\langle \boldsymbol{x}' \rangle}$. Then the following holds:*

(1) $g\big(\boldsymbol{x}', a\big) = 0$ iff $g = h \cdot (x_n - a)$ for some $h \in K\langle \boldsymbol{x} \rangle$.
(2) $\langle g_1, \dots, g_m, x_n - a \rangle = \langle g_1(\boldsymbol{x}', a), \dots, g_m(\boldsymbol{x}', a), x_n - a \rangle$ as ideals of $K\langle \boldsymbol{x} \rangle$.

Proof. $x_n - a$ is x_n-general of order 1. Thus, we may apply the division theorem and get $g = h_i(x_n - a) + r$ with $r \in \mathfrak{m}_{K\langle \boldsymbol{x}' \rangle}$. Substituting x_n by a on both sides gives $g(\boldsymbol{x}', a) = r$, and the two statements follow easily. □

Proof of "WDT ⇒ WPT". Let $g = x_n^b$ and apply the Weierstraß division theorem to obtain $x_n^b = fh + r$ with $h \in K\langle \boldsymbol{x} \rangle$, $\deg_{x_n}(r) < b$. We have

$$\begin{aligned} x_n^b &= (fh + r)\big|_{\boldsymbol{x}'=0} \\ &= \big(cx_n^b + \text{higher terms in } x_n\big) \cdot h\big|_{\boldsymbol{x}'=0} + \big(\text{terms in } x_n \text{ of degree } < b\big), \end{aligned}$$

and comparing coefficients shows that $h(0) \neq 0$. It follows that h is a unit and $f = h^{-1}(x_n^b - r)$. Uniqueness, respectively the supplement of the WDT implies uniqueness, respectively the supplement of the WPT. □

Example 1.9.1. $f = y - xy^2 + x^2$ is y-general of order 1. Division of $g = y$ by f gives $g = (1 + xy - x^3 + x^2y^2 - 2x^4y + \ldots) \cdot f + (-x^2 + x^5 + \ldots)$, which is correct up to degree 6.

The Weierstraß division theorem can also be deduced from the preparation theorem, at least in characteristic 0 (cf. [GrR, I.4, Supplement 3]).

We first prove the uniqueness statement of the Weierstraß division theorem, the existence statement follows from the Weierstraß finiteness theorem, which we formulate and prove below.

Proof of WDT, uniqueness. Suppose $g = fh + r = fh' + r'$ with power series $h, h' \in K\langle \boldsymbol{x} \rangle$, and $r, r' \in K\langle \boldsymbol{x}' \rangle[x_n]$ of x_n-degree at most $b-1$. Then

$$f \cdot (h - h') = r' - r \in K\langle \boldsymbol{x}' \rangle[x_n] , \quad \deg_{x_n}\big(r' - r\big) \le b - 1 .$$

It therefore suffices to show that from $fh = r$ with $\deg_{x_n}(r) \le b - 1$, it follows that $h = r = 0$. Write

$$f = \sum_{i=0}^{\infty} f_i(\boldsymbol{x}')x_n^i , \quad h = \sum_{i=0}^{\infty} h_i(\boldsymbol{x}')x_n^i , \quad r = \sum_{i=0}^{b-1} r_i(\boldsymbol{x}')x_n^i .$$

As f is x_n-general of order b, the coefficient f_b of f is a unit in $K\langle \boldsymbol{x}' \rangle$ ow and $\operatorname{ord}(f_i) \ge 1$ for $i = 0, \ldots, b-1$. Assuming that $h \ne 0$, there is a minimal k such that $\operatorname{ord}(h_k) \le \operatorname{ord}(h_i)$ for all $i \in \mathbb{N}$. Then, the coefficient of x_n^{b+k} in $fh - r$ equals

$$f_{b+k}h_0 + \ldots + f_{b+1}h_{k-1} + f_b h_k + f_{b-1}h_{k+1} + \ldots + f_0 h_{k+b} . \tag{1.1.3}$$

We have $\operatorname{ord}(f_b h_k) = \operatorname{ord}(h_k)$ (since f_b is a unit), while for $i > 0$,

$$\begin{aligned} \operatorname{ord}(f_{b+i}h_{k-i}) &\ge \operatorname{ord}(h_{k-i}) > \operatorname{ord}(h_k) , \\ \operatorname{ord}(f_{b-i}h_{k+i}) &> \operatorname{ord}(h_{k+i}) \ge \operatorname{ord}(h_k) . \end{aligned}$$

Thus, the sum (1.1.3) cannot vanish, contradicting the assumption that $fh - r = 0$. We conclude that $h = 0$, which immediately implies $r = 0$. □

Remark 1.9.2. (1) The existence part of the Weierstraß division theorem also holds for C^∞-functions, but not the uniqueness part (because of the existence of flat functions being non-zero but with vanishing Taylor series), cf. [Mat, Mal].

(2) For $f, g \in K\langle \boldsymbol{x} \rangle$ and a decomposition as in (1.1.2) with $h, r \in K[[\boldsymbol{x}]]$, the uniqueness statement in the Weierstraß division theorem implies that h and r are convergent, too. The same remark applies to the Weierstraß preparation theorem.

Theorem 1.10 (Weierstraß finiteness theorem – WFT).
Let $\varphi : A \to B$ be a morphism of analytic K-algebras, and let M be a finite B-module. Then M is finite over A iff $M/\mathfrak{m}_A M$ is finite over K.

Applying Nakayama's lemma B.3.6, we can specify the finiteness theorem as a statement on generating sets:

Corollary 1.11. *With the assumptions of the Weierstraß finiteness theorem, elements* $e_1, \dots, e_n \in M$ *generate* M *over* A *iff the corresponding residue classes* $\overline{e}_1, \dots, \overline{e}_n$ *generate* $M/\mathfrak{m}_A M$ *over* K.

We show below that the finiteness theorem and the Weierstraß division theorem are equivalent: first, we show that the WFT implies the WDT and give a proof of the WDT for formal power series. After some reduction, this proof is almost straightforward, inductively constructing power series of increasing order whose sum defines a formal power series. Then we show that the WDT implies the WFT and, afterwards, give the proof of Grauert and Remmert for the Weierstraß division theorem (with estimates to cover the convergent case, too).

Proof of "WFT ⇒ WDT, existence". Let $A = K\langle \boldsymbol{x}' \rangle$, $M = B = K\langle \boldsymbol{x} \rangle / \langle f \rangle$, with f being x_n-general of order b, and let $\varphi : A \to B$ be induced by the inclusion $K\langle \boldsymbol{x}' \rangle \hookrightarrow K\langle \boldsymbol{x} \rangle$. Then we have isomorphisms of K-vector spaces

$$\begin{aligned} M/\mathfrak{m}_A M = K\langle \boldsymbol{x} \rangle / \langle x_1, \dots, x_{n-1}, f \rangle \ &= \ K\langle \boldsymbol{x} \rangle / \langle x_1, \dots, x_{n-1}, x_n^b \rangle \\ &\cong K \oplus K \cdot x_n \oplus \dots \oplus K \cdot x_n^{b-1} . \end{aligned}$$

By the WFT, M is a finitely generated A-module, hence Nakayama's lemma is applicable and $1, \dots, x_n^{b-1}$ generate $K\langle \boldsymbol{x} \rangle / \langle f \rangle$ as a $K\langle \boldsymbol{x}' \rangle$-module. This means that $g = hf + r$ as required in the WDT. □

In terms of finite and quasifinite morphisms, we can reformulate the finiteness theorem:

Definition 1.12. A morphism $\varphi : A \to B$ of local K-algebras is called *quasifinite* iff $\dim_K B/\mathfrak{m}_A B < \infty$. It is called *finite* if B is a finite A-module (via φ).

Corollary 1.13. *Let* $\varphi : A \to B$ *be a morphism of analytic* K*-algebras. Then*

$$\varphi \text{ is finite} \iff \varphi \text{ is quasifinite.}$$

Proof of WFT, formal case. We proceed in two steps:

Step 1. Assume $A = K\langle \boldsymbol{x} \rangle = K\langle x_1, \dots, x_n \rangle$, $B = K\langle \boldsymbol{y} \rangle = K\langle y_1, \dots, y_m \rangle$.

Set $f_i := \varphi(x_i) \in \mathfrak{m}_B$, and let $e_1, \dots, e_p \in M$ be such that the corresponding residue classes generate $M/\mathfrak{m}_A M$ over K, that is, for any $e \in M$ there are $c_i \in K$, and $a_j \in M$ with

$$e = \sum_{i=1}^{p} c_i e_i + \sum_{j=1}^{n} f_j a_j .$$

Applying this to $a_j \in M$, we obtain the existence of $a_{j\nu} \in M$, $c_{ji} \in K$ such that

$$\begin{aligned} e &= \sum_{i=1}^{p} c_i e_i + \sum_{j=1}^{n} f_j \left(\sum_{i=1}^{p} c_{ji} e_i \; + \sum_{\nu=1}^{n} f_\nu a_{j\nu} \right) \\ &= \sum_{i=1}^{p} \left(c_i + \sum_{j=1}^{n} c_{ji} f_j \right) \cdot e_i \; + \sum_{j,\nu=1}^{n} f_j f_\nu a_{j\nu} \,, \end{aligned}$$

where the last sum is in $\mathfrak{m}_A^2 M$. Now, replace $a_{j\nu}$ by decompositions of the same kind, and repeat this process. After k steps we have

$$e = \sum_{i=1}^{p} \left(c_i^{(0)} + c_i^{(1)} + \cdots + c_i^{(k-1)} \right) \cdot e_i \; + \; d^{(k)}$$

with $c_i^{(j)} \in \mathfrak{m}_A^j B \subset \mathfrak{m}_B^j$ and $d^{(k)} \in \mathfrak{m}_A^k M \subset \mathfrak{m}_B^k M$. Since M is finite over B, Lemma 1.3 implies $\bigcap_{k=0}^{\infty} \mathfrak{m}_B^k M = 0$. Moreover, $\sum_j c_i^{(j)}$ is formally convergent. Hence, we obtain

$$e \; = \sum_{i=1}^{p} \left(\sum_{j=0}^{\infty} c_i^{(j)} \right) \cdot e_i,$$

which proves the WFT in this special case for formal power series.

Step 2. Let $A = K\langle \boldsymbol{x} \rangle / I$, $B = K\langle \boldsymbol{y} \rangle / J$ for some ideals I and J.

If M is a finite B-module, then it is also a finite $K\langle \boldsymbol{y} \rangle$-module. By Lemma 1.14, below, there exists a lifting

$$\begin{array}{ccc} K\langle \boldsymbol{x} \rangle & \xrightarrow{\widetilde{\varphi}} & K\langle \boldsymbol{y} \rangle \\ \downarrow & & \downarrow \\ A & \xrightarrow[\varphi]{} & B. \end{array}$$

Applying Step 1 to $\widetilde{\varphi}$ and using the fact that $M/\mathfrak{m}_A M = M/\mathfrak{m}_{K\langle \boldsymbol{x} \rangle} M$, it follows that M is finite over $K\langle \boldsymbol{x} \rangle$ and hence over A. □

The following *lifting lemma* will be strengthened in Lemmas 1.23 and 1.27.

Lemma 1.14. *Let $\varphi : K\langle \boldsymbol{x} \rangle / I \to K\langle \boldsymbol{y} \rangle / J$ be a morphism of analytic K-algebras. Then there exists a lifting $\widetilde{\varphi} : K\langle \boldsymbol{x} \rangle \to K\langle \boldsymbol{y} \rangle$ of φ with $\widetilde{\varphi}(I) \subset J$, that is, we have a commutative diagram*

$$\begin{array}{ccc} K\langle \boldsymbol{x} \rangle & \xrightarrow{\widetilde{\varphi}} & K\langle \boldsymbol{y} \rangle \\ \downarrow & & \downarrow \\ K\langle \boldsymbol{x} \rangle / I & \xrightarrow[\varphi]{} & K\langle \boldsymbol{y} \rangle / J. \end{array}$$

Proof. Let $\overline{x}_i \in K\langle \boldsymbol{x} \rangle / I$ be the image of x_i under the canonical projection $K\langle \boldsymbol{x} \rangle \twoheadrightarrow K\langle \boldsymbol{x} \rangle / I$. Choose $\widetilde{f}_i \in K\langle \boldsymbol{y} \rangle$ to be any preimage of $\varphi(\overline{x}_i)$ under the projection $K\langle \boldsymbol{y} \rangle \twoheadrightarrow K\langle \boldsymbol{y} \rangle / J$. Then we can define a lifting $\widetilde{\varphi}$ by setting $\widetilde{\varphi}(x_i) := \widetilde{f}_i$, which is well-defined according to Remark 1.1.1 (5). □

Proof of "WDT ⇒ WFT". Using Step 2 in the proof of the WFT in the formal case, it suffices to consider a morphism

$$\varphi : A = K\langle x_1, \ldots, x_m\rangle \to K\langle y_1, \ldots, y_n\rangle = B\,.$$

We can factorize φ,

$$\begin{tikzcd} & C = K\langle x_1, \ldots, x_m, y_1, \ldots, y_n\rangle \arrow[dr, two heads, "\widetilde{\varphi}"] & \\ A = K\langle x_1, \ldots, x_m\rangle \arrow[ur, hook, "i"] \arrow[rr, "\varphi"] & & K\langle y_1, \ldots, y_n\rangle = B, \end{tikzcd}$$

where $\widetilde{\varphi}$ is given by $\widetilde{\varphi}(x_i) := \varphi(x_i)$ and $\widetilde{\varphi}(y_j) := y_j$.

If M is a finite B-module, it is finite as a C-module, too. Hence, it suffices to prove the theorem for an injection $i : A \hookrightarrow C$. Furthermore, we can consider the chain of inclusions

$$A \subset C_1 \subset C_2 \subset \cdots \subset C_m = C\,, \quad C_i := K\langle \boldsymbol{x}, y_1, \ldots, y_i\rangle\,.$$

Hence, it suffices to consider the situation that one variable is added. That is, we are left with

$$\varphi \,:\, A = K\langle \boldsymbol{x}'\rangle = K\langle x_1, \ldots, x_{n-1}\rangle \hookrightarrow K\langle x_1, \ldots, x_n\rangle = K\langle \boldsymbol{x}\rangle = B\,.$$

Suppose that M is finite over B and that $M/\mathfrak{m}_A M$ is finite over K. Then there exist $e_1, \ldots, e_p \in M$ such that $M = e_1K + \ldots + e_pK + \mathfrak{m}_A M$ and $e_{p+1}, \ldots, e_q \in M$ such that $M = e_{p+1}B + \ldots + e_qB$. It follows that for any $e \in M$ there exist $b_j \in K + \mathfrak{m}_A B$ such that $e = b_1e_1 + \ldots + b_qe_q$. In particular, there exist $b_{ij} \in K + \mathfrak{m}_A B$ such that

$$x_n \cdot e_i \;=\; \sum_{j=1}^{q} b_{ij} \cdot e_j\,, \qquad i = 1, \ldots, q\,. \tag{1.1.4}$$

Consider the matrix $Z := x_n \cdot \mathbf{1}_q - (b_{ij})$. By Cramer's rule $f \cdot \mathbf{1}_q = Z^\sharp \cdot Z$, where $Z^\sharp$ is the adjoint matrix of Z and $f = \det Z$. We obtain

$$\begin{pmatrix} f \cdot e_1 \\ \vdots \\ f \cdot e_q \end{pmatrix} = Z^\sharp \cdot Z \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_q \end{pmatrix} \overset{(1.1.4)}{=} 0\,,$$

which means that $f \cdot M = 0$, and, hence, M is a finite $B/\langle f\rangle$-module. As $f(0, x_n)$ is a polynomial of degree q, f is x_n-general of order $b \leq q$. Hence, by the WDT, $B/\langle f\rangle$ is a finite $A = K\langle \boldsymbol{x}'\rangle$-module. Together with the above, we get that M is finite over A, generated by $x_n^j e_i$ with $0 \leq j \leq b-1$, $1 \leq i \leq q$. $\square$

Proof of WDT. As the statement of the WDT is obviously satisfied for $n=1$, we may assume that $n \geq 2$, and we set $B := K\langle \boldsymbol{x} \rangle$, $A := K\langle \boldsymbol{x}' \rangle$. Each $h \in K\langle \boldsymbol{x} \rangle$ decomposes as

$$h \;=\; \sum_{i=0}^{b-1} h_i(\boldsymbol{x}') \cdot x_n^i \;+\; x_n^b \cdot \sum_{i=0}^{\infty} h_{b+i}(\boldsymbol{x}') \cdot x_n^i \;=:\; \widehat{h} + x_n^b \widetilde{h}\,.$$

Since h converges, there is a $\boldsymbol{\rho} = (\boldsymbol{\rho}', \rho_n) \in (\mathbb{R}_{>0})^n$ such that

$$\|h\|_{\boldsymbol{\rho}} \;=\; \sum_{i=0}^{\infty} \|h_i\|_{\boldsymbol{\rho}'} \cdot \rho_n^i \;=\; \big\|\widehat{h}\big\|_{\boldsymbol{\rho}} + \rho_n^b \cdot \big\|\widetilde{h}\big\|_{\boldsymbol{\rho}} \;<\; \infty\,.$$

It follows that

$$\big\|\widetilde{h}\big\|_{\boldsymbol{\rho}} \;\leq\; \rho_n^{-b} \cdot \|h\|_{\boldsymbol{\rho}} \;<\; \infty\,. \tag{1.1.5}$$

In particular, $\widetilde{h} \in K\langle \boldsymbol{x} \rangle$. In this way, we decompose an x_n-general $f \in K\langle \boldsymbol{x} \rangle$ of order b as $f = \widehat{f} + x_n^b \widetilde{f}$, where $\widetilde{f} \in K\langle \boldsymbol{x} \rangle$ is a unit, and where $\widehat{f} = \sum_{i=0}^{b-1} f_i x_n^i$ with $f_i \in \mathfrak{m}_A$. Since f converges, $\|f_i\|_{\boldsymbol{\rho}'} \to 0$ for $\boldsymbol{\rho}' \to 0$. Hence, we can choose $\boldsymbol{\rho}$ such that:

- $\big\|f\big\|_{\boldsymbol{\rho}} < \infty\,,$
- $\big\|\widetilde{f}^{-1}\big\|_{\boldsymbol{\rho}} < \infty\,,$
- $\big\|\widetilde{f}^{-1}\big\|_{\boldsymbol{\rho}} \cdot \big\|f_i\big\|_{\boldsymbol{\rho}'} \;\leq\; \dfrac{1}{2b} \cdot \rho_n^{b-i}$ for $0 \leq i \leq b-1\,.$

Using Exercise 1.1.3, we obtain

$$\big\|\widetilde{f}^{-1} \cdot \widehat{f}\big\|_{\boldsymbol{\rho}} \;\leq\; \big\|\widetilde{f}^{-1}\big\|_{\boldsymbol{\rho}} \cdot \sum_{i=0}^{b-1} \|f_i\|_{\boldsymbol{\rho}'} \cdot \rho_n^i \;\leq\; \sum_{i=0}^{b-1} \frac{1}{2b} \cdot \rho_n^b \;=\; \frac{1}{2} \cdot \rho_n^b\,. \tag{1.1.6}$$

Now let $g = \widehat{g} + x_n^b \widetilde{g} \in K\langle \boldsymbol{x} \rangle$ be any element such that $\|g\|_{\boldsymbol{\rho}} < \infty$. We want to divide g by f with remainder of x_n-degree less than b. The idea is to take $\widehat{g}$ as part of the remainder and to recursively add correction terms.

Since $x_n^b = \widetilde{f}^{-1} f - \widetilde{f}^{-1} \widehat{f}$, we can write

$$g \;=\; \widehat{g} + \widetilde{f}^{-1} f \widetilde{g} - \widetilde{f}^{-1} \widehat{f} \widetilde{g}\,.$$

Note that $k_1 := -\widetilde{f}^{-1} \widehat{f} \widetilde{g} \in \mathfrak{m}_A B$, since $\widehat{f} \in \mathfrak{m}_A B$. Writing $k_1 = \widehat{k}_1 + x_n^b \widetilde{k}_1$, we get that $\widehat{k}_1$ and $\widetilde{k}_1$ both belong to $\mathfrak{m}_A B$. Now, we proceed recursively, defining

$$k_0 := g\,, \quad k_{i+1} := -\widetilde{f}^{-1} \widehat{f} \widetilde{k}_i\,, \quad i \geq 0\,.$$

We obtain $k_i = \widehat{k}_i + x_n^b \widetilde{k}_i \in \mathfrak{m}_A^i B$, hence, $\widehat{k}_i, \widetilde{k}_i \in \mathfrak{m}_A^i B$. An obvious induction shows that

$$g = \sum_{i=0}^{j} \widehat{k}_i + \left(\widetilde{f}^{-1} \cdot \sum_{i=0}^{j} \widetilde{k}_i \right) \cdot f + k_{j+1}$$

for all $j \geq 1$, and, as $\widehat{k}_i, \widetilde{k}_i \in \mathfrak{m}_A^i B$, $\sum_{i=0}^{\infty} \widehat{k}_i$ and $\sum_{i=0}^{\infty} \widetilde{k}_i$ define formal power series. As $\bigcap_{i=0}^{\infty} \mathfrak{m}_A^i B = 0$, we have

$$g = \sum_{i=0}^{\infty} \widehat{k}_i + \left(\widetilde{f}^{-1} \cdot \sum_{i=0}^{\infty} \widetilde{k}_i \right) \cdot f = r + h \cdot f \,,$$

which is the statement of the WDT for formal power series.

It remains to show that h is convergent. Then the convergence of g implies that r is convergent, too. The inequalities (1.1.5) and (1.1.6) yield

$$\big\|\widetilde{k}_{i+1}\big\|_{\boldsymbol{\rho}} \leq \rho_n^{-b} \big\|k_{i+1}\big\|_{\boldsymbol{\rho}} \leq \frac{1}{2} \cdot \big\|\widetilde{k}_i\big\|_{\boldsymbol{\rho}},$$

and

$$\left\| \sum_{i=0}^{\infty} \widetilde{k}_i \right\|_{\boldsymbol{\rho}} \leq \sum_{i=0}^{\infty} \frac{1}{2^i} \cdot \big\|\widetilde{k}_0\big\|_{\boldsymbol{\rho}} = 2 \big\|\widetilde{g}\big\|_{\boldsymbol{\rho}} < \infty \,.$$

As also $\big\|\widetilde{f}\big\|_{\boldsymbol{\rho}} < \infty$, Exercise 1.1.3 gives that h converges. $\square$

Remarks and Exercises

(A) *Discrete and Real Valuations.* In general terms, a *valuation* of a field K is a map $v : K^* \to G$ from the multiplicative group K^* of K to a totally ordered semiring $(G, \odot, \oplus)$, such that the conditions

$$v(ab) = v(a) \odot v(b) \,, \quad v(a+b) \leq v(a) \oplus v(b)$$

are satisfied for each $a, b \in K^*$. $(G, \odot)$ is called the *group of values* of v, and $v(a)$ is called the *value* of a.

A valuation of K is called a *real valuation* if $(G, \odot, \oplus) = (\mathbb{R}_{>0}, \cdot, +)$. Usually, we denote a real valuation by $|\;|$ instead of v, and extend it to a map $|\;| : K \to \mathbb{R}_{\geq 0}$ by setting $|0| := 0$. For $\mathbb{C}$ and each of its subfields, there is an obvious real valuation given by the usual absolute value. On the other hand, every field has the *trivial (real) valuation* assigning the value 1 to each $a \neq 0$. A sequence $(a_n)_{n \in \mathbb{N}}$ in K is called a *Cauchy sequence* (with respect to the valuation $|\;|$) if for each $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $|a_m - a_n| \leq \varepsilon$ for all $m, n \geq N$. We say that K is a *complete real valued field (with valuation $|\;|$)* iff every Cauchy sequence with respect to $|\;|$ converges in K.

Exercise 1.1.1. Prove the following statements:

(1) For finite fields, the trivial valuation is the only real valuation. Moreover, with the trivial valuation, each field is a complete real valued field.

(2) Let p be a prime number, then the map

$$v : \mathbb{Z} \setminus \{0\} \to \mathbb{R}_{>0}\,, \quad a \mapsto p^{-m} \text{ with } m := \max\{k \in \mathbb{N} \mid p^k \text{ divides } a\}$$

extends to a unique real valuation of $\mathbb{Q}$. With this valuation, $\mathbb{Q}$ is a real valued field that is not complete.

The completion of $\mathbb{Q}$ with respect to the valuation in (2) is called the field of *p-adic numbers.*

Let $\mathbb{Z}^{\text{inv}}$ denote $\mathbb{Z}$ equipped with the inverse of the natural order. Then a *discrete valuation (of rank* 1*)* of K is a valuation with values in $(\mathbb{Z}^{\text{inv}}, +, \min)$. That is, a discrete valuation on K is a map $v : K^* \to \mathbb{Z}$ such that

$$v(ab) = v(a) + v(b)\,, \quad v(a+b) \geq \min\{v(a), v(b)\}\,.$$

Usually, a discrete valuation is extended to a map $v : K \to \mathbb{Z} \cup \{\infty\}$ by setting $v(0) := \infty$. Then the set $R := \{a \in K \mid v(a) \geq 0\}$ defines a subring of K whose quotient field is K, and $\{a \in K \mid v(a) > 0\}$ defines a proper ideal of R, which is also called the *centre* of the discrete valuation. Note that the restriction of the valuation v to R uniquely determines v. R is also called a *discrete valuation ring.*

Clearly, the order function $\operatorname{ord} : K[[\boldsymbol{x}]] \to \mathbb{Z}_{\geq 0}$ defines a discrete valuation on $K[[\boldsymbol{x}]]$ that extends (in a unique way) to a discrete valuation of the quotient field

$$\operatorname{Quot}(K[[\boldsymbol{x}]]) = K[[\boldsymbol{x}]][\boldsymbol{x}^{-1}] = \left\{ \sum_{|\boldsymbol{\alpha}|=m}^{\infty} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \;\middle|\; m \in \mathbb{Z},\ c_{\boldsymbol{\alpha}} \in K \right\}.$$

Exercise 1.1.2. Let K be a field, and let $v : K^* \to \mathbb{Z}$ be a discrete valuation of K. Prove that

$$|a| := \begin{cases} 0 & \text{if } a = 0\,, \\ e^{-v(a)} & \text{otherwise}\,. \end{cases}$$

defines a real valuation $|\ | : K \to \mathbb{R}_{\geq 0}$ of K.

Exercise 1.1.3. (1) Let $\boldsymbol{\varepsilon} \in (\mathbb{R}_{>0})^n$, $f, g \in K[[\boldsymbol{x}]] = K[[x_1, \ldots, x_n]]$. Prove that $\|f \cdot g\|_{\boldsymbol{\varepsilon}} \leq \|f\|_{\boldsymbol{\varepsilon}} \cdot \|g\|_{\boldsymbol{\varepsilon}}$.

(2) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $K\langle\boldsymbol{x}\rangle$ with $\operatorname{ord}(f_{k+1}) > \operatorname{ord}(f_k)$ for all k. Show that $\sum_{k\in\mathbb{N}} f_k$ is a well-defined convergent power series.

Exercise 1.1.4. Prove Remark 1.1.1 (1), (2), (5).

Hint. To show (2), you may use the geometric series and Exercise 1.1.3 (2). For the proof of the uniqueness statement in (5), consider the difference $\varphi(g) - g(\varphi(\boldsymbol{x}))$ and prove by induction on m that it lies in each $\mathfrak{m}^m_{K\langle\boldsymbol{y}\rangle}$. To show convergence you may use a straightforward estimate.

Exercise 1.1.5. Show that each ideal $I \subset K\langle x_1, \dots, x_n\rangle$ which is generated by monomials can be generated by finitely many monomials.

More precisely, let $\leq$ be the natural partial ordering on $\mathbb{N}^n$ given by $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ iff $\alpha_i \leq \beta_i$ for all $i = 1, \dots, n$. If $I = \langle \boldsymbol{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Lambda\rangle$, show by induction on n that $\mathrm{Min}(\Lambda) := \{\boldsymbol{\alpha} \in \Lambda \mid \boldsymbol{\alpha}$ is minimal w.r.t. $\leq\}$ is finite and that the monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ $\boldsymbol{\alpha} \in \mathrm{Min}(\Lambda)$ generate I.

(B) *The x_n-Generality Assumption.* Lemma 1.5 shows that the x_n-generality assumption on $f \in K\langle \boldsymbol{x}\rangle = K\langle x_1, \dots, x_n\rangle$ in the Weierstraß preparation theorem can always be achieved after a polynomial change of coordinates. For large fields (as compared to the order of f), even a *linear* change of coordinates is sufficient:

Exercise 1.1.6. (1) Let $\boldsymbol{c} = (c_1, \dots, c_{n-1}) \in K^{n-1}$, and let φ be the linear automorphism of $K\langle \boldsymbol{x}\rangle = K\langle x_1, \dots, x_n\rangle$ given by

$$x_i \longmapsto \begin{cases} x_i + c_i x_n & \text{if } i \in \{1, \dots, n-1\}\,, \\ x_n & \text{if } i = n\,. \end{cases} \tag{1.1.7}$$

Show that $\varphi(f)$ is x_n-general of order $b = \mathrm{ord}(f)$ iff $f^{(b)}(\boldsymbol{c}, 1) \neq 0$, where $f^{(b)}$ denotes the sum of terms of degree b in f.

(2) Show that there exists a linear automorphism (1.1.7) with $\varphi(f)$ being x_n-general of order b if $\#(K) > b$ (in particular, if K is infinite).

(3) Show that for $\#(K) = b$ there exists still a linear automorphism φ (maybe of a different kind) such that $\varphi(f)$ is x_n-general of order b.

Exercise 1.1.7. Let K be a finite field, and let $n \geq 2$. Show that for each $d > \#(K)$ there exists a polynomial f of degree d such that for each linear automorphism $\varphi : K[[x_1, \dots, x_n]] \to K[[x_1, \dots, x_n]]$ the image $\varphi(f)$ is not x_n-general of finite order.

(C) *Local Rings and Localization.* Let R be a ring. An element $u \in R$ is called a *unit* if it is invertible in R. The ring R is said to be *local* if it has an ideal $\mathfrak{m}$ such that all elements of $R \setminus \mathfrak{m}$ are units. Then $\mathfrak{m}$ is the unique maximal ideal of R. On the other hand, each ring with a unique maximal ideal is local. A *local K-algebra* is a K-algebra which is a local ring.

For R any ring and $\mathfrak{p} \subset R$ a prime ideal, the *localization* of R at $\mathfrak{p}$ is defined to be

$$R_{\mathfrak{p}} := \left\{ \frac{p}{q} \,\middle|\, p, q \in R,\ q \notin \mathfrak{p} \right\},$$

where $\frac{p}{q}$ denotes the equivalence class of (p, q) with $(p, q) \sim (p', q')$ iff there exists some $s \in R \setminus \mathfrak{p}$ such that $s(pq' - p'q) = 0$. With the obvious ring structure, $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$.

(D) *Analytic vs. Algebraic Local Rings.* Let $R = K[\boldsymbol{x}]/I = K[x_1, \dots, x_n]/I$, let $\boldsymbol{p} = (p_1, \dots, p_n) \in K^n$, and let $\mathfrak{m} = \langle \boldsymbol{x} - \boldsymbol{p}\rangle$ be the corresponding maximal ideal of R. Then the localization $R_{\mathfrak{m}}$ of R at $\mathfrak{m}$ (which is a local K-algebra) is called the *algebraic local ring* of

$$V_K(I) := \bigl\{q \in K^n \bigm| f(q) = 0 \ \forall\, f \in I\bigr\}$$

at $\boldsymbol{p}$, while the analytic K-algebra $K\langle\boldsymbol{x}-\boldsymbol{p}\rangle/I \cdot K\langle\boldsymbol{x}-\boldsymbol{p}\rangle$ is called the *analytic local ring* of $V_K(I)$ at p. Since any quotient of polynomials has a power series expansion at points where the denominator does not vanish, the algebraic local ring is a K-subalgebra of the analytic one.

The following exercise is about to show that neither the Weierstraß preparation theorem, nor the division theorem, nor the finiteness theorem hold for algebraic local rings (in place of analytic ones):

Exercise 1.1.8. Let $A := \mathbb{C}[x]_{\langle x\rangle}$, $B := \mathbb{C}[x,y]_{\langle x,y\rangle}$, and $f := x^2 + y^2 + y^3$. Prove the following statements:
(1) There are no unit $u \in B$, and no $a_0, a_1 \in A$ such that $uf = y^2 + a_1 y + a_0$.
(2) Let $M := B/fB$. Then $M/\mathfrak{m}_A M$ is a finite dimensional $\mathbb{C}$-vector space, but M is not finite over A (Nakayama's lemma).

(E) *Computational Remark.* The proof of the Weierstraß division theorem given above is due to Grauert and Remmert (see [GrR]). The argument is less straightforward than the proof of the finiteness theorem in the formal case, but it has the advantage to provide a very elegant and short proof of the convergence (with respect to the valuation of K). Thus, it is nowadays the preferred proof. Moreover, the proof is constructive in the sense that it gives, in the i-th step, power series which formally converge to r, respectively h, as $i \to \infty$. One can verify that the convergence with respect to the $\langle x\rangle$-adic topology is also faster than the one in the first proof given for the WFT in the formal case.

The resulting algorithm for computing r and h in the division theorem *up to a given degree* is a follows: we have

$$f \;=\; \sum_{i=0}^{b-1} f_i(\boldsymbol{x}')x_n^i + x_n^b \sum_{i=0}^{\infty} f_{b+i}(\boldsymbol{x}')x_n^i \;=\; \widehat{f} + x_n^b \widetilde{f}$$

with $\widehat{f} \in \mathfrak{m}_{K\langle\boldsymbol{x}'\rangle}K\langle\boldsymbol{x}\rangle$ and $\widetilde{f} = f_b(1-f_*)$ a unit, $f_* \in \mathfrak{m}_{K\langle\boldsymbol{x}\rangle}$. Using the geometric series, we can easily compute

$$\widetilde{f}^{-1} \;=\; \frac{1}{f_b}\sum_{i=0}^{\infty}(f_*)^i$$

up to a given order. Thus, starting with $k_0 = g$, we can compute the power series $k_{i+1} = -\widetilde{f}^{-1}\widehat{f}\widetilde{k}_i$ up to any given order, too. The decomposition $k_i = \widehat{k}_i + x_n^b \widetilde{k}_i$ is almost without costs. We obtain

$$r = \sum_{i=0}^{\infty} \widehat{k}_i , \qquad h = \widetilde{f}^{-1} \sum_{i=0}^{\infty} \widetilde{k}_i .$$

If $\widehat{f} \in \mathfrak{m}^p_{K\langle \boldsymbol{x}' \rangle} K\langle \boldsymbol{x} \rangle$ for some $p > 1$ then $k_i \in \mathfrak{m}^{ip}_{K\langle \boldsymbol{x}' \rangle} K\langle \boldsymbol{x} \rangle$, and the formal convergence proceeds in steps of size p.

For the Weierstraß preparation theorem, take $g = x_n^b$. Then h is a unit and $hf = x_n^b - r$ is a Weierstraß polynomial.

Exercise 1.1.9. Let $f := x^2 + y^2 + y^3$. Find, up to terms of order 5, a unit $u \in \mathbb{R}\langle x, y \rangle$, and $a_0, a_1 \in \mathbb{R}\langle x, y \rangle$ such that $uf = y^2 + a_1 y + a_0$.

The algorithm described above is implemented in Singular in the library `weierstr.lib`[2]. We give two examples, one for the division theorem and one for the preparation theorem:

```
LIB "weierstr.lib";
ring R = 0,(x,y,z),ds;
poly f = (x2+y3+yz3)*z;
generalOrder(f);  //checks whether f is z-general (z=last variable)
//-> -1
```

The output shows that f is not z-general. Thus, we must apply a coordinate change in order to make f z-general:

```
f = lastvarGeneral(f);
f;
//-> x2z+2xz2+z3+y3z+yz4
```

Now, f is z-general (of order 3). We apply the algorithm for the Weierstraß preparation theorem up to order 5:

```
list P = weierstrPrep(f,5);
P;
//-> [1]:
//->    1+2xy-yz+7x2y2-4xy2z+y2z2-y5
//-> [2]:
//->    x2z+2xz2+z3+y3z+2x3yz+3x2yz2
//-> [3]:
//->    4
```

The returned list provides the unit u as first entry `P[1]`, the Weierstraß polynomial as second entry `P[2]` and the needed number of iterations (here, 4) as last entry. We check that `P[1]*f-P[2]` has order 6:

```
ord(P[1]*f-P[2]);
//-> 6
```

[2] This library is distributed with Singular, version 3-0-2 or higher.

Next, we divide the z-general polynomial f by $g = z^2 + y^2$, applying the algorithm for the WDT described above (up to degree 10):

```
poly g = z2+y2;
list D = weierstrDiv(f,g,10);
D;
//-> [1]:
//->    2x+z-y3+yz2
//-> [2]:
//->    -2xy2+x2z-y2z+y3z+y5
//-> [3]:
//->    3
```

We check that `f=D[1]*g+D[2]` up to total degree 10 (again, the third entry 3 of the output is the number of iterations needed):

```
ord(f-D[1]*g-D[2]);
//-> -1
```

The return value -1 indicates that `f=D[1]*g+D[2]` up to any degree, that is, we are in the special situation that the division of f by g results in polynomials.

1.2 Application to Analytic Algebras

The importance of the Weierstraß theorems will become clear in this section. Again, let K denote a complete real valued field.

Theorem 1.15 (Noether property). *Any analytic algebra A is Noetherian, that is, every ideal of A is finitely generated.*

Proof. Any quotient ring of a Noetherian ring is Noetherian. Therefore, it suffices to show the theorem for $A = K\langle \boldsymbol{x} \rangle = K\langle x_1, \dots, x_n \rangle$.

We proceed by induction on n. The case $n = 0$ being trivial, we may assume that $K\langle \boldsymbol{x}' \rangle = K\langle x_1, \dots, x_{n-1} \rangle$ is Noetherian.

Let $I \subset K\langle \boldsymbol{x} \rangle$ be a non-zero ideal and $f \in I$, $f \neq 0$. After a coordinate change, we may assume that f is x_n-general, that is, by the Weierstraß preparation theorem, $f \in I_0 := I \cap K\langle \boldsymbol{x}' \rangle[x_n]$. We claim that $I = I_0 \cdot K\langle \boldsymbol{x} \rangle$. Indeed, given $g \in I$, the Weierstraß division theorem implies a decomposition $g = fh + r$ with $r \in I \cap K\langle \boldsymbol{x}' \rangle[x_n] = I_0$, $h \in K\langle \boldsymbol{x} \rangle$.

$K\langle \boldsymbol{x}' \rangle$ being Noetherian by induction hypothesis, Hilbert's basis theorem[3] implies that $K\langle \boldsymbol{x}' \rangle[x_n]$ is Noetherian, too. Hence, I_0 is finitely generated in $K\langle \boldsymbol{x}' \rangle[x_n]$, and therefore also $I = I_0 \cdot K\langle \boldsymbol{x} \rangle$ is finitely generated in $K\langle \boldsymbol{x} \rangle$. □

Theorem 1.16 (Factoriality). *The power series ring $K\langle x_1, \dots, x_n \rangle$ is a factorial ring*[4].

[3] *Hilbert's basis theorem* says that for a Noetherian ring R the polynomial ring $R[x]$ is Noetherian, too.

[4] See Remarks and Exercises (A) on page 31.

Supplement. *If $f \in K\langle \boldsymbol{x}' \rangle[x_n]$ is a Weierstraß polynomial then there are uniquely determined irreducible Weierstraß polynomials $g_i \in K\langle \boldsymbol{x}' \rangle[x_n]$, $i = 1, \ldots, s$, such that $f = g_1 \cdot \ldots \cdot g_s$. This is also the prime decomposition of f in $K\langle \boldsymbol{x} \rangle$. Moreover, $K\langle \boldsymbol{x}' \rangle[x_n]$ is factorial.*

Proof. $K\langle \boldsymbol{x} \rangle$ is an integral domain. Hence, it suffices to show that each non-unit in $K\langle \boldsymbol{x} \rangle$ can be written as a product of prime elements of $K\langle \boldsymbol{x} \rangle$.

Again, we use induction on n. Let $f \in K\langle \boldsymbol{x} \rangle \setminus \{0\}$ be a non-unit. Without loss of generality, f is x_n-general of order $b > 0$ (Lemma 1.5). By the preparation theorem, $f = ug$ with u a unit, and with $g \in K\langle \boldsymbol{x}' \rangle[x_n]$ a Weierstraß polynomial of degree b.

But $K\langle \boldsymbol{x}' \rangle[x_n]$ is factorial (by the induction hypothesis and the lemma of Gauß). Therefore, g has a decomposition $g = g_1 \cdot \ldots \cdot g_s$ into prime factors $g_i \in K\langle \boldsymbol{x}' \rangle[x_n]$. Since g is a Weierstraß polynomial, we can normalize the g_i, and it easily follows that the g_i are Weierstraß polynomials, too. With this extra assumption, the g_i are uniquely determined (not only up to units).

Applying the division theorem to g_i yields an isomorphism of $K\langle \boldsymbol{x}' \rangle$-modules at the bottom of the following diagram (which is even a ring isomorphism)

$$\begin{tikzcd}
K\langle \boldsymbol{x}' \rangle[x_n] \arrow[r, hook] \arrow[d, two heads] & K\langle \boldsymbol{x} \rangle \arrow[d, two heads] \\
K\langle \boldsymbol{x}' \rangle[x_n]/\langle g_i \rangle \arrow[r, "\cong"] & K\langle \boldsymbol{x} \rangle/\langle g_i \rangle .
\end{tikzcd}$$

Since g_i is prime in $K\langle \boldsymbol{x}' \rangle[x_n]$, the quotient $K\langle \boldsymbol{x}' \rangle[x_n]/\langle g_i \rangle \cong K\langle \boldsymbol{x} \rangle/\langle g_i \rangle$ is an integral domain. Hence, $\langle g_i \rangle$ is a prime ideal of $K\langle \boldsymbol{x} \rangle$, $i = 1, \ldots, s$, and thus $g = g_1 \cdot \ldots \cdot g_s$ is a prime decomposition of g in $K\langle \boldsymbol{x} \rangle$. $\square$

Note that there are analytic algebras $K\langle \boldsymbol{x} \rangle / I$ which are integral domains but not factorial (Exercise 1.2.1).

Theorem 1.17 (Hensel's lemma). *Let $f \in K\langle \boldsymbol{x}' \rangle[x_n]$ be a monic polynomial of degree $b \geq 1$, and let*

$$f(\mathbf{0}, x_n) = (x_n - c_1)^{b_1} \cdot \ldots \cdot (x_n - c_s)^{b_s},$$

where the $c_i \in K$ are pairwise different. Then there exist uniquely determined monic polynomials $f_i \in K\langle \boldsymbol{x}' \rangle[x_n]$ of degree b_i, $i = 1, \ldots, s$, such that

$$f = f_1 \cdot \ldots \cdot f_s\,, \qquad f_i(\mathbf{0}, x_n) = (x_n - c_i)^{b_i}.$$

Proof. By induction on s, the case $s = 1$ being trivial.

Set $g(x_1, \ldots, x_n) := f(\boldsymbol{x}', x_n + c_s)$. Since, by assumption, the c_i are pairwise different, we have $c_s \neq c_i$ for $i = 1, \ldots, s-1$, and, therefore, g is x_n-general of order b_s. By the Weierstraß preparation theorem, we get $g = u \cdot h$ with $u \in K\langle \boldsymbol{x} \rangle$ a unit and h a Weierstraß polynomial of degree b_s. In particular, h is monic and $h(\mathbf{0}, x_n) = x_n^{b_s}$. Since g is monic in x_n, of degree

$b = b_1 + \ldots + b_s$, the supplement of the Weierstraß preparation theorem implies that the unit u is monic in x_n too, of degree $b - b_s = b_1 + \ldots + b_{s-1}$. Moreover,

$$\begin{aligned} g(\mathbf{0}, x_n - c_s) &= f(\mathbf{0}, x_n) \;=\; (x_n - c_1)^{b_1} \cdot \ldots \cdot (x_n - c_s)^{b_s}\,, \\ h(\mathbf{0}, x_n - c_s) &= (x_n - c_s)^{b_s}. \end{aligned}$$

It follows that $u(\mathbf{0}, x_n - c_s) \;=\; (x_n - c_1)^{b_1} \cdot \ldots \cdot (x_n - c_{s-1})^{b_{s-1}}$. By the induction hypothesis, the monic polynomial $f'(\boldsymbol{x}) := u(\boldsymbol{x}', x_n - c_s)$ decomposes into monic polynomials $f_1, \ldots, f_{s-1} \in K\langle \boldsymbol{x}' \rangle[x_n]$ of degrees $b_1, \ldots, b_{s-1}$ such that $f_i(\mathbf{0}, x_n) = (x_n - c_i)^{b_i}$. Setting $f_s(\boldsymbol{x}) = h(\boldsymbol{x}', x_n - c_s)$, we get the claimed decomposition $f = f_1 \cdot \ldots \cdot f_s$. The uniqueness of the decomposition is implied by the supplement to Theorem 1.16. □

Theorem 1.18 (Implicit function theorem).
Let $f_i \in A = K\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$, $i = 1, \ldots, m$, satisfy $f_i(\mathbf{0}) = 0$, and

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\mathbf{0}) & \ldots & \frac{\partial f_1}{\partial y_m}(\mathbf{0}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial y_1}(\mathbf{0}) & \ldots & \frac{\partial f_m}{\partial y_m}(\mathbf{0}) \end{pmatrix} \neq 0\,.$$

Then $A/\langle f_1, \ldots, f_m \rangle \cong K\langle x_1, \ldots, x_n \rangle$, and there exist unique power series $Y_1, \ldots, Y_m \in \mathfrak{m}_{K\langle \boldsymbol{x} \rangle}$ solving the implicit system of equations

$$f_1(\boldsymbol{x}, \boldsymbol{y}) = \cdots = f_m(\boldsymbol{x}, \boldsymbol{y}) = 0$$

in $\boldsymbol{y}$, that is, satisfying

$$f_i\big(\boldsymbol{x}, Y_1(\boldsymbol{x}), \ldots, Y_m(\boldsymbol{x})\big) = 0\,, \quad i = 1, \ldots, m\,.$$

Moreover, $\langle f_1, \ldots, f_m \rangle = \langle y_1 - Y_1, \ldots, y_m - Y_m \rangle$.

Proof. Step 1. Existence. We proceed by induction on m. For $m = 0$, there is nothing to show. Thus, let $m \geq 1$.

Since, by assumption, the matrix $\big(\frac{\partial f_i}{\partial y_j}(\mathbf{0})\big)_{i,j=1\ldots m}$ is invertible, we may, after a linear coordinate transformation, assume that

$$f_i(\boldsymbol{x}, \boldsymbol{y}) \;=\; y_i + c_i(\boldsymbol{x}) + (\text{terms in } \boldsymbol{x}, \boldsymbol{y} \text{ of order} \geq 2)\,, \quad c_i(\mathbf{0}) = 0\,.$$

Then f_i is y_i-general of order 1, and the Weierstraß preparation theorem implies the existence of a unit $u \in K\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ such that

$$u f_m = y_m + a\,, \quad a \in \mathfrak{m}_{K\langle \boldsymbol{x}, \boldsymbol{y}' \rangle}\,,$$

where $\boldsymbol{y}' = (y_1, \ldots, y_{m-1})$. Setting $\widetilde{Y}_m := -a \in \mathfrak{m}_{K\langle \boldsymbol{x}, \boldsymbol{y}' \rangle}$, we get

$$f_m\big(\boldsymbol{x}, \boldsymbol{y}', \widetilde{Y}_m\big) = 0\,, \tag{1.2.1}$$

and we define $\widetilde{f}_i := f_i(\boldsymbol{x}, \boldsymbol{y}', \widetilde{Y}_m) \in K\langle \boldsymbol{x}, \boldsymbol{y}'\rangle$, $i = 1, \ldots, m-1$. Then

$$\langle f_1, \ldots, f_m\rangle = \langle \widetilde{f}_1, \ldots, \widetilde{f}_{m-1}, y_m - \widetilde{Y}_m\rangle \tag{1.2.2}$$

(due to Corollary 1.9), and $\widetilde{f}_i(\boldsymbol{x}, \boldsymbol{y}') = y_i + c_i(\boldsymbol{x}) + (\text{terms in of order} \geq 2)$.

Thus, the induction hypothesis applies to $\widetilde{f}_1, \ldots, \widetilde{f}_{m-1}$ and there exist power series $Y_1, \ldots, Y_{m-1} \in \mathfrak{m}_{K\langle \boldsymbol{x}\rangle}$ such that

$$\widetilde{f}_i(\boldsymbol{x}, Y_1, \ldots, Y_{m-1}) = 0, \quad i = 1, \ldots, m-1, \tag{1.2.3}$$

and $\langle \widetilde{f}_1, \ldots, \widetilde{f}_{m-1}\rangle = \langle y_1 - Y_1, \ldots, y_{m-1} - Y_{m-1}\rangle \subset K\langle \boldsymbol{x}, \boldsymbol{y}'\rangle$. Setting

$$Y_m := \widetilde{Y}_m(\boldsymbol{x}, Y_1, \ldots, Y_{m-1}),$$

(1.2.3) and (1.2.1) give $f_i(\boldsymbol{x}, Y_1, \ldots, Y_m) = 0$ for $i = 1, \ldots, m$, and

$$\begin{aligned}\langle f_1, \ldots, f_m\rangle &= \langle y_1 - Y_1, \ldots, y_{m-1} - Y_{m-1}, y_m - \widetilde{Y}_m\rangle \\ &= \langle y_1 - Y_1, \ldots, y_{m-1} - Y_{m-1}, y_m - Y_m\rangle .\end{aligned}$$

Step 2. Uniqueness. Let $Y_1', \ldots, Y_m' \in \mathfrak{m}_{K\langle \boldsymbol{x}\rangle}$ satisfy

$$f_i(\boldsymbol{x}, Y_1'(\boldsymbol{x}), \ldots, Y_m'(\boldsymbol{x})) = 0, \quad i = 1, \ldots, m.$$

Writing $y_i - Y_i \in \langle f_1, \ldots, f_m\rangle$ as a linear combination of $f_1, \ldots, f_m$ and substituting y_i by Y_i' gives $Y_i' - Y_i = 0$ for all i. □

Definition 1.19. Let A be an analytic K-algebra with maximal ideal $\mathfrak{m}_A$, let $I \subset A$ be an ideal, and let M be a finitely generated A-module.

(1) $\operatorname{mng}(M) := \dim_K M/\mathfrak{m}M$ denotes the *minimal number of generators* of M.
(2) $\mathfrak{m}_A/\mathfrak{m}_A^2$ is called the *cotangent space of A*.
(3) $(\mathfrak{m}_A/\mathfrak{m}_A^2)^* = \operatorname{Hom}_K(\mathfrak{m}_A/\mathfrak{m}_A^2, K)$ is called the *(Zariski) tangent space of A*.
(4) $\operatorname{edim}(A) := \dim_K(\mathfrak{m}_A/\mathfrak{m}_A^2)$ is called the *embedding dimension of A*.
(5) If $\varphi : A \to B$ is a morphism of analytic K-algebras then the induced linear map

$$\dot{\varphi} : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$$

is called the *cotangent map of* φ.
(6) $\operatorname{jrk}(I) := \dim_K(I/I \cap \mathfrak{m}_A^2)$ is called the *Jacobian rank* of I.

Remark 1.19.1. (1) The cotangent map $\dot{\varphi}$ has a familiar description if $A = K\langle y_1, \ldots, y_k\rangle$, $B = K\langle x_1, \ldots, x_n\rangle$ and if the map $\varphi : A \to B$ is given by $\varphi(y_i) = f_i$, $i = 1, \ldots, k$. Then $f_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\mathbf{0}) x_j + g_i$ with $g_i \in \langle \boldsymbol{x}\rangle^2$ and $\dot{\varphi}$ maps y_i to $\sum_{j=1}^n \frac{\partial f_j}{\partial x_j}(\mathbf{0}) x_j$. Hence, with respect to the bases $\{\overline{y}_1, \ldots, \overline{y}_k\}$ of

$\mathfrak{m}_A/\mathfrak{m}_A^2$ and $\{\overline{x}_1, \dots, \overline{x}_n\}$ of $\mathfrak{m}_B/\mathfrak{m}_B^2$, the linear map $\dot{\varphi}$ is given by the transpose of the *Jacobian matrix* $J(f_1, \dots, f_k)$ at $\mathbf{0}$,

$$\dot{\varphi} = J(f_1, \dots, f_k)^t\big|_{\boldsymbol{x}=\mathbf{0}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{0}) & \dots & \frac{\partial f_k}{\partial x_1}(\mathbf{0}) \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n}(\mathbf{0}) & \dots & \frac{\partial f_k}{\partial x_n}(\mathbf{0}) \end{pmatrix}.$$

In general, if $\varphi : K\langle \boldsymbol{y}\rangle/I \to K\langle \boldsymbol{x}\rangle/J$, then φ can be lifted to $\widetilde{\varphi} : K\langle \boldsymbol{y}\rangle \to K\langle \boldsymbol{x}\rangle$ by Lemma 1.14 and $\dot{\varphi}$ is induced by $\dot{\widetilde{\varphi}}$. If $I \subset \langle \boldsymbol{y}\rangle^2$ and $J \subset \langle \boldsymbol{x}\rangle^2$ then $\dot{\varphi} = \dot{\widetilde{\varphi}}$; in general, we have to mod out the linear parts of I, respectively of J.

(2) If $A = K\langle x_1, \dots, x_n\rangle$ and $I = \langle f_1, \dots, f_k\rangle \subset \mathfrak{m}_A$, then $\operatorname{jrk}(I)$ is the rank of the linear part $I/I \cap \mathfrak{m}_A^2$ of I and this is just the rank of the Jacobian matrix $\big(\frac{\partial f_i}{\partial x_j}(\mathbf{0})\big)_{i=1\dots k, j=1\dots n}$. In particular, this rank depends only on I, but not on the chosen generators.

Theorem 1.20 (Epimorphism theorem). *Let $\varphi : A \to B$ be a morphism of analytic K-algebras. Then the following are equivalent:*

(a) φ is surjective.
(b) $\mathfrak{m}_A B = \mathfrak{m}_B$.
(c) $\dot{\varphi} : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.

Proof. φ being surjective means that $\varphi(\mathfrak{m}_A) = \mathfrak{m}_B$, while (b) means that $\varphi(\mathfrak{m}_A)$ generates $\mathfrak{m}_B$ as B-module. Of course, (a) implies (b) and (c).

If (b) is satisfied then $1 \in B$ generates $B/\mathfrak{m}_A B$ over K. Hence, by Corollary 1.11, it generates B as A-module, that is, $B = A \cdot 1 = \varphi(A)$, proving (a). If (c) is satisfied then $\varphi(\mathfrak{m}_A)$ generates $\mathfrak{m}_B/\mathfrak{m}_B^2$ over K. Applying Corollary 1.11 with $M = \mathfrak{m}_B$, we conclude that $\varphi(\mathfrak{m}_A)$ generates $\mathfrak{m}_B$ over B, that is, $\mathfrak{m}_A B = \mathfrak{m}_B$, and we obtain (b). $\square$

In particular, if $\varphi : K\langle x_1, \dots, x_n\rangle \to A$ is a morphism of analytic algebras such that the images $\varphi(x_i)$, $i = 1, \dots, n$, generate $\mathfrak{m}_A$ then φ is surjective.

Note that it is not true that $\dot{\varphi}$ bijective implies φ bijective. For example, the residue class map $\varphi : A \to A/\mathfrak{m}_A^2$ is not injective if $\mathfrak{m}_A^2 \neq 0$, but $\dot{\varphi}$ is an isomorphism. However, we have:

Theorem 1.21 (Inverse function theorem).
Let $\varphi : A \to K\langle x_1, \dots, x_n\rangle$ be an analytic morphism, and let $\mathfrak{m}_A \subset A$ be the maximal ideal. Then the following are equivalent:

(a) φ is an isomorphism.
(b) $\dot{\varphi} : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_{K\langle \boldsymbol{x}\rangle}/\mathfrak{m}_{K\langle \boldsymbol{x}\rangle}^2$ is an isomorphism.
(c) $\operatorname{edim} A = \operatorname{rank}(\dot{\varphi}) = n$.

Proof. (c) just says that $\dot{\varphi}$ is a surjection of vector spaces of the same dimension. Hence, (b) and (c) are equivalent. Since, the implication (a) $\Rightarrow$ (b) is obvious, it remains only to prove (b) $\Rightarrow$ (a).

By the epimorphism Theorem 1.20, φ is surjective. Since $\dot\varphi$ is an isomorphism, there are $g_i \in A$ inducing a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$ such that $\varphi(g_i) = x_i$, $i = 1, \ldots, n$. The g_i define a morphism $\psi : K\langle x_1, \ldots, x_n\rangle \to A$, $x_i \mapsto g_i$, with $\dot\psi = \dot\varphi^{-1}$. Hence, $\dot\psi$ is surjective and therefore, again by the epimorphism theorem, ψ is surjective. Since $\varphi \circ \psi = \mathrm{id}$, it follows that φ is injective, hence bijective (with inverse ψ). □

If $A = K\langle y_1, \ldots, y_n\rangle$ and $\varphi(y_i) = f_i$ then the linear (cotangent) map $\dot\varphi$ is, in the bases $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$, given by the transpose of the Jacobian matrix at $\mathbf{0}$, that is, φ is an isomorphism iff $\det\big(\frac{\partial f_i}{\partial x_j}(\mathbf{0})\big)_{i,j=1\ldots n} \neq 0$. This can then be rephrased by saying that a morphism of power series rings is an isomorphism iff the induced map on the cotangent spaces, or, equivalently, the induced map on the Zariski tangent spaces (given by the Jacobian matrix at $\mathbf{0}$), is an isomorphism. This is the usual form of the inverse function theorem.

For later use we state three lemmas.

Lemma 1.22 (Jacobian rank lemma). *Let A be an analytic K-algebra with maximal ideal $\mathfrak{m}_A$, and let $I \subset \mathfrak{m}_A$ be an ideal. Then*

$$\mathrm{jrk}(I) \;=\; \mathrm{edim}(A) - \mathrm{edim}(A/I)\,.$$

Proof. This follows from the exact sequence of K-vector spaces

$$0 \longrightarrow (I + \mathfrak{m}_A^2)/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_A/(I + \mathfrak{m}_A^2) \longrightarrow 0\,,$$

noting that $(I + \mathfrak{m}_A^2)/\mathfrak{m}_A^2 \cong I/(I \cap \mathfrak{m}_A^2)$ and $\mathfrak{m}_A/(I + \mathfrak{m}_A^2) \cong \mathfrak{m}_{A/I}/\mathfrak{m}_{A/I}^2$. □

Lemma 1.23 (Lifting lemma). *Let φ be a morphism of analytic K-algebras, $\varphi :\ A = K\langle x_1, \ldots, x_n\rangle/I \to B = K\langle y_1, \ldots, y_m\rangle/J$.*

Then φ has a lifting $\widetilde\varphi : K\langle \boldsymbol{x}\rangle \to K\langle \boldsymbol{y}\rangle$ which can be chosen as an isomorphism in the case that φ is an isomorphism and $n = m$, respectively as an epimorphism in the case that φ is an epimorphism and $n \geq m$.

For a generalization of this lemma, see Lemma 1.27.

Proof. The existence of $\widetilde\varphi$ was already shown in Lemma 1.14. For the additional properties, we need special choices. We may assume that $I \subset \langle \boldsymbol{x}\rangle$. Then the proof of Lemma 1.22 shows that we can choose $g_1, \ldots, g_n \in \langle \boldsymbol{x}\rangle$ inducing a K-basis of $\langle \boldsymbol{x}\rangle/\langle \boldsymbol{x}\rangle^2$ such that $g_1, \ldots, g_e$ are a K-basis of $\mathfrak{m}_A/\mathfrak{m}_A^2 = \langle \boldsymbol{x}\rangle/(I + \langle \boldsymbol{x}\rangle^2)$ and $g_{e+1}, \ldots, g_n \in I$.

By the inverse function theorem, the morphism $K\langle \boldsymbol{x}\rangle \to K\langle \boldsymbol{x}\rangle$, $x_i \mapsto g_i$, is an isomorphism. Hence, prescribing the images in $K\langle \boldsymbol{y}\rangle$ of $g_1, \ldots, g_n$ defines a unique K-algebra homomorphism $K\langle \boldsymbol{x}\rangle \to K\langle \boldsymbol{y}\rangle$.

Let $\overline{g}_i \in K\langle \boldsymbol{x}\rangle/I$ be the class mod I of g_i and set $\overline{h}_i := \varphi(\overline{g}_i)$. Then $\overline{g}_i$ and, hence, $\overline{h}_i$ are zero for $i > e$, and $\overline{h}_1, \ldots, \overline{h}_e$ is a basis (respectively a generating system) of $\langle \boldsymbol{y}\rangle/(J + \langle \boldsymbol{y}\rangle^2) = \mathfrak{m}_B/\mathfrak{m}_B^2$ if φ is bijective (respectively surjective).

Lift $\overline{h}_1, \ldots, \overline{h}_e$ arbitrarily to $h_1, \ldots, h_e \in \langle \boldsymbol{y} \rangle$ and choose $h_{e+1}, \ldots, h_n \in J$ such that $h_1, \ldots, h_n$ are a basis of $\langle \boldsymbol{y} \rangle / \langle \boldsymbol{y} \rangle^2$ in the first case (respectively a generating system in the second case). Clearly, mapping $g_i \mapsto h_i$, $i = 1, \ldots, n$, defines a lifting $\widetilde{\varphi} : K\langle \boldsymbol{x} \rangle \to K\langle \boldsymbol{y} \rangle$ of φ. The inverse function theorem (respectively the epimorphism theorem) implies that $\widetilde{\varphi}$ is an isomorphism (respectively an epimorphism). □

Lemma 1.24 (Embedding lemma). *Let $A = K\langle x_1, \ldots, x_n \rangle / I$ be an analytic algebra, and let $e = \mathrm{edim}(A)$. Then we have*

(1) $A \cong K\langle y_1, \ldots, y_e \rangle / J$ with $J \subset \langle \boldsymbol{y} \rangle^2$;
(2) $n \geq e$, and $n = e$ iff $I \subset \langle \boldsymbol{x} \rangle^2$.

Proof. (1) After renumbering the x_i, we may assume that $\boldsymbol{x}' = \{x_1, \ldots, x_e\}$ is a basis of $\mathfrak{m}_A / \mathfrak{m}_A^2$. Set $J := \langle \boldsymbol{x}' \rangle \cap I$, and consider the canonical map $\varphi : K\langle \boldsymbol{x}' \rangle / J \to K\langle \boldsymbol{x} \rangle / I = A$. It induces an isomorphism

$$\langle \boldsymbol{x}' \rangle / \big(J + \langle \boldsymbol{x}' \rangle^2\big) \xrightarrow{\cong} \mathfrak{m}_A / \mathfrak{m}_A^2 \,.$$

By the inverse function theorem, φ is an isomorphism and, for dimension reasons, $J \subset \langle \boldsymbol{x}' \rangle^2$.

(2) is a consequence of the Jacobian rank Lemma 1.22. □

Next, we come to another important finiteness theorem for analytic K-algebras, the Noether normalization theorem, which states that each analytic algebra A is a finite module over a free power series algebra $K\langle \boldsymbol{y} \rangle \subset A$.

Theorem 1.25 (Noether normalization theorem). *Let A be an analytic K-algebra. Then there exists an analytic subalgebra $B \subset A$ such that:*

(1) $B \cong K\langle y_1, \ldots, y_d \rangle$,
(2) A is a finitely generated B-module.

The subalgebra $B \subset A$ is called a *Noether normalization* of A.

Supplement. *If $A = K\langle x_1, \ldots, x_n \rangle / I$, then the y_i can be chosen of the form*

$$y_i \;=\; x_i + \sum_{j=i+1}^{n} c_{ij} x_j \,, \qquad c_{ij} \in A.$$

If the field K is infinite then we can even choose $c_{ij} \in K$.

Proof. If $A = K\langle x_1, \ldots, x_n \rangle$, the statement is trivial (setting $B := A$). Hence, let $A = K\langle x_1, \ldots, x_n \rangle / I$, and let $f \in I$ be a non-zero element. By Lemma 1.5 (respectively Exercise 1.1.6), we know that f becomes x_n-general after a coordinate change of type $x_i \mapsto x_i + c_{in} x_n$, $x_n \mapsto x_n$, with $c_{in} = x_n^{\nu_i}$ (or $c_{in} \in K$ if K is infinite).

By the Weierstraß preparation theorem, $K\langle x_1, \ldots, x_n \rangle / \langle f \rangle$ is finite over $K\langle \boldsymbol{x}' \rangle = K\langle x_1, \ldots, x_{n-1} \rangle$. There are two possible cases:

Case 1. If $I \cap K\langle \boldsymbol{x}' \rangle = 0$, then set $B := K\langle \boldsymbol{x}' \rangle$ and the statement of the theorem follows.

Case 2. If $I \cap K\langle \boldsymbol{x}' \rangle \neq 0$, then we apply induction on n.

If $n = 1$, then $I \cap K \neq 0$ means $I = \langle 1 \rangle$, hence A is the zero ring, and there is nothing to prove. If $n > 1$, by induction there exists a Noether normalization $K\langle y_1, \dots, y_d \rangle \hookrightarrow K\langle \boldsymbol{x}' \rangle/(I \cap K\langle \boldsymbol{x}' \rangle)$. Consider the diagram

$$\begin{tikzcd}
K\langle y_1,\dots,y_d\rangle \arrow[r, hook, "\text{finite}"'] & K\langle \boldsymbol{x}'\rangle/(I\cap K\langle \boldsymbol{x}'\rangle) \arrow[r, hook] & K\langle \boldsymbol{x}\rangle/I \\
 & K\langle \boldsymbol{x}'\rangle \arrow[u, two heads] \arrow[r, hook, "\text{finite}"'] & K\langle \boldsymbol{x}\rangle/\langle f\rangle \arrow[u, two heads]
\end{tikzcd}$$

It follows that the upper inclusion of the commutative square is finite, too. Hence, the composition $K\langle y_1, \dots, y_d \rangle \hookrightarrow K\langle \boldsymbol{x} \rangle/I$ is finite, and the theorem, together with the supplement, is proven. $\square$

For $A \in \mathscr{A}_K$ a fixed analytic K-algebra, we introduce now the category of analytic A-algebras. While the geometric counterpart of analytic K-algebras (for $K = \mathbb{C}$) are complex space germs (see Section 1.4), analytic A-algebras correspond to families of complex space germs over the complex germ corresponding to A. Such families are a central object of investigation in deformation theory (see Chapter II).

Definition 1.26. (1) An analytic K-algebra B together with a morphism $A \to B$ of analytic K-algebras is called an *analytic A-algebra*. A *morphism $\varphi : B \to C$ of analytic A-algebras* (or simply an *A-morphism*) is a morphism of K-algebras fitting in the commutative diagram

$$\begin{tikzcd}
B \arrow[rr, "\varphi"] & & C \\
 & A\,. \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The *category of analytic A-algebras* is denoted by $\mathscr{A}_A$.

(2) Let $A = K\langle \boldsymbol{t} \rangle/I \in \mathscr{A}_K$. Then an A-algebra B is called a *free power series algebra over A* if, for some $n \geq 0$, B is A-isomorphic to

$$A\langle \boldsymbol{x} \rangle := A\langle x_1, \dots, x_n \rangle := K\langle \boldsymbol{t}, \boldsymbol{x} \rangle / IK\langle \boldsymbol{t}, \boldsymbol{x} \rangle\,,$$

where $A\langle \boldsymbol{x} \rangle \in \mathscr{A}_A$ via the canonical morphism $A \hookrightarrow A\langle \boldsymbol{x} \rangle$.

Remark 1.26.1. Elements g of $A\langle \boldsymbol{x} \rangle$ can be written uniquely as power series $g = \sum_{\boldsymbol{\nu} \in \mathbb{N}^n} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$ with coefficients $a_{\boldsymbol{\nu}} \in A$. Any morphism $\varphi : A\langle \boldsymbol{x} \rangle \to B$ in $\mathscr{A}_A$ is uniquely determined by $\varphi(x_i) \in B$, $i = 1, \dots, n$.

Conversely, given $b_1, \dots, b_n \in \mathfrak{m}_B$, there is a unique morphism $A\langle \boldsymbol{x} \rangle \xrightarrow{\varphi} B$ in $\mathscr{A}_A$ such that $\varphi(x_i) = b_i$ for all i. This morphism maps $\sum_{\boldsymbol{\nu} \in \mathbb{N}^n} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$ to the power series $\sum_{\boldsymbol{\nu} \in \mathbb{N}^n} h(a_{\boldsymbol{\nu}}) b_1^{\nu_1} \cdots b_n^{\nu_n}$, where $h : A \to B$ is the map defining the A-algebra structure of B. We leave the proof of these simple facts as Exercise 1.2.8.

For morphisms of analytic A-algebras, we can generalize Lemma 1.23:

Lemma 1.27 (Relative lifting lemma). *Any morphism of analytic A-algebras, $\varphi : A\langle x_1, \dots, x_n\rangle/I \to A\langle y_1, \dots, y_m\rangle/J$, can be lifted to a morphism $\widetilde{\varphi} : A\langle \boldsymbol{x}\rangle \to A\langle \boldsymbol{y}\rangle$ of free power series algebras over A.*

If φ is an isomorphism and if $n = m$, then $\widetilde{\varphi}$ can be chosen as an isomorphism. If φ is an epimorphism and if $n \geq m$, then $\widetilde{\varphi}$ can be chosen as an epimorphism.

Proof. Since the proof is a slight variation of the proof of Lemma 1.23, we only sketch it. We may assume that $A = K\langle \boldsymbol{t}\rangle/H$, $\boldsymbol{t} = (t_1, \dots, t_k)$, with $H \subset \langle \boldsymbol{t}\rangle^2$ (by Lemma 1.24), and that $B := K\langle \boldsymbol{x}, \boldsymbol{t}\rangle/(I + \langle H\rangle)$ with $I \subset \langle \boldsymbol{x}, \boldsymbol{t}\rangle$. The exact sequence

$$0 \longrightarrow I/\big(I \cap \langle \boldsymbol{x}, \boldsymbol{t}\rangle^2\big) \longrightarrow \langle \boldsymbol{x}, \boldsymbol{t}\rangle/\langle \boldsymbol{x}, \boldsymbol{t}\rangle^2 \longrightarrow \langle \boldsymbol{x}, \boldsymbol{t}\rangle/\big(I + \langle \boldsymbol{x}, \boldsymbol{t}\rangle^2\big) \longrightarrow 0$$

shows that $t_1, \dots, t_k$ can be extended to $g_1, \dots, g_{n+k} \in K\langle \boldsymbol{x}, \boldsymbol{t}\rangle$, representing a K-basis of $\langle \boldsymbol{x}, \boldsymbol{t}\rangle/\langle \boldsymbol{x}, \boldsymbol{t}\rangle^2$, such that $g_{e+1}, \dots, g_{n+k} \in I$ and $g_1, \dots, g_e$ represent a $\mathbb{C}$-basis of $\langle \boldsymbol{x}, \boldsymbol{t}\rangle/(I + \langle \boldsymbol{x}, \boldsymbol{t}\rangle^2) = \mathfrak{m}_B/\mathfrak{m}_B^2$. Let $\overline{g}_i \in B$ be the image of g_i, and set $\overline{h}_i := \varphi(\overline{g}_i)$. Note that for $g_i = t_j$ the image $\overline{h}_i$ is the class $\overline{t}_j$ of t_j in $C := \mathbb{C}\langle \boldsymbol{y}, \boldsymbol{t}\rangle/(J + \langle H\rangle)$.

We lift $\overline{h}_i$ to $h_i \in \langle \boldsymbol{y}, \boldsymbol{t}\rangle$ as follows: if $\overline{h}_i = \overline{t}_j$ then we set $h_i = t_j$; if $i > e$ then we choose $h_i \in J$. The remaining h_i are chosen arbitrarily. Then the arguments of the proof of Lemma 1.23 show that the unique K-algebra homomorphism $\widetilde{\varphi} : K\langle \boldsymbol{x}, \boldsymbol{t}\rangle \to K\langle \boldsymbol{x}, \boldsymbol{t}\rangle$ defined by $g_i \mapsto h_i$ lifts φ and is an isomorphism (respectively epimorphism) if φ is. Moreover, since $\widetilde{\varphi}$ is the identity on $K\langle \boldsymbol{t}\rangle$, it is a morphism of A-algebras. □

We conclude this section by introducing the analytic tensor product in $\mathscr{A}_K$. This will be needed when introducing the Cartesian product of complex spaces and germs.

Definition 1.28. Let $A = K\langle \boldsymbol{y}\rangle/I$ and $B = K\langle \boldsymbol{x}\rangle/J$, with $\boldsymbol{y} = (y_1, \dots, y_m)$, $\boldsymbol{x} = (x_1, \dots, x_n)$, be analytic K-algebras. Then the analytic K-algebra

$$A \mathbin{\widehat{\otimes}_K} B := A \mathbin{\widehat{\otimes}} B := K\langle \boldsymbol{x}, \boldsymbol{y}\rangle/\big(IK\langle \boldsymbol{x}, \boldsymbol{y}\rangle + JK\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big)$$

is called the *analytic tensor product* of A and B.

In particular, $A\langle \boldsymbol{x}\rangle = A \mathbin{\widehat{\otimes}} K\langle \boldsymbol{x}\rangle$.

Remarks and Exercises

(A) *Factorial Rings.* Let R be a ring. An element $f \in R$ is called *irreducible* if f is not a unit and if $f = f_1 f_2$ with $f_1, f_2 \in R$ implies that f_1 or f_2 is a unit. $f \in R$ is called a *prime* element if the ideal generated by f is a prime ideal. If R is an integral domain, each non-zero prime element is irreducible; in a factorial ring the converse is also true.

Here, R is a called a *factorial ring*, or a *unique factorization domain*, if it is an integral domain and each $f \in R \setminus \{0\}$ can be written as a finite product $f = f_1 \cdot \ldots \cdot f_r$ of prime elements $f_i \in R$ (called *prime decomposition of* f). Each factorial ring has the *unique factorization property*: if $f = f_1 \cdot \ldots \cdot f_r$ and $f = g_1 \cdot \ldots \cdot g_s$ are factorizations of f into prime elements, then $r = s$ and, up to a permutation of the factors, $f_i = u_i g_i$ with u_i a unit in R.

The *lemma of Gauß* says that, if R is a unique factorization domain, then the polynomial ring $R[x]$ is a unique factorization domain, too. In particular, all polynomial rings $K[x_1, \ldots, x_n]$ with coefficients in a field K are factorial.

Exercise 1.2.1. Let $R = K\langle x, y, z\rangle / \langle x^2 - yz\rangle$. Prove the following statements:
(1) R is an integral domain.
(2) The residue class $\overline{y}$ of y is irreducible in R but not prime. In particular, $\overline{y}$ cannot be written as a finite product of finitely many prime elements in R.

(B) *Comparing Factorizations in* $\mathbb{C}[[\boldsymbol{x}]]$ *and in* $\mathbb{C}\{\boldsymbol{x}\}$. A convergent power series $g \in \mathbb{C}\{\boldsymbol{x}\}$ is irreducible as an element of $\mathbb{C}\{\boldsymbol{x}\}$ iff it is irreducible as an element of $\mathbb{C}[[\boldsymbol{x}]]$. In particular, each irreducible factorization $g = g_1 \cdot \ldots \cdot g_s$ of g in $\mathbb{C}\{\boldsymbol{x}\}$ is an irreducible factorization of f in $\mathbb{C}[[\boldsymbol{x}]]$.

This fact can be deduced as an immediate consequence of *Artin's approximation theorem*, which states that, for given $f_1, \ldots, f_m \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}$, and for given formal power series $\overline{Y}_1, \ldots, \overline{Y}_m \in \mathfrak{m}_{\mathbb{C}[[\boldsymbol{x}]]}$ satisfying

$$f_i\big(\boldsymbol{x}, \overline{Y}_1(\boldsymbol{x}), \ldots, \overline{Y}_m(\boldsymbol{x})\big) = 0\,, \quad i = 1, \ldots, m\,,$$

there exist convergent power series $Y_1, \ldots, Y_m \in \mathfrak{m}_{\mathbb{C}\{\boldsymbol{x}\}}$ such that

$$f_i\big(\boldsymbol{x}, Y_1(\boldsymbol{x}), \ldots, Y_m(\boldsymbol{x})\big) = 0\,, \quad i = 1, \ldots, m\,.$$

Moreover, it says that, for each fixed $k > 0$, we may find convergent solutions $Y_1, \ldots, Y_m$ as above with the additional property that $Y_i \equiv \overline{Y}_i \mod \langle \boldsymbol{x}\rangle^k$. See [Art, KPR, DJP] for a proof.

Now, g being reducible over $\mathbb{C}[[\boldsymbol{x}]]$ means that $f(\boldsymbol{x}, y_1, y_2) := g(\boldsymbol{x}) - y_1 y_2$ has a formal solution $Y_1, Y_2 \in \mathfrak{m}_{\mathbb{C}[[\boldsymbol{x}]]}$. Artin's approximation says then that it necessarily has a convergent solution, too. Hence, g is reducible over $\mathbb{C}\{\boldsymbol{x}\}$.

(C) *Henselian Local K-Algebras.* Neither the implicit function theorem, nor the epimorphism theorem, nor the inverse function theorem hold for the localization $K[\boldsymbol{x}]_{\langle \boldsymbol{x}\rangle}$ in place of $K\langle \boldsymbol{x}\rangle$:

Exercise 1.2.2. Let $\varphi : K[x]_{\langle x\rangle} \to K[x]_{\langle x\rangle}$ be given by $x \mapsto x + x^2$. Show that $\dot{\varphi} : \langle x\rangle / \langle x\rangle^2 \to \langle x\rangle / \langle x\rangle^2$ is an isomorphism, but φ is not surjective. In particular, there is no $Y \in \langle x\rangle \subset K[x]_{\langle x\rangle}$ such that $x - Y - Y^2 = 0$.

But all these theorems hold for Henselian local K-algebras. Here, a local K-algebra A with $K = A/\mathfrak{m}_A$ is called *Henselian* if the following holds: Given a monic polynomial $f \in A[t]$ and a factorization $\overline{f} = g_1 \cdot \ldots \cdot g_s$ of the image in $K[t]$, with $g_i \in K[t]$ monic and pairwise coprime, there exist monic polynomials $f_1, \ldots, f_s \in A[t]$ such that $f = f_1 \cdot \ldots \cdot f_s$ and $\overline{f}_i = g_i$ for $i = 1, \ldots, s$. Here, for $f \in A[t]$, $\overline{f}$ denotes the image of f in $(A/\mathfrak{m}_A)[t] = K[t]$.

Note that, if K is algebraically closed and if $A = K\langle x_1, \ldots, x_n\rangle$, the assumption on $\overline{f}$ just means that $f(\mathbf{0}, t) = (t - c_1)^{b_1} \cdot \ldots \cdot (t - c_n)^{b_n}$ with b_i, c_i as in Theorem 1.17. Hence, $K\langle x_1, \ldots, x_n\rangle$ is Henselian by Hensel's lemma. This holds for arbitrary fields K, as can be deduced from Hensel's lemma by passing to the algebraic closure. For details, we refer to [KPR].

Exercise 1.2.3. Show that each analytic K-algebra $K\langle \boldsymbol{x}\rangle/I$ is Henselian.

Exercise 1.2.4. Let $f, g, h \in K\langle \boldsymbol{x}'\rangle[x_n]$ be such that $\langle \boldsymbol{x}', g, h\rangle = K\langle \boldsymbol{x}'\rangle[x_n]$ and

$$f \equiv g \cdot h \mod \langle \boldsymbol{x}'\rangle \,.$$

Show that then there exist polynomials $g_1, h_1 \in K\langle \boldsymbol{x}'\rangle[x_n]$ such that

$$f = g_1 \cdot h_1 \,, \quad g_1 \equiv g \mod \langle \boldsymbol{x}'\rangle \,, \quad h_1 \equiv h \mod \langle \boldsymbol{x}'\rangle \,.$$

Moreover, show that the statement is no longer true if we omit the condition $\langle \boldsymbol{x}', g, h\rangle = K\langle \boldsymbol{x}'\rangle[x_n]$.

(D) *Computing Implicit Functions: Newton's Lemma.* To compute the solution $Y \in \mathfrak{m}_{K\langle \boldsymbol{x}\rangle}$ of an implicit equation $f(\boldsymbol{x}, y) = 0$ (with $f \in K\langle \boldsymbol{x}, y\rangle$ satisfying $f(\mathbf{0}, 0) = 0$ and $\frac{\partial f}{\partial y}(\mathbf{0}) \neq 0$), we may use a variant of the well-known *Newton method* for approximating the zeros of a differentiable function. For instance, starting with the initial solution $Y^{(0)} = 0$, we may set

$$Y^{(j+1)}(\boldsymbol{x}) := Y^{(j)}(\boldsymbol{x}) - \frac{f\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)}{\frac{\partial f}{\partial y}\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)} \,.$$

Note that the denominator $\frac{\partial f}{\partial y}\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)$ is a unit in $K\langle \boldsymbol{x}\rangle$ as $\frac{\partial f}{\partial y}$ has a non-zero constant term, and as $Y^{(j)} \in \langle \boldsymbol{x}\rangle$. Moreover, we get

$$\begin{aligned} f\big(\boldsymbol{x}, Y^{(j+1)}(\boldsymbol{x})\big) = f\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big) &- \frac{\partial f}{\partial y}\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big) \cdot \frac{f\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)}{\frac{\partial f}{\partial y}\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)} \\ &+ h(\boldsymbol{x}) \cdot \left(\frac{f\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)}{\frac{\partial f}{\partial y}\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)} \right)^2 \end{aligned}$$

for some $h \in K\langle \boldsymbol{x}\rangle$. Thus,

$$f\big(\boldsymbol{x}, Y^{(j+1)}(\boldsymbol{x})\big) \in \big\langle f\big(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\big)\big\rangle^2 \,,$$

and Newton's lemma (see Exercise 1.2.6) shows that the sequence of power series $Y^{(j)}$, $j \in \mathbb{N}$, is formally convergent to Y.

For instance, we may compute $\sqrt{1+x}-1$ along the above lines: consider $f(x,y) := (y+1)^2 - (1+x) = y^2 + 2y - x$. Then we get $Y^{(0)} = 0$,

$$Y^{(1)} = \frac{x}{2}, \quad Y^{(2)} = \frac{x}{2} - \frac{\frac{x^2}{4}}{x+2} = \frac{x}{2} - \frac{x^2}{8} \cdot \sum_{k=0}^{\infty} \left(\frac{-x}{2}\right)^k, \quad \dots .$$

Plugging in, we get $f(x, Y^{(2)}) = \frac{1}{64}x^4 + \text{higher terms in } x$. Thus, Newton's lemma gives that

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

is correct up to degree 3.

Exercise 1.2.5. Let $f \in K\langle \boldsymbol{x}, y\rangle$ satisfy $f \in \langle y\rangle + \langle \boldsymbol{x}\rangle^k$ and $\frac{\partial f}{\partial y}(\mathbf{0}) \neq 0$. Show that there exists some $Y \in \langle \boldsymbol{x}\rangle^k$ such that $f(\boldsymbol{x}, Y) = 0$.

Exercise 1.2.6 (Newton's lemma). Let $f \in K\langle \boldsymbol{x}, y\rangle$, and let $\overline{Y} \in K\langle \boldsymbol{x}\rangle$ be such that, for $D := \frac{\partial f}{\partial y}\big(\boldsymbol{x}, \overline{Y}(\boldsymbol{x})\big)$, we have

$$f\big(\boldsymbol{x}, \overline{Y}(\boldsymbol{x})\big) \in \langle \boldsymbol{x}\rangle^k \cdot \langle D\rangle^2 \subset K\langle \boldsymbol{x}\rangle, \quad k \geq 1 .$$

Show that there exists a $Y \in K\langle \boldsymbol{x}\rangle$ with $Y - \overline{Y} \in \langle \boldsymbol{x}\rangle^k \cdot \langle D\rangle$ such that

$$f\big(\boldsymbol{x}, Y(\boldsymbol{x})\big) = 0 .$$

HINT. Introduce a new variable t, and develop $F(\boldsymbol{x}, t) := f\big(\boldsymbol{x}, \overline{Y} + tD\big)$ as a power series in t (with coefficients in $K\langle \boldsymbol{x}\rangle$). Then use Exercise 1.2.5 to deduce that there is a $T \in \langle x\rangle^k$ solving the equation $F(\boldsymbol{x}, T) = 0$.

Exercise 1.2.7 (Jacobian criterion). Let $A = K\langle x_1, \dots, x_n\rangle / \langle f_1, \dots, f_k\rangle$. Moreover, let r be the minimal cardinality for a system of generators of $I = \langle f_1, \dots, f_k\rangle$. Show that $A \cong K\langle x_1, \dots, x_{n-r}\rangle$ iff the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{0}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{0}) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(\mathbf{0}) & \dots & \frac{\partial f_k}{\partial x_n}(\mathbf{0}) \end{pmatrix}$$

has rank r.

Exercise 1.2.8. Prove the claims in Remark 1.26.1.

We do not go further into the theory of analytic algebras, but refer to the textbook by Grauert and Remmert [GrR]. Moreover, since much of the theory can be developed in the general framework of local commutative algebra, we also refer to [AtM, Mat2, Eis].

Still missing is the analogue of Hilbert's Nullstellensatz for analytic algebras, the Hilbert-Rückert Nullstellensatz, which provides a strong relation between algebra and geometry. To formulate it, we need to be able to talk about the zero set of an ideal $I \subset K\langle x_1, \dots, x_n\rangle$ in a neighbourhood of $\mathbf{0} \in K^n$. So, the topology of K^n comes into play. Moreover, we have to assume that K is algebraically closed. Therefore, as always when we treat geometric questions, we restrict ourselves to the case $K = \mathbb{C}$. The next sections provide us with the needed geometric notions. We then formulate and prove the Hilbert-Rückert Nullstellensatz in Section 1.6 (see Theorem 1.72 on p. 76).

1.3 Complex Spaces

In this section we introduce complex spaces, the basic objects of this book, by using the notion and elementary properties of sheaves from Appendix A. Moreover, we introduce some basic constructions such as subspaces, image spaces and fibre products.

In order to provide, besides the formal definition, geometric understanding for the notion of a complex space we begin with analytic sets and a definition of reduced complex spaces which is modeled on the definition of a complex manifold, and which naturally leads to the concept of a (reduced) $\mathbb{C}$-analytic ringed space. Then, it is only a short step to give the general definition of a complex space via structure sheaves with nilpotent elements.

From now on we are working with the field $K = \mathbb{C}$, and we endow $\mathbb{C}^n$ with the usual Euclidean topology.

Definition 1.29. (1) Let $U \subset \mathbb{C}^n$ be an open subset. A complex valued function $f : U \to \mathbb{C}$ is called *(complex) analytic*, or *holomorphic*, if it is *holomorphic at* $\boldsymbol{p}$ for all $\boldsymbol{p} \in U$. That is, for all $\boldsymbol{p} = (p_1, \dots, p_n) \in U$ there is an open neighbourhood $V \subset U$ and a power series

$$\sum_{|\boldsymbol{\alpha}|=0}^{\infty} c_{\boldsymbol{\alpha}} (x_1 - p_1)^{\alpha_1} \cdot \ldots \cdot (x_n - p_n)^{\alpha_n}$$

which converges in V to $f|_V$. In particular, the *coordinate functions* $x_1, \dots, x_n$ of $\mathbb{C}^n$, $x_i : \mathbb{C}^n \to \mathbb{C}$, $\boldsymbol{p} \mapsto p_i$, are holomorphic.

A map $f = (f_1, \dots, f_m) : U \to \mathbb{C}^m$ is called *holomorphic* or *analytic* if the *component functions* $f_i = x_i \circ f$ are.

A holomorphic map $f : U \to V$, $V \subset \mathbb{C}^m$ open, is called *biholomorphic* if f is bijective, and if the inverse $f^{-1} : V \to U$ is holomorphic, too. By the inverse function Theorem 1.21, we have necessarily $m = n$.

We call functions $f_1, \dots, f_n : U \to \mathbb{C}$ *(local) analytic coordinates at* $\boldsymbol{p}$, if each f_i is holomorphic at $\boldsymbol{p}$ with $f_i(\boldsymbol{p}) = 0$, and with $\det\big(\frac{\partial f_i}{\partial x_j}(\boldsymbol{p})\big)_{i,j=1\dots n} \neq 0$. In other words, $f_1, \dots, f_n$ are analytic coordinates at $\boldsymbol{p}$ iff $f = (f_1, \dots, f_n)$

defines a biholomorphic map between an open neighbourhood of $\boldsymbol{p}$ in $\mathbb{C}^n$ and an open neighbourhood of $\mathbf{0}$ in $\mathbb{C}^n$, mapping $\boldsymbol{p}$ to $\mathbf{0}$ (again, by the inverse function theorem).

Recall that a complex power series that converges in V converges uniformly on every compact subset of V, and that the terms can be summed up in any order. Differentiation and summation commute, that is, we can differentiate (respectively integrate) a power series term by term, and the radius of convergence does not change with differentiation (integration). The power series expansion of a holomorphic function f at $\boldsymbol{p}$ is given by its *Taylor series*

$$f(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=0}^{\infty} \frac{1}{\boldsymbol{\alpha}!} \cdot \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}(\boldsymbol{p}) \cdot (x_1 - p_1)^{\alpha_1} \cdot \ldots \cdot (x_n - p_n)^{\alpha_n} ,$$

$\boldsymbol{\alpha}! := \alpha_1! \cdot \ldots \cdot \alpha_n!$, which converges in some open neighbourhood V of $\boldsymbol{p}$.

(2) Let $U \subset \mathbb{C}^n$ be an open subset. If $U \neq \emptyset$, we denote by $\mathcal{O}(U)$ the $\mathbb{C}$-algebra of holomorphic functions on U,

$$\mathcal{O}(U) := \bigl\{ f : U \to \mathbb{C} \text{ holomorphic} \bigr\} .$$

Moreover, we set $\mathcal{O}(\emptyset) := \{0\}$. The association $U \mapsto \mathcal{O}(U)$ defined in this way, together with the restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$, $f \mapsto f|_V$, for $V \subset U$ open, defines a presheaf $\mathcal{O}_{\mathbb{C}^n}$, which is, in fact, a sheaf on $\mathbb{C}^n$. We identify $\mathcal{O}(U)$ with $\Gamma(U, \mathcal{O}_{\mathbb{C}^n})$.

$\mathcal{O}_{\mathbb{C}^n}$ is called the *sheaf of holomorphic functions* on $\mathbb{C}^n$. The *sheaf of holomorphic functions on* U is $\mathcal{O}_U := \mathcal{O}_{\mathbb{C}^n}|_U = i^{-1}\mathcal{O}_{\mathbb{C}^n}$, where $i : U \hookrightarrow \mathbb{C}^n$ is the inclusion map, and $i^{-1}\mathcal{O}_{\mathbb{C}^n}$ denotes the topological preimage sheaf.

We refer to the elements of the stalks $\mathcal{O}_{\mathbb{C}^n,\boldsymbol{p}}$, $\boldsymbol{p} \in \mathbb{C}^n$, also as *germs of holomorphic functions at* $\boldsymbol{p}$ (see A.1). That is, a germ of a holomorphic function at $\boldsymbol{p}$ is the equivalence class of a holomorphic function f defined in an open neighbourhood of $\boldsymbol{p}$, where two functions, defined in open neighbourhoods of $\boldsymbol{p}$, are equivalent if they coincide in some, usually smaller, common neighbourhood of $\boldsymbol{p}$. We write $f_{\boldsymbol{p}}$ for the class of f under this relation, and call it the *germ* of f at $\boldsymbol{p}$.

Note that, for $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{C}^n$, the Taylor series expansion of holomorphic functions at $\boldsymbol{p}$ provides an isomorphism

$$\mathcal{O}_{\mathbb{C}^n,\boldsymbol{p}} \cong \mathbb{C}\{x_1 - p_1, \ldots, x_n - p_n\} \cong \mathbb{C}\{x_1, \ldots, x_n\} .$$

In particular, $\mathcal{O}_{\mathbb{C}^n,\boldsymbol{p}}$ is an analytic $\mathbb{C}$-algebra.

Definition 1.30. Let $D \subset \mathbb{C}^n$ be an open subset. Then a subset $A \subset D$ is called *analytic at* $\boldsymbol{p} \in D$ if there are an open neighbourhood $U \subset D$ of $\boldsymbol{p}$ and holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$ such that

$$U \cap A = V(f_1, \ldots, f_k) := \bigl\{ \boldsymbol{a} \in U \bigm| f_1(\boldsymbol{a}) = \ldots = f_k(\boldsymbol{a}) = 0 \bigr\} .$$

A is called an *analytic* subset of D if it is analytic at every $\boldsymbol{p} \in D$. It is called a *locally analytic* subset if it is analytic at every $\boldsymbol{p} \in A$.

An analytic subset of D is closed in D, and a locally analytic subset is *locally closed*, that is, it is the intersection of an open and a closed subset of D.

In particular, $\mathbb{C}^n = V(0)$ and $\emptyset = V(1)$ are analytic sets in $\mathbb{C}^n$. If $A \subset D$ is analytic and $B \subset A$ is open, then B is locally analytic in D, hence analytic in some open subset of D.

A map $f : A \to B$ between analytic sets is called *holomorphic* (or *analytic*, or a *morphism*) if it is locally the restriction of a holomorphic map between open subsets of some $\mathbb{C}^n$. If $A \subset \mathbb{C}^n$, $B \subset \mathbb{C}^m$ are locally analytic, this means that each $\boldsymbol{p} \in A$ has an open neighbourhood $U \subset \mathbb{C}^n$ such that $f|_{U\cap A} = \widetilde{f}|_{U\cap A}$ for some holomorphic map $\widetilde{f} = \big(\widetilde{f}_1, \dots, \widetilde{f}_m\big) : U \to \mathbb{C}^m$, $\widetilde{f}_i \in \mathcal{O}(U)$. Note that holomorphic maps between analytic sets in $\mathbb{C}^n$ are automatically continuous, and that the composition of holomorphic maps is again holomorphic.

f is called *biholomorphic* (or an *isomorphism*) if it is bijective, and if the inverse $f^{-1} : B \to A$ is also holomorphic. f is called *biholomorphic at* $\boldsymbol{p}$, if there exist open neighbourhoods $U \subset \mathbb{C}^n$ of $\boldsymbol{p}$ and $V \subset \mathbb{C}^m$ of $f(\boldsymbol{p})$ such that $f|_{U\cap A} : U \cap A \to V \cap B$ is biholomorphic.

Remark 1.30.1. Let $f : A \to B$ be a holomorphic map between analytic sets $A \subset \mathbb{C}^n$, $B \subset \mathbb{C}^m$ which is biholomorphic at $\boldsymbol{p} \in A$. If $m = n$, then f can be lifted to a biholomorphic map $\widetilde{f} : U \to V$ for open neighbourhoods U of $\boldsymbol{p}$, and V of $\boldsymbol{q} = f(\boldsymbol{p})$ (by the lifting Lemma 1.23). If $m \neq n$, then f can, of course, not be lifted to a biholomorphic map $U \to V$. But, by the embedding Lemma 1.24, there exists some $e \leq \min\{m, n\}$ and analytic coordinates $u_1, \dots, u_n$ at $\boldsymbol{p} \in \mathbb{C}^n$, and $v_1, \dots, v_m$ at $\boldsymbol{q} \in \mathbb{C}^m$, such that the projections $\pi_1 : (u_1, \dots, u_n) \to (u_1, \dots, u_e)$, and $\pi_2 : (v_1, \dots, v_m) \to (v_1, \dots, v_e)$, map U, V to open neighbourhoods U' of $\pi_1(\boldsymbol{p})$ and V' of $\pi_2(\boldsymbol{q})$ in $\mathbb{C}^e$, and such that the restrictions of π_1, π_2, $\pi_1' : A \cap U \to \pi_1(A \cap U) =: A'$ and $\pi_2' : B \cap V \to \pi_1(B \cap V) =: B'$ are both biholomorphic. Now, the biholomorphic map $\pi_2' \circ f \circ (\pi_1')^{-1} : A' \to B'$ can be lifted locally to a biholomorphic map between open neighbourhoods in $\mathbb{C}^e$.

In the following, we present three possible definitions of a reduced complex space. The first definition is modeled on that of a complex manifold, while the other two will be sheaf theoretic definitions. The advantage of the first definition is that it provides an easier access to reduced complex spaces and their geometry. However, the sheaf theoretic description is needed later for the more general notion of a complex space. That both definitions are equivalent is a consequence of Cartan's coherence Theorem 1.75.

Definition 1.31 (Reduced complex spaces I). Let X be a Hausdorff topological space. Then a set of pairs $\big\{(U_i, \varphi_i) \bigm| i \in I\big\}$ is called an *analytic atlas* (or a *holomorphic atlas*) for X if $\{U_i \mid i \in I\}$ is an open covering of X, and if, for each $i \in I$, $\varphi_i : U_i \to A_i$ is a homeomorphism onto a locally closed analytic set $A_i \subset \mathbb{C}^{n_i}$ such that, for all $(i, j) \in I \times I$ with $U_i \cap U_j \neq \emptyset$, the transition functions

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

are morphisms of analytic sets (hence, isomorphisms with $\varphi_{ij}^{-1} = \varphi_{ji}$).

Each element (U_i, φ_i) of an analytic atlas is called an *analytic* (or *holomorphic*) *chart*, and two analytic atlases are called *equivalent* if their union defines an analytic atlas for X, too.

The topological space X together with an equivalence class of analytic atlases is called a *reduced complex space.*

A reduced complex space X is a *complex manifold of dimension* n iff there exists an analytic atlas $\{(U_i, \varphi_i) \mid i \in I\}$ such that each φ_i is a homeomorphism onto an open subset $D_i \subset \mathbb{C}^n$.

Example 1.31.1. Each local analytic subset $A \subset \mathbb{C}^n$ with the (class of the) *standard atlas*, that is, the atlas consisting of the global chart (A, id_A), is a reduced complex space.

In particular, we always consider $\mathbb{C}^n$ as a reduced complex space, equipped with the standard atlas $\{(\mathbb{C}^n, \mathrm{id}_{\mathbb{C}^n})\}$.

Definition 1.32. A *morphism of reduced complex spaces* X, Y with analytic atlases $\big\{(U_i, \varphi_i) \,\big|\, i \in I\big\}$, $\big\{(V_j, \psi_j) \,\big|\, j \in J\big\}$, is a continuous map $f : X \to Y$ such that for all $(i,j) \in I \times J$ with $f^{-1}(V_j) \cap U_i \neq \emptyset$ the composition

$$\varphi_i\big(f^{-1}(V_j) \cap U_i\big) \xrightarrow{\varphi_i^{-1}} f^{-1}(V_j) \cap U_i \xrightarrow{f} V_j \xrightarrow{\psi_j} \psi_j(V_j)$$

is a morphism of analytic sets.

Such an f is called an *isomorphism of reduced complex spaces* if it is a bijection, and if the inverse f^{-1} is a morphism of reduced complex spaces, too.

A morphism $f : X \to \mathbb{C}$ is called an *analytic* (or *holomorphic*) *function* on X. We denote by $\mathcal{O}(X)$ the set of analytic functions on X, which is obviously a $\mathbb{C}$-algebra.

If X is a reduced complex space, and if $U \subset X$ is an open subset, then U is a complex space, too, with atlas $\big\{(U \cap U_i, \varphi_i|_{U \cap U_i})\big\}$. Such a U is called an *open subspace* of X. For $V \subset U$ open in X, the restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is a morphism of $\mathbb{C}$-algebras. Thus, we get a presheaf $\mathcal{O}_X$ on X, which is in fact a sheaf, called the *sheaf of analytic* (or *holomorphic*) *functions* on X. Note that, by definition, each analytic function $U \to \mathbb{C}$ is continuous. Thus, $\mathcal{O}_X$ is a subsheaf of the sheaf $\mathscr{C}_X$ of continuous complex valued functions on X.

Similar to the above, we refer to the elements of the stalks $\mathcal{O}_{X,p}$, $p \in X$, as *germs at* p *of holomorphic functions on* X. Each such germ is represented by a holomorphic function $f \in \mathcal{O}_X(U)$, defined on an open neighbourhood U of p. Conversely, each $f \in \mathcal{O}_X(U)$ defines a unique germ at $p \in U$, which is denoted by f_p.

If (U, φ) is an analytic chart with $p \in U$, and with φ a homeomorphism from U onto a locally closed analytic set $A \subset \mathbb{C}^n$, then $f \mapsto f \circ \varphi^{-1}$ defines an isomorphism of sheaves $\mathcal{O}_A \cong \varphi_*(\mathcal{O}_X|_U)$. In particular, we get

$$\mathcal{O}_{X,p} \cong \mathcal{O}_{A,\varphi(p)} .$$

Next, we show that $\mathcal{O}_{A,\varphi(p)}$ (hence $\mathcal{O}_{X,p}$) is an analytic $\mathbb{C}$-algebra. Without restriction, assume that $\varphi(p) = \mathbf{0}$, and let A be an analytic subset of some open neighbourhood $V \subset \mathbb{C}^n$ of $\mathbf{0}$. Then each germ at $\mathbf{0}$ of an analytic function on A is the restriction to A of a germ in $\mathcal{O}_{V,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$. Moreover, two germs $f_{\mathbf{0}}, g_{\mathbf{0}} \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ induce the same element of $\mathcal{O}_{A,\mathbf{0}}$ iff they have small holomorphic representatives $f, g : W \to \mathbb{C}$ satisfying $f|_{A\cap W} - g|_{A\cap W} = 0$. Thus, $\mathcal{O}_{A,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I(A)$, where $I(A)$ denotes the ideal

$$I(A) := \left\{ f_{\mathbf{0}} \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} \;\middle|\; \exists\, f \in \mathcal{O}_{\mathbb{C}^n}(W) \text{ representing } f_{\mathbf{0}} \text{ and } f|_{A\cap W} = 0 \right\} .$$

Since $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ is Noetherian, $I(A)$ is finitely generated, thus $\mathcal{O}_{A,\mathbf{0}}$ is an analytic $\mathbb{C}$-algebra.

These considerations show that each reduced complex space in the sense of Definition 1.31 is, in a natural way, a reduced complex space in the sense of the following definition:

Definition 1.33 (Reduced complex spaces II). A *reduced complex space* is a $\mathbb{C}$-analytic ringed space $(X, \mathcal{O}_X)$, where X is a Hausdorff topological space, and where $\mathcal{O}_X$ is a subsheaf of $\mathscr{C}_X$ satisfying

> each point $p \in X$ has an open neighbourhood $U \subset X$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(A, \mathcal{O}_A)$ as $\mathbb{C}$-analytic ringed space, where A is a locally closed analytic subset in some $\mathbb{C}^n$ and where $\mathcal{O}_A$ is the sheaf of holomorphic functions on A. (1.3.1)

A *morphism* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of reduced complex spaces is just a morphism of $\mathbb{C}$-analytic ringed spaces, that is, $f : X \to Y$ is continuous, and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of local $\mathbb{C}$-algebras (A.6).

The equivalence between Definitions 1.31, 1.32 and Definition 1.33 is specified by the following proposition:

Proposition 1.34. *Associating to a complex space X (in the sense of Definition 1.31) the $\mathbb{C}$-analytic ringed space $(X, \mathcal{O}_X)$, with $\mathcal{O}_X$ the sheaf of holomorphic functions on X, and associating to a morphism $f : X \to Y$ of complex spaces the morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$, with*

$$f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X\,, \quad g \longmapsto g \circ f \quad \textit{for } g \in \mathcal{O}_Y(V)\,,\ V \subset Y \textit{ open}\,,$$

defines a functor from the category of reduced complex spaces to the full[5] *subcategory of $\mathbb{C}$-analytic ringed spaces satisfying the conditions of Definition 1.33. This functor is an equivalence of categories. In particular, two reduced complex spaces X, Y are isomorphic iff $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are isomorphic as $\mathbb{C}$-analytic ringed spaces.*

[5] Let $\mathscr{C}$ be a category. Then a subcategory $\mathscr{B}$ of $\mathscr{C}$ is called a *full subcategory* if $\mathrm{Hom}_{\mathscr{B}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$ for any two objects A, B of $\mathscr{B}$.

Proof. Let $f : X \to Y$ be a morphism of reduced complex spaces (in the sense of Definition 1.32). Then, using local charts, it follows that the induced maps $f_p^\sharp : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ are morphisms of $\mathbb{C}$-analytic algebras. Hence, $(f, f^\sharp)$ is a morphism of $\mathbb{C}$-analytic ringed spaces. The functor properties are obvious.

It remains to see that this functor defines an equivalence of categories. The key point is that for a morphism $(f, f^\sharp) : (A, \mathcal{O}_A) \to (B, \mathcal{O}_B)$, with A, B analytic subsets of some open sets $V \subset \mathbb{C}^n$, $W \subset \mathbb{C}^m$, the continuous map $f : A \to B$ uniquely determines $f^\sharp$. Indeed, as for each $\boldsymbol{p} \in A$, the induced map of stalks $f_{\boldsymbol{p}}^\sharp$ is a morphism of $\mathbb{C}$-algebras, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{B,f(\boldsymbol{p})} & \xrightarrow{\quad f_{\boldsymbol{p}}^\sharp \quad} & \mathcal{O}_{A,\boldsymbol{p}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{B,f(\boldsymbol{p})}/\mathfrak{m}_{B,f(\boldsymbol{p})} = \mathbb{C} & \xrightarrow{\ \mathrm{id}\ } & \mathbb{C} = \mathcal{O}_{A,\boldsymbol{p}}/\mathfrak{m}_{A,\boldsymbol{p}}\,. \end{array} \tag{1.3.2}$$

If $f = (f_1, \dots, f_n)$, and if $x_i \in \mathcal{O}_B(B)$, $i = 1, \dots, m$, are induced by the coordinate functions on $W \subset \mathbb{C}^m$, we read from this diagram that

$$f_k(\boldsymbol{p}) = \big((x_k)_{f(\boldsymbol{p})} \bmod \mathfrak{m}_{B,f(\boldsymbol{p})}\big) = \big(f_{\boldsymbol{p}}^\sharp(x_k)_{\boldsymbol{p}} \bmod \mathfrak{m}_{A,\boldsymbol{p}}\big) = f^\sharp(x_k)(\boldsymbol{p})\,.$$

Since each (continuous) map $A \to B$ is uniquely determined by the values at all points $\boldsymbol{p} \in A$, we get $f_k = f^\sharp(x_k)$ and it follows from Remark 1.1.1 (5) and Lemma 1.14 that $f^\sharp$ is uniquely determined by the images $f^\sharp(x_k)$, $k = 1, \dots, m$.

Now, we can define the inverse functor: let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of $\mathbb{C}$-analytic ringed spaces satisfying the requirements of Definition 1.33. Then the property (1.3.1) implies that there is an open covering $\{U_i \mid i \in I\}$ of X and isomorphisms $(\varphi_i, \varphi_i^\sharp) : (U_i, \mathcal{O}_X|_{U_i}) \to (A_i, \mathcal{O}_{A_i})$ of $\mathbb{C}$-analytic ringed spaces with $A_i \subset \mathbb{C}^{n_i}$ locally analytic.

By the above, the components of the transition functions $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1}$ are given by $\varphi_{ij}^\sharp(x_k) = \big((\varphi_i^{-1})^\sharp \circ \varphi_j^\sharp\big)(x_k)$, $k = 1, \dots, n_i$. Thus, they are holomorphic functions. It follows that $\big\{(U_i, \varphi_i) \mid i \in I\big\}$ is an analytic atlas for X, and the equivalence class of this atlas is independent of the chosen covering and isomorphisms. Equipping X and Y in this way with analytic atlases, it is clear that $f : X \to Y$ is a morphism of reduced complex spaces in the sense of Definition 1.32. □

Next, we come to the definition of a general complex space. The definition is similar to Definition 1.33, except that $\mathcal{O}_X$ may have nilpotent elements and, hence, cannot be a subsheaf of the sheaf of continuous functions on X. Nilpotent elements appear naturally when we consider fibres of holomorphic maps. Indeed, the behaviour of the fibres of a morphism can only be understood if we take nilpotent elements into account.

Definition 1.35 (Complex Spaces). Let $D \subset \mathbb{C}^n$ be an open subset.

(1) An ideal sheaf $\mathcal{J} \subset \mathcal{O}_D$ is called *of finite type* if, for every point $\boldsymbol{p} \in D$, there exists an open neighbourhood U of $\boldsymbol{p}$ in D, and holomorphic functions $f_1, \dots, f_k \in \mathcal{O}(U)$ generating $\mathcal{J}$ over U, that is, such that

$$\mathcal{J}|_U = f_1\mathcal{O}_U + \ldots + f_k\mathcal{O}_U \,.$$

Then $\mathcal{O}_D/\mathcal{J}$ is a sheaf of rings on D, and we define

$$V(\mathcal{J}) := \big\{\boldsymbol{p} \in D \,\big|\, \mathcal{J}_{\boldsymbol{p}} \neq \mathcal{O}_{D,\boldsymbol{p}}\big\} = \big\{\boldsymbol{p} \in D \,\big|\, (\mathcal{O}_D/\mathcal{J})_{\boldsymbol{p}} \neq 0\big\} \,.$$

to be the *analytic set in D defined by $\mathcal{J}$*. This is also the *support* of $\mathcal{O}_D/\mathcal{J}$:

$$V(\mathcal{J}) = \operatorname{supp}(\mathcal{O}_D/\mathcal{J}) \,.$$

For each $\boldsymbol{p} \in D$ we have $\mathcal{J}_{\boldsymbol{p}} \neq \mathcal{O}_{D,\boldsymbol{p}}$ iff $f(\boldsymbol{p}) = 0$ for all $f \in \mathcal{J}_{\boldsymbol{p}}$ and, hence, for a neighbourhood U as above

$$V(\mathcal{J}) \cap U \;=\; V(f_1, \dots, f_k) \;=\; \big\{\boldsymbol{p} \in U \,\big|\, f_1(\boldsymbol{p}) = \ldots = f_k(\boldsymbol{p}) = 0\big\} \,.$$

In particular, for $\mathcal{J}$ of finite type, $V(\mathcal{J})$ is an analytic subset of D.

(2) For $\mathcal{J} \subset \mathcal{O}_D$ an ideal sheaf of finite type and $X := V(\mathcal{J})$, we set

$$\mathcal{O}_X \;:=\; (\mathcal{O}_D/\mathcal{J})\big|_X \,.$$

Then $(X, \mathcal{O}_X) = \big(V(\mathcal{J}),\, (\mathcal{O}_D/\mathcal{J})|_X\big)$ is a $\mathbb{C}$-analytic ringed space, called a *complex model space* or the *complex model space defined by $\mathcal{J}$*.

(3) A *complex space*, or *complex analytic space*, is a $\mathbb{C}$-analytic ringed space $(X, \mathcal{O}_X)$ such that X is Hausdorff and, for every $p \in X$, there exists a neighbourhood U of p such that $(U, \mathcal{O}_X|_U)$ is isomorphic to a complex model space as $\mathbb{C}$-analytic ringed space.

We usually write X instead of $(X, \mathcal{O}_X)$. $\mathcal{O}_X$ is called the *structure sheaf* of X, and, for $U \subset X$ open, each section $f \in \Gamma(U, \mathcal{O}_X)$ is called a *holomorphic function* on U.

A *morphism* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *of complex spaces* is just a morphism of $\mathbb{C}$-analytic ringed spaces. Such a morphism is also called a *holomorphic map.* We write $\mathrm{Mor}(X, Y)$ for the set of morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. An isomorphism of complex spaces is also called a *biholomorphic map.*

Let $(X, \mathcal{O}_X)$ be a complex model space, and $p \in X$. Then

$$\mathcal{O}_{X,p} \;\cong\; \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/\mathcal{J}_{\mathbf{0}} \;\cong\; \mathbb{C}\{x_1, \dots, x_n\}/\langle f_1, \dots, f_k\rangle$$

for some $f_1, \dots, f_k \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$. We say that $x_1, \dots, x_n$ are *local (analytic) coordinates* and that $f_1, \dots, f_k$ are *local equations* for X at p.

On the other hand, given convergent power series $f_1, \dots, f_k \in \mathbb{C}\{\boldsymbol{x}\}$, there is an open neighbourhood $U \subset \mathbb{C}^n$ of $\mathbf{0}$ such that each f_i defines a holomorphic function $f_i : U \to \mathbb{C}$. Setting $\mathcal{J} := f_1\mathcal{O}_U + \ldots + f_k\mathcal{O}_U$, the complex (model) space

$$(X, \mathcal{O}_X) := \big(V(\mathcal{J}), (\mathcal{O}_U/\mathcal{J})|_{V(\mathcal{J})}\big)$$

satisfies $\mathcal{O}_{X,\mathbf{0}} \cong \mathbb{C}\{\boldsymbol{x}\}/\langle f_1, \ldots, f_k\rangle$. Thus, each analytic $\mathbb{C}$-algebra appears as the stalk of the structure sheaf of a complex space.

Definition 1.36 (Reduced Complex Spaces III). A complex space $(X, \mathcal{O}_X)$ is called *reduced* if, for each $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a reduced ring, that is, has no nilpotent elements.

Remark 1.36.1. The three definitions of a reduced complex space coincide. The equivalence of Definitions 1.31 and 1.33 was already shown (Proposition 1.34). Definition 1.33 implies 1.36 by the fact that $\mathcal{O}_A \cong \mathcal{O}_D/\mathcal{J}(A)$ where $\mathcal{J}(A) \subset \mathcal{O}_D$ is the full ideal sheaf of $A \subset D$ (see Definition 1.37 below), which is of finite type by Cartan's Theorem 1.75. On the other hand, if X is a reduced complex space in the sense of Definition 1.36 then, locally, $\mathcal{O}_X$ is isomorphic to $(\mathcal{O}_D/\mathcal{J})|_X$ with $X = V(\mathcal{J}) \subset D$. Then $\mathcal{J}$ is contained in the full ideal sheaf $\mathcal{J}(X)$ and, since $\mathcal{J}_p$ is a radical ideal for $p \in X$, the Hilbert-Rückert Nullstellensatz (Theorem 1.72) implies that $\mathcal{J} = \mathcal{J}(X)$.

Let $D \subset \mathbb{C}^n$ be any open set such that A is an analytic subset of D. Then each holomorphic function on A locally lifts to a holomorphic function on an open set $U \subset D$. Moreover, two holomorphic functions f, g on U induce the same holomorphic function on $A \cap U$ iff $(f - g)(\boldsymbol{p}) = 0$ for all $\boldsymbol{p} \in A$. Thus,

$$\mathcal{O}_A \cong \big(\mathcal{O}_D/\mathcal{J}(A)\big)\big|_A\,,$$

where $\mathcal{J}(A) \subset \mathcal{O}_D$ is the full ideal sheaf of $A \subset D$:

Definition 1.37. Let $(X, \mathcal{O}_X)$ be a complex space and $M \subset X$ any subset. Then the *full ideal sheaf* or the *vanishing ideal sheaf* $\mathcal{J}(M)$ of M is the sheaf defined by

$$\mathcal{J}(M)(U) \;=\; \big\{f \in \mathcal{O}_X(U) \,\big|\, M \cap U \subset V(f)\big\}\,,$$

for $U \subset X$ open.

Remark 1.37.1. (1) It is easy to see that $\mathcal{J}(M)$ is a radical sheaf (A.5), and that for an analytic set $A \subset D$ we have $A = V(\mathcal{J}(A))$.

Cartan's Theorem 1.75 says that the full ideal sheaf $\mathcal{J}(A)$ is coherent and the Hilbert-Rückert Nullstellensatz (Theorem 1.72) says that $\mathcal{J}(A) = \sqrt{\mathcal{J}}$ for each ideal sheaf $\mathcal{J}$ such that $A = V(\mathcal{J})$.

(2) The full ideal $\mathcal{J}(X)$ coincides with the nilradical $\mathcal{N}il(\mathcal{O}_X) = \sqrt{\langle 0\rangle}$ of $\mathcal{O}_X$.

Definition 1.38. (1) A *closed complex subspace* of a complex space $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-analytic ringed space $(Y, \mathcal{O}_Y)$, given by an ideal sheaf of finite type $\mathcal{J}_Y \subset \mathcal{O}_X$ such that $Y = V\big(\mathcal{J}_Y\big) := \operatorname{supp}\big(\mathcal{O}_X/\mathcal{J}_Y\big)$ and $\mathcal{O}_Y = \big(\mathcal{O}_X/\mathcal{J}_Y\big)|_Y$.

In particular, we define the *reduction of a complex space* $(X, \mathcal{O}_X)$ to be the (reduced) closed complex subspace

$$X_{red} := (X, \mathcal{O}_X/\mathcal{J}(X))$$

defined by the full ideal sheaf $\mathcal{J}(X)$ of X. We also say that $\mathcal{O}_X/\mathcal{J}(X)$ is the *reduced structure (sheaf)* on X.

An *open complex subspace* $(U, \mathcal{O}_U)$ of $(X, \mathcal{O}_X)$ is given by an open subset $U \subset X$ and $\mathcal{O}_U = \mathcal{O}_X|_U$.

(2) A morphism $f : X \to Y$ of complex spaces is called an *open* (resp. *closed*) *embedding* if there exists an open (resp. closed) subspace $Z \subset Y$ and an isomorphism $g : X \xrightarrow{\cong} Z$ such that $f = i \circ g$, where $i : Z \hookrightarrow Y$ is the inclusion map.

Remark 1.38.1. (1) If $(Y, \mathcal{O}_Y)$ is a closed complex subspace of a complex space $(X, \mathcal{O}_X)$, then Y is closed in X and $(Y, \mathcal{O}_Y)$ is a complex space.

Indeed, by the coherence theorem of Oka (Theorem 1.63, below), the structure sheaf of any complex space is coherent. As the ideal sheaf $\mathcal{J}_Y \subset \mathcal{O}_X$ is supposed to be of finite type, it is also coherent, and the same holds for the quotient $\mathcal{O}_X/\mathcal{J}_Y$ (A.7, Fact 3). Hence, Y is the support of a coherent $\mathcal{O}_X$-sheaf, thus closed in X (A.7, Fact 1).

To see that $(Y, \mathcal{O}_Y)$ is a complex space, we may assume that $(X, \mathcal{O}_X)$ is a complex model space, defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_D$ of finite type (with $D \subset \mathbb{C}^n$ an open subset). Let $p \in Y$. Since $\mathcal{J}_Y$ is of finite type, there is an open neighbourhood $U \subset D$ of p and functions $f_1, \ldots, f_k \in \mathcal{O}_D(U)$ such that the corresponding residue classes $\overline{f}_1, \ldots, \overline{f}_k \in \mathcal{O}_X(X \cap U)$ generate $\mathcal{J}_Y|_{X \cap U}$. Then $(Y \cap U, \mathcal{O}_Y|_{Y \cap U})$ is the complex model space in U defined by the finitely generated ideal $\mathcal{I}|_U + f_1\mathcal{O}_U + \ldots + f_k\mathcal{O}_U \subset \mathcal{O}_U$.

(2) A complex space $(X, \mathcal{O}_X)$ is reduced iff $X_{red} = (X, \mathcal{O}_X)$.

Example 1.38.2. (1) An important example of non-reduced complex spaces are *fat points* (or *Artinian complex space germs*). As such we denote non-reduced complex spaces X satisfying $X_{red} = \{\mathrm{pt}\}$. That is, the underlying topological space of a fat point consists only of one point, and the structure sheaf of X is uniquely determined by the stalk at this point. When defining a fat point, we usually specify only this stalk. For instance, we call $T_\varepsilon := \big(\{\mathrm{pt}\}, \mathbb{C}[\varepsilon]\big)$ the *fat point of length two*, since the defining $\mathbb{C}$-algebra is non-reduced (as $\varepsilon^2 = 0$) and a two-dimensional complex vector space. More generally, each analytic $\mathbb{C}$-algebra A with $1 < \dim_{\mathbb{C}} A < \infty$ defines a fat point. Note that T_ε may be embedded as a closed complex subspace in each fat point.

(2) Let $(X, \mathcal{O}_X) = \big(V(y), \mathcal{O}_{\mathbb{C}^2}/\langle xy, y^2\rangle\big) \subset \big(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}\big)$ then the reduction of X is $X_{red} = \big(V(y), \mathcal{O}_{\mathbb{C}^2}/\langle y\rangle\big) \cong (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$. Here, X is the union of the x-axis and a fat point with support $\{\mathbf{0}\}$. Indeed, the primary decomposition of $I = \langle xy, y^2\rangle = \langle y\rangle \cap \langle x, y^2\rangle$ yields $X = V(y) \cup V(x, y^2)$ with $V(x, y^2) \cong T_\varepsilon$.

Definition 1.39. Let X be a complex space, $p \in X$ and $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{X,p}$. Then we define

$$\begin{aligned} \dim_p X &:= \text{Krull dimension of } \mathcal{O}_{X,p}, \text{ the } \textit{dimension of } X \textit{ at } p, \\ \dim X &:= \sup_{p \in X} \dim(X,p), \text{ the } \textit{dimension of } X, \\ \operatorname{edim}_p X &:= \dim_{\mathbb{C}} \mathfrak{m}_p/\mathfrak{m}_p^2, \text{ the } \textit{embedding dimension of } X \textit{ at } p. \end{aligned}$$

Note that $\dim_p X = \dim_p X_{red}$ (see B.2), while the embedding dimension of X_{red} at p may be strictly smaller than the embedding dimension of X at p.

We refer to a reduced complex space X as a *curve* (respectively as a *surface*) if $\dim_p X = 1$ (respectively $\dim_p X = 2$) for all $p \in X$.

Remark 1.39.1. Locally at a point $p \in X$, we can identify each complex space $(X, \mathcal{O}_X)$ with a complex model space $\bigl(V(\mathcal{J}), \mathcal{O}_D/\mathcal{J}\bigr)$, where D is an open set in $\mathbb{C}^n$, and where $\mathcal{J} = f_1\mathcal{O}_D + \ldots + f_k\mathcal{O}_D \subset \mathcal{O}_D$. While $\mathcal{O}_D/\mathcal{J}$ is part of the structure, the embedding $X \subset D \subset \mathbb{C}^n$ and, hence, $\mathcal{J}$ is not part of the structure. Indeed, we may embed $(X, \mathcal{O}_X)$ in different ways as a subspace of $\mathbb{C}^m$ for various m. By the embedding Lemma 1.24, the minimal possible m is $\operatorname{edim}_p X$, which is the reason for calling $\operatorname{edim}_p X$ the embedding dimension of X at p (Exercise 1.3.3).

Definition 1.40. A complex space X is called *regular at* $p \in X$, if

$$\dim_p X = \operatorname{edim}_p X \,,$$

that is, if $\mathcal{O}_{X,p}$ is a regular local ring. Then p is also called a *regular point* of X. A point of X is called *singular* if it is not a regular point of X.

By Proposition 1.48 below, a complex space X is a complex manifold iff X is regular at each $p \in X$.

Definition 1.41. A morphism $\bigl(f, f^\sharp\bigr) : X \to Y$ of complex spaces is called *regular at* $p \in X$ if the induced ring map $f_p^\sharp : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a regular morphism of analytic K-algebras.

Here, a morphism $\varphi : A = K\langle \boldsymbol{x} \rangle / I \to B$ of analytic K-algebras is called *regular*, or B is called a *regular A-algebra*, if B is isomorphic (as A-algebra) to a free power series algebra over A, that is, if $B \cong A\langle \boldsymbol{y} \rangle := K\langle \boldsymbol{x}, \boldsymbol{y} \rangle / IK\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_m)$ are disjoint sets of variables.

Instead of "regular", the notions *smooth* or *non-singular* are used as well.

Remark 1.41.1. Recall from the proof of Proposition 1.34 that a morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of *reduced* complex spaces is uniquely determined by f. If $(X, \mathcal{O}_X)$ is not reduced, this is no longer true. As a concrete example, consider the fat point $T_\varepsilon = \bigl(\{\mathrm{pt}\}, \mathbb{C}[\varepsilon]\bigr)$. We may supplement the continuous map $T_\varepsilon \ni \mathrm{pt} \mapsto 0 \in \mathbb{C}$ to a morphism $\bigl(f, f^\sharp\bigr) : T_\varepsilon \to \mathbb{C}$ by setting $f^\sharp(x) := a\varepsilon$, where $a \in \mathbb{C}$ is arbitraily chosen.

The next lemma describes morphisms to $\mathbb{C}^n$.

Lemma 1.42. *Let X be a complex space, $U \subset X$ an open subspace, and let $x_1, \dots, x_n$ denote the coordinate functions of $\mathbb{C}^n$. Then, for each $n \geq 1$,*

$$\Phi_U : \mathrm{Mor}(U, \mathbb{C}^n) \longrightarrow \Gamma(U, \mathcal{O}_X)^n, \quad \big(f, f^\sharp\big) \longmapsto \big(f^\sharp(x_1), \dots, f^\sharp(x_n)\big)\,,$$

is a bijective map.

The sections $f_i := f^\sharp(x_i) \in \Gamma(U, \mathcal{O}_X)$ can thus be considered as holomorphic functions $f_i : U \to \mathbb{C}$. We call $f_1, \dots, f_n$ the *component functions* of $(f, f^\sharp)$.

Proof. By considering the component functions, we may restrict ourselves to the case $n = 1$. We define the inverse map of Φ_U,

$$\Psi_U : \Gamma(U, \mathcal{O}_X) \longrightarrow \mathrm{Mor}(U, \mathbb{C})\,, \quad g \longmapsto \big(\overline{g}, \overline{g}^\sharp\big)\,. \tag{1.3.3}$$

Here, $\overline{g} : U \to \mathbb{C}$ is the *evaluation map* associated to $g \in \Gamma(U, \mathcal{O}_X)$,

$$\overline{g} : U \to \mathbb{C}\,, \quad \overline{g}(p) := (g_p \bmod \mathfrak{m}_{X,p})\,.$$

Note that $\overline{g}$ is continuous. Indeed, locally, we may assume that $\big(X, \mathcal{O}_X\big) = \big(V(\mathcal{J}), (\mathcal{O}_V/\mathcal{J})|_{V(\mathcal{J})}\big)$ is a complex model space, with $V \subset \mathbb{C}^m$ open and $V(\mathcal{J}) \subset V$. Then, by definition, we may lift g to a section $\widetilde{g} \in \Gamma(V, \mathcal{O}_{\mathbb{C}^m})$ (after shrinking V), that is, to a holomorphic, hence continuous, function $\widetilde{g} : V \to \mathbb{C}$ such that $\overline{g}(p) = \widetilde{g}(p)$. for all $p \in V$.

The existence of a local lifting of g to a holomorphic function $\widetilde{g}$ on some open subset of $\mathbb{C}^n$ allows us to define the sheaf map $\overline{g}^\sharp : \mathcal{O}_{\mathbb{C}} \to g_* \mathcal{O}_X$ by composition (see also the definition of $f^\sharp$ in Proposition 1.34): for each open $W \subset \mathbb{C}$, define

$$\overline{g}^\sharp : \Gamma\big(W, \mathcal{O}_{\mathbb{C}}\big) \to \Gamma\big(\widetilde{g}^{-1}(W), \mathcal{O}_V\big) \to \Gamma\big(\widetilde{g}^{-1}(W),\, \mathcal{O}_V/\mathcal{J}\big) = \Gamma\big(W, \overline{g}_* \mathcal{O}_X\big)\,,$$

$$(z : W \to \mathbb{C}) \;\longmapsto\; (z \circ \widetilde{g} : \widetilde{g}^{-1}(W) \to \mathbb{C}) \;\longmapsto\; (z \circ \widetilde{g} \bmod \mathcal{J})\,.$$

Clearly, for each $p \in U$, the induced map of germs $\overline{g}^\sharp_p : \mathcal{O}_{\mathbb{C}, \overline{g}(p)} \to \mathcal{O}_{X,p}$ is a morphism of local $\mathbb{C}$-algebras.

It remains to show that Φ_U is the inverse of Ψ_U. That is, for each morphism of complex spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and for each $g \in \Gamma(U, \mathcal{O}_X)$, we have to show that

$$\overline{f^\sharp(x)} = f\,, \qquad \overline{g}^\sharp(x) = g\,,$$

where x is a coordinate of $\mathbb{C}$. The analogue of the commutative diagram (1.3.2) implies that the first equality holds. To see the second, note that a map $h^\sharp : \mathcal{O}_{\mathbb{C}}(W) \to \mathcal{O}_X\big(f^{-1}(W)\big)$ is already determined by $h^\sharp(x)$, since the induced maps of germs $\mathcal{O}_{\mathbb{C}, f(p)} \to \mathcal{O}_{X,p}$, $p \in W$, are determined by the image of x (Remark 1.1.1 (5) and Lemma 1.14). From the definitions, we get

$$\overline{g}^\sharp(x) = \big(x \circ \widetilde{g} \bmod \mathcal{J}\big) = g\,.$$

$\square$

If Y is a complex subspace of $\mathbb{C}^n$, any morphism $(f, f^\sharp) : X \to Y$ is determined by the continuous map f and by the component functions $f_i = f^\sharp(x_i)$, as follows from Lemma 1.42. Moreover, if $(f, f^\sharp) : X \to Y$ is a morphism of arbitrary complex spaces, then locally at $p \in X$ we may embed X into $\mathbb{C}^n$, and locally at $q = f(p) \in Y$ we may embed Y into $\mathbb{C}^m$ (via model spaces). Then there exist open neighbourhoods $U \subset \mathbb{C}^n$ of p and $V \subset \mathbb{C}^m$ of q and holomorphic functions $\widetilde{f}_1, \ldots, \widetilde{f}_m : U \to \mathbb{C}$ such that the diagram

$$\begin{tikzcd} \mathbb{C}^n \supset U \arrow[r, "\widetilde{f}=(\widetilde{f}_1,\ldots,\widetilde{f}_m)"] & V \subset \mathbb{C}^m \\ X \cap U \arrow[u, hook] \arrow[r, "f|U"'] & Y \cap V \arrow[u, hook] \end{tikzcd}$$

commutes, and $\widetilde{f}^{\,\sharp}(\mathcal{J}_{Y,\widetilde{f}(p')}) \subset \mathcal{J}_{X,p'}$ for all $p' \in U$ (use Lemma 1.14 and A.7, Fact 2). That is, locally, holomorphic maps are just restrictions of holomorphic maps between open sets of complex number spaces mapping the corresponding ideal sheaves into each other.

If $n = m$, and if $(f, f^\sharp)$ is an isomorphism, we may choose embeddings with $n = m$ and with $\bigl(\widetilde{f}, \widetilde{f}^{\,\sharp}\bigr)$ being an isomorphism. This follows from the embedding Lemma 1.24, the lifting Lemma 1.14 and the inverse function Theorem 1.21.

Definition 1.43. Let $(X, \mathcal{O}_X)$ be a complex space. A subset $A \subset X$ is called *analytic at* a point $p \in X$ if there exist a neighbourhood U of p and $f_1, \ldots, f_k \in \mathcal{O}_X(U)$ such that

$$A \cap U = V(f_1, \ldots, f_k) := \operatorname{supp}(\mathcal{O}_U/\mathcal{J})$$

with $\mathcal{J} := f_1\mathcal{O}_U + \ldots + f_k\mathcal{O}_U$. If A is analytic at every point $p \in A$, then it is called a *locally closed analytic set in* X. If A is analytic at every $p \in X$, then it is called a *(closed) analytic set in* X.

Remark 1.43.1. (1) Analytic sets in X are just the supports of coherent $\mathcal{O}_X$-sheaves. This follows from Oka's Theorem 1.63 and A.7, Fact 5.

(2) The underlying set of a complex subspace of $(X, \mathcal{O}_X)$ is an analytic set in X. On the other hand, there is a canonical way to identify an analytic set $A \subset X$ with a (reduced) closed complex subspace of $(X, \mathcal{O}_X)$, setting $\mathcal{O}_A := (\mathcal{O}_X/\mathcal{J}(A))|_A$, where $\mathcal{J}(A)$ denotes the full ideal sheaf of $A \subset X$.

(3) Finite unions and arbitrary intersections of analytic sets in a complex space X are analytic sets in X (Exercise 1.3.5).

(4) If $(Y, \mathcal{O}_Y)$ is a closed complex subspace of $(X, \mathcal{O}_X)$, and if A is an analytic set in Y, then A is also an analytic set in X (Exercise 1.3.6).

In general, for a morphism $(f, f^\sharp) : X \to Y$ of complex spaces, the image $f(X)$ is not an analytic set. Consider, for example, the projection

$$\mathbb{C}^3 \supset X = V(xy-1) \cup V(z) \xrightarrow{f} \mathbb{C}^2\,, \quad (x,y,z) \longmapsto (y,z)\,.$$

Then $f(X) = \big(\mathbb{C}^2 \setminus V(y)\big) \cup \{\mathbf{0}\}$, which is not analytic at $\mathbf{0}$ (see Figure 1.1).

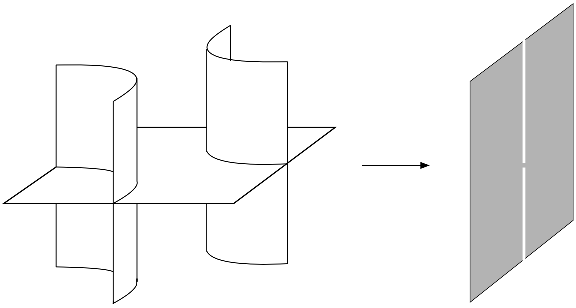


Fig. 1.1. The projection $V(xy-1) \cup V(z) \xrightarrow{f} \mathbb{C}^2\,,\ (x,y,z) \mapsto (y,z)$.

We shall see later (Theorem 1.67) that the image of a *finite* morphism is always analytic. At this point, we consider a formal property under which $f(X)$ is analytic.

Lemma 1.44. *Let $f : X \to Y$ be a morphism of complex spaces with $f_*\mathcal{O}_X$ coherent. Then the closure of $f(X)$ in the Euclidean topology satisfies*

$$\overline{f(X)} = \operatorname{supp}\big(\mathcal{O}_Y / \mathcal{A}nn_{\mathcal{O}_Y}(f_*\mathcal{O}_X)\big)\,,$$

which is a closed analytic set in Y. In particular, if $f_*\mathcal{O}_X$ is coherent and if $f(X)$ is closed, then $f(X)$ is analytic in Y.

Proof. The ideal sheaf $\mathcal{J} = \mathcal{A}nn_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$ is coherent (A.7, Fact 5). It follows that $V(\mathcal{J}) = \operatorname{supp}(\mathcal{O}_Y/\mathcal{J})$ is a closed analytic set. If $y \in f(X)$ then $(f_*\mathcal{O}_X)_y \neq 0$, hence, $\mathcal{J}_y \neq \mathcal{O}_{Y,y}$ and, therefore, $y \in V(\mathcal{J})$.

If $y \notin \overline{f(X)}$ then $(f_*\mathcal{O}_X)_y = 0$ and, therefore, $y \notin V(\mathcal{J})$. Hence, $f(X) \subset V(\mathcal{J}) \subset \overline{f(X)}$. As $V(\mathcal{J})$ is closed, the result follows. $\square$

The fact that $f_*\mathcal{O}_X$ is coherent does not yet imply that the image $f(X) \subset Y$ is closed. For example, consider the inclusion $i : \mathbb{C}^2 \setminus \{0\} \hookrightarrow \mathbb{C}^2$. Then the Riemann removable singularity theorem (Theorem 1.98) yields that $i_*\mathcal{O}_{\mathbb{C}^2\setminus\{0\}} = \mathcal{O}_{\mathbb{C}^2}$, which is a coherent sheaf. However, the image of i is not closed in $\mathbb{C}^2$.

If $f(X)$ is closed (in the Euclidean topology) and $f_*\mathcal{O}_X$ is coherent then $f(X) = V\big(\mathcal{A}nn_{\mathcal{O}_Y}(f_*\mathcal{O}_X)\big)$ is analytic, by Lemma 1.44. We may equip the image $f(X)$ with different structure sheaves:

Definition 1.45. Let $(f, f^\sharp) : X \to Y$ be a morphism of complex spaces with $f(X)$ closed in Y and $f_*\mathcal{O}_X$ coherent. We call the complex space

(1) $\big(f(X),\, \mathcal{O}_Y/\mathcal{A}nn_{\mathcal{O}_Y}(f_*\mathcal{O}_X)\big)$ the *image* of f with *annihilator structure*,
(2) $\big(f(X),\, \mathcal{O}_Y/\mathcal{J}(f(X))\big)$ the *image* of f with *reduced structure*, and
(3) $\big(f(X),\, \mathcal{O}_Y/\mathcal{F}itt(f_*\mathcal{O}_X)\big)$ the *image* of f with *Fitting structure.*

Here, $\mathcal{J}(f(X))$ denotes the full ideal sheaf of $f(X) \subset Y$, and $\mathcal{F}itt(f_*\mathcal{O}_X)$ denotes the 0-th Fitting ideal sheaf of $f_*\mathcal{O}_X$ (see below).

These definitions make sense. For the annihilator structure this follows from Lemma 1.44. For the reduced structure, see Remark 1.37.1. To define the *0-th Fitting ideal sheaf*, note that, since $f_*\mathcal{O}_X$ is coherent, Y can be covered by open sets $U \subset Y$ such that on each U we have an exact sequence

$$\mathcal{O}_U^q \xrightarrow{A} \mathcal{O}_U^p \longrightarrow f_*\mathcal{O}_X\big|_U \longrightarrow 0\,.$$

The maximal *minors* of A, that is, the determinants of $p\times p$ submatrices of A, define ideals $\mathcal{F}(V) \subset \Gamma(V, \mathcal{O}_Y)$, $V \subset U$ open, which are independent of the chosen presentation (cf. [Eis, Lan]). These define, locally, a coherent ideal sheaf $\mathcal{F} =: \mathcal{F}itt_{\mathcal{O}_Y}(f_*\mathcal{O}_X) \subset \mathcal{O}_Y$, the 0-th *Fitting ideal sheaf* of $f_*\mathcal{O}_X$. Note that $\mathcal{F}_y = \mathcal{O}_{Y,y}$ iff a $p\times p$-minor of A_y is a unit in $\mathcal{O}_{Y,y}$, which is equivalent to the surjectivity of A_y, that is, to $(f_*\mathcal{O}_X)_y = 0$. Hence, we obtain

$$V\big(\mathcal{F}itt_{\mathcal{O}_Y}(f_*\mathcal{O}_X)\big) \;=\; \mathrm{supp}(f_*\mathcal{O}_X) \;=\; f(X)\,.$$

Remark 1.45.1. (1) Of course, we can take any ideal sheaf $\mathcal{J}$ of finite type with $V(\mathcal{J}) = f(X)$ to define a complex structure on the analytic set $f(X)$, but the above three are the most important.

(2) The annihilator structure on $f(X)$ is closely related to the structure map $f^\sharp$, as $\mathcal{A}nn_{\mathcal{O}_Y}(f_*\mathcal{O}_X) = \mathcal{K}er(f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X)$ (using that $f_*\mathcal{O}_X$ contains the unit section). The reduced structure turns the image into a reduced complex space.

Compared to these two structures, the Fitting structure has the advantage that it is *compatible with base change* in the following sense: if $\varphi : Z \to Y$ is a morphism of complex spaces then

$$\mathcal{F}itt_{\mathcal{O}_Z}\big(\varphi^*(f_*\mathcal{O}_X)\big) \;=\; \varphi^{-1}\big(\mathcal{F}itt_{\mathcal{O}_Y}(f_*\mathcal{O}_X)\big)\cdot \mathcal{O}_Z\,.$$

To see this fact, apply the right exact functor φ^* to a local presentation of $f_*\mathcal{O}_X|_U$ as above and note that computing determinants is compatible with base change (computing the determinant and then substituting the variables is the same as first substituting the variables in the matrix and then computing the determinant). See also Exercise 1.5.3.

(3) Using Cramer's rule, it is easy to see that the Fitting ideal is contained in the annihilator ideal, hence,

$$\mathcal{F}itt_{\mathcal{O}_Y}(f_*\mathcal{O}_X) \;\subset\; \mathcal{A}nn_{\mathcal{O}_Y}(f_*\mathcal{O}_X) \;\subset\; \mathcal{J}\big(f(X)\big)\,.$$

Next, we provide an example, where the structures are pairwise different:

Example 1.45.2. Consider $X = \{\mathbf{0}\} \subset \mathbb{C}^2$ as complex space equipped with the structure sheaf $\mathcal{O}_X = (\mathcal{O}_{\mathbb{C}^2}/\mathcal{J}(X)^2)|_X$, with stalk $\mathcal{O}_{X,\mathbf{0}} = \mathbb{C}\{x,y\}/\mathfrak{m}^2$. Let $f : X \to \mathbb{C}$ be the projection on the first coordinate. Then $f(X) = \{0\}$. Hence, the structure sheaf on $f(X)$ is uniquely determined by its stalk at 0. We compute (see also Exercise 1.3.7)

$$\mathcal{F}itt_{\mathcal{O}_{\mathbb{C}}}(f_*\mathcal{O}_X)_0 = \langle x^3 \rangle \subsetneq \mathcal{A}nn_{\mathcal{O}_{\mathbb{C}}}(f_*\mathcal{O}_X)_0 = \langle x^2 \rangle \subsetneq \mathcal{J}(f(X))_0 = \langle x \rangle \,.$$

Notation. From now on, we preferably denote a morphism of complex spaces simply by f instead of $(f, f^\sharp)$. Moreover, we write $f = (f_1, \dots, f_n)$, where $f_1, \dots, f_n$ are the component functions of $(f, f^\sharp)$. Note however, that the structure map $f^\sharp$ is always part of the data (see also Remark 1.41.1).

We conclude this section by considering the Cartesian product of two complex spaces and, more generally, the fibre product of two morphisms of complex spaces.

Definition 1.46. Let $f : X \to T$, $g : Y \to T$ be two morphisms of complex spaces. Then the *(analytic) fibre product* of X and Y over T is a triple $(X \times_T Y, \pi_X, \pi_Y)$ consisting of a complex space $X \times_T Y$ and two morphisms $\pi_X : X \times_T Y \to X$, $\pi_Y : X \times_T Y \to Y$ such that $f \circ \pi_X = g \circ \pi_Y$, satisfying the following *universal property*: for any complex space Z and any two morphisms $h : Z \to X$, $h' : Z \to Y$ satisfying $g \circ h' = f \circ h$ there exists a unique morphism $\varphi : Z \to X \times_T Y$ such that the following diagram commutes

$$\begin{tikzcd}
Z \arrow[dr, "\varphi"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "h'"'] & & \\
& X \times_T Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & T\,.
\end{tikzcd}$$

The usual diagram chase shows that, if the fibre product exists, then it is unique up to a unique isomorphism. We use the notation

$$\begin{tikzcd}
W \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{f}"'] \arrow[dr, phantom, "\Box"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & T
\end{tikzcd}$$

for a commutative diagram providing the universal property of the fibre product and call it a *Cartesian diagram*.

Proof of the existence of the fibre product (sketched).

Step 1. As a topological space, set

$$X \times_T Y := \big\{(x,y) \in X \times Y \,\big|\, f(x) = g(y)\big\}\,.$$

Step 2. It is easy to see $\mathbb{C}^n \times_{\{\mathrm{pt}\}} \mathbb{C}^m \cong \mathbb{C}^{n+m}$ with structure sheaf $\mathcal{O}_{\mathbb{C}^{n+m}}$.

Step 3. Let $X \subset U \subset \mathbb{C}^n$ and $Y \subset V \subset \mathbb{C}^m$ be two complex model spaces with ideal sheaves $\mathcal{J}_X$, $\mathcal{J}_Y$. Then the topological space $X \times_{\{\mathrm{pt}\}} Y \subset \mathbb{C}^{n+m}$ is just the Cartesian product $X \times Y$ with the product topology. The ideal sheaf is given by

$$\mathcal{J}_{X\times_{\{\mathrm{pt}\}}Y} = \mathcal{J}_X \cdot \mathcal{O}_{\mathbb{C}^{n+m}} + \mathcal{J}_Y \cdot \mathcal{O}_{\mathbb{C}^{n+m}}$$

and the struture sheaf is $\mathcal{O}_{X\times_{\{\mathrm{pt}\}}Y} = \mathcal{O}_{\mathbb{C}^{n+m}}\big/\mathcal{J}_{X\times_{\{\mathrm{pt}\}}Y}$.

Step 4. Assume that $T \subset W \subset \mathbb{C}^p$ is a complex model space and

$$f = (f_1,\dots,f_p) : X \to T \subset \mathbb{C}^p\,, \quad g = (g_1,\dots,g_p) : Y \to T \subset \mathbb{C}^p\,.$$

Then define the ideal sheaf to be $\mathcal{J}_{X\times_T Y} := \big\langle f_i - g_i \bigm| i = 1,\dots,p\big\rangle \cdot \mathcal{O}_{X\times_{\{\mathrm{pt}\}}Y}$, and set $\mathcal{O}_{X\times_T Y} = \mathcal{O}_{X\times_{\{\mathrm{pt}\}}Y}\big/\mathcal{J}_{X\times_T Y}$.

Step 5. In general, we cover X, Y, T by complex model spaces and apply the above construction. By the universal property, we have exactly one way to glue and, by A.2, we get a uniquely defined structure sheaf on $X \times_T Y$.

Example 1.46.1. (1) If $T = \{\mathrm{pt}\}$ is a reduced point then the fibre product $X \times_{\{\mathrm{pt}\}} Y$ is called the *Cartesian product* $X \times Y$ of the complex spaces X and Y. If $x \in X$, $y \in Y$, and if $\mathcal{O}_{X,x} = \mathbb{C}\{\boldsymbol{x}\}/I$, $\mathcal{O}_{Y,y} = \mathbb{C}\{\boldsymbol{y}\}/J$, then

$$\mathcal{O}_{X\times Y,(x,y)} = \mathbb{C}\{\boldsymbol{x},\boldsymbol{y}\}\big/\big(I\mathbb{C}\{\boldsymbol{x},\boldsymbol{y}\} + J\mathbb{C}\{\boldsymbol{x},\boldsymbol{y}\}\big)\,.$$

This local ring is the analytic tensor product $\mathcal{O}_{X,x} \mathbin{\widehat{\otimes}} \mathcal{O}_{Y,y}$ of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ (see Definition 1.28 on p. 31).

(2) If $g : Y \to T$ is an inclusion, that is, if Y is an analytic subspace of T, then $X \times_T Y$ is called the *preimage* $f^{-1}(Y)$ of Y under the morphism f. If $\mathcal{J}$ is the ideal sheaf of $Y \subset T$ then $\mathcal{O}_{f^{-1}(Y)} = \mathcal{O}_X/\mathcal{J}\mathcal{O}_X$, where $\mathcal{J}\mathcal{O}_X$ denotes the image of $f^*\mathcal{J} = f^{-1}\mathcal{J} \otimes \mathcal{O}_X$ (see A.6) in $\mathcal{O}_X$ under the multiplication $a \otimes b \mapsto ab$. In particular, if $x \in X$ then $\mathcal{O}_{f^{-1}(Y),x} = \mathcal{O}_{X,x}/\mathcal{J}_{f(x)}\mathcal{O}_{X,x}$.

(3) If $p \in T$ is a point then $X \times_T \{p\}$ is called *fibre* $f^{-1}(p)$ of f over p. Let $T \subset \mathbb{C}^k$, and let $f = (f_1,\dots,f_k)$, $\boldsymbol{p} = (p_1,\dots,p_k) \in T$. Then it follows from the construction of the fibre product that

$$\mathcal{O}_{f^{-1}(\boldsymbol{p})} = \big(\mathcal{O}_X/\langle f_1 - p_1,\dots,f_k - p_k\rangle\mathcal{O}_X\big)\big|_{f^{-1}(\boldsymbol{p})}\,.$$

(4) If $Y = T$ and $g = \mathrm{id}_T$ then $X \times_T T$ is called the *graph* $\Gamma(f)$ of f. Note that the obvious map $X \times_T T \to X \times T$ embeds $\Gamma(f)$ as a closed subspace in $X \times T$. We have a commutative diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & \Gamma(f) \arrow[l, "\cong"', "\pi_1"] \arrow[d, "\pi_2"] \subset X \times T \\ & T\,, \end{tikzcd}$$

where π_1, π_2 are the two projections, and where π_1 is an isomorphism with $\pi_1^{-1} = (\mathrm{id}_X, f)$. If $f = (f_1, \dots, f_k) : X \to T \subset \mathbb{C}^k$, and if $\boldsymbol{y} = (y_1, \dots, y_k)$ are coordinates of $\mathbb{C}^k$, then

$$\mathcal{O}_{\Gamma(f)} = \mathcal{O}_{X\times T}/\langle f_1 - y_1, \dots, f_k - y_k\rangle \mathcal{O}_{X\times T}\,.$$

As a concrete example, consider the graph of the morphism $f : T_\varepsilon \to \mathbb{C}$ with component function $f_1 = a\varepsilon$, $a \in \mathbb{C}$, in Remark 1.41.1. Then $\mathcal{O}_{\Gamma(f),\mathbf{0}} = \mathbb{C}\{x, \varepsilon\}/\langle \varepsilon^2, a\varepsilon - x\rangle \cong \mathbb{C}\{\varepsilon\}/\langle\varepsilon\rangle^2 = \mathcal{O}_{T_\varepsilon,0}$.

(5) If X and Y are subspaces of T with $i : X \hookrightarrow T$ and $j : Y \hookrightarrow T$ the inclusion morphisms, then $X \times_T Y = i^{-1}(Y)$ is the *intersection* $X \cap Y$ of X and Y. If $x \in X \cap Y$, and if $\mathcal{I}$, resp. $\mathcal{J}$, denotes the ideal sheaf of X, resp. Y, in T, then $\mathcal{O}_{X\cap Y,x} = \mathcal{O}_{T,x}/(\mathcal{I}_x + \mathcal{J}_x)$.

Note that, unlike in the case of algebraic varieties, the Cartesian product is stalkwise not given by the (algebraic) tensor product of rings but by the analytic tensor product. The analytic tensor product usually contains the algebraic tensor product as a proper subring:

$$\mathcal{O}_{\mathbb{C}\times\mathbb{C},(0,0)} = \mathbb{C}\{x, y\} \supsetneq \mathbb{C}\{x\} \otimes_{\mathbb{C}} \mathbb{C}\{y\} = \mathcal{O}_{\mathbb{C},0} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0}\,.$$

However, as we shall see later, if $g : Y \to T$ is a finite morphism of complex spaces, then we may restrict ourselves on considering the algebraic tensor product when computing the fibre product $X \times_T Y$ (Lemma 1.89).

Remarks and Exercises

(A) *Projective n-Space.* An important example of a complex manifold is the *complex projective n-space* $\mathbb{P}^n$. The underlying topological space is defined as the set of lines through the origin $\mathbf{0}$ in $\mathbb{C}^{n+1}$, endowed with the quotient topology with respect to the map $\pi : \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \twoheadrightarrow \mathbb{P}^n$ sending $\boldsymbol{p} = (p_0, \dots, p_n)$ to the line through $\boldsymbol{p}$ and $\mathbf{0}$. More formally, we may define $\mathbb{P}^n$ as the set of orbits of the natural $\mathbb{C}^*$-action $\lambda \cdot \boldsymbol{p} = (\lambda p_0, \dots, \lambda p_n)$ on $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$, that is,

$$\mathbb{P}^n = \left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right)/\mathbb{C}^*$$

is the set of equivalence classes of $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ where two points are equivalent if they are both on the same line through the origin $\mathbf{0}$ in $\mathbb{C}^{n+1}$. We write $(p_0 : \ldots : p_n)$ for the image of $\boldsymbol{p}$ under π, and call $p_0, \dots, p_n$ *homogeneous coordinates* of the point $\pi(\boldsymbol{p}) \in \mathbb{P}^n$.

The complex manifold structure on $\mathbb{P}^n$ (according to Definition 1.31) is defined by the holomorphic atlas $\left\{(U_i, \varphi_i) \,\middle|\, i = 0, \dots, n\right\}$ with

$$U_i := \{(p_0 : \ldots : p_n) \in \mathbb{P}^n \mid p_i \neq 0\}\,,$$

and with

$$\varphi_i : U_i \longrightarrow \mathbb{C}^n , \quad (p_0 : \ldots : p_n) \longmapsto \left(\frac{p_0}{p_i}, \ldots, \frac{\widehat{p_i}}{p_i}, \ldots, \frac{p_n}{p_i} \right)$$

(Here, $\widehat{\ }$ means that the corresponding entry is omitted). Indeed, φ_i is a homeomorphism, and the transition functions are given by

$$\varphi_{ij} : (q_1, \ldots, q_n) \longmapsto \frac{1}{q_j} (q_1, \ldots, q_i, 1, q_{i+1}, \ldots, \widehat{q_j}, \ldots, q_n) ,$$

thus holomorphic. According to Definition 1.33, the charts U_i define the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$ with $\mathcal{O}_{\mathbb{P}^n}|_{U_i} \cong \mathcal{O}_{\mathbb{C}^n}$.

If $f \in \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous polynomial of degree d, then $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$ for each $\lambda \in \mathbb{C}$. Hence, the zero-set

$$V(f) = \{\boldsymbol{p} \in \mathbb{P}^n \mid f(\boldsymbol{p}) = 0\}$$

is well-defined (although f does not define a function from $\mathbb{P}^n$ to $\mathbb{C}$). More generally, if $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous[6] ideal, then the zero-set

$$V(I) = \big\{\boldsymbol{p} \in \mathbb{P}^n \;\big|\; f(\boldsymbol{p}) = 0 \text{ for all } f \in I\big\}$$

is well-defined and called the *projective algebraic set* defined by I.

Substituting $x_i = 1$ in f, we get a polynomial $f|_{x_i=1}$ and doing this for all $f \in I$, we get an ideal $I|_{x_i=1} \subset \mathbb{C}[x_0, \ldots, \widehat{x_i}, \ldots, x_n] = \mathbb{C}[y_1, \ldots, y_n]$. Since $f \circ \varphi_i^{-1}(y_1, \ldots, y_n) = f|_{x_i=1}(y_1, \ldots, y_n)$, we see that

$$\varphi_i\big(V(I) \cap U_i\big) = V\big(I\big|_{x_i=1}\big) \subset \mathbb{C}^n$$

is an analytic subset of $\mathbb{C}^n$ and, therefore, $V(I)$ is an analytic subset of the complex space $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$.

If $h = \sum_{i=0}^n c_i x_i$, $c_i \in \mathbb{C}$, is a homogeneous linear form, then the analytic set $H := V(h) \subset \mathbb{C}^{n+1}$ is isomorphic to $\mathbb{C}^n$, and the image $\pi(H) \subset \mathbb{P}^n$ (with the induced structure of a complex manifold) is isomorphic to $\mathbb{P}^{n-1}$. Moreover, there exists a linear coordinate change $\psi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ inducing an isomorphism $\overline{\psi} : \mathbb{P}^n \xrightarrow{\cong} \mathbb{P}^n$, and mapping H to $V(x_0)$. Hence, we get an isomorphism

$$\mathbb{P}^n \setminus \pi(H) \xrightarrow[\overline{\psi}]{\cong} \{(p_0 : \ldots : p_n) \in \mathbb{P}^n \mid p_0 \neq 0\} \xrightarrow[\varphi_0]{\cong} \mathbb{C}^n .$$

$\mathbb{P}^n \setminus \pi(H)$ is called an *affine chart* of $\mathbb{P}^n$, and ψ is called an *affine coordinate map* on $\mathbb{P}^n$. The inverse $\big(\varphi_0 \circ \overline{\psi}\big)^{-1}$ is sometimes denoted by

$$\boldsymbol{q} = (q_1, \ldots, q_n) \longmapsto \big(x_0(\boldsymbol{q}) : \ldots : x_n(\boldsymbol{q})\big) .$$

---

[6] The ring $P = \mathbb{C}[x_0, \ldots, x_n]$ is *graded* (that is, $P = \bigoplus_{d \geq 0} P_d$ as Abelian group with $P_{d_1} \cdot P_{d_2} \subset P_{d_1+d_2}$) with P_d being the $\mathbb{C}$-vector space of homogeneous polynomials of degree d. Each $f \in P$ can be uniquely written as $f = \sum_{d \geq 0} f_d$ with $f_d \in P_d$. We call f_d the *homogeneous component* of f *of degree* d. An ideal $I \subset P$ is called *homogeneous* if it can be generated by homogeneous elements. This is equivalent to the fact that $f \in I$ implies that each homogeneous component f_d is in I.

(B) *Analytic versus Algebraic Sets.* If $f_1, \ldots, f_k \in \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$ are polynomials, then the *affine algebraic set* $V(f_1, \ldots, f_k) \subset \mathbb{C}^n$ coincides (set-theoretically) with the analytic subset $V(f_1, \ldots, f_k)$ of $\mathbb{C}^n$. Hence, any affine algebraic set is an analytic subset of some $\mathbb{C}^n$. However, there are more analytic sets than algebraic ones. For instance, $V(y - \sin(x)) \subset \mathbb{C}^2$ is analytic but not algebraic, as it intersects the line $V(y)$ in infinitely many points. Moreover, analytic sets are usually defined only on a proper open subset of $\mathbb{C}^n$ and not on all of $\mathbb{C}^n$.

For analytic subsets of a complex projective space, however, the situation is different: as shown by Chow [Cho], each analytic subset X of $\mathbb{P}^n$ is algebraic, that is, of the form $X = V(I)$ for $I \subset \mathbb{C}[x_0, \ldots, x_n]$ a homogeneous ideal (see [Fis, 4.3] for a more sophisticated version of Chow's theorem).

(C) *Algebraic Varieties versus Complex Spaces.* Let $\big(X^{\mathrm{alg}}, \mathcal{O}_X^{\mathrm{alg}}\big)$ be an algebraic variety over $\mathbb{C}$, that is, a separated scheme of finite type over $\mathbb{C}$ (see [Har]). Then, by definition, $(X^{\mathrm{alg}}, \mathcal{O}_X^{\mathrm{alg}})$ is a locally ringed space where X^{alg} can be covered by affine open sets U such that

$$\big(U, \mathcal{O}_X^{\mathrm{alg}}|_U\big) \cong \big(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}^{\mathrm{alg}}/I\mathcal{O}_{\mathbb{C}^n}^{\mathrm{alg}}\big)$$

as locally ringed spaces, where $\mathcal{O}_{\mathbb{C}^n}^{\mathrm{alg}}$ is the sheaf of algebraic (regular) functions on $\mathbb{C}^n$, and where I is an ideal of $\mathbb{C}[\boldsymbol{x}] = \Gamma\big(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}^{\mathrm{alg}}\big)$.

We can associate to the algebraic variety $(X^{\mathrm{alg}}, \mathcal{O}_X^{\mathrm{alg}})$, in a natural way, a complex space $(X, \mathcal{O}_X)$: equip the affine open sets U with the structure corresponding to $\mathcal{O}_{\mathbb{C}^n}/I\mathcal{O}_{\mathbb{C}^n}$, where $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf of holomorphic functions on $\mathbb{C}^n$. These structures can be glued to obtain a complex space structure on X^{alg} (this basically follows since the algebraic structures could be glued).

For example, let $X^{\mathrm{alg}} \subset \mathbb{P}^n$ be the projective scheme defined by a homogeneous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$. That is, X^{alg} is given by $V(I)$ as a set (endowed with the Zariski topology) and with structure sheaf $\mathcal{O}_X^{\mathrm{alg}}$ defined by the covering $U_i = \{x_i \neq 0\}$, $i = 0, \ldots, n$, with

$$\mathcal{O}_X^{\mathrm{alg}}\big|_{U_i \cap X} \cong \big(\mathcal{O}_{\mathbb{C}^n}^{\mathrm{alg}}/(I|_{x_i=1})\big)\big|_{V(I|_{x_i=1})}$$

via φ_i. The complex analytic space associated to X^{alg} is given by the structure sheaf $\mathcal{O}_X$ with

$$\mathcal{O}_X\big|_{U_i \cap X} \cong \big(\mathcal{O}_{\mathbb{C}^n}/(I|_{x_i=1})\big)\big|_{V(I|_{x_i=1})}.$$

The theorem of Chow says that any closed complex subspace of $\mathbb{P}^n$ arises in this way from some projective algebraic subscheme of $\mathbb{P}^n$.

There are, however, two important differences between $\big(X^{\mathrm{alg}}, \mathcal{O}_X^{\mathrm{alg}}\big)$ and $(X, \mathcal{O}_X)$. First of all, X^{alg} carries the *Zariski topology* (that is, the open sets are the complements of algebraic subsets of X^{alg}), while X carries the Euclidean topology. Moreover, the local rings of the structure sheaves are different. For $X \subset \mathbb{C}^n$ a complex model space, and $\boldsymbol{p} = (p_1, \ldots, p_n) \in X$ a (closed) point, we get that

$$\mathcal{O}^{\mathrm{alg}}_{X,\boldsymbol{p}} \cong \mathbb{C}[\boldsymbol{x}]_{\langle \boldsymbol{x}-\boldsymbol{p}\rangle}/I\mathbb{C}[\boldsymbol{x}]_{\langle \boldsymbol{x}-\boldsymbol{p}\rangle}$$

is the *algebraic local ring* of X at $\boldsymbol{p}$, where

$$\mathbb{C}[\boldsymbol{x}]_{\langle \boldsymbol{x}-\boldsymbol{p}\rangle} = \left\{ \frac{f}{g} \;\middle|\; f,g \in \mathbb{C}[\boldsymbol{x}],\ g(\boldsymbol{p}) \neq 0 \right\}$$

is the localization of the polynomial ring $\mathbb{C}[\boldsymbol{x}]$ at the maximal ideal $\langle \boldsymbol{x}-\boldsymbol{p}\rangle$. On the other hand,

$$\mathcal{O}_{X,\boldsymbol{p}} \cong \mathbb{C}\{\boldsymbol{x}-\boldsymbol{p}\}/I \cdot \mathbb{C}\{\boldsymbol{x}-\boldsymbol{p}\}$$

is the *analytic local ring* of X at $\boldsymbol{p}$, containing $\mathcal{O}^{\mathrm{alg}}_{X,\boldsymbol{p}}$ as a subring. Note that $\mathcal{O}^{\mathrm{alg}}_{X,\boldsymbol{p}}$ is not an analytic $\mathbb{C}$-algebra, and that the Weierstraß theorems do not hold in $\mathcal{O}^{\mathrm{alg}}_{X,\boldsymbol{p}}$. For a more detailed comparison, we refer to [Har, App. B] and [Ser2].

(D) *Dimension Theory.* There are different concepts of dimension which all lead to the same local dimension theory for complex spaces. Our definition of $\dim_p X$, based on the Krull dimension is purely algebraic (see Appendix B.2). Alternatively, we may consider

- the *Weierstraß dimension* of X at p, which is the least number d such that there exists a Noether normalization $\mathbb{C}\{y_1,\dots,y_d\} \hookrightarrow \mathcal{O}_{X,p}$ of $\mathcal{O}_{X,p}$.
- the *Chevalley dimension* of X at p, which is the least number of generators for an $\mathfrak{m}_{X,p}$-primary ideal ($\mathfrak{m}_{X,p} \subset \mathcal{O}_{X,p}$ the maximal ideal). Or, in geometric terms, the least number d for which there are $f_1,\dots,f_d \in \mathcal{O}_X(U)$, defined on an open neighbourhood $U \subset X$ of p, such that p is an isolated point of the analytic set $V(f_1,\dots,f_d) \subset U$.

Using some of the results on the Krull dimension collected in Appendix B.2, it is not difficult to show that these notions coincide:

Exercise 1.3.1. Let X be a complex space, $p \in X$. Show that the following holds:
(1) If $\mathbb{C}\{y_1,\dots,y_d\} \hookrightarrow \mathcal{O}_{X,p}$ is a Noether normalization, then $\dim_p X = d$.
(2) $\dim_p X = 0$ iff p is an isolated point of X.
(3) $\dim_p X$ is the minimal integer d for which there are $f_1,\dots,f_d \in \mathcal{O}_X(U)$, defined on an open neighbourhood $U \subset X$ of p, such that p is an isolated point of the analytic set $V(f_1,\dots,f_d) \subset U$.

We refer to [GrR2] (resp. [DJP]) for a self-contained discussion of dimension theory for complex spaces from a geometric (resp. algebraic) point of view.

There are two different effective approaches to computing dimension: either use the characterization of the dimension as the degree of the Hilbert-Samuel polynomial (see B.2) or use the theory of standard bases. We refer to [GrP, DeL] for details and Singular examples.

The next two exercises provide additional geometric intuition for the local dimension, respectively embedding dimension, of complex spaces:

Exercise 1.3.2. Let $(X, \mathcal{O}_X) = \big(V(\mathcal{J}), (\mathcal{O}_D/\mathcal{J})|_X\big)$ be a complex model space (with $D \subset \mathbb{C}^n$ an open subset), and let $p \in X$. Prove the following statements:

(1) If X contains no neighbourhood of p in D, then there exists a complex line $L \subset \mathbb{C}^n$ through p such that p is an isolated point of $X \cap L$.

(2) $\dim_p X \leq e$ iff there exists a complex plane $H \subset X$ of dimension $n - e$ such that $p \in X \cap H$ is an isolated point of $X \cap H$.

Exercise 1.3.3. Let X be a complex space, $p \in X$. Deduce from the embedding Lemma 1.24 that $m = \operatorname{edim}_p X$ is the minimal possible dimension such that locally at p we may identify X with a complex model space defined by an ideal $\mathcal{J} \subset \mathcal{O}_D$ of finite type, where $D \subset \mathbb{C}^m$ is an open subset.

The remaining exercises are independent of the above remarks (A) – (D):

Exercise 1.3.4. Let $f : X \to Y$ be a morphism of complex spaces. Prove that f is a closed embedding iff, for all $x \in X$, the induced morphism of stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective.

Exercise 1.3.5. Let A, B be analytic sets in a complex space X, and let $\mathcal{I}, \mathcal{J}, \mathcal{I}_i \subset \mathcal{O}_X$ be ideal sheaves of finite type. Prove the following statements:

(1) $V(\mathcal{I} \cdot \mathcal{J}) = V(\mathcal{I} \cap \mathcal{J}) = V(\mathcal{I}) \cup V(\mathcal{J})$.

(2) $V(\sum_{i \in I} \mathcal{I}_i) = \bigcap_{i \in I} V(\mathcal{I}_i)$.

(3) $\mathcal{J}(A \cup B) = \mathcal{J}(A) \cap \mathcal{J}(B)$.

(4) $\mathcal{J}(A \cap B) = \sqrt{\mathcal{J}(A) + \mathcal{J}(B)}$ (use the Hilbert-Rückert Nullstellensatz).

Moreover, give an example for $\mathcal{J}(A \cap B) \supsetneq \mathcal{J}(A) + \mathcal{J}(B)$.

Exercise 1.3.6. Prove Remark 1.43.1 (4).

Exercise 1.3.7. Prove the claimed equalities in Example 1.45.2.

1.4 Complex Space Germs and Singularities

Many problems in this book concern singularities, that is, local properties of complex spaces. The appropriate notion is the notion of a germ. Most of the notions and properties of complex space germs can be deduced directly from those of complex spaces and, conversely, properties of germs describe properties of complex spaces in a neighbourhood of a given point.

Definition 1.47. (1) A *pointed complex space* is a pair (X, x) consisting of a complex space X and a point $x \in X$. A *morphism* $f : (X, x) \to (Y, y)$ *of pointed complex spaces* is a morphism $f : X \to Y$ of complex spaces such that $f(x) = y$.

The *category of complex space germs* has as objects pointed complex spaces and as morphisms equivalence classes of morphisms of pointed complex spaces defined in some open neighbourhood of the distinguished point. Explicitly, if

U and V are open neighbourhoods of x in X and if $f : (U, x) \to (Y, y)$ and $g : (V, x) \to (Y, y)$ are morphisms of pointed complex spaces, then f and g are equivalent if there exists an open neighbourhood $W \subset U \cap V$ of x in X such that $f|_W = g|_W$.

In the category of complex space germs, the objects are called *(complex) space germs* and the morphisms *(holomorphic) map germs*. A complex space germ is also called a *singularity.*

(2) If $f : (X, x) \to (Y, y)$, resp. $g : (Y, y) \to (Z, z)$, are holomorphic map germs, then they are represented by morphisms $f : (U, x) \to (Y, y)$, resp. $g : (V, y) \to (Z, z)$, of pointed complex spaces, where U, V are open neighbourhoods of x, y, respectively. Then the *composition* $g \circ f : (X, x) \to (Z, z)$ is the holomorphic map germ represented by $g \circ (f|_{f^{-1}(V)\cap U})$. The map germ $f : (X, x) \to (Y, y)$ is an *isomorphism* if there exists a map germ $h : (Y, y) \to (X, x)$ such that $f \circ h = \mathrm{id}_{(Y,y)}$ and $h \circ f = \mathrm{id}_{(X,x)}$.

(3) If $U \subset X$ is an open neighbourhood of x, then the germs (U, x) and (X, x) are isomorphic via the inclusion map $U \hookrightarrow X$. We identify the complex space germ (U, x) with (X, x) and call U a *representative of the germ* (X, x). Similarly, if $f : (X, x) \to (Y, y)$ is a holomorphic map germ and if $U \subset X$, resp. $V \subset Y$, are representatives of (X, x), resp. (Y, y), such that $f(U) \subset V$, then we call $f : U \to V$ a *representative of the map germ* f. If $X \subset Y$ is a complex subspace and $x \in X$, then (X, x) is called a *subgerm* of (Y, x).

It follows that properties of complex space germs and map germs hold for sufficiently small neighbourhoods of the distinguished points, where "sufficiently small" depends on the context.

(4) If (X, x) is a germ, represented by the complex space X with structure sheaf $\mathcal{O}_X$, then the stalk $\mathcal{O}_{X,x}$ is called the *(analytic) local ring* of the germ (X, x) and also denoted by $\mathcal{O}_{(X,x)}$.

We call (X, x) *reduced* if the local ring $\mathcal{O}_{X,x}$ is reduced. Then we also say that X is *reduced at* x. Moreover, we set

$$\dim(X, x) := \dim_x X\,, \quad \mathrm{edim}(X, x) := \mathrm{edim}_x X\,.$$

Complex space germs of dimension 1 (resp. 2) are called *curve singularities* (resp. *surface singularities*).

Of course, the notions of germs (resp. map germs) can be defined in the same manner for pointed topological spaces (resp. continuous maps of pointed topological spaces), for pointed differential manifolds (resp. differential maps of pointed differential manifolds), etc.

Let (X, x) be a complex space germ, and let $I \subset \mathcal{O}_{X,x}$ be an ideal. Let $(U, \mathcal{O}_U)$ be a representative of (X, x) and $f_1, \dots, f_s \in \mathcal{O}_U(U)$ such that I is generated by the germs of $f_1, \dots, f_s$ at x. The closed complex subspace of U defined by $\mathcal{I} = \sum_{i=1}^{s} f_i \mathcal{O}_U$ defines a closed (analytic) subgerm

$$\big(V(I), x\big) := \big(V(\mathcal{J}), x\big) \subset (U, x) = (X, x)$$

of (X,x), called the *closed (analytic) subgerm defined by I*.

If $I = \langle f\rangle \subset \mathcal{O}_{\mathbb{C}^n,\boldsymbol{p}}$ is a principal ideal with $f \neq 0$, then the germ

$$\big(V(f),\boldsymbol{p}\big) := \big(V(I),\boldsymbol{p}\big) \subset (\mathbb{C}^n,\boldsymbol{p})$$

is called a *hypersurface singularity*. Hypersurface singularities in $(\mathbb{C}^2,\boldsymbol{p})$ are called *plane curve singularities.*

Note that a morphism $(X,x) \to (Y,y)$ of complex space germs determines a tuple $(f_x, f_x^\sharp)$ consisting of a germ f_x of a continuous map and of a morphism $f_x^\sharp$ of analytic $\mathbb{C}$-algebras, the tuple being induced by a morphism of pointed complex spaces $(f,f^\sharp): U \to V$, $f(x)=y$. Here, $f_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is the morphism of stalks induced by $f^\sharp : \mathcal{O}_V \to f_*\mathcal{O}_U$ (see A.6).

Remark 1.47.1. (1) We usually write $f = f_x : (X,x) \to (Y,y)$ to denote a morphism of complex space germs. Note, however, that the morphism of analytic $\mathbb{C}$-algebras $f_x^\sharp$ is always part of the data. Indeed, for non-reduced germs, the morphism $(f_x, f_x^\sharp)$ is not uniquely determined by f_x (see the example in Remark 1.41.1).

(2) Conversely, given pointed complex spaces (X,x) and (Y,y) and a morphism $\varphi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of analytic $\mathbb{C}$-algebras, then φ determines a holomorphic map germ $(f_x, f_x^\sharp) : (X,x) \to (Y,y)$ (as in the proof of Proposition 1.34). In particular, all properties of a complex space germ are encoded in the local ring. This may be formalized by saying that the functor

$$\begin{array}{ccc}
\text{(complex space germs)} & \longrightarrow & (\text{analytic } \mathbb{C}\text{-algebras}) \\
(X,x) & \longmapsto & \mathcal{O}_{X,x} \\
f_x : (X,x) \to (Y,y) & \longmapsto & f_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}
\end{array}$$

is an *(anti-)equivalence of categories.* In other words, the following holds:

(i) If $f_x, g_x : (X,x) \to (Y,y)$ satisfy $f_x^\sharp = g_x^\sharp$, then $f_x = g_x$.

(ii) If A is an analytic $\mathbb{C}$-algebra, then there is a complex space germ (X,x) such that $A \cong \mathcal{O}_{X,x}$.

(iii) If $\varphi : B \to A$ is a morphism of analytic $\mathbb{C}$-algebras, then there are isomorphisms $\psi : A \xrightarrow{\cong} \mathcal{O}_{(X,x)}$, $\phi : B \xrightarrow{\cong} \mathcal{O}_{(Y,y)}$ and a holomorphic map germ $f_x : (X,x) \to (Y,y)$ such that $\varphi = \psi^{-1} \circ f_x^\sharp \circ \phi$.

In particular, two complex space germs are isomorphic iff their local rings are isomorphic.

Indeed, statement (i) follows from Lemma 1.42, and (ii) follows from the considerations right after Definition 1.35. To see (iii), note that each morphism $\varphi : \mathbb{C}\{y_1,\dots,y_m\}/J = B \to A = \mathbb{C}\{x_1,\dots,x_n\}/I$ of analytic $\mathbb{C}$-algebras can be lifted to a morphism $\widetilde{\varphi} : \mathbb{C}\{\boldsymbol{y}\} \to \mathbb{C}\{\boldsymbol{x}\}$ (Lemma 1.14). The images $\phi_i := \widetilde{\varphi}(y_i) \in \mathbb{C}\{\boldsymbol{x}\}$, $i = 1,\dots,m$, converge in some neighbourhood U of $\boldsymbol{0} \in \mathbb{C}^n$. Thus, they define a holomorphic map $\phi = (\phi_1,\dots,\phi_m) : U \to \mathbb{C}^m$ (Lemma 1.42) which induces a unique morphism between the complex space germs associated to A and B, having the required property.

We define the *fibre* $\big(f^{-1}(y),x\big)$ of a morphism $f:(X,x)\to(Y,y)$ of complex space germs to be the germ of the fibre of a representative. The local ring is $\mathcal{O}_{f^{-1}(y),x}=\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$. More generally, the *fibre product* of two morphisms $f:(X,x)\to(T,t)$ and $g:(Y,y)\to(T,t)$ is the germ at (x,y) of the fibre product of two representatives.

In this way, constructions for complex spaces usually induce constructions for germs. However, the image of a morphism of germs $f:(X,x)\to(Y,y)$ is, in general, not defined (even as a set). For example, the morphism of germs $f:(\mathbb{C}^2,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})$, $(x_1,x_2)\mapsto(x_1,x_1x_2)$, has no well-defined image germ: small balls in $\mathbb{C}^2$ around $\mathbf{0}$ are mapped to sectors in $\mathbb{C}^2$, which become thinner if the ball becomes smaller. In Section 1.5, we shall see that the germ of the image is well-defined if f is a finite morphism, that is, if the germ of the fibre $\big(f^{-1}(y),x\big)$ consists of only one point.

We begin the study of properties of germs with a characterization of regular complex space germs. We say that (X,x) is a *regular* (or *non-singular*, or *smooth*) *germ* if there is a representative X which is regular at x. A germ which is not regular is called *singular*.

Proposition 1.48 (Rank theorem). *Let X be a complex space, $x\in X$, and let $\mathcal{O}_{X,x}\cong\mathbb{C}\{x_1,\dots,x_m\}/I$ with $I=\langle f_1,\dots,f_k\rangle$. Then the following conditions are equivalent:*

(a) (X,x) is regular and $\dim(X,x)=n$.
(b) $\mathcal{O}_{X,x}\cong\mathbb{C}\{x_1,\dots,x_n\}$.
(c) There is an open subset $U\subset X$, $x\in U$, such that $(U,\mathcal{O}_X|_U)$ is a complex manifold of dimension n.
(d) There is an open neighbourhood D of $\mathbf{0}$ in $\mathbb{C}^m$ such that the f_i converge in D and

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(\boldsymbol{p})\right)_{\substack{i=1\dots k\\ j=1\dots m}}=m-n$$

for all $\boldsymbol{p}\in D$.

Moreover, if these conditions hold, the ideal I is generated by $m-n$ of the f_i, say $I=\langle f_1,\dots,f_{m-n}\rangle$, and there is an isomorphism $\widetilde{\varphi}:\mathbb{C}\{\boldsymbol{x}\}\to\mathbb{C}\{\boldsymbol{x}\}$ sending f_i to x_{n+i}, $i=1,\dots,m-n$, which induces an isomorphism

$$\varphi:\mathbb{C}\{\boldsymbol{x}\}/I\xrightarrow{\cong}\mathbb{C}\{\boldsymbol{x}\}/\langle x_{n+1},\dots,x_m\rangle\cong\mathbb{C}\{x_1,\dots,x_n\}\,.$$

Proof. (a) $\Rightarrow$ (b) If $n=\dim(X,x)=\operatorname{edim}(X,x)$, the embedding Lemma 1.24 implies that $\mathcal{O}_{X,x}\cong\mathbb{C}\{x_1,\dots,x_n\}/J$ for some ideal J, and Krull's principal ideal theorem implies $J=\langle 0\rangle$ since $\mathbb{C}\{x_1,\dots,x_n\}$ is an integral domain.

(b) $\Rightarrow$ (c) If $\mathcal{O}_{X,x}\cong\mathbb{C}\{x_1,\dots,x_n\}$, then Remark 1.47.1 (2) yields that the germ (X,x) is isomorphic to $(\mathbb{C}^n,0)$. That is, we may assume that, locally at x, the complex space X is isomorphic to an open subspace $D\subset\mathbb{C}^n$.

(c) ⇒ (a) If X is a complex manifold of dimension n, then, by definition, X is reduced and locally homeomorphic to an open subset of $\mathbb{C}^n$. From the proof of Proposition 1.34, we know that this implies that the complex space germ (X,x) is isomorphic to $(\mathbb{C}^n,\mathbf{0})$. In particular, $\mathcal{O}_{X,x} \cong \mathbb{C}\{x_1,\dots,x_n\}$, thus $\dim(X,x) = n = \operatorname{edim}(X,x)$.

(b) ⇒ (d) By Lemma 1.23, the isomorphism

$$\varphi : \mathbb{C}\{\boldsymbol{x}\}/I \xrightarrow{\cong} \mathcal{O}_{X,x} \xrightarrow{\cong} \mathbb{C}\{x_1,\dots,x_n\} \xrightarrow{\cong} \mathbb{C}\{\boldsymbol{x}\}/\langle \boldsymbol{x}''\rangle\,,$$

$\langle \boldsymbol{x}''\rangle = \langle x_{n+1},\dots,x_m\rangle$, lifts to an isomorphism

$$\widetilde{\varphi} : \mathbb{C}\{\boldsymbol{x}\} \xrightarrow{\cong} \mathbb{C}\{\boldsymbol{x}\}\,, \quad \widetilde{\varphi}(I) = \langle \boldsymbol{x}''\rangle\,.$$

Setting $\phi_i := \widetilde{\varphi}(x_i) \in \mathbb{C}\{\boldsymbol{x}\}$, we obtain $\phi_i(\mathbf{0}) = 0$ and $\phi_1,\dots,\phi_m$ converge in some open neighbourhood $D' \subset \mathbb{C}^m$ of the origin. Then $\phi := (\phi_1,\dots,\phi_m)$ defines a holomorphic map $\phi : D' \to \mathbb{C}^m$ with $\phi(\mathbf{0}) = \mathbf{0}$. Indeed, we may assume that it defines a biholomorphic map $\phi : D' \to D$ of open neighbourhoods of the origin in $\mathbb{C}^m$, the inverse map being given by $\psi = (\psi_1,\dots,\psi_m)$, $\psi_i := \widetilde{\varphi}^{-1}(x_i)$. We may also assume that $f_1,\dots,f_k$ converge on D. Then, for each $\boldsymbol{p} = \phi(\boldsymbol{q}) \in D$,

$$\begin{aligned}\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(\boldsymbol{p})\right)_{\substack{i=1\dots k\\ j=1\dots m}} &= \operatorname{rank}\left(\frac{\partial (f_i\circ\phi)}{\partial x_j}(\boldsymbol{q})\right)_{\substack{i=1\dots k\\ j=1\dots m}}\\ &= \operatorname{rank}\left(\frac{\partial (\widetilde{\varphi}(f_i))}{\partial x_j}(\boldsymbol{q})\right)_{\substack{i=1\dots k\\ j=1\dots m}} = \operatorname{rank}\left(\frac{\partial x_\ell}{\partial x_j}(\boldsymbol{q})\right)_{\substack{\ell=n+1\dots m\\ j=1\dots m}} = m-n\,.\end{aligned}$$

The first equality is obtained by applying the chain rule and the inverse function Theorem 1.21; the last one since the rank of the Jacobian matrix is independent of the chosen generators of the ideal (see p. 27).

(d) ⇒ (b) We may assume, after renumerating the f_i and x_j, that

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(\boldsymbol{p})\right)_{\substack{i=1,\dots,m-n\\ j=n+1,\dots,m}} = m-n$$

for all $\boldsymbol{p} \in U$. Then the implicit function Theorem 1.18 yields the existence of power series $g_i \in \langle \boldsymbol{x}'\rangle\mathbb{C}\{\boldsymbol{x}'\}$, $\boldsymbol{x}' = (x_1,\dots,x_n)$, such that

$$\langle f_1,\dots,f_{m-n}\rangle \;=\; \langle x_{n+1} - g_1(\boldsymbol{x}'),\dots,x_m - g_{m-n}(\boldsymbol{x}')\rangle\,.$$

The isomorphism $\widetilde{\varphi} : \mathbb{C}\{\boldsymbol{x}\} \to \mathbb{C}\{\boldsymbol{x}\}$ given by $\boldsymbol{x}' \mapsto \boldsymbol{x}'$, $x_{n+i} \mapsto x_{n+i} + g_i(\boldsymbol{x}')$, $i = 1,\dots,m-n$, maps the ideal $\langle f_1,\dots,f_{m-n}\rangle$ to $\langle \boldsymbol{x}''\rangle$. The same argument as above shows that the Jacobian rank condition implies

$$\frac{\partial(\widetilde{\varphi}(f_i))}{\partial x_j}(\boldsymbol{q}) = 0, \quad i = m-n+1,\dots,m\,,\;\; j = 1,\dots,n\,,$$

for each point $\boldsymbol{q}$ in a sufficiently small neighbourhood $D' \subset \mathbb{C}^m$ of the origin. Since $\widetilde{\varphi}(f_i)(\mathbf{0}) = 0$, this implies that $\widetilde{\varphi}(f_i)$ vanishes on $V(\boldsymbol{x}'')$, hence $\widetilde{\varphi}(f_i) \in \langle \boldsymbol{x}'' \rangle$, for each $i \geq m-n+1$. Altogether, we get that $\widetilde{\varphi}$ induces an isomorphism

$$\varphi : \mathbb{C}\{\boldsymbol{x}\}/I \xrightarrow{\cong} \mathbb{C}\{\boldsymbol{x}\}/\langle \boldsymbol{x}'' \rangle \cong \mathbb{C}\{\boldsymbol{x}'\}\,.$$

□

Remark 1.48.1. If a complex space X is smooth at x, then Proposition 1.48 yields that X is smooth in a whole neighbourhood of x. More generally, we shall show that the *singular locus of X*

$$\operatorname{Sing}(X) := \bigl\{x \in X \bigm| X \text{ is not smooth at } x\bigr\}$$

is a closed analytic subset of X (Proposition 1.104 and Corollary 1.111) and, thus, $\bigl(\operatorname{Sing}(X), x\bigr)$ is a closed subgerm of (X, x).

We close this section by discussing the decomposition of complex space germs into irreducible components. We restrict ourselves to the decomposition of germs of analytic sets (that is, of reduced closed subgerms) which is the geometric counterpart of the prime decomposition of radical ideals in analytic algebras. This concept generalizes in an obvious way to non-reduced closed subgerms, using the existence of a (minimal) primary decomposition for analytic algebras.

Definition 1.49. Let X be a complex space, let $A \subset X$ be an analytic subset, and let $x \in X$. Then (A, x) is called *irreducible* if $\mathcal{J}(A)_x \subset \mathcal{O}_{X,x}$ is a prime ideal. Otherwise (A, x) is called *reducible*. We also say that A is *irreducible* (resp. *reducible*) *at* x.

Note that the Hilbert-Rückert Nullstellensatz (Theorem 1.72) implies that the analytic set germ defined by an ideal $I \subset \mathcal{O}_{X,x}$ is irreducible iff $\sqrt{I}$ is a prime ideal.

In particular, the identification of germs of analytic sets with reduced closed subgerms leads to the following definition:

Definition 1.50. A reduced complex space germ (X, x) is called *irreducible* iff $\mathcal{O}_{X,x}$ is an integral domain.

Note that each regular germ is irreducible by Proposition 1.48.

Proposition 1.51 (Irreducible decomposition). *Let X be a complex space, let $A \subset X$ be an analytic set, and let $x \in A$. Then there is a decomposition*

$$(A, x) = (A_1, x) \cup \ldots \cup (A_r, x)\,, \tag{1.4.1}$$

where $(A_1, x), \ldots, (A_r, x) \subset (X, x)$ are irreducible germs of analytic sets such that $(A_i, x) \not\subset (A_j, x)$ for $i \neq j$. This decomposition is unique, up to a permutation of the germs (A_i, x).

We call (1.4.1) the *irreducible decomposition* of the analytic germ (A, x), and we refer to $(A_1, x), \ldots, (A_r, x)$ as the *irreducible components* of (A, x).

Proof. For the *existence* of an irreducible decomposition, note that $\mathcal{O}_{X,x}$ is a Noetherian ring (Theorem 1.15), and that $\mathcal{J}(A)_x$ is a radical ideal in $\mathcal{O}_{X,x}$. Hence, $\mathcal{J}(A)_x$ has a minimal prime decomposition $\mathcal{J}(A)_x = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ (B.1). We define (A_i, x) to be the germ of an analytic set defined by the prime ideal $\mathfrak{p}_i$, $i = 1, \ldots, r$. That is, (A_i, x) is the germ at x of an analytic set $A_i = \operatorname{supp}(\mathcal{O}_U/\mathcal{I}_i)$, where $U \subset X$ is an open subspace, and $\mathcal{I}_i$ is a $\mathcal{O}_U$-ideal of finite type with stalk $\mathcal{I}_{i,x} = \mathfrak{p}_i$ according to Remark 1.47.1 (2). Then Remark 1.43.1 (3) and Exercise 1.4.3 imply that

$$(A_1, x) \cup \ldots \cup (A_r, x) = \bigl(V(\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r), x\bigr) = \bigl(V(\mathcal{J}(A)_x), x\bigr) = (A, x)\,.$$

It remains to show that the $\mathcal{J}(A_i)_x$ are prime ideals and that $(A_i, x) \not\subset (A_j, x)$ for $i \neq j$. For this, it is sufficient to show that $\mathfrak{p}_i = \mathcal{J}(A_i)_x$ for all $i = 1, \ldots, r$, which is an immediate consequence of the Nullstellensatz.

To show the *uniqueness* of the irreducible decomposition, assume that $(A, x) = (A'_1, x) \cup \ldots \cup (A'_s, x)$ is another irreducible decomposition of (A, x). By definition, this leads to a prime decomposition

$$\mathcal{J}(A)_x = \mathcal{J}(A'_1)_x \cap \ldots \cap \mathcal{J}(A'_s)_x$$

with $\mathcal{J}(A'_i)_x \not\subset \mathcal{J}(A'_j)_x$ for $i \neq j$. The latter means that the given decomposition is a minimal prime decomposition. The uniqueness of the associated primes gives $s = r$ and $\mathcal{J}(A'_i)_x = \mathfrak{p}_i = \mathcal{J}(A_i)_x$ (after renumbering). Thus, $(A'_i, x) = (A_i, x)$ for all $i = 1, \ldots, r$ (see Exercise 1.4.3). $\square$

As an immediate consequence of Proposition 1.51, we obtain:

Corollary 1.52. *Let X be a complex space, $A \subset X$ be an analytic set, and $x \in A$. Then the following are equivalent:*

(a) (A, x) is irreducible.
(b) There are no germs (A_1, x), (A_2, x) of analytic sets in (X, x) such that $(A, x) = (A_1, x) \cup (A_2, x)$ and $(A_1, x) \neq (A, x) \neq (A_2, x)$.

Remarks and Exercises

(A) *Irreducible Decomposition and Dimension.* Let (X, x) be a reduced complex space germ, and let $(X, x) = (X_1, x) \cup \ldots \cup (X_r, x)$ be its irreducible decomposition. Then

$$\dim(X, x) = \max\{\dim(X_i, x) \mid i = 1, \ldots, r\}$$

(see Appendix B.2). We call (X, x) *pure dimensional* or *equidimensional* if all its irreducible components have the same dimension. We call a complex space

X *locally pure dimensional* or *locally equidimensional* if each germ (X, x), $x \in X$, is equidimensional or, equivalently, if the function $x \mapsto \dim_x X$ is constant on each connected component of X.

Besides the irreducible decomposition of germs, one also has the concept of a (global) *irreducible decomposition of complex spaces* (resp. of analytic sets in complex spaces) which we shortly discuss next.

(B) *Irreducible Decomposition of Complex Spaces.* A reduced complex space X is called *irreducible* if there are no proper (closed) analytic subsets $A_1, A_2 \subset X$ such that $X = A_1 \cup A_2$. An arbitrary complex space X is called irreducible if its reduction X_{red} is irreducible. Otherwise, X is called *reducible.*

A reduced germ (X, x) is irreducible if its local ring $\mathcal{O}_{X,x}$ is an integral domain. A similar characterization, using the structure sheaf, does not hold for reduced complex spaces: if X is irreducible, the ring $\mathcal{O}_X(X)$ of global sections in the structure sheaf is an integral domain. But, the converse implication does not hold. For instance, if $X \subset \mathbb{P}^2$ is the union of two lines in $\mathbb{P}^2$, then $\mathcal{O}_X(X) = \mathbb{C}$ is a field, but X is reducible. Similarly, if all rings $\mathcal{O}_X(U)$, $U \subset X$ open, are integral domains, then X is irreducible. But this is only sufficient and not necessary for irreducibility: the hypersurface $V(x_1^2(1-x_1^2) - x_2^2) \subset \mathbb{C}^2$ is irreducible though, for U a small neighbourhood of the origin, $\mathcal{O}_X(U)$ is not an integral domain (see Figure 1.2[7]).

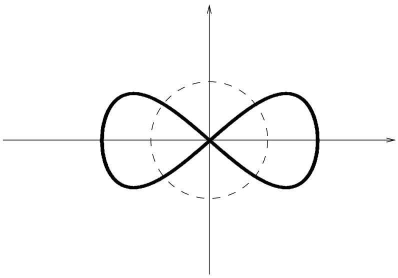


Fig. 1.2. The hypersurface $V(x_1^2(1-x_1^2) - x_2^2) \subset \mathbb{C}^2$ (real picture).

To give a sheaf theoretic characterization for irreducibility, one has to consider the *sheaf of meromorphic functions* $\mathcal{M}_X$. In fact, X is irreducible iff $\mathcal{M}_X(X)$ is a field. An important geometric characterization is the following: let X is a reduced complex space. Then X is irreducible iff $X \setminus \mathrm{Sing}(X)$ is connected and this holds iff every proper analytic subset of X is nowhere dense in X. We refer to [GrR2, Ch. 9, §1] for these and for further characterizations of irreducible complex spaces.

[7] Note that the real pictures are misleading if one considers the characterization (ii) for an irreducible complex space. In our example, the two connected components of the real part of $X \setminus \{\mathbf{0}\}$ are connected by a path in the complex domain.

A family $\{A_j \mid j \in J\}$ of irreducible closed complex subspaces of a reduced complex space X is called an *irreducible decomposition of* X if $\{A_j \mid j \in J\}$ is a locally finite covering of X such that, for each $j \in J$, A_j is not contained in any $A_{j'}$, $j' \neq j$. Such a family exists and it is uniquely determined (see [GrR2, Ch. 9, §2]). We refer to A_j, $j \in J$, as the *irreducible components of* X. Note that a compact complex space has only finitely many irreducible components. On the other hand, $V(\sin(x)) \subset \mathbb{C}$ decomposes into infinitely many irreducible (zero-dimensional) components.

Exercise 1.4.1. Let X be an irreducible reduced complex space X. Prove that all germs (X, x), $x \in X$, are pure dimensional of the same dimension n.

Exercise 1.4.2. Let $I = \langle f_1, \ldots, f_k \rangle \subset \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, and let X be the closed complex subspace of $\mathbb{C}^n$ defined by $I \cdot \mathcal{O}_{\mathbb{C}^n}$. Prove that the reduction X_{red} is irreducible iff $\sqrt{I} \subset \mathbb{C}[\boldsymbol{x}]$ is a prime ideal.

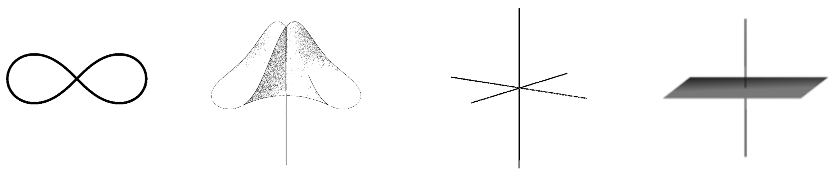
The remaining exercises are independent of remarks (A) and (B):

Exercise 1.4.3. Let X be a complex space, A, A' analytic sets in X, and $x \in X$. Prove that the following are equivalent:

(a) $(A, x) \supset (A', x)$.
(b) $\mathcal{J}(A)|_U \subset \mathcal{J}(A')|_U$ for some open neighbourhood $U \subset X$ of x.
(c) $\mathcal{J}(A)_x \subset \mathcal{J}(A')_x$.

Exercise 1.4.4. Determine the singular locus of the complex spaces defined by the following $\mathcal{O}_{\mathbb{C}^n}$-ideals:

(a) $\langle (x_1^2 + x_2^2)^2 - x_1^2 + x_2^2 \rangle \subset \mathcal{O}_{\mathbb{C}^2}$ (*"Bernoulli's lemniscate"*);
(b) $\langle x_1^2 - x_2^2 x_3 \rangle \subset \mathcal{O}_{\mathbb{C}^2}$ (*"Whitney's umbrella"*);
(c) $\langle x_1x_2, x_2x_3, x_1x_3 \rangle \subset \mathcal{O}_{\mathbb{C}^3}$ (*"coordinate cross"*);
(d) $\langle x_1x_3, x_2x_3 \rangle \subset \mathcal{O}_{\mathbb{C}^3}$.



(a) (b) (c) (d)

Exercise 1.4.5. Let X be a complex space, $x \in X$, and $(Y_1, x), \ldots, (Y_\ell, x)$ irreducible (reduced) closed subgerms of (X, x) of dimension at least 1. Moreover, let $f_1, \ldots, f_s \in \mathcal{O}_X(U)$, $U \subset X$ an open neighbourhood of x, and assume that x is an isolated point of the analytic set $V(f_1, \ldots, f_s)$. Show that there is a $\mathbb{C}$-linear combination g of the f_i such that $g \notin \mathcal{J}(Y_i)_x$ for all $j = 1, \ldots, \ell$.

Exercise 1.4.6. Let (X, x) be a reduced complex space germ, and let $(A, x) \subset (X, x)$ be a closed subgerm such that $\dim(A, x) = \dim(X, x)$. Prove that (A, x) contains an irreducible component of (X, x).

Exercise 1.4.7. Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{Z}^r$ be integer vectors. Define a complex space germ $(X, \boldsymbol{0}) \subset (\mathbb{C}^m, \boldsymbol{0})$ by the ideal $I \subset \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_m\}$, generated by the binomials $\boldsymbol{x}^{\boldsymbol{k}} - \boldsymbol{x}^{\boldsymbol{\ell}}$ for all $\boldsymbol{k} = (k_1, \ldots, k_m)$, $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_m) \in \mathbb{N}^m$ satisfying $\sum_{i=1}^m k_i \boldsymbol{a}_i = \sum_{i=1}^m \ell_i \boldsymbol{a}_i$.

Prove that $(X, \boldsymbol{0})$ is non-singular if there are $1 \leq i_1 < \cdots < i_n \leq m$ such that $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ are linearly independent over $\mathbb{Z}$ and each other vector $\boldsymbol{a}_j$ admits a (unique) representation $\boldsymbol{a}_j = u_1 \boldsymbol{a}_{i_1} + \ldots + u_n \boldsymbol{a}_{i_n}$ with $u_1, \ldots, u_n \in \mathbb{N}$. Show further that such a non-singular germ has dimension n.

1.5 Finite Morphisms and Finite Coherence Theorem

In this section, we focus on finite morphisms. The key statement which we are going to prove is that the direct image of a coherent sheaf under a finite morphism is coherent (Theorem 1.67). In particular, we get that the image of an analytic set under a finite morphism is again an analytic set (Corollary 1.68).

On our way, we prove Oka's theorem saying that the structure sheaf of a complex space is coherent (Theorem 1.63).

Definition 1.53. A continuous map $f : X \to Y$ of topological spaces is called *finite* if f is closed and if all fibres $f^{-1}(y)$, $y \in Y$, are finite sets. The map f is called *finite at* $x \in X$ if there are neighbourhoods U of x and V of $f(x)$, such that $f(U) \subset V$ and the restriction $f_{U,V} : U \to V$ is finite.

Note that compositions of finite maps are finite and that the restriction of a finite map to a closed subspace is finite. Closed embeddings of topological spaces are finite maps. The inclusion map $\mathbb{C} \setminus \{0\} \hookrightarrow \mathbb{C}$, however, is *not* finite, since it is not closed.

Lemma 1.54. *Let $f : X \to Y$ be a finite map of topological spaces where X is Hausdorff, let $y \in Y$, and let $f^{-1}(y) = \{x_1, \ldots, x_s\}$. Further, let $U_i' \subset X$ be pairwise disjoint open neighbourhoods of x_i, $i = 1, \ldots, s$. Then, for each open neighbourhood V' of y, there exists an open neighbourhood $V \subset V'$ of y such that*

(1) $U_i := U_i' \cap f^{-1}(V)$, $i = 1, \ldots, s$, are pairwise disjoint open neighbourhoods of the x_i,
(2) $f^{-1}(V) = U_1 \cup \ldots \cup U_s$, and
(3) the restrictions $f_{U_i,V} : U_i \to V$, $i = 1, \ldots, s$, are closed (hence, finite).

Proof. The union $U = U_1' \cup \ldots \cup U_s'$ is open in X, hence the image of its complement, $f(X \setminus U)$, is closed in Y (as f is closed). By construction, we have $f^{-1}(y) \subset U$, that is, $y \notin f(X \setminus U)$. It follows that $V := V' \cap \big(Y \setminus f(X \setminus U)\big)$

is an open neighbourhood of y in Y. As (1) and (2) are obvious, it only remains to show that the restrictions $f_{U_i,V} : U_i \to V$ are closed maps.

For this, let $A \subset U_i$ be closed. Then, by (1) and (2), A is also closed in $f^{-1}(V)$, that is, there is a closed subset $A' \subset X$ such that $A = A' \cap f^{-1}(V)$. Since f is closed, the image $f(A') \subset Y$ is closed. Therefore, $f(A) = f(A') \cap V$ is closed in V. □

Definition 1.55. A morphism $f : X \to Y$ of complex spaces is called *finite (at $x \in X$)* if the underlying map $f : X \to Y$ of topological spaces is finite (at x). A morphism of germs $f : (X, x) \to (Y, y)$ is called *finite* if it has a finite representative $f : U \to V$ (or, equivalently, if each representative of f is finite at x).

Proposition 1.56. *Let $f : X \to Y$ be a finite morphism of complex spaces, $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_s\}$. Further, let $V \subset Y$ and $U_1, \dots, U_s \subset X$ be open subspaces satisfying the conditions of Lemma 1.54, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then there are isomorphisms*

(1) $f_*\mathcal{F}|_V \cong \bigoplus_{i=1}^{s} (f_{U_i,V})_*(\mathcal{F}|_{U_i})$ *of $\mathcal{O}_V$-modules,*

(2) $(f_*\mathcal{F})_y \cong \bigoplus_{i=1}^{s} \mathcal{F}_{x_i}$ *of $\mathcal{O}_{Y,y}$-modules.*

Proof. Let $W \subset V$ be an open subspace. Then

$$f^{-1}(W) \subset f^{-1}(V) = U_1 \cup \ldots \cup U_s \,.$$

Since the U_i are pairwise disjoint, we get isomorphisms of $\Gamma(W, \mathcal{O}_Y)$-modules

$$\begin{aligned} \Gamma(W, f_*\mathcal{F}) = \Gamma\big(f^{-1}(W), \mathcal{F}\big) &\cong \bigoplus_{i=1}^{s} \Gamma\big(f^{-1}(W) \cap U_i, \mathcal{F}\big) \\ &\cong \bigoplus_{i=1}^{s} \Gamma\big(f_{U_i,V}^{-1}(W), \mathcal{F}|_{U_i}\big) = \bigoplus_{i=1}^{s} \Gamma\big(W, (f_{U_i,V})_*\mathcal{F}|_{U_i}\big) \end{aligned}$$

(using that $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module via $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$). Since the isomorphisms are compatible with the restriction maps to open subsets, we obtain (1) and (2). □

The proposition has the following important corollary:

Corollary 1.57. *If $f : X \to Y$ is a finite morphism of complex spaces, then the direct image functor f_* is an exact functor.*

Proof. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules, $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_s\}$. Then, by Proposition 1.56, we obtain a commutative diagram of $\mathcal{O}_{Y,y}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^{s} \mathcal{F}'_{x_i} & \longrightarrow & \bigoplus_{i=1}^{s} \mathcal{F}_{x_i} & \longrightarrow & \bigoplus_{i=1}^{s} \mathcal{F}''_{x_i} & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & (f_*\mathcal{F}')_y & \longrightarrow & (f_*\mathcal{F})_y & \longrightarrow & (f_*\mathcal{F}'')_y & \longrightarrow & 0\,. \end{array}$$

Since the upper sequence is exact, the lower is exact, too. □

Next, we consider special finite maps, the so-called Weierstraß maps, and prove the finite coherence theorem for these maps. Later in this section, we will reduce the general case to this special case.

Definition 1.58. Let $B \subset \mathbb{C}^n$ be an open subset, and let

$$f(\boldsymbol{y}, z) = z^b + a_1(\boldsymbol{y})z^{b-1} + \ldots + a_b(\boldsymbol{y}) \in \Gamma(B, \mathcal{O}_{\mathbb{C}^n})[z]\,.$$

Set $A := V(f) \subset B \times \mathbb{C}$ and $\mathcal{O}_A := \mathcal{O}_{B\times\mathbb{C}}/\langle f\rangle$. We refer to the map $A \xrightarrow{\pi} B$ induced by the projection $B \times \mathbb{C} \to B$ as a *Weierstraß map* of degree b.

Note that, for each $\boldsymbol{y} \in B$, the fibre $\pi^{-1}(\boldsymbol{y}) = \{\boldsymbol{y}\} \times \{z \in \mathbb{C} \mid f(\boldsymbol{y}, z) = 0\}$ is finite. Indeed, for $\boldsymbol{y}$ fixed, $f(\boldsymbol{y}, z)$ is a polynomial in z of degree b and has, therefore, at most b roots (see Figure 1.3).

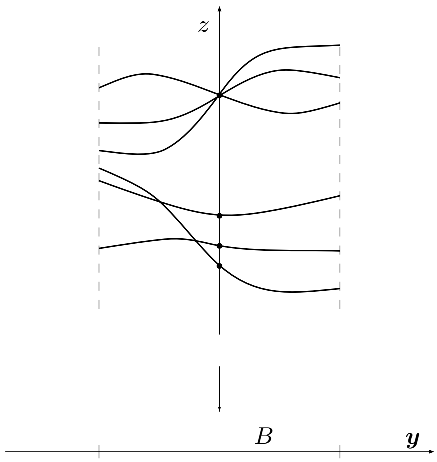


Fig. 1.3. The (local) zero-set of a Weierstraß polynomial.

Lemma 1.59. *Each Weierstraß map $\pi : A \to B$ is a finite holomorphic map.*

Proof. It suffices to show that π is a closed map. Let $M \subset A$ be closed, let $\boldsymbol{y}$ be a point in the closure of the image $\pi(M)$ in $B \subset \mathbb{C}^n$, and let $(\boldsymbol{y}_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $\pi(M)$ which converges to $\boldsymbol{y}$. For each $\nu \in \mathbb{N}$ choose $z_\nu \in \mathbb{C}$ such that $(\boldsymbol{y}_\nu, z_\nu) \in M \subset A$, that is, such that

$$-z_\nu^b = a_1(\boldsymbol{y}_\nu)z_\nu^{b-1} + \ldots + a_b(\boldsymbol{y}_\nu)\,.$$

Either $|z_\nu| < 1$ or, by the above expression,

$$|z_\nu| \;=\; \frac{|z_\nu|^b}{|z_\nu|^{b-1}} \;\le\; \sum_{i=1}^{b} |a_i(\boldsymbol{y}_\nu)|\,.$$

Since $(\boldsymbol{y}_\nu)_{\nu\in\mathbb{N}}$ is convergent, and since the a_i are continuous functions, the sequences $\big(|a_i(\boldsymbol{y}_\nu)|\big)_{\nu\in\mathbb{N}}$, $i = 1,\ldots,b$, are bounded. Thus, $(|z_\nu|)_{\nu\in\mathbb{N}}$ is also bounded. It follows that there is a subsequence $(z_{\nu_k})_{k\in\mathbb{N}}$ which converges to some $z \in \mathbb{C}$. Since M is closed, the limit $(\boldsymbol{y}, z)$ of the sequence $(\boldsymbol{y}_{\nu_k}, z_{\nu_k})_{k\in\mathbb{N}}$ is a point of M. As π is continuous, $\pi(\boldsymbol{y}, z) = \lim_{k\to\infty} \pi(\boldsymbol{y}_{\nu_k}, z_{\nu_k}) = \boldsymbol{y}$. Hence, $\boldsymbol{y} \in \pi(M)$, and we conclude that $\pi(M)$ is closed. □

Remark 1.59.1. Let $(\boldsymbol{y}_\nu)_{\nu\in\mathbb{N}} \subset B$ be a sequence converging to $\boldsymbol{y}$ and consider the sequence of polynomials $\big(f(\boldsymbol{y}_\nu, z)\big)_{\nu\in\mathbb{N}} \subset \mathbb{C}[z]$. If we choose z_ν to be any root of $f(\boldsymbol{y}_\nu, z)$, $\nu \in \mathbb{N}$, then the proof of Lemma 1.59 shows that there exists a subsequence of $(z_\nu)_{\nu\in\mathbb{N}}$ converging to a root of $f(\boldsymbol{y}, z) \in \mathbb{C}[z]$. This fact is sometimes referred to as the *continuity of the roots* of a Weierstraß polynomial (see also Exercise 1.5.5).

Theorem 1.60 (General Weierstraß division theorem). *Let*

$$f(\boldsymbol{y}, z) \;=\; z^b + a_1(\boldsymbol{y})z^{b-1} + \ldots + a_b(\boldsymbol{y}) \in \varGamma(B, \mathcal{O}_{\mathbb{C}^n})[z]$$

be a Weierstraß polynomial, with $B \subset \mathbb{C}^n$ open, and let $\pi : A = V(f) \to B$ be the corresponding Weierstraß map. Fix $\boldsymbol{y} \in B$, and let $\pi^{-1}(\boldsymbol{y}) = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s\}$.

Then, for each $g_i \in \mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$, $i = 1,\ldots,s$, there exist unique $r \in \mathcal{O}_{B,\boldsymbol{y}}[z]$ and $h_i \in \mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$ such that

$$\begin{aligned} g_1 &= h_1 f + r\,, \\ &\;\vdots \\ g_s &= h_s f + r\,, \end{aligned} \qquad\qquad \deg_z(r) \le b - 1\,.$$

The theorem says that we can *simultaneously* divide the germs $g_i \in \mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$ by the germ defined by f in $\mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$, with a *common remainder* $r \in \mathcal{O}_{B,\boldsymbol{y}}[z]$.

Proof. Without loss of generality, we can assume $\boldsymbol{y} = \mathbf{0}$.

The case $s = 1$ is just the usual Weierstraß division Theorem 1.8. Let $s \ge 2$, and let $\boldsymbol{x}_i = (\mathbf{0}, z_i)$, $i = 1,\ldots,s$. Then

$$f(\mathbf{0}, z) \;=\; (z - z_1)^{b_1} \cdot \ldots \cdot (z - z_s)^{b_s}\,, \qquad b_i > 0\,, \quad \sum_{i=1}^{s} b_i = b\,.$$

By Hensel's lemma, there are monic polynomials $f_i \in \mathcal{O}_{B,\mathbf{0}}[z]$ of degree b_i, $i = 1,\ldots,s$, such that $f = f_1 \cdot \ldots \cdot f_s$ and $f_i(\mathbf{0}, z) = (z - z_i)^{b_i}$. In particular, each f_i is $(z - z_i)$-general of order b_i.

We set $e_i := f_1 \cdot \ldots \cdot f_{i-1} \cdot f_{i+1} \cdot \ldots \cdot f_s \in \mathcal{O}_{B,\mathbf{0}}[z]$, which has degree $b - b_i$ in z, and which is a unit in the local ring $\mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$. Applying the Weierstraß division theorem to $e_i^{-1} g_i \in \mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$ gives the existence of $r_i \in \mathcal{O}_{B,\mathbf{0}}[z]$, $h_i' \in \mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$ such that $e_i^{-1} g_i = h_i' f_i + r_i$ with $\deg_z(r_i) < b_i$.

Defining $r := \sum_{i=1}^{s} e_i r_i \in \mathcal{O}_{B,\mathbf{0}}[z]$, we get $\deg_z(r) < b$. Moreover, as f_j, $j \neq i$, is a unit in $\mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}$, we obtain in this ring

$$g_i \;=\; h_i' e_i f_i + r - \sum_{j\neq i} \frac{r_j}{f_j} f_j e_j \;=\; \underbrace{\left(h_i' - \sum_{j\neq i} \frac{r_j}{f_j} \right)}_{=:\, h_i} f + r \,,$$

and the existence part is proven. The uniqueness is left as Exercise 1.5.6. $\square$

Now, we are well-prepared to prove an isomorphism of sheaves which will be the basis for the proof of Oka's coherence theorem.

Theorem 1.61 (Weierstraß isomorphism). *Let $\pi : A \to B$ be a Weierstraß map of degree b. Then $\pi_*\mathcal{O}_A$ is a locally free $\mathcal{O}_B$-module of rank b. More precisely, the map $\pi^0 : \mathcal{O}_B^b \to \pi_*\mathcal{O}_A$ defined by*

$$\begin{aligned} \mathit{\Gamma}(U, \mathcal{O}_B^b) = \mathit{\Gamma}\big(U, \mathcal{O}_B\big)^b &\longrightarrow \mathit{\Gamma}\big(\pi^{-1}(U), \mathcal{O}_A\big) \\ (r_1, \ldots, r_b) &\longmapsto \big(r_1 y^{b-1} + \ldots + r_b \ \mathit{mod}\ \langle f \rangle\big) \end{aligned}$$

is an isomorphism of $\mathcal{O}_B$-modules.

Proof. Since π^0 is an $\mathcal{O}_B$-linear morphism of sheaves, we have to show that for each $\boldsymbol{y} \in B$ the morphism of stalks $\pi^0_{\boldsymbol{y}} : \mathcal{O}^b_{B,\boldsymbol{y}} \to (\pi_*\mathcal{O}_A)_{\boldsymbol{y}}$ is an isomorphism.

If $\pi^{-1}(\boldsymbol{y}) = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s\}$, Proposition 1.56 gives an isomorphism of $\mathcal{O}_{B,\boldsymbol{y}}$-modules $(\pi_*\mathcal{O}_A)_{\boldsymbol{y}} \cong \mathcal{O}_{A,\boldsymbol{x}_1} \oplus \ldots \oplus \mathcal{O}_{A,\boldsymbol{x}_s}$. Given $g_i \in \mathcal{O}_{A,\boldsymbol{x}_i} = \mathcal{O}_{B\times\mathbb{C},\boldsymbol{x}_i}/\langle f \rangle$, $i = 1, \ldots, s$, we deduce from the general Weierstraß division theorem that there is a uniquely determined polynomial $r \in \mathcal{O}_{B,\boldsymbol{y}}[z]$ of degree at most $b-1$ such that $g_i = (r_{\boldsymbol{x}_i} \bmod \langle f \rangle)$ for each $i = 1, \ldots, s$.

Writing $r = r_1 z^{b-1} + \ldots + r_b$, we conclude that $(g_1, \ldots, g_s)$ has the unique preimage $(r_1, \ldots, r_b) \in \mathcal{O}^b_{B,\boldsymbol{y}}$ under $\pi^0_{\boldsymbol{y}}$. $\square$

Lemma 1.62. *Let $\pi : A \to B$ be a Weierstraß map, and let $\mathcal{F}$ be an $\mathcal{O}_A$-module such that $\pi_*\mathcal{F}$ is a finite (resp. coherent) $\mathcal{O}_B$-module. Then $\mathcal{F}$ is a finite (resp. coherent) $\mathcal{O}_A$-module.*

Proof. Step 1. Since $\pi_*\mathcal{F}$ is a finite $\mathcal{O}_B$-module, B can be covered by open sets $V \subset B$ such that, locally on V, the direct image sheaf $f_*\mathcal{F}$ is generated by $g_1, \ldots, g_k \in \mathit{\Gamma}(V, \pi_*\mathcal{F}) = \mathit{\Gamma}(\pi^{-1}(V), \mathcal{F})$. We claim that $g_1, \ldots, g_k$ generate also $\mathcal{F}|_{\pi^{-1}(V)}$ as $\mathcal{O}_{\pi^{-1}(V)}$-module (which yields, in particular, that $\mathcal{F}$ is locally a finite $\mathcal{O}_A$-module). Indeed, for each $\boldsymbol{y} \in V$, the stalk $(\pi_*\mathcal{F})_{\boldsymbol{y}}$ is generated by the germs of $g_1, \ldots, g_k$ as $\mathcal{O}_{B,\boldsymbol{y}}$-module. Thus, Proposition 1.56 (2) yields that for each point $\boldsymbol{x} \in \pi^{-1}(\boldsymbol{y})$ the stalk $\mathcal{F}_{\boldsymbol{x}}$ is generated by $g_1, \dots, g_k$ as $\mathcal{O}_{B,\boldsymbol{y}}$-module, hence also as $\mathcal{O}_{A,\boldsymbol{x}}$-module.

Step 2. Let $\pi_*\mathcal{F}$ be a coherent $\mathcal{O}_B$-module. We have to show that, for each open subset $U \subset A$ and for each surjection $\varphi : \mathcal{O}_U^q \to \mathcal{F}|_U$, the kernel is an $\mathcal{O}_U$-module of finite type. Let $\boldsymbol{x} \in U$ and $\boldsymbol{y} = \pi(\boldsymbol{x})$. Then Lemma 1.54 yields an open neighbourhood $V \subset B$ of $\boldsymbol{y}$ such that $\pi^{-1}(V)$ is the disjoint union of $U_1, \dots, U_s$, where each U_i contains precisely one point of the fibre $\pi^{-1}(\boldsymbol{y})$. We may assume that $\boldsymbol{x} \in U_1 = U$ and extend φ to a map $\widetilde{\varphi} : \mathcal{O}^q_{\pi^{-1}(V)} \to \mathcal{F}|_{\pi^{-1}(V)}$ by setting $\widetilde{\varphi}_{U_i} = 0$ for all $i = 2, \dots, s$. Since the direct image functor $\overline{\pi}_*$ for the restriction $\overline{\pi} = \pi|_{\pi^{-1}(V)}$ is exact (Lemma 1.57), we get an exact sequence of $\mathcal{O}_V$-modules $0 \to \overline{\pi}_* \mathcal{K}er(\widetilde{\varphi}) \to (\pi_*\mathcal{O}_A|_V)^q \to \pi_*\mathcal{F}|_V \to 0$.

The Weierstraß isomorphism Theorem 1.61 yields that $\pi_*\mathcal{O}_A|_V$ is a free $\mathcal{O}_V$-module. Thus, the coherence of $\pi_*\mathcal{F}$ implies that $\overline{\pi}_* \mathcal{K}er(\widetilde{\varphi})$ is an $\mathcal{O}_V$-module of finite type. Since $\overline{\pi}$ is a Weierstraß map, Step 1 applies, showing that $\mathcal{K}er(\widetilde{\varphi})$ is an $\mathcal{O}_{\pi^{-1}(V)}$-sheaf of finite type. In particular, $\mathcal{K}er(\widetilde{\varphi})|_U = \mathcal{K}er(\varphi)$ is an $\mathcal{O}_U$-module of finite type. □

Based on the results for Weierstraß maps obtained so far, we can give a proof of Oka's coherence theorem [Oka]:

Theorem 1.63 (Coherence of the structure sheaf). *The structure sheaf $\mathcal{O}_X$ of a complex space X is coherent.*

Proof. Coherence being a local property, we may suppose that X is a complex model space defined by an ideal sheaf $\mathcal{J} \subset \mathcal{O}_D$ of finite type, $D \subset \mathbb{C}^n$ open (see also A.7, Fact 6). By A.7, Facts 2 and 6, $\mathcal{O}_D/\mathcal{J}$ is coherent if $\mathcal{O}_D$ is coherent. Therefore, we can assume $(X, \mathcal{O}_X) = (D, \mathcal{O}_D)$.

We use induction on n, the case $n = 0$ being trivial. To show the coherence of $\mathcal{O}_D$, we have to show that for each morphism

$$\varphi : \mathcal{O}_D^k \longrightarrow \mathcal{O}_D\,, \quad (a_1, \dots, a_k) \longmapsto a_1 f_1 + \ldots + a_k f_k\,,$$

$\mathcal{K}er(\varphi)$ is of finite type. Since $\mathcal{O}_D^k$ is of finite type, we may assume $\varphi \neq 0$ and, without loss of generality, $f := f_1 \neq 0$.

We claim that the sheaf $\mathcal{O}_D/f\mathcal{O}_D$ is a coherent sheaf of rings at any point $\boldsymbol{x} \in D$. If $f(\boldsymbol{x}) \neq 0$ then f has no zero in some neighbourhood of $\boldsymbol{x}$ and $\mathcal{O}_D/f\mathcal{O}_D$ is locally the zero sheaf, hence coherent. Thus, we may assume $f(\boldsymbol{x}) = 0$. We may also assume that $\boldsymbol{x} = \boldsymbol{0}$ and that f is x_n-general of order b. By the Weierstraß preparation theorem, there exists a Weierstraß polynomial $g_{\boldsymbol{0}} \in \mathcal{O}_{\mathbb{C}^{n-1},0}[x_n]$ such that $g_{\boldsymbol{0}}\mathcal{O}_D = f_{\boldsymbol{0}}\mathcal{O}_D$. We choose a neighbourhood $B \subset \mathbb{C}^{n-1}$ of $\boldsymbol{0}$ such that the germ $g_{\boldsymbol{0}}$ has a representative $g \in \Gamma(B, \mathcal{O}_{\mathbb{C}^{n-1}})[x_n]$ with $g\mathcal{O}_U = f\mathcal{O}_U$ for a sufficiently small neighbourhood $U \subset D$ of $\boldsymbol{0}$. We consider the Weierstraß map

$$\pi :\ A = \big\{(\boldsymbol{y}, z) \in B \times \mathbb{C} \,\big|\, g(\boldsymbol{y}, z) = 0\big\} \longrightarrow B\,.$$

The Weierstraß isomorphism Theorem 1.61 yields $\pi_*\mathcal{O}_A \cong \mathcal{O}_B^b$ with $\mathcal{O}_B$ being coherent by the induction hypothesis. Hence, $\pi_*\mathcal{O}_A$ is $\mathcal{O}_B$-coherent. Now, Lemma 1.62 applies, showing that $\mathcal{O}_A = (\mathcal{O}_{B\times\mathbb{C}}/g\,\mathcal{O}_{B\times\mathbb{C}})|_A$ is $\mathcal{O}_A$-coherent. Then the trivial extension to $B\times\mathbb{C}$, $i_*\mathcal{O}_A$ is also a coherent sheaf of rings. Since the sheaf $\mathcal{O}_D/f\mathcal{O}_D$ locally coincides with $i_*\mathcal{O}_A$ near $\mathbf{0}$, we get the claim.

During the following construction, let $\mathcal{O} = \mathcal{O}_D|_U$ for a sufficiently small neighbourhood $U \subset D$ of $\mathbf{0}$ which we allow to shrink. We consider the following commutative diagram of sheaf morphisms with exact bottom row

$$\begin{tikzcd}
\mathcal{O}^q \oplus \mathcal{O}^k \arrow[drr, dashed, "\phi"] & & \mathcal{O}^k \arrow[ll, hook'] \arrow[r, "\varphi"] \arrow[d, hook, "f"] & \mathcal{O} \arrow[d, hook, "f"] \\
\mathcal{O}^q \arrow[u, hook] \arrow[rr, dashed, "\psi"'] \arrow[d, two heads] & & \mathcal{O}^k \arrow[r, "\varphi"] \arrow[d, two heads, "\pi"] & \mathcal{O} \arrow[d, two heads, "\pi"] \\
(\mathcal{O}/f\mathcal{O})^q \arrow[rr, "\overline{\psi}"'] & & (\mathcal{O}/f\mathcal{O})^k \arrow[r, "\overline{\varphi}"'] & \mathcal{O}/f\mathcal{O}\,.
\end{tikzcd}$$

Since $f_{\mathbf{0}} \neq 0$, the multiplication map f is injective. π is the canonical projection, $\overline{\varphi}$ is the $\mathcal{O}/f\mathcal{O}$-linear map induced by φ, and $\overline{\psi}$ exists since $\mathcal{O}/f\mathcal{O}$ is a coherent $\mathcal{O}/f\mathcal{O}$-module. ψ is an $\mathcal{O}$-linear lift of $\overline{\psi}$, which exists since $\mathcal{O}^q$ is free and generated by $\Gamma(U,\mathcal{O})^q$. We define

$$\phi : \mathcal{O}^q \oplus \mathcal{O}^k \longrightarrow \mathcal{O}^k\,, \quad (\boldsymbol{a},\boldsymbol{b}) \longmapsto \psi(\boldsymbol{a}) + f\boldsymbol{b}\,.$$

By diagram chasing, we see that ϕ surjects onto $\mathcal{K} := \mathcal{K}er(\pi\circ\varphi) \subset \mathcal{O}^k$. In particular, $\mathcal{K}$ is finitely generated.

Since f is injective, for each $\boldsymbol{a} \in \mathcal{K}$, there is a unique $h(\boldsymbol{a}) \in \mathcal{O}$ such that $f\circ h(\boldsymbol{a}) = \varphi(\boldsymbol{a})$. This obviously defines a splitting $\varphi|_{\mathcal{K}} = f\circ h$ of $\varphi|_{\mathcal{K}}$ through an $\mathcal{O}$-linear map $h : \mathcal{K} \to \mathcal{O}$ with $h|_{\mathcal{K}er(\varphi)} = 0$. Define

$$\chi : \mathcal{K} \longrightarrow \mathcal{O}^k\,, \quad \boldsymbol{a} \longmapsto \boldsymbol{a} - \big(h(\boldsymbol{a}),\mathbf{0}\big)$$

Then $\varphi\circ\chi(\boldsymbol{a}) = 0$, that is, $\chi(\mathcal{K}) \subset \mathcal{K}er(\varphi)$. Since $\mathcal{K}er(\varphi) \subset \mathcal{K}$ and since $\chi|_{\mathcal{K}er(\varphi)} = \mathrm{id}_{\mathcal{K}er(\varphi)}$, we get that χ surjects onto $\mathcal{K}er(\varphi)$. Therefore, $\chi\circ\phi$ defines a surjection $\mathcal{O}^q \oplus \mathcal{O}^k \twoheadrightarrow \mathcal{K}er(\varphi)$, proving that $\mathcal{K}er(\varphi)$ is of finite type. □

Corollary 1.64. *Let X be a complex space.*

(1) If Y is a complex subspace of X, given by the ideal sheaf $\mathcal{J}_Y \subset \mathcal{O}_X$, then $\mathcal{J}_Y$ and $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{J}_Y)|_Y$ are coherent.
(2) A closed subset $A \subset X$ is analytic iff there exists a coherent sheaf $\mathcal{F}$ such that $A = \mathrm{supp}(\mathcal{F})$.

Proof. A subsheaf of a coherent sheaf is coherent iff it is of finite type. Hence, (1) follows from Oka's Theorem 1.63. For (2), note that if $A \subset X$ is an analytic set, then $A = \mathrm{supp}(\mathcal{O}_X/\mathcal{J})$ for some ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$ of finite type. By Oka's theorem, $\mathcal{O}_X$ and $\mathcal{J}$ are coherent, thus $\mathcal{O}_X/\mathcal{J}$ is coherent, too (A.7, Fact 2). Conversely, if $A = \mathrm{supp}(\mathcal{F})$ then $A = \mathrm{supp}\big(\mathcal{O}_X/\mathcal{A}nn(\mathcal{F})\big)$, and $\mathcal{F}$ being coherent implies that $\mathcal{A}nn(\mathcal{F})$ is coherent, too (A.7, Fact 5). $\square$

Corollary 1.65. *Let $\pi : A \to B$ be a Weierstraß map. If $\mathcal{F}$ is a coherent $\mathcal{O}_A$-module then $\pi_*\mathcal{F}$ is a coherent $\mathcal{O}_B$-module.*

Proof. Let $\boldsymbol{y} \in B$, $\pi^{-1}(\boldsymbol{y}) = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_s\}$. By Lemma 1.54, there are an open neighbourhood $V \subset B$ of $\boldsymbol{y}$ and pairwise disjoint open neighbourhoods $U_i \subset A$ of $\boldsymbol{x}_i$, $i = 1, \dots, s$, such that $\pi^{-1}(V) = U_1 \cup \ldots \cup U_s$ and such that the restrictions $\pi_{U_i,V} : U_i \to V$ are finite maps. Since $\mathcal{F}$ is coherent, we may assume that, for each $i = 1, \dots, s$, there is an exact sequence

$$\mathcal{O}_{U_i}^{q_i} \longrightarrow \mathcal{O}_{U_i}^{k_i} \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0 \tag{1.5.1}$$

(shrinking V if necessary). By adding direct summands, we may assume that $q_i = q$, $k_i = k$ for each i.

We set $U = \pi^{-1}(V) \subset A$. As U is the disjoint union of $U_1, \dots, U_s$, the exact sequences (1.5.1) yield an exact sequence $\mathcal{O}_U^q \to \mathcal{O}_U^k \to \mathcal{F}|_U \to 0$. As $\pi : U \to V$ is finite, the direct image functor π_* is exact (Corollary 1.57). Thus, the induced sequence

$$(\pi_*\mathcal{O}_A^q)|_V \longrightarrow (\pi_*\mathcal{O}_A^k)|_V \longrightarrow \pi_*\mathcal{F}|_V \longrightarrow 0\,,$$

is exact, too. Applying the Weierstraß isomorphism Theorem 1.61, we get an exact sequence of $\mathcal{O}_V$-modules $\mathcal{O}_V^{qb} \to \mathcal{O}_V^{kb} \to \pi_*\mathcal{F}|_V \to 0$, where b is the degree of π. Finally, as $\mathcal{O}_B$ is coherent, the existence of such an exact sequence (for each $\boldsymbol{y} \in B$) implies that $\pi_*\mathcal{F}$ is a coherent $\mathcal{O}_B$-module (see A.7). $\square$

The following theorem appears to be the main result about finite holomorphic maps. It has numerous applications, in particular in singularity theory. Its main advantage is that the assumption is purely topologically and very easy to verify.

Theorem 1.66 (Local finiteness theorem). *Let $f : X \to Y$ be a morphism of complex spaces, let $y \in Y$, and let x be an isolated point of the fibre $f^{-1}(y)$. Then there exist open neighbourhoods $U \subset X$ of x and $V \subset Y$ of y such that $f(U) \subset V$ and*

(1) $f_{U,V} : U \to V$ is finite.
(2) For each coherent $\mathcal{O}_U$-module $\mathcal{F}$ the direct image $(f_{U,V})_*\mathcal{F}$ is a coherent $\mathcal{O}_V$-module.

Proof. All statements being local, it suffices to consider the case that X and Y are complex model spaces. Further, it suffices to consider the case that f is a projection: consider the graph of f,

$$\begin{tikzcd} X \arrow[r, "\cong", "\varphi"'] \arrow[drr, "f"'] & \Gamma(f) \arrow[r, hook] \arrow[dr] & X \times Y \arrow[d, "\mathrm{pr}"] \\ & & Y\,. \end{tikzcd}$$

Then an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent iff the direct image $\varphi_*\mathcal{F}$ is a coherent $\mathcal{O}_{\Gamma(f)}$-module.

Thus, altogether, assume that there are (sufficiently small) open subsets $B \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^k$ such that X is a closed subspace of $B \times D$, given by the coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{B\times D}$, and f is the projection

$$f = \mathrm{pr}_{X,Y} : B \times D \supset X \to Y \subset B\,.$$

Further, assume that $y = \mathbf{0} \in B$ and that $x = (\mathbf{0}, \mathbf{0})$. Since x is an isolated point of the fibre $f^{-1}(\mathbf{0}) = X \cap (\{\mathbf{0}\} \times D)$, after shrinking $D \subset \mathbb{C}^k$, we have $X \cap (\{\mathbf{0}\} \times D) = \{(\mathbf{0}, \mathbf{0})\}$.

Now, we prove the theorem by induction on k, starting with $k = 1$. Since $X \cap (\{\mathbf{0}\} \times D) = \{(\mathbf{0}, 0)\}$, there exists a germ $\widetilde{g} \in \mathcal{I}_{(\mathbf{0},0)}$ such that $\widetilde{g}(\mathbf{0}, 0) = 0$ and $z \mapsto \widetilde{g}(\mathbf{0}, z)$ is not the zero map on D. By the Weierstraß preparation theorem, there exists a Weierstraß polynomial g and a unit $u \in \mathcal{O}_{D\times B,(\mathbf{0},0)}$ such that

$$u\widetilde{g} \,=\, g \,=\, z^b + a_1 z^{b-1} + \ldots + a_b \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}[z]\,, \quad a_i(\mathbf{0}) = 0\,,$$

$i = 1, \ldots b$. After shrinking B and D, we may assume that $a_i \in \Gamma(B, \mathcal{O}_{\mathbb{C}^n})$, and we may consider the Weierstraß map defined by the projection on B,

$$A = \big\{(\boldsymbol{y}, z) \in B \times D \;\big|\; g(\boldsymbol{y}, z) = 0\big\} \xrightarrow{\;p\;} B \subset \mathbb{C}^n\,.$$

Due to Lemma 1.59, p is finite. Since $i : X \hookrightarrow A$ is a closed embedding, the restriction $p_{X,Y} = f : X \to Y$ is finite, too. Moreover, if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the trivial extension $i_*\mathcal{F}$ of $\mathcal{F}$ to A is a coherent $\mathcal{O}_A$-module (A.7, Fact 6). Thus, Corollary 1.65 yields that $p_*i_*\mathcal{F} \cong f_*\mathcal{F}$ is a coherent $\mathcal{O}_B$-module.

For the induction step, let $k > 1$ and assume that $D = D'' \times D'$, where $D'' \subset \mathbb{C}^{k-1}$ and $D' \subset \mathbb{C}$ are open neighbourhoods of the origin. Then $f = \mathrm{pr}_{X,Y}$ is induced by the composition of two projections:

$$B \times D = B \times (D'' \times D') \xrightarrow{\;p'\;} B \times D'' \xrightarrow{\;p''\;} B\,.$$

As the statement holds for $k = 1$, we may assume that (after shrinking B, D'' and D') the restriction $p'|_X : X \to B \times D''$ is finite and that for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ the direct image $(p'|_X)_*\mathcal{F}$ is a coherent $\mathcal{O}_{B\times D''}$-module. In particular, by Lemma 1.44, the image $X_1 := p'(X)$ is a closed complex subspace of $B \times D''$, endowed with one of the structure sheaves of Definition 1.45. Note that, in each case, the restriction $\pi' = p'_{X,X_1} : X \to X_1$ is finite and $\pi'_*\mathcal{F}$ is a coherent $\mathcal{O}_{X_1}$-module.

Since $X \cap (\{\mathbf{0}\} \times D) = \{(\mathbf{0}, \mathbf{0})\}$, we also have $X_1 \cap (\{\mathbf{0}\} \times D'') = \{(\mathbf{0}, \mathbf{0})\}$. Thus, the induction hypothesis implies that (after shrinking B and D'') we may assume that the restriction $\pi'' = p''_{X_1,Y}$ is finite and that the direct image $\pi''_* \mathcal{G}$ of a coherent $\mathcal{O}_{X_1}$-module $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-sheaf. Together with the above, we get that $f = \pi'' \circ \pi'$ is finite and that the direct image $f_*\mathcal{F} \cong \pi''_* \pi'_* \mathcal{F}$ of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module. □

Taking into account the considerations on finite maps at the beginning of this section, the local finiteness theorem implies the finite coherence theorem which succinctly says that for a finite morphism f the direct image functor f_* preserves coherence:

Theorem 1.67 (Finite coherence theorem, FCT). *Let $f : X \to Y$ be a finite morphism of complex spaces, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module.*

Proof. Let $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_s\}$. By Lemma 1.54 and Proposition 1.56, there are open neighbourhoods $V \subset Y$ of y and $U_i \subset X$ of x_i, $i = 1, \dots, s$, such that the restrictions $f_{U_i,V} : U_i \to V$ are finite and

$$f_*\mathcal{F}|_V \;\cong\; \bigoplus_{i=1}^{s} (f_{U_i,V})_*(\mathcal{F}|_{U_i})\,.$$

The local finiteness theorem implies that (after shrinking V and U_i) we may assume that $(f_{U_i,V})_*(\mathcal{F}|_{U_i})$ is a coherent $\mathcal{O}_V$-module. Since direct sums of coherent sheaves are coherent (A.7, Fact 2) and since coherence is a local property, we deduce that $f_*\mathcal{F}$ is coherent, as claimed in the theorem. □

Together with Corollary 1.64 (2), the finite coherence theorem shows that the image of a finite morphism of complex spaces is analytically closed. More precisely, we obtain:

Corollary 1.68 (Finite mapping theorem). *Let $f : X \to Y$ be a finite morphism of complex spaces and $Z \subset X$ a closed complex subspace of X. Then $f(Z) \subset Y$ is an analytic subset of Y (which can be endowed with one of the structure sheaves of Definition 1.45).*

The finite coherence theorem and the local finiteness theorem are due to Grauert and Remmert (cf. [GrR2]). We emphasize again that, in both theorems, the assumptions are of a purely topological nature (thus, independent of the structure sheaf).

Remarks and Exercises

(A) *Proper Maps and the Proper Coherence Theorem.* Recall that a continuous map is called *proper* if the preimage of any compact set is compact.

Much deeper than the finite coherence theorem (and much more difficult to prove) is the coherence theorem for proper maps due to Grauert [Gra], which says that *for a proper morphism of complex spaces the direct image functor preserves coherence.* A proof can be found in [FoK], respectively in [GrR2, Ch. 10]. As for finite morphisms, one can deduce as a corollary that the image of an analytic set under a proper morphism of complex spaces is analytically closed. This statement is also referred to as the *proper mapping theorem.* It was proved first by Remmert [Rem].

Let X, Y be complex spaces with X being compact. Then the projection $X \times Y \to Y$ is a proper morphism. In particular, the projection $\mathbb{P}^n \times Y \to Y$ is proper.

Exercise 1.5.1. Show that finite maps between complex spaces are proper.

Exercise 1.5.2. Let X be a complex space which is compact and connected, and let $f : X \to \mathbb{C}$ be a holomorphic map. Prove that f is constant.

HINT. Apply the proper mapping theorem.

(B) *Computing the Image by Elimination.* Let $X = V(g_1, \dots, g_k) \subset \mathbb{P}^m(\mathbb{C})$ be defined by homogeneous polynomials $g_1, \dots, g_k \in \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_0, \dots, x_n]$, and let $f_0, \dots, f_n \in \mathbb{C}[\boldsymbol{x}]$ be homogeneous polynomials of the same degree d with $V(f_0, \dots, f_n) \cap X = \emptyset$. Then we get a (proper) morphism $f : X \to \mathbb{P}^n(\mathbb{C})$, $\boldsymbol{x} \mapsto (f_0(\boldsymbol{x}) : \ldots : f_n(\boldsymbol{x}))$. The *annihilator structure* on the image of f can be computed effectively in a computer algebra system like SINGULAR by eliminating $\boldsymbol{x}$ from the ideal

$$J = \langle y_0 - f_0, \dots, y_n - f_n, g_1, \dots, g_k \rangle_{\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]}$$

that is, by computing the elimination ideal $J \cap \mathbb{C}[\boldsymbol{y}]$ (see [GrP, App. A.7] for a much broader discussion of the geometric meaning of elimination). For instance, the following SINGULAR session computes the image of the morphism $f : \mathbb{P}^1 \to \mathbb{P}^2$, $(x_0 : x_1) \mapsto (x_0^3 : x_1^2 x_0 : x_1^3)$:

```
ring r=0,(x(0),x(1),y(0),y(1),y(2)),dp;
poly f(0),f(1),f(2) = x(0)^3,x(1)^2*x(0),x(1)^3;
ideal J=y(0)-f(0),y(1)-f(1),y(2)-f(2);
eliminate(J,x(0)*x(1));
//-> _[1]=y(1)^3-y(0)*y(2)^2
```

Hence, $f(\mathbb{P}^1) = V(y_1^3 - y_0 y_2^2) \subset \mathbb{P}^2$.

For details on how to compute elimination ideals using SINGULAR (and on the implemented algorithms), we refer to [GrP, Sect. 1.8.2], resp. [DeL, Sects. 3.6.2 and 9.5].

Exercise 1.5.3. Let $X = T = \mathbb{C}$, and $Y = \mathbb{C}^2$, each equipped with the reduced structure. Moreover, let $f : X \to Y$ be given by $t \mapsto (t^2, t^3)$. Show that f is a finite morphism and that the Fitting, annihilator, and reduced structure of the image $f(X)$ coincide (see Exercise 1.6.4 for a more general statement).

Moreover, show that the annihilator structure and the reduced structure are not compatible with the base change $g : T \hookrightarrow Y$, $x \mapsto (x, 0)$ (see Remark 1.45.1 (2)).

Exercise 1.5.4. Let $f : X \to Y$ be a finite morphism, let $A \subset X$ be an analytic set, and let $y \in Y$. Moreover, let $\pi^{-1}(y) = \{x_1, \ldots, x_s\}$ and assume that the germ (A, x_i) decomposes into r_i irreducible components, $i = 1, \ldots, s$. Prove that the germ of the image $f(A)$ at y decomposes into at most $\sum_{i=1}^{s} r_i$ irreducible components.

Exercise 1.5.5. Let $\pi : A \to B$ be a Weierstraß map and $(\boldsymbol{y}, z) \in A$. Prove the following statements:

(1) π is an *open* map, that is, it maps open sets in A to open sets in B.
(2) To every sequence $(\boldsymbol{y}_\nu)_{\nu \in \mathbb{N}} \subset B$ converging to $\boldsymbol{y}$ there exists a sequence $(z_\nu)_{\nu \in \mathbb{N}} \subset \mathbb{C}$ such that $(\boldsymbol{y}_\nu, z_\nu) \in A$ and $(z_\nu)_{\nu \in \mathbb{N}}$ converges to z.

HINT FOR (1). Use Hensel's lemma to reduce the statement to the case that $\pi^{-1}(\boldsymbol{y}) = \{(\boldsymbol{y}, z)\}$.

Exercise 1.5.6. Prove the uniqueness statement in the general Weierstraß division Theorem 1.60.

1.6 Applications of the Finite Coherence Theorem

The finite coherence theorem (in particular, the local finiteness theorem) has many applications. In this section, we apply it to prove the Hilbert-Rückert Nullstellensatz. Moreover, we sketch a proof (based on the the Hilbert-Rückert Nullstellensatz) of Cartan's theorem that the full ideal sheaf of an analytic set is coherent.

Definition 1.69. A map germ $f : (X, x) \to (Y, y)$ is called *finite* if it has a finite representative $f : U \to V$.

Proposition 1.70. *Let $f = (f, f^\sharp) : (X, x) \to (Y, y)$ be a morphism of complex space germs. Then the following conditions are equivalent:*

(a) $f : (X, x) \to (Y, y)$ is finite.
(b) The fibre $\big(f^{-1}(y), x\big)$ consists of one point $\{x\}$ (as a set).
(c) The ring morphism $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite.
(d) The ring morphism $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is quasifinite.

Proof. The equivalence of (a) and (b) follows from the local finiteness Theorem 1.66.

To prove (a) $\Rightarrow$ (c), choose a finite representative $f : X \to Y$ such that $f^{-1}(y) = \{x\}$. Then Proposition 1.56 yields that $\mathcal{O}_{X,x} \cong (f_*\mathcal{O}_X)_y$, where $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-sheaf (due to the finite coherence Theorem 1.67).

In particular, $\mathcal{O}_{X,x}$ is a finitely generated $\mathcal{O}_{Y,y}$-module, which is precisely the meaning of $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ being finite.

By Corollary 1.13, (c) is equivalent to (d). Thus, we are left with the proof of (d) $\Rightarrow$ (b). By definition, $\mathcal{O}_{f^{-1}(y)} = \mathcal{O}_X / \mathfrak{m}_y \mathcal{O}_X$ where $\mathfrak{m}_y \subset \mathcal{O}_Y$ is the ideal sheaf of the (reduced) point $\{y\}$. If $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is quasifinite then $\dim_{\mathbb{C}} \mathcal{O}_{f^{-1}(y),x} = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x} < \infty$. Nakayama's lemma implies that $\mathfrak{m}_{X,x}^p \mathcal{O}_{f^{-1}(y),x} = 0$, hence $\mathfrak{m}_{X,x}^p \subset \mathfrak{m}_y \mathcal{O}_{X,x}$, for some $p > 0$. It follows that, locally at x, we have an inclusion of sets $f^{-1}(y) = V(\mathfrak{m}_y \mathcal{O}_X) \subset V(\mathfrak{m}_{X,x}^p) = \{x\}$. □

Lemma 1.71. *Let $(f, f^\sharp) : (X, x) \to (Y, y)$ be a finite morphism of germs such that $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. Then f is surjective (that is, has a surjective representative $f : U \to V$).*

Proof. By the local finiteness Theorem 1.66, there is a finite representative $f : U \to V$ such that $f(U)$ is closed in V and $f_*\mathcal{O}_U$ is a coherent $\mathcal{O}_V$-sheaf. Then, for sufficiently small V and U,

$$\mathcal{A}nn_{\mathcal{O}_V}(f_*\mathcal{O}_U) \;=\; \mathcal{K}er(f^\sharp : \mathcal{O}_V \to f_*\mathcal{O}_U) \;=\; 0\,,$$

since the stalk at y is zero by assumption, and since the annihilator sheaf is coherent, too (A.7, Fact 5). Therefore, $f(U) = V\big(\mathcal{A}nn_{\mathcal{O}_V}(f_*\mathcal{O}_U)\big) = V$. □

Remark 1.71.1. Lemma 1.71 applies, in particular, to a Noether normalization: let $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ be a complex space germ with $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$, and let $\varphi : \mathbb{C}\{y_1, \ldots, y_d\} \hookrightarrow \mathcal{O}_{X,\mathbf{0}}$ be a Noether normalization (Theorem 1.25). Setting $f^\sharp = \varphi$ and $f = \big(\varphi(y_1), \ldots, \varphi(y_d)\big)$, we obtain a finite and surjective morphism $(f, f^\sharp) : (X, x) \to (\mathbb{C}^d, 0)$, which we refer to as a *Noether normalization (of complex space germs)*.

The following theorem, due to Rückert, is the analytic counterpart to the Hilbert Nullstellensatz for polynomial rings.

Theorem 1.72 (Hilbert-Rückert Nullstellensatz). *Let X be a complex space, $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf. Then*

$$\mathcal{J}\big(V(\mathcal{I})\big) \;=\; \sqrt{\mathcal{I}}\,,$$

where $\mathcal{J}\big(V(\mathcal{I})\big)$ is the full ideal sheaf of $V(\mathcal{I})$.

Proof. Since, obviously, $\sqrt{\mathcal{I}} \subset \mathcal{J}\big(V(\mathcal{I})\big)$, and since both sheaves have the same support, we have to show that for each $x \in V(\mathcal{I})$ the inclusion map

$$\big(\sqrt{\mathcal{I}}\,\big)_x \;=\; \sqrt{\mathcal{I}_x} \;\hookrightarrow\; \mathcal{J}\big(V(\mathcal{I})\big)_x$$

is surjective.

Consider a primary decomposition of $\mathcal{I}_x$, $\mathcal{I}_x = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$, with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ prime ideals. Then

$$\sqrt{\mathcal{I}_x} = \bigcap_{i=1}^{r} \mathfrak{p}_i\,, \qquad \mathcal{J}\big(V(\mathcal{I}_x)\big) = \bigcap_{i=1}^{r} \mathcal{J}\big(V(\mathfrak{q}_i)\big) = \bigcap_{i=1}^{r} \mathcal{J}\big(V(\mathfrak{p}_i)\big)$$

(see Exercise 1.3.5). Thus, it suffices to show that for a prime ideal $\mathfrak{p} \subset \mathcal{O}_{X,x}$ we have $\mathcal{J}\big(V(\mathfrak{p})\big) = \sqrt{\mathfrak{p}}$.

Choose a Noether normalization

$$\varphi : \mathbb{C}\{\boldsymbol{y}\} = \mathbb{C}\{y_1, \dots, y_d\} \hookrightarrow \mathcal{O}_{X,x}/\mathcal{J}\big(V(\mathfrak{p})\big)$$

and a lifting $\widetilde{\varphi} : \mathbb{C}\{\boldsymbol{y}\} \hookrightarrow \mathcal{O}_{X,x}$, which induces a morphism $\mathbb{C}\{\boldsymbol{y}\} \hookrightarrow \mathcal{O}_{X,x}/\mathfrak{p}$. Since $V(\mathfrak{p}) = V\big(\mathcal{J}\big(V(\mathfrak{p})\big)\big)$ as topological spaces, the induced morphism of germs $V(\mathfrak{p}) \to (\mathbb{C}^d, 0)$ is finite. By Proposition 1.70, it follows that $\mathcal{O}_{X,x}/\mathfrak{p}$ is finite over $\mathbb{C}\{\boldsymbol{y}\}$, in particular, $\mathcal{O}_{X,x}/\mathfrak{p}$ is integral over $\mathbb{C}\{\boldsymbol{y}\}$ (via $\widetilde{\varphi}$). Thus, each $f \in \mathcal{J}\big(V(\mathfrak{p})\big) \subset \mathcal{O}_{X,x}$ satisfies a relation (of minimal degree)

$$f^r + a_1 f^{r-1} + \ldots + a_r \,\in\, \mathfrak{p}$$

with $a_i \in \widetilde{\varphi}(\mathbb{C}\{\boldsymbol{y}\})$. Since $\mathfrak{p} \subset \mathcal{J}\big(V(\mathfrak{p})\big)$, we have $a_r \in \mathcal{J}\big(V(\mathfrak{p})\big) \cap \widetilde{\varphi}(\mathbb{C}\{\boldsymbol{y}\})$ which is 0 as φ is injective. It follows that

$$f \cdot (f^{r-1} + a_1 f^{r-2} + \ldots + a_{r-1}) \,\in\, \mathfrak{p}$$

and $f^{r-1} + a_1 f^{r-2} + \ldots + a_{r-1} \notin \mathfrak{p}$, because we started with a relation of minimal degree. As $\mathfrak{p}$ is a prime ideal, we get $f \in \mathfrak{p}$, which proves the theorem. □

Corollary 1.73. *Let $\mathcal{F}$ be a coherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$. Suppose that f, considered as a morphism $f : X \to \mathbb{C}$ satisfies $f|_{\operatorname{supp}(\mathcal{F})} = 0$. Then, for each $x \in X$, there exists a neighbourhood U of x and a positive integer r such that $f^r \mathcal{F}|_U = 0$.*

In particular, if $f(x) = 0$ for all $x \in X$ then all germs $f_x \in \mathcal{O}_{X,x}$ are nilpotent.

Proof. Apply the Hilbert-Rückert Nullstellensatz to $\mathcal{I} = \mathcal{A}nn_{\mathcal{O}_X}(\mathcal{F})$. For the second statement take $\mathcal{F} = \mathcal{O}_X$. □

Corollary 1.74. *Let $\mathcal{F}$ be a coherent sheaf on X, $x \in \operatorname{supp}(\mathcal{F})$. Then the following are equivalent:*

(a) x is an isolated point of the support of $\mathcal{F}$.
(b) $\mathfrak{m}_{X,x}^r \mathcal{F}_x = 0$ for some $r > 0$.
(c) $\dim_{\mathbb{C}} \mathcal{F}_x < \infty$.

Proof. (a) ⇒ (b) If x is an isolated point of the support of $\mathcal{F}$, then for each $f \in \mathfrak{m}_{X,x}$ there exists a neighbourhood U of x such that $f|_{\operatorname{supp}(\mathcal{F}) \cap U} = 0$. By Corollary 1.73, there exists some $r > 0$ such that $f^r \mathcal{F}_x = 0$. Since $\mathfrak{m}_{X,x}$ is a finitely generated $\mathcal{O}_{X,x}$-module, we easily deduce (b).

(b) ⇒ (c) If $\mathfrak{m}_{X,x}^r \mathcal{F}_x = 0$, then

$$\dim_{\mathbb{C}} \mathcal{F}_x = \dim_{\mathbb{C}} \mathcal{F}_x/\mathfrak{m}_{X,x}^r \mathcal{F}_x = \sum_{i=1}^{r} \dim_{\mathbb{C}} \mathfrak{m}_{X,x}^{i-1}\mathcal{F}_x/\mathfrak{m}_{X,x}^{i}\mathcal{F}_x\,,$$

which is finite as $\mathcal{O}_{X,x}$ is Noetherian and $\mathcal{F}_x$ a finitely generated $\mathcal{O}_{X,x}$-module.

(c) $\Rightarrow$ (a) Let $\dim_{\mathbb{C}} \mathcal{F}_x < \infty$. Then, by Nakayama's lemma there exists an integer $s > 0$ such that $\mathfrak{m}_{X,x}^s \mathcal{F}_x = 0$, that is, $\mathfrak{m}_{X,x}^s \subset \mathrm{Ann}_{\mathcal{O}_{X,x}} \mathcal{F}_x$. Hence, locally at x, $\mathrm{supp}(\mathcal{F}) = V(\mathrm{Ann}_{\mathcal{O}_X} \mathcal{F}) \subset V(\mathfrak{m}_{X,x}^s) = \{x\}$. □

We close this section with Cartan's theorem on the coherence of the full ideal sheaf. Since the proof is slightly more involved than the proofs of the previous fundamental coherence Theorems 1.63 and 1.67, we only sketch the proof given by Grauert and Remmert. For details, we refer to [GrR2, Section 4.2] or [DJP, Theorem 6.3.2].

Theorem 1.75 (Coherence of the full ideal sheaf). *Let A be an analytic set in the complex space X. Then the full ideal sheaf $\mathcal{J}(A)$ of all holomorphic functions on X vanishing on A is a coherent $\mathcal{O}_X$-sheaf.*

In view of Oka's coherence Theorem 1.63, Cartan's theorem may be rephrased as follows: let $f_1, \dots, f_r \in \mathcal{O}_X(U)$, $U \subset X$ open, $x \in U$, represent a set of generators for the stalk $\mathcal{J}(A)_x \subset \mathcal{O}_{X,x}$. Then $f_1, \dots, f_r$ generate $\mathcal{J}(A)$ on a whole neighbourhood of x in X (A.7, Fact 1). In other words, locally at x, the full ideal sheaf of A coincides with the $\mathcal{O}_U$-module $\mathcal{I} = \sum_{i=1}^r f_i \mathcal{O}_U$.

Note that, a priori, it is clear that $\mathcal{I}_{x'} \subset \mathcal{J}(A)_{x'}$ for all $x' \in U$; but it is not clear that the opposite inclusion holds, that is, that $\mathcal{I}_{x'}$ is a radical ideal (Hilbert-Rückert Nullstellensatz).

Sketch of proof. We may use general facts on coherent sheaves (see Appendix A.7) to reduce the proof to the case that $X = D \subset \mathbb{C}^n$ is an open neighbourhood of $\mathbf{0}$ and to showing coherence locally at $\mathbf{0}$. Using the existence of an irreducible decomposition of analytic set germs and Exercise 1.3.5 (3), we may assume additionally that $A \subset D$ is *irreducible* at $\mathbf{0}$.

The proof requires now a closer analysis of the structure of locally irreducible analytic sets as given by [GrR2, Lemmas 3.3.4, 3.4.1]: locally at $\mathbf{0}$, there is a finite and open surjection $h : A \to B$, with $B \subset \mathbb{C}^d$ open, $d = \dim(A)$, which is locally biholomorphic outside (the preimage of) some analytic hypersurface $V(\Delta) \subsetneq B$, the discriminant of h, where Δ is a holomorphic function on B. The proof of this fact uses the Weierstraß preparation theorem and Hensel's lemma and it gives a precise description of h (and its local inverse) on $X \setminus h^{-1}(V(\Delta))$. From this description, we get that there are Weierstraß polynomials $f_1, \dots, f_{n-d}$ at $\mathbf{0}$ vanishing on $(A, \mathbf{0})$ such that for $x \in D \setminus h^{-1}(V(\Delta))$ close to $\mathbf{0}$, $\mathcal{J}(A)_x = \sum_{i=1}^{n-d} f_i \mathcal{O}_{D,x}$.

For the fibre at $\mathbf{0}$, we know that $\mathcal{J}(A)_{\mathbf{0}} \supset \sum_{i=1}^{n-d} f_i \mathcal{O}_{D,\mathbf{0}}$. We complement $f_1, \dots, f_{n-d}$ to a generating set $f_1, \dots, f_r$ of $\mathcal{J}(A)_{\mathbf{0}}$. After shrinking D, we may assume that $f_1, \dots, f_r$ converge on D and consider the finitely generated (hence, coherent) ideal sheaf $\mathcal{I} = \sum_{i=1}^{r} f_i \mathcal{O}_D$. From our construction, we know that $\mathcal{J}(A)_x = \mathcal{I}_x$ for $x = \mathbf{0}$ and for all $x \in D \setminus h^{-1}(V(\Delta))$.

It remains to extend this statement to $x \in h^{-1}(V(\Delta)) \setminus \{\mathbf{0}\}$. For this, let $\widetilde{\Delta} = \Delta \circ h$ and consider the ideal quotient

$$\mathcal{I} : \widetilde{\Delta} := \operatorname{Ker}\left(\mathcal{O}_D \xrightarrow{\cdot \widetilde{\Delta}} \mathcal{O}_D/\mathcal{I}\right),$$

which is a coherent $\mathcal{O}_D$-sheaf (A.7, Fact 3). Since $\mathcal{I}_{\mathbf{0}} = \mathcal{J}(A)_{\mathbf{0}}$ is prime and $\widetilde{\Delta} \notin \mathcal{I}_{\mathbf{0}}$, we may assume that $\mathcal{I} : \widetilde{\Delta} = \mathcal{I}$ (shrinking D is necessary). Now, let $g \in \mathcal{J}(A)_x$ for x close to $\mathbf{0}$. Then, locally at x, the ideal quotient $\mathcal{I} : g$ is coherent and $V(\mathcal{I} : g) \subset h^{-1}(V(\Delta)) = V(\widetilde{\Delta})$. By the Hilbert-Rückert Nullstellensatz, this implies that $\widetilde{\Delta}^r \in \mathcal{I} : g$ for some $r \geq 0$. If $r > 0$, this means that $\widetilde{\Delta}^{r-1} g \in \mathcal{I} : \widetilde{\Delta} = \mathcal{I}$, that is, $\widetilde{\Delta}^{r-1} \in \mathcal{I} : g$. By induction on r we obtain that $1 = \widetilde{\Delta}^0 \in \mathcal{I} : g$, that is, $g \in \mathcal{I}_x$. □

Theorem 1.76 (Coherence of the radical). *Let $\mathcal{I}$ be a coherent ideal sheaf on the complex space X. Then the radical $\sqrt{\mathcal{I}}$ is coherent. In particular, the sheaf $\mathcal{N}il(\mathcal{O}_X)$ of nilpotent elements of $\mathcal{O}_X$ is coherent.*

Proof. Since $A \subset X$ is analytic, there exists a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $A = V(\mathcal{I})$. By the Hilbert-Rückert Nullstellensatz, $\mathcal{J}(A) = \sqrt{\mathcal{I}}$ and the result follows from Cartan's Theorem 1.75. □

Exercises

Exercise 1.6.1. Let $(f, f^\sharp) : (X, x) \to (Y, y)$ be a finite morphism of complex space germs and assume that (Y, y) is *reduced*. Show that f is surjective (that is, has a surjective representative $f : U \to V$) iff $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. Show that this statement does not generalize to morphisms of non-reduced complex space germs.

Exercise 1.6.2. Let $f : (X, x) \to (Y, y)$ be a finite morphism of complex space germs. Prove the following statements:

(1) $\dim\big(f(X), y\big) = \dim(X, x)$.
(2) If f is *open*, that is, if it has an open representative $f : U \to V$, then $\dim\big(Y, y\big) = \dim(X, x)$.

HINT: Use Exercise 1.3.1.

Exercise 1.6.3. Let $f : (X, x) \to (Y, y)$ be a morphism of reduced complex space germs and assume that (Y, y) is irreducible. Prove the following statements:

(1) If f is open, then all elements of the kernel of $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ are nilpotent.
(2) If f is finite and if $(f_*\mathcal{O}_X)_y$ is a torsion free $\mathcal{O}_{Y,y}$-module then f is open.

Exercise 1.6.4. Let $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0})$ be a finite morphism of germs such that, for a sufficiently small representative $f : U \to V$, the restriction $f|_{U \setminus \{0\}}$ induces an isomorphism $f|_{U\setminus\{0\}} : U \setminus \{0\} \xrightarrow{\cong} C \setminus \{0\}$, where $C \subset V$ is a curve. Prove the following statements:

(1) The Fitting ideal $\mathcal{F}itt(f_*\mathcal{O}_X))_{\mathbf{0}}$ is a principal ideal of $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$.
(2) The Fitting, annihilator, and reduced structure of the germ of the image of f at $\mathbf{0}$ coincide.

Hint for (1). Use the Auslander Buchsbaum formula (Corollary B.9.4).

1.7 Finite Morphisms and Flatness

In the same manner as for modules (cf. Appendix B), we define flatness for sheaves of modules on a ringed space $(X, \mathcal{A})$. An $\mathcal{A}$-module $\mathcal{M}$ is called *flat*, if for each exact sequence $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ of $\mathcal{A}$-modules, the induced sequence

$$0 \longrightarrow \mathcal{N}' \otimes_{\mathcal{A}} \mathcal{M} \longrightarrow \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \longrightarrow \mathcal{N}'' \otimes_{\mathcal{A}} \mathcal{M} \longrightarrow 0$$

is also exact, or, equivalently, if for all points $x \in X$ the stalk $\mathcal{M}_x$ is a flat $\mathcal{A}_x$-module.

Definition 1.77. A morphism $f : X \to Y$ of complex spaces is called *flat at* $x \in X$ if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module (via $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$). It is called *flat* if f is flat at each point $x \in X$, or, equivalently, if $\mathcal{O}_X$ is a flat $f^{-1}\mathcal{O}_Y$-module. A morphism of germs $f : (X, x) \to (Y, y)$ is called flat if it has a flat representative.

Example 1.77.1. Let $X \subset \mathbb{C}^2$ be the subspace defined by $y^2 - x$ and $Y \subset \mathbb{C}^2$ defined by xy. Let $f : X \to \mathbb{C}$, resp. $g : Y \to \mathbb{C}$, be the projections to the x-axis (cf. Figure 1.4).

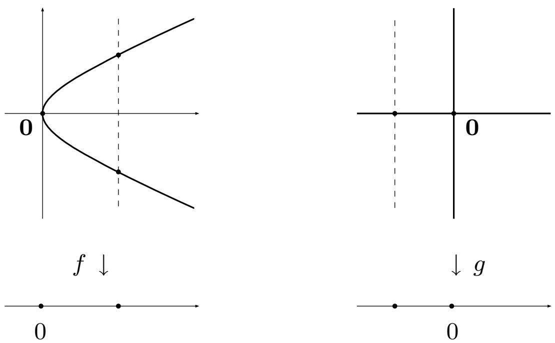


Fig. 1.4. Projections of $V(y^2 - x)$, resp. $V(xy)$ to the x-axis.

The Weierstraß division Theorem 1.8 yields that $\mathbb{C}\{x,y\}/\langle y^2-x\rangle$ is a free, hence flat, $\mathbb{C}\{x\}$-module of rank 2 (with basis $1, y$). Thus, f is flat at $\mathbf{0}$. On the other hand, g is not flat at $\mathbf{0}$. In fact, to see that $M=\mathbb{C}\{x,y\}/\langle xy\rangle$ is not flat over $\mathbb{C}\{x\}$, tensor the exact sequence of $\mathbb{C}\{x\}$-modules

$$0 \longrightarrow \langle x\rangle \longrightarrow \mathbb{C}\{x\} \longrightarrow \mathbb{C} \longrightarrow 0$$

with M. The induced map $\langle x\rangle \otimes_{\mathbb{C}\{x\}} M \to \mathbb{C}\{x\} \otimes_{\mathbb{C}\{x\}} M = M$ is not injective, since $x \otimes [y] \neq 0$ is mapped to $[xy]=0$.

A priori, flatness is a purely algebraic concept. But it turns out to have a geometrical meaning which can be roughly formulated as a *continuous behaviour of the fibres*. For instance, looking at the fibres of f and g in Example 1.77.1, we get $f^{-1}(x)=\big\{(x,\sqrt{x}),(x,-\sqrt{x})\big\}$ if $x\neq 0$, and $f^{-1}(0)=\big\{(0,0)\big\}$ with multiplicity 2. Hence, the fibres of f behave "continuously" at 0 if we count them with multiplicity. On the other hand, $g^{-1}(x)=\big\{(x,0)\big\}$ if $x\neq 0$, and $g^{-1}(0)=\{0\}\times\mathbb{C}$. In particular, the fibre dimension of g jumps locally at 0.

For finite maps, flatness has a particularly nice geometric interpretation. As shown below, the finite coherence Theorem 1.67 implies that all numerical invariants which can be described as the fibre dimension of coherent sheaves behave *semicontinuously* in a family. If the family is flat, then the invariants vary even *continuously*, which means that they are locally constant. Hence, for finite morphisms, flatness is the precise algebraic formulation of what has been, somehow mysteriously, called the *"principle of conservation of numbers"*.

The following theorem can be understood as a sheafified version of the flatness criterion (Proposition B.3.5) for finite maps:

Theorem 1.78. *Let $f: X\to Y$ be a finite morphism of complex spaces and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then the following conditions are equivalent:*

(a) $\mathcal{F}$ is f-flat, that is, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x\in X$.
(b) $(f_*\mathcal{F})_y$ is a flat $\mathcal{O}_{Y,y}$-module for all $y\in Y$.
(c) $f_*\mathcal{F}$ is a locally free sheaf on Y.

In particular, f is flat iff $f_*\mathcal{O}_X$ is a locally free sheaf on Y.

Proof. Since f is finite, Proposition 1.56 yields that $(f_*\mathcal{F})_y \cong \bigoplus_{x\in f^{-1}(y)}\mathcal{F}_x$, hence the equivalence of (a) and (b). The finite coherence Theorem 1.67 implies that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-sheaf, in particular, $(f_*\mathcal{F})_y$ is a finitely generated $\mathcal{O}_{Y,y}$-module for each $y\in Y$. By the flatness criterion of Proposition B.3.5, $(f_*\mathcal{F})_y$ is flat iff it is a free $\mathcal{O}_{Y,y}$-module. The equivalence of (b) and (c) follows now from Theorem 1.80 (1) below. $\square$

Definition 1.79. A subset of a complex space X is called *analytically closed* if it is a closed analytic subset of X; it is called *analytically open* if it is the complement of a closed analytic subset of X.

Theorem 1.80. *Let X be a complex space, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and set*

$$\mathcal{F}(x) := \mathcal{F}_x/\mathfrak{m}_{X,x}\mathcal{F}_x\,, \qquad \nu(\mathcal{F},x) := \dim_{\mathbb{C}} \mathcal{F}(x)\,.$$

(1) The following are equivalent
(a) $\mathcal{F}$ is locally free on X.
(b) $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for each $x \in X$.
(2) If X is reduced, then (a) and (b) are also equivalent to
(c) The function $x \mapsto \nu(\mathcal{F},x)$ is locally constant on X.
(3) The sets

$$S_d(\mathcal{F}) := \bigl\{x \in X \bigm| \nu(\mathcal{F},x) > d\bigr\}\,, \quad d \in \mathbb{Z}\,,$$
$$\mathrm{NFree}(\mathcal{F}) := \bigl\{x \in X \bigm| \mathcal{F}_x \textit{ is not free}\bigr\}$$

are analytically closed in X.
(4) If X is reduced, then $\mathrm{Free}(\mathcal{F}) := X \setminus \mathrm{NFree}(\mathcal{F})$ is dense in X. If X is reduced and irreducible, then

$$\mathrm{NFree}(\mathcal{F}) = S_{d_0}(\mathcal{F}) \quad \textit{with} \quad d_0 = \min\bigl\{\nu(\mathcal{F},x) \bigm| x \in X\bigr\}\,.$$

Note that $\nu(\mathcal{F},x) = \mathrm{mng}(\mathcal{F}_x)$ is the minimum number of generators of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. Further note that the assumption that X is reduced in (2) and (4) is necessary: take X the non-reduced point T_ε, $\mathcal{O}_{T_\varepsilon} = \mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle$ and $\mathcal{F} = \langle\varepsilon\rangle \cdot \mathcal{O}_{T_\varepsilon}$.

It follows from (3) that the set $\mathrm{Free}(\mathcal{F}) = X \setminus \mathrm{NFree}(\mathcal{F})$ is analytically open in X. On the other hand, $\mathrm{Free}(\mathcal{F})$ is the disjoint union of the sets

$$\mathrm{Free}_d(\mathcal{F}) = \bigl\{x \in X \bigm| \mathcal{F}_x \text{ is free of rank } d\bigr\}\,.$$

Thus, each of the sets $\mathrm{Free}_d(\mathcal{F})$ is analytically open in X, too. Finally, note that $\mathrm{Free}(\mathcal{F})$ is also the flat locus of $\mathcal{F}$ (Proposition B.3.5).

Proof. (1), (2): $\mathcal{F}$ being locally free means that each point $x_0 \in X$ has a neighbourhood U such that $\mathcal{F}|_U \cong \mathcal{O}_U^\nu$ for some ν. Clearly (a) implies (b) and (c), even if X is non-reduced.

As $\mathcal{F}$ is coherent, for each $x_0 \in X$ there exists a connected open neighbourhood U of x_0 and an exact sequence

$$\mathcal{O}_U^q \xrightarrow{A} \mathcal{O}_U^p \xrightarrow{\pi} \mathcal{F}|_U \longrightarrow 0\,.$$

Hence, for each $x \in U$ the sequence $\mathcal{O}_{U,x}^q \xrightarrow{A} \mathcal{O}_{U,x}^p \to \mathcal{F}_x \to 0$ is exact, and, after tensoring with $\mathbb{C}$ as $\mathcal{O}_{U,x}$-module, we get an exact sequence of finite dimensional $\mathbb{C}$-vector spaces

$$\mathbb{C}^q \xrightarrow{A(x)} \mathbb{C}^p \longrightarrow \mathcal{F}(x) \longrightarrow 0\,,$$

with $\mathrm{rank}(A(x)) = p - \nu(\mathcal{F},x)$.

By Nakayama's lemma, we may choose finitely many sections in $\Gamma(U,\mathcal{F})$ which represent a basis of the fibre $\mathcal{F}(x_0)$ and which generate $\mathcal{F}_x$ for all $x \in U$ (shrinking U if necessary). Hence, we may assume $p = \nu(\mathcal{F}, x_0)$. In this situation

$$S_d(\mathcal{F}) \cap U \;=\; \bigl\{x \in U \bigm| \operatorname{rank}(A(x)) < p - d\bigr\}$$

is the zero set of the ideal generated by all $(p-d)$-minors of A. In particular, it is analytic. We use this setting in the following.

Supposing (c), we may assume that $\nu(\mathcal{F}, x)$ is constant on U and, hence, $\operatorname{rank}(A(x)) = 0$ on U, that is, we may assume that each entry $a_{ij} \in \mathcal{O}_U(U)$ of A satisfies $a_{ij}(x) = 0$ for all $x \in U$. By Corollary 1.73, this means that each a_{ij} is nilpotent. If X is reduced, this implies that each a_{ij} is zero. Thus, $\mathcal{O}_U^p \cong \mathcal{F}|_U$ which implies (a).

Now, assume that X is not necessarily reduced and that (b) is satisfied. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}m\,(A) \longrightarrow \mathcal{O}_U^p \xrightarrow{\;\pi\;} \mathcal{F}|_U \longrightarrow 0\,. \tag{1.7.1}$$

Since $\mathcal{F}_{x_0}$ is free, the induced sequence $0 \to \mathcal{I}m\,(A)(x_0) \to \mathbb{C}^p \to \mathcal{F}(x_0) \to 0$ is exact. Hence, $\nu(\mathcal{I}m\,(A), x_0) = p - \nu(\mathcal{F}, x_0) = 0$, that is, $\mathcal{I}m\,(A)(x_0) = 0$. By Nakayama's lemma, this implies $\mathcal{I}m\,(A)_{x_0} = 0$. Since $\mathcal{I}m\,(A)$ is coherent, $\mathcal{I}m\,(A) = 0$ in a neighbourhood of x_0, which implies (a).

(3) To show that $\operatorname{NFree}(\mathcal{F})$ is analytically closed in X, consider again the exact sequence (1.7.1). The stalk $\mathcal{F}_{x_0}$ is free iff this sequence splits at x_0 (Exercise 1.7.1), that is, iff there is a morphism $\sigma : \mathcal{F}_{x_0} \to \mathcal{O}_{U,x_0}^p$ with $\pi_{x_0} \circ \sigma = \mathrm{id}$. Now consider the map

$$\widetilde{\pi} : \mathcal{H}om(\mathcal{F}|_U, \mathcal{O}_U^p) \longrightarrow \mathcal{H}om(\mathcal{F}|_U, \mathcal{F}|_U)\,, \quad \psi \longmapsto \pi \circ \psi\,.$$

If the sequence (1.7.1) splits at x_0, we have $\varphi = \widetilde{\pi}_{x_0}(\sigma \circ \varphi)$ for each homomorphism $\varphi : \mathcal{F}_{x_0} \to \mathcal{F}_{x_0}$. Thus, $\widetilde{\pi}_{x_0}$ is surjective. Conversely, if $\widetilde{\pi}_{x_0}$ is surjective, then the identity map $\mathrm{id} : \mathcal{F}_{x_0} \to \mathcal{F}_{x_0}$ has a preimage $\sigma : \mathcal{F}_{x_0} \to \mathcal{O}_{U,x_0}^p$, which is a splitting.

We have shown that the stalk $\mathcal{F}_{x_0}$ is free iff $\widetilde{\pi}_{x_0}$ is surjective, that is, iff $\operatorname{Coker}(\widetilde{\pi}_{x_0}) = 0$. Since $\mathcal{C}oker\,(\widetilde{\pi})$ is a coherent $\mathcal{O}_U$-sheaf, we get that $\operatorname{NFree}(\mathcal{F}|_U) = \operatorname{supp}\bigl(\mathcal{C}oker\,(\widetilde{\pi})\bigr)$ is analytically closed in U.

(4) Let X be reduced, $x_0 \in X$ and U any neighbourhood of x_0. Let d_0 be the minimum value of $\nu(\mathcal{F}, x)$ on U. Then $S_{d_0}(\mathcal{F}) \cap U$ is analytically closed in U by (3) and its complement $U \setminus S_{d_0}(\mathcal{F})$ is non-empty. By (2), $\mathcal{F}$ is locally free on $U \setminus S_{d_0}(\mathcal{F})$. Hence, any neighbourhood of x_0 contains points of $X \setminus \operatorname{NFree}(\mathcal{F})$. Thus, $\operatorname{Free}(\mathcal{F}) = X \setminus \operatorname{NFree}(\mathcal{F})$ is dense in X. If, additionally, X is irreducible then, as a proper closed analytic subset of X, $S_{d_0}(\mathcal{F})$ is nowhere dense in X. Hence, its complement $X \setminus S_{d_0}(\mathcal{F})$ is dense in X. This shows that $\mathcal{F}$ cannot be locally free at any point $x \in S_{d_0}(\mathcal{F})$. $\square$

The theorem says, in particular, if the stalk $\mathcal{F}_x$ is free then $\mathcal{F}$ is locally free in a neighbourhood of x. Moreover, the open set $X \setminus \mathrm{NFree}(\mathcal{F})$ decomposes in connected components and $\mathcal{F}$ has constant rank on each component. An alternative proof of the analyticity of $\mathrm{NFree}(\mathcal{F})$ is given in Exercise 1.7.5.

As a corollary, we obtain the main result of this section, which provides the promised geometric interpretation of flatness for finite morphisms:

Theorem 1.81 (Semicontinuity of fibre functions). *Let $f : X \to Y$ be a finite morphism of complex spaces and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) The function

$$y \longmapsto \nu(f_*\mathcal{F}, y) = \sum_{x \in f^{-1}(y)} \dim_{\mathbb{C}} \mathcal{F}_x/\mathfrak{m}_y\mathcal{F}_x$$

is upper semicontinuous[8] *on Y (here, $\mathfrak{m}_y$ denotes the maximal ideal of $\mathcal{O}_{Y,y}$).*

(2) If $\mathcal{F}$ is f-flat then $\nu(f_*\mathcal{F}, y)$ is locally constant on Y.

(3) If Y is reduced then $\nu(f_*\mathcal{F}, y)$ is locally constant on Y iff $\mathcal{F}$ is f-flat.

Statement (2) is called the *principle of conservation of numbers.*

Proof. (1) We get from Proposition 1.56 that

$$(f_*\mathcal{F})_y \cong \bigoplus_{x \in f^{-1}(y)} \mathcal{F}_x/\mathfrak{m}_y\mathcal{F}_x.$$

Since $f_*\mathcal{F}$ is coherent on Y (by the finite coherence Theorem 1.67), the result follows from Theorem 1.80 (3).

(2) If $\mathcal{F}$ is f-flat then $f_*\mathcal{F}$ is locally free by Theorem 1.78, hence locally at $y_0 \in Y$ of constant rank equal to $\nu(f_*\mathcal{F}, y_0)$.

(3) This follows from Theorem 1.80 (4). □

Another corollary is the following

Theorem 1.82 (Openness of flatness). *Let $f : X \to Y$ be finite and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then the set of points $x \in X$ where $\mathcal{F}_x$ is $\mathcal{O}_{Y,f(x)}$-flat is analytically open in X. In particular, the set of points in which f is flat is analytically open.*

Proof. By Theorem 1.80, $\mathrm{Free}(f_*\mathcal{F})$ is analytically open in Y. Since

$$(f_*\mathcal{F})_y \cong \bigoplus_{x \in f^{-1}(y)} \mathcal{F}_x$$

[8] A function $\varphi : Y \to \mathbb{R}$, Y a topological space, is called *upper semicontinuous* if for each $y_0 \in Y$ there is a neighbourhood V of y_0 such that $\varphi(y) \le \varphi(y_0)$ for all $y \in V$.

as $\mathcal{O}_{Y,y}$-module, $(f_*\mathcal{F})_y$ is $\mathcal{O}_{Y,y}$-free iff $\mathcal{F}_x$ is $\mathcal{O}_{Y,y}$-free for all $x \in f^{-1}(y)$. By Proposition B.3.5 this is equivalent to $\mathcal{F}_x$ being $\mathcal{O}_{Y,y}$-flat for all $x \in f^{-1}(y)$.

Now, given $x_0 \in X$ there are neighbourhoods $U = U(x_0)$ and $V = V(y_0)$, $y_0 = f(x_0)$, such that $f_{U,V} : U \to V$ is finite and $f_{U,V}^{-1}(y_0) = \{x_0\}$. If the stalk $\mathcal{F}_{x_0}$ is $\mathcal{O}_{Y,y_0}$-flat then $(f_{U,V})_*\mathcal{F}|_U$ is free in a neighbourhood $V'(y_0) \subset V$. Thus, $\mathcal{F}_x$ is $\mathcal{O}_{Y,f(x)}$-flat for all $x \in f^{-1}(V') \cap U$. □

Remark 1.82.1. A much stronger theorem due to Frisch says that Theorem 1.82 holds for *each* holomorphic map $f : X \to Y$ (see Theorem 1.83).

Remarks and Exercises

Using $\mathcal{E}xt$ *sheaves*, we can give a more conceptual description of the non-free locus NFree($\mathcal{F}$).

Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{J}$ is called *injective* if the functor $\mathcal{F} \mapsto \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J})$ is exact on the category of $\mathcal{O}_X$-modules. It is a fact that each $\mathcal{O}_X$-module $\mathcal{F}$ has an *injective resolution*

$$0 \to \mathcal{F} \to \mathcal{L}^0(\mathcal{F}) \xrightarrow{\varphi_0} \mathcal{L}^1(\mathcal{F}) \xrightarrow{\varphi_1} \mathcal{L}^2(\mathcal{F}) \xrightarrow{\varphi_2} \dots$$

(that is, the sequence is exact and the modules $\mathcal{L}^i(\mathcal{F})$ are injective).

For a second $\mathcal{O}_X$-module $\mathcal{M}$, we have an induced sequence of sheaves which is a complex

$$\begin{aligned} 0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F}) \;\to\; \mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{M}, \mathcal{L}^0(\mathcal{F})\big) &\xrightarrow{\varphi'_0} \mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{M}, \mathcal{L}^1(\mathcal{F})\big) \\ &\xrightarrow{\varphi'_1} \mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{M}, \mathcal{L}^2(\mathcal{F})\big) \xrightarrow{\varphi'_2} \dots . \end{aligned}$$

Then $\mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F}) := \mathscr{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F})$ and, for all $i \geq 1$,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F}) := \mathcal{H}^i\big(\mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{M}, \mathcal{L}^\bullet(\mathcal{F})\big)\big) := \mathcal{K}er(\varphi'_{i+1}) / \mathcal{I}m(\varphi'_i) .$$

For details and further properties of $\mathcal{E}xt$, in particular for the long exact $\mathcal{E}xt$ sequences, we refer to [God].

If it happens that $\mathcal{M}$ has a resolution by locally free sheaves $\mathcal{M}_i$ of finite rank, $\dots \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0 \to \mathcal{M} \to 0$, then

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F}) \cong \mathcal{H}^i\big(\mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{M}_\bullet, \mathcal{F}\big) .$$

However, such a locally free resolution of $\mathcal{M}$ may not exist.

Now, let $(X, \mathcal{O}_X)$ be a complex space and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. Then $\mathcal{M}$ has locally a locally free resolution and, if $\mathcal{F}$ is coherent, then $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F})$ is coherent, too (Exercise 1.7.7). Moreover, for all $x \in X$, we have then

$$\big(\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F})\big)_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{F}_x) ,$$

which can be computed by a free resolution of the $\mathcal{O}_{X,x}$-module $\mathcal{M}_x$. This allows us to compute NFree($\mathcal{M}$) via Ext (see Exercise 1.7.8).

Exercise 1.7.1. Let A be a ring, and let $\pi : F \to M$ be a surjection of A-modules with F free. Prove that the following are equivalent

(a) M is projective.
(b) There is a morphism $\sigma : M \to F$ with $\pi \circ \sigma = \mathrm{id}_M$.
(c) The map $\mathrm{Hom}_A(M, F) \to \mathrm{Hom}_A(M, M)$, $\psi \mapsto \pi \circ \psi$, is surjective.

Exercise 1.7.2. Let A be a ring and

$$0 \longrightarrow M' \xrightarrow{\alpha} F \xrightarrow{\beta} M'' \longrightarrow 0 \tag{1.7.2}$$

an exact sequence of A-modules with F free. Show that the following are equivalent:

(a) The sequence (1.7.2) is *left split*, that is, there exists a morphism $\tau : F \to M'$ with $\tau \circ \alpha = \mathrm{id}_{M'}$.
(b) The sequence (1.7.2) is *right split*, that is, there exists a morphism $\sigma : M'' \to F$ with $\beta \circ \sigma = \mathrm{id}_{M''}$.
(c) $F \cong M' \oplus M''$.
(d) M' is projective.
(e) M'' is projective.

Exercise 1.7.3. Let A be a Noetherian local ring, and let N, M be finite A-modules. Denote by $\mathrm{mng}(M)$, the minimal number of generators of M (see Definition 1.19). Show that the following holds:

(1) $\mathrm{mng}(M \otimes_A N) = \mathrm{mng}(M) \cdot \mathrm{mng}(N)$ and $\mathrm{mng}(\bigwedge^p M) = \binom{\mathrm{mng}(M)}{p}$.
(2) M is free of rank d iff $\bigwedge^d M$ is free of rank 1.
(3) M is free of rank 1 iff the canonical map

$$\phi : M^* \otimes_A M \to A\,, \quad \varphi \otimes x \mapsto \varphi(x)\,,$$

is an isomorphism.

Exercise 1.7.4. Let X be a complex space and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank n. Let $\mathcal{E}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ denote the *dual $\mathcal{O}_X$-module*.

(1) Show that $\mathcal{E}^*$ is a locally free $\mathcal{O}_X$-module of rank n.
(2) Prove that there is a canonical isomorphism $(\mathcal{E}^*)^* \cong \mathcal{E}$.
(3) Prove that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{F}$ for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$.

Exercise 1.7.5. Let $\mathcal{F}$ be a coherent sheaf on a complex space X and let $\phi : \mathcal{F}^* \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X$ be given by $\varphi \otimes f \mapsto \varphi(f)$. Prove that

$$X \setminus \mathrm{Free}_d(\mathcal{F}) \;=\; \mathrm{supp}(\mathcal{K}er\,(\phi)) \cup \mathrm{supp}(\mathcal{C}oker\,(\phi))\,.$$

Conclude that $\mathrm{Free}_d(\mathcal{F})$ is analytically open and that $\mathrm{NFree}(\mathcal{F})$ is analytically closed in X.

Exercise 1.7.6. Let $f : X \to Y$ be a finite morphism of complex spaces which is flat. Prove that f is open (see also Theorem 1.84).

Exercise 1.7.7. Let X be a complex space and $\mathcal{F}, \mathcal{G}$ coherent $\mathcal{O}_X$-modules.

(1) Show that for any $x_0 \in X$ and $i \in \mathbb{N}$ there exists an open neighbourhood U of x_0 and an exact sequence

$$0 \to \mathcal{R} \to \mathcal{L}_{i-1} \to \ldots \to \mathcal{L}_1 \xrightarrow{\varphi_1} \mathcal{L}_0 \xrightarrow{\varphi_0} \mathcal{F}|_U \to 0$$

with $\mathcal{L}_j \cong \mathcal{O}_U^{n_j}$ and $\mathcal{R}$ a coherent $\mathcal{O}_U$-module. This module $\mathcal{R}$ is called the *i-th syzygy module* of $\mathcal{F}|_U$ and denoted by $\mathcal{S}yz_i(\mathcal{F}|_U)$. The sheaves $\mathcal{S}yz_i(\mathcal{F}|_U)$ can be glued to the *i-th syzygy sheaf* $\mathcal{S}yz_i(\mathcal{F})$ which is a coherent $\mathcal{O}_X$-module.

(2) Prove that, for $i \geq 0$, the $\mathcal{O}_X$-module $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent by showing that there is an exact sequence

$$\mathcal{H}om_{\mathcal{O}_U}(\mathcal{L}_{i-1}, \mathcal{G}|_U) \to \mathcal{H}om_{\mathcal{O}_U}(\mathcal{R}, \mathcal{G}|_U) \to \mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to 0\,.$$

(3) Use (2) to show that, for $x \in X$,

$$\big(\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F})\big)_x \cong \operatorname{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{F}_x)\,.$$

Exercise 1.7.8. Let $\mathcal{F}$ be a coherent sheaf on the complex space X.

(1) Show that $\mathcal{F}$ is locally free iff $\mathcal{E}xt^1_{\mathcal{O}_X}\big(\mathcal{F}, \mathcal{S}yz_1(\mathcal{F})\big) = 0$.

Hint. Use the argument in the proof of Theorem 1.80.

(2) Let $\mathcal{J} \subset \mathcal{O}_U$ denote either the 0-th Fitting ideal or the annihilator ideal of $\mathcal{E}xt^1_{\mathcal{O}_X}\big(\mathcal{F}, \mathcal{S}yz_1(\mathcal{F})\big)$. Show that

$$\operatorname{NFree}(\mathcal{F}) = \operatorname{supp}\big(\mathcal{E}xt^1_{\mathcal{O}_X}\big(\mathcal{F}, \mathcal{S}yz_1(\mathcal{F})\big)\big) = V(\mathcal{J})\,.$$

1.8 Flat Morphisms and Fibres

The aim of this section is to collect some of the most important properties of flat morphisms $f : X \to S$ of complex spaces to provide an easy reference, in particular, for the sections dealing with deformation theory.

Recall from Section 1.7 that, for finite morphisms, flatness implies locally the constancy of the total multiplicity of the fibres. If, additionally, the base is reduced then flatness can even be characterized by this numerical condition (Theorem 1.80). Moreover, we proved that flatness is an open property (Theorem 1.82).

In the following, we do not assume that f is finite. As we shall see, also in this general situation, flatness implies strong continuity conditions on the fibres. Moreover, flatness is used to study the singular locus of an arbitrary morphism of complex spaces.

Before we state geometric consequences of the algebraic properties of flatness treated in Appendix B, let us cite the following two important theorems. The first one is due to Frisch and generalizes Theorem 1.82:

Theorem 1.83 (Frisch). *Let $f : X \to S$ be a morphism of complex spaces. Then the* flat locus *of f, that is, the set of all points $x \in X$ such that f is flat at x, is analytically open in X.*

Proof. See [Fri]. □

The second theorem is due to Douady. It provides another openness result for flat morphisms (generalizing Exercise 1.7.6):

Theorem 1.84 (Douady). *Every flat morphism $f : X \to S$ of complex spaces is* open, *that is, it maps open sets in X to open sets in S.*

Proof. See [Dou] or [Fis, Prop. 3.19]. □

Together with Frisch's Theorem 1.83, Douady's theorem implies that if a morphism $f : X \to S$ is flat at $x \in X$, then it is locally surjective onto some neighbourhood of $f(x)$ in S. In particular, closed embeddings of proper subspaces are never flat.

The next proposition is the geometric version of Theorems B.8.13 and B.8.11:

Proposition 1.85. *Let $f : X \to S$ be a morphism of complex spaces, and let $x \in X$. Then, for $s = f(x)$ and $X_s = f^{-1}(s)$, the following holds:*

(1) $\dim(X, x) \leq \dim(X_s, x) + \dim(S, s)$ *with equality if f is flat at x.*
(2) If $S = \mathbb{C}^d$ and $f = (f_1, \dots, f_d)$, then f is flat at x iff $f_1, \dots, f_d$ is an $\mathcal{O}_{X,x}$-regular sequence.
(3) If X is a complete intersection at x, or, more generally, Cohen-Macaulay at x,[9] *and $S = \mathbb{C}^d$, then f is flat at x iff* $\dim(X, x) = \dim(X_s, x) + d$.

Proposition B.5.3 yields a criterion for checking whether a morphism of complex space germs is an isomorphism.

Lemma 1.86. *Let*

$$\begin{tikzcd} (X,x) \arrow[rr, "f"] \arrow[dr, "\phi"'] & & (Y,y) \arrow[dl, "\psi"] \\ & (S,s) & \end{tikzcd}$$

be a commutative diagram with ϕ flat. Then f is an isomorphism iff f induces an isomorphism of the special fibres,

$$f\colon (\phi^{-1}(s), x) \xrightarrow{\cong} (\psi^{-1}(s), y)\,.$$

[9] X is called a *complete intersection* (resp. *Cohen-Macaulay*) *at x* if the local ring $\mathcal{O}_{X,x}$ is a complete intersection (resp. Cohen-Macaulay). We also say that the germ (X, x) is a *complete intersection singularity* (resp. Cohen-Macaulay). Note that smooth germs and hypersurface singularities are complete intersection singularities, hence Cohen-Macaulay. Further, every reduced curve singularity and every normal surface singularity are Cohen-Macaulay (see Exercise 1.8.5).

Proof. We only need to show the "if" direction. For this consider the induced maps of local rings, $\phi^\sharp\colon \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ and $f^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

Since f induces an isomorphism of the special fibres and $y \in \psi^{-1}(s)$, the germ of $f^{-1}(y)$ consists of the point x only. Hence, f is a finite morphism of germs (Theorem 1.70) and, therefore, $\mathcal{O}_{Y,y}$ is a finite $\mathcal{O}_{X,x}$-module. That f induces an isomorphism of the special fibres means algebraically that

$$f^\sharp \otimes \mathrm{id}\colon \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}/\mathfrak{m}_{S,s} \longrightarrow \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}/\mathfrak{m}_{S,s}$$

is an isomorphism. Therefore, the assumptions of Proposition B.5.3 are fulfilled and $f^\sharp$ is an isomorphism. $\square$

We are now going to prove that flatness is preserved under base change. The proof in analytic geometry is slightly more complicated than in algebraic geometry where it follows directly from properties of the tensor product.

Proposition 1.87 (Preservation of flatness under base change). *If*

$$\begin{tikzcd} Z \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{f}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

is a Cartesian diagram of morphisms of complex spaces with f flat, then $\widetilde{f}$ is also flat.

Since the fibre product reduces to the Cartesian product if $S = \{\mathrm{pt}\}$ is a (reduced) point, and since a map to $\{\mathrm{pt}\}$ is certainly flat, we deduce the following corollary:

Corollary 1.88 (Flatness of projection). *If X, T are complex spaces then the projection $X \times T \to T$ is flat. Equivalently, for every $x \in X$ and $t \in T$, the analytic tensor product $\mathcal{O}_{X,x} \widehat{\otimes} \mathcal{O}_{T,t}$ is a flat $\mathcal{O}_{T,t}$-module.*

For the proof of Proposition 1.87 we need two lemmas:

Lemma 1.89. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of complex spaces. Moreover, let g be finite. Then, using the notations of Definition 1.46 and A.6, there is a natural isomorphism*

$$\pi_X^{-1}\mathcal{O}_X \otimes_{\pi_Y^{-1}g^{-1}\mathcal{O}_S} \pi_Y^{-1}\mathcal{O}_Y \xrightarrow{\cong} \mathcal{O}_{X\times_S Y}\,,$$

induced by the map $a \otimes b \mapsto ab := \widehat{\pi}_X(a) \cdot \widehat{\pi}_Y(b)$.

In particular, for $S = \{pt\}$ the reduced point and for Y a fat point, we get that $\mathcal{O}_{X,x} \widehat{\otimes} \mathcal{O}_Y = \mathcal{O}_{X,x} \otimes_{\mathbb{C}} \mathcal{O}_Y$.

Proof. We have to show that the morphism is stalkwise an isomorphism at each point $p = (x, y) \in X \times Y$ such that $f(x) = s = g(y)$ (cf. A.5). Further, we may suppose that $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$, $S \subset \mathbb{C}^k$ and that x, y, s are the origins of $\mathbb{C}^n, \mathbb{C}^m, \mathbb{C}^k$, respectively. Hence, what we actually have to show is that the map

$$\mathbb{C}\{\boldsymbol{x}\}/I_X \otimes_{\mathbb{C}\{\boldsymbol{s}\}/I_S} \mathbb{C}\{\boldsymbol{y}\}/I_Y \longrightarrow \mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}/J\,,$$

$$J = I_X\mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\} + I_Y\mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\} + \langle f_1 - g_1, \ldots, f_k - g_k\rangle\mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}\,,$$

induced by the multiplication $\psi : \mathbb{C}\{\boldsymbol{x}\} \otimes_{\mathbb{C}} \mathbb{C}\{\boldsymbol{y}\} \to \mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}$, $a \otimes b \mapsto ab$, is an isomorphism.

Let $J_0 \subset \mathbb{C}\{\boldsymbol{x}\} \otimes_{\mathbb{C}} \mathbb{C}\{\boldsymbol{y}\}$ denote the ideal generated by $h \otimes 1$, $h \in I_X$, $1 \otimes h'$, $h' \in I_Y$, and the differences $f_i \otimes 1 - 1 \otimes g_i$, $i = 1, \ldots, k$. Then we have to show that ψ induces an isomorphism

$$(\mathbb{C}\{\boldsymbol{x}\} \otimes_{\mathbb{C}} \mathbb{C}\{\boldsymbol{y}\})/J_0 \xrightarrow{\cong} \mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}/J\,.$$

The latter map is always injective (even if g is not finite): it is faithfully flat by Propositions B.3.3 (5),(8) and B.3.5, applied to

$$(\mathbb{C}\{\boldsymbol{x}\} \otimes_{\mathbb{C}} \mathbb{C}\{\boldsymbol{y}\})/J_0 \to \mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}/J \to \mathbb{C}[[\boldsymbol{x}, \boldsymbol{y}]]/J\mathbb{C}[[\boldsymbol{x}, \boldsymbol{y}]]\,.$$

By Proposition B.3.3 (10)(ii), $\psi^{-1}(J) = J_0$. Hence, we get injectivity.

To see the surjectivity, we use that $\mathbb{C}\{\boldsymbol{s}\}/I_S \to \mathbb{C}\{\boldsymbol{y}\}/I_Y$, $s_i \mapsto g_i(\boldsymbol{y})$, is finite. The finiteness implies that $\langle \boldsymbol{y}\rangle^m \subset I_Y + \langle g_1, \ldots, g_k\rangle$ for m sufficiently large. This further implies that in $\mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}/J$ we can replace high powers of $\boldsymbol{y}$ by polynomials in the f_i (since $f_i \equiv g_i \mod J$). Hence, each element of $\mathbb{C}\{\boldsymbol{x}, \boldsymbol{y}\}/J$ can be represented as a finite sum $\sum_i a_i(\boldsymbol{x})b_i(\boldsymbol{y})$ with $a_i \in \mathbb{C}\{\boldsymbol{x}\}$ and $b_i \in \mathbb{C}\{\boldsymbol{y}\}$. This completes the proof. $\square$

Lemma 1.90 (Finite-submersive factorization lemma). *Each morphism $f\colon (X, x) \to (Y, y)$ of complex germs factors through a finite map and a submersion, that is, there exists a commutative diagram*

$$\begin{tikzcd} (X,x) \arrow[r, "\varphi"] \arrow[dr, "f"'] & (Y,y) \times (\mathbb{C}^n, \mathbf{0}) \arrow[d, "p"] \\ & (Y,y) \end{tikzcd}$$

with $n = \dim(f^{-1}(y), x)$, φ finite and p the projection on the first factor. Moreover, if $(f^{-1}(y), x)$ is smooth, then φ can be chosen such that it induces an isomorphism $(f^{-1}(y), x) \xrightarrow{\cong} (\mathbb{C}^n, \mathbf{0})$.

Proof. Choose a Noether normalization $\varphi'\colon (f^{-1}(y), x) \to (\mathbb{C}^n, \mathbf{0})$ of the fibre $(f^{-1}(y), x)$ (Theorem 1.25). Then φ' is finite and, by the lifting Lemma 1.14, we can extend φ' to a map $\varphi''\colon (X, x) \to (\mathbb{C}^n, \mathbf{0})$. Setting

$$\varphi := f \times \varphi'' : (X,x) \to (Y,y) \times (\mathbb{C}^n, \mathbf{0})\,,$$

we get $\varphi^{-1}(y,\mathbf{0}) = f^{-1}(y) \cap \varphi''^{\,-1}(\mathbf{0}) = \varphi'^{\,-1}(\mathbf{0}) = \{x\}$, hence φ is finite and the result follows. $\square$

Proof of Proposition 1.87. The statement is local in T, hence we may consider morphisms of germs.

Since the base change map $g\colon (T,t) \to (S,s)$ factors through a finite map and a submersion (Lemma 1.90) we have to show that flatness is preserved by finite and submersive base changes. For this, we consider the dual base change diagram on the level of local rings

$$\begin{tikzcd}
\mathcal{O}_{Z,z} & \mathcal{O}_{X,x} \arrow[l, "\widetilde{g}^\sharp"'] \\
\mathcal{O}_{T,t} \arrow[u, "\widetilde{f}^\sharp"] & \mathcal{O}_{S,s}\,, \arrow[l, "g^\sharp"'] \arrow[u, "f^\sharp"']
\end{tikzcd}$$

where $\mathcal{O}_{Z,z} \cong \mathcal{O}_{X\times_S T,(x,t)}$.

If g is finite, then Lemma 1.89 yields $\mathcal{O}_{X\times_S T,(x,t)} \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{T,t}$, and Proposition B.3.3 (3) implies that $\mathcal{O}_{Z,z}$ is $\mathcal{O}_{T,t}$-flat.

Now, let g be a submersion, that is, g is the projection

$$g\colon (T,t) = (S,s) \times (\mathbb{C}^n,\mathbf{0}) \to (S,s)\,, \quad n = \dim(g^{-1}(s),t)\,.$$

Let $(S,s) \subset (\mathbb{C}^r,\mathbf{0})$, and denote by $f_1,\dots,f_r$ and $g_1,\dots,g_r$ the component functions of f and g, respectively. Then, set theoretically,

$$X \times_S T = \{(x,\boldsymbol{s},\boldsymbol{y}) \in X \times S \times \mathbb{C}^n \mid f_i(x) = g_i(\boldsymbol{s},\boldsymbol{y}) \text{ for all } i\}$$

where $X, \mathbb{C}^n, S$ are small representatives of the corresponding germs. Since $g_i(\boldsymbol{s},\boldsymbol{y}) = s_i$, we have $X \times_S T = \varGamma(f) \times \mathbb{C}^n$, and

$$\mathcal{O}_Z = \mathcal{O}_{X\times_S T} = \mathcal{O}_{X\times S\times\mathbb{C}^n}/\langle f_i - s_i\rangle = \mathcal{O}_{\varGamma(f)\times\mathbb{C}^n}\,.$$

Finally, tensoring the left-hand side of the dual base change diagram by $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/\mathfrak{m}_{\mathbb{C}^n,\mathbf{0}}^{k+1}$, we get the diagram

$$\begin{tikzcd}
\mathcal{O}_{\varGamma(f)\times\mathbb{C}^n,(x,s,\mathbf{0})}/\mathfrak{m}_{\mathbb{C}^n,\mathbf{0}}^{k+1} & \mathcal{O}_{X,x} \arrow[l] \\
\mathcal{O}_{S\times\mathbb{C}^n,(s,\mathbf{0})}/\mathfrak{m}_{\mathbb{C}^n,\mathbf{0}}^{k+1} \arrow[u, "\widetilde{f}^\sharp_{(k)}"] & \mathcal{O}_{S,s}\,. \arrow[l, "g^\sharp_{(k)}"'] \arrow[u, "f^\sharp"']
\end{tikzcd}$$

Since $g^\sharp_{(k)}$ is finite, the above reasoning gives that $\widetilde{f}^\sharp_{(k)}$ is flat.

This holds for each $k \geq 0$. Hence, $\widetilde{f}^{\,\sharp}$ is flat by the local criterion for flatness (Theorem B.5.1 (4)). $\square$

We state now a theoretically *and* computationally useful criterion for flatness due to Grothendieck:

Proposition 1.91 (Flatness by relations). *Let $I = \langle f_1, \ldots, f_k\rangle \subset \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ be an ideal, (S,s) a complex space germ and $\widetilde{I} = \langle F_1, \ldots, F_k\rangle \subset \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}$ a lifting of I, that is, F_i is a preimage of f_i under the surjection*

$$\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)} \twoheadrightarrow \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)} \otimes_{\mathcal{O}_{S,s}} \mathbb{C} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}\,.$$

Then the following are equivalent:

(a) $\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}/\widetilde{I}$ is $\mathcal{O}_{S,s}$-flat;

(b) any relation $(r_1, \ldots, r_k)$ among $f_1, \ldots, f_k$ lifts to a relation $(R_1, \ldots, R_k)$ among $F_1, \ldots, F_k$. That is, for each $(r_1, \ldots, r_k)$ satisfying

$$\sum_{i=1}^{k} r_i f_i = 0\,, \quad r_i \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}\,,$$

there exists $(R_1, \ldots, R_k)$ such that

$$\sum_{i=1}^{k} R_i F_i = 0\,, \text{ with } R_i \in \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}$$

and the image of R_i in $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ is r_i;

(c) any free resolution of $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$

$$\ldots \to \mathcal{O}^{p_2}_{\mathbb{C}^n,\mathbf{0}} \to \mathcal{O}^{p_1}_{\mathbb{C}^n,\mathbf{0}} \to \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} \to \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I \to 0$$

lifts to a free resolution of $\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}/\widetilde{I}$,

$$\ldots \to \mathcal{O}^{p_2}_{\mathbb{C}^n\times S,(\mathbf{0},s)} \to \mathcal{O}^{p_1}_{\mathbb{C}^n\times S,(\mathbf{0},s)} \to \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)} \to \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}/\widetilde{I} \to 0\,.$$

That is, the latter sequence tensored with $\otimes_{\mathcal{O}_{S,s}}\mathbb{C}$ yields the first sequence.

Proof. Set $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$, $\widetilde{\mathcal{O}} = \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}$ and consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Ker}(\widetilde{d}_1) & \longrightarrow & \widetilde{\mathcal{O}}^k & \xrightarrow{\widetilde{d}_1} & \widetilde{\mathcal{O}} & \longrightarrow & \widetilde{\mathcal{O}}/\widetilde{I} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Ker}(d_1) & \longrightarrow & \mathcal{O}^k & \xrightarrow{d_1} & \mathcal{O} & \longrightarrow & \mathcal{O}/I & \longrightarrow 0\,,
\end{array}$$

where $\widetilde{d}_1$ (respectively d_1) maps the i-th canonical generator of $\widetilde{\mathcal{O}}^k$ (respectively $\mathcal{O}^k$) to F_i (respectively f_i) and the vertical maps are the canonical surjections.

The set of all relations among $F_1, \ldots, F_k$ (respectively $f_1, \ldots, f_k$) is the submodule $\widetilde{K} := \mathrm{Ker}(\widetilde{d}_1)$ (respectively $K := \mathrm{Ker}(d_1)$) and, hence, condition (b) is equivalent to the canonical map $\widetilde{K} \to K$ being surjective.

By the local criterion for flatness (Theorem B.5.1) applied to $\mathcal{O}_{S,s} \to \widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{O}}/\widetilde{I}$ and since $\widetilde{\mathcal{O}}$ is flat over $\mathcal{O}_{S,s}$ by Corollary 1.88, we get that $\widetilde{\mathcal{O}}/\widetilde{I}$ is $\mathcal{O}_{S,s}$-flat iff $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\widetilde{\mathcal{O}}/\widetilde{I}, \mathbb{C}) = 0$. Moreover, tensoring the exact sequence

$$0 \longrightarrow \widetilde{I} \longrightarrow \widetilde{\mathcal{O}} \longrightarrow \widetilde{\mathcal{O}}/\widetilde{I} \longrightarrow 0$$

with $\otimes_{\mathcal{O}_{S,s}}\mathbb{C}$, we deduce that $\widetilde{\mathcal{O}}/\widetilde{I}$ is $\mathcal{O}_{S,s}$-flat iff $\widetilde{I} \otimes \mathbb{C} \to \mathcal{O}$ is injective or, equivalently, that $\widetilde{I} \otimes \mathbb{C} \to I$ is an isomorphism.

Note that, moreover, the flatness of $\widetilde{\mathcal{O}}/\widetilde{I}$ implies $\mathrm{Tor}_i^{\mathcal{O}_{S,s}}(\widetilde{\mathcal{O}}/\widetilde{I}, \mathbb{C}) = 0$ for $i \geq 1$, hence $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\widetilde{I}, \mathbb{C}) = 0$, and therefore that $\widetilde{I}$ is $\mathcal{O}_{S,s}$-flat.

After these preparations, we can show the equivalence of (a),(b) and (c). To see (a) $\Longleftrightarrow$ (b) consider the diagram with exact rows (the tensor products being over $\mathcal{O}_{S,s}$)

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K} \arrow[r] \arrow[d, two heads] & \widetilde{\mathcal{O}}^k \arrow[r] \arrow[d, two heads] & \widetilde{I} \arrow[r] \arrow[d, two heads] & 0 \\
 & \widetilde{K} \otimes \mathbb{C} \arrow[r] \arrow[d] & \widetilde{\mathcal{O}}^k \otimes \mathbb{C} \arrow[r] \arrow[d, "\cong"] & \widetilde{I} \otimes \mathbb{C} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & K \arrow[r] & \mathcal{O}^k \arrow[r] & I \arrow[r] & 0\,.
\end{tikzcd}$$

If $\widetilde{\mathcal{O}}/\widetilde{I}$ is flat, then by the above arguments $\widetilde{I}$ is flat, which implies that $\widetilde{K} \otimes \mathbb{C} \to \widetilde{\mathcal{O}}^k \otimes \mathbb{C}$ is injective, and $\widetilde{I} \otimes \mathbb{C} \to I$ is bijective. Then $\widetilde{K} \otimes \mathbb{C} \to K$ is bijective and $\widetilde{K} \to K$ surjective, which is equivalent to (b).

If $\widetilde{K} \to K$ is surjective, then $\widetilde{K} \otimes \mathbb{C} \to K$ is surjective and a diagram chase shows that $\widetilde{I} \otimes \mathbb{C} \to I$ is bijective, which is equivalent to (a).

Since (b) is a special case of (c), we have to show that (b) implies (c). As the diagram chase arguments work for any number of generators of I we may assume $k = p_1$. Then $\widetilde{K} \to K$ surjective implies that any surjection $\mathcal{O}^{p_2} \to K$ lifts to $\widetilde{\mathcal{O}}^{p_2} \to \widetilde{K}$. That is, we have a commutative diagram with exact rows

$$\begin{tikzcd}
\widetilde{\mathcal{O}}^{p_2} \arrow[r, "\widetilde{d}_2"] \arrow[d] & \widetilde{\mathcal{O}}^{p_1} \arrow[r] \arrow[d] & \widetilde{I} \arrow[r] \arrow[d, two heads] & 0 \\
\mathcal{O}^{p_2} \arrow[r, "d_2"] & \mathcal{O}^{p_1} \arrow[r] & I \arrow[r] & 0
\end{tikzcd}$$

and $\widetilde{I}$ is flat over $\mathcal{O}_{S,s}$. Then, by the same arguments as above we obtain that $\mathrm{Ker}(\widetilde{d}_2) \to \mathrm{Ker}(d_2)$ is surjective and statement (c) follows by induction. □

Exercises

Exercise 1.8.1. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Show that $\mathcal{F} \oplus \mathcal{G}$ is flat if and only if $\mathcal{F}$ and $\mathcal{G}$ are flat.

Exercise 1.8.2. Let $(X, \mathcal{O}_X)$ be the non-reduced complex space given by $X = \{(x, y, z) \in \mathbb{C}^3 \mid z = 0\}$ and the structure sheaf $\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^3}/\langle xz, yz, z^2\rangle$. Show that $\mathcal{O}_{X,\mathbf{0}}$ is not Cohen-Macaulay. Is the projection map $X \to \mathbb{C}^2$, $(x, y, z) \mapsto (x, y)$, flat ?

Exercise 1.8.3. Prove the following theorem of *Hilbert-Burch* (see [Bur]): Let R be a Noetherian ring and $I = \langle f_1, \dots, f_n\rangle \subset R$ an ideal. Assume that R/I has a free resolution of the form

$$0 \to R^{n-1} \xrightarrow{A} R^n \xrightarrow{(f_1,\dots,f_n)} R \to R/I \to 0\,.$$

Let $A^{(k)}$ denote the $(n-1) \times (n-1)$-submatrix of A obtained by deleting the k-th row and let $d^{(k)} = (-1)^{n-k} \det(A^{(k)})$. Then there exists a unique non-zerodivisor $f \in R$ such that $f_k = f d^{(k)}$ for $k = 1, \dots, n$.

Exercise 1.8.4. Use Exercise 1.8.3 to prove the following statement: Let A be an $n \times (n-1)$-matrix with entries $a_{ij} \in \mathcal{O}_{\mathbb{C}^m,\mathbf{0}}$ and $f_k = \det A^{(k)}$, $k = 1, \dots, n$. Let (S, s) be any complex germ and let $\widetilde{A}$ be a matrix with entries $\widetilde{a}_{ij} \in \mathcal{O}_{\mathbb{C}^m \times S,(\mathbf{0},s)}$ such that $\widetilde{a}_{ij} (\operatorname{mod} \mathfrak{m}_{S,s}) = a_{ij}$. If $\widetilde{f}_k = \det(\widetilde{A}^{(k)})$ and $\widetilde{I} = \langle \widetilde{f}_1, \dots, \widetilde{f}_n\rangle$, then $\mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)}/\widetilde{I}$ is $\mathcal{O}_{S,s}$-flat.

Exercise 1.8.5. Prove that every reduced curve singularity and every normal surface singularity are Cohen-Macaulay.

1.9 Normalization and Non-Normal Locus

We study now flat morphisms whose special fibre is reduced, resp. normal. Our goal is to show that in each case the property of the special fibre carries over to the nearby fibres. The same result holds if the special fibre is regular, as will be shown in the Section 1.10.

Recall that a (germ of) a complex space is called reduced, resp. regular, if (the stalk of) the structure sheaf has this property. Similarly:

Definition 1.92. Let X be a complex space and $x \in X$. Then X is called *normal* at x if the local ring $\mathcal{O}_{X,x}$ is normal[10]. If this is the case, we also say that the complex space germ (X, x) is *normal*. X is called *normal* if it is normal at every $x \in X$.

Given a complex space X, we introduce the *non-reduced locus* of X,

$$\operatorname{NRed}(X) := \bigl\{x \in X \bigm| X \text{ is not reduced at } x\bigr\}\,,$$

and the *non-normal locus* of X,

[10] Recall that a ring A is called *normal* if it is reduced and integrally closed in its total ring of fractions $\operatorname{Quot}(A)$. The integral closure of a reduced ring A in $\operatorname{Quot}(A)$ is called the *normalization* of A, and it is denoted by $\overline{A}$.

$$\mathrm{NNor}(X) := \left\{ x \in X \;\middle|\; X \text{ is not normal at } x \right\},$$

Points in $\mathrm{NRed}(X)$, resp. in $\mathrm{NNor}(X)$, are also called *non-reduced*, resp. *non-normal, points of* X. Accordingly, we refer to points in $X \setminus \mathrm{NRed}(X)$, resp. in $X \setminus \mathrm{NNor}(X)$ as *reduced*, resp. *normal, points of* X.

As regular local rings are normal and as normal rings are reduced, we have inclusions

$$\mathrm{NRed}(X) \subset \mathrm{NNor}(X) \subset \mathrm{Sing}(X)\,.$$

Moreover, if X is normal at x, then it is irreducible at x, that is, $\mathcal{O}_{X,x}$ is an integral domain.

Proposition 1.93. *Let X be a complex space. Then the non-reduced locus* $\mathrm{NRed}(X)$ *and the non-normal locus* $\mathrm{NNor}(X)$ *of X are analytically closed.*

Proof. Since $\mathrm{NRed}(X) = \mathrm{supp}(\mathcal{N}il(\mathcal{O}_X))$, the non-reduced locus is a closed analytic subset of X by Theorem 1.76. An elegant proof for the fact that $\mathrm{NNor}(X)$ is analytic (originally due to Oka), which we recall, was given by Grauert and Remmert [GrR2, § 5]: Let $\mathcal{S}_X = \mathcal{J}(\mathrm{Sing}(X))$ be the full ideal sheaf of the singular locus. As we will show in Corollary 1.111, $\mathrm{Sing}(X)$ is analytic, hence, $\mathcal{S}_X$ is coherent by Cartan's Theorem 1.75. Multiplication by elements of $\mathcal{O}_X$ induces an injection

$$\sigma : \mathcal{O}_X \hookrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{S}_X, \mathcal{S}_X)\,.$$

By Remark 1.93.1 below the non-normal locus of X equals

$$\mathrm{NNor}(X) = \mathrm{supp}\big(\mathcal{C}oker(\sigma)\big)\,, \tag{1.9.1}$$

which is analytic, since $\mathcal{C}oker(\sigma)$ is coherent by the three lemma (A.7, Fact 2). □

Remark 1.93.1. The equality (1.9.1) is based on the *Grauert-Remmert criterion for normality*: Let $\mathcal{J} \subset \mathcal{O}_X$ be a radical ideal such that, locally at a point $x \in X$, $V(\mathcal{J})$ contains the non-normal locus of X, and such that the stalk $\mathcal{J}_x$ contains a non-zerodivisor of $\mathcal{O}_{X,x}$. Then X is normal at x iff $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{J}_x, \mathcal{J}_x) = \mathcal{O}_{X,x}$.

Since $\mathrm{NNor}(X) \subset \mathrm{Sing}(X)$ and $\mathrm{Sing}(X)$ is nowhere dense in X if X is reduced (see Corollary 1.111 below), also the set of non-normal points is nowhere dense in a reduced complex space X.

Definition 1.94. A *normalization* of a reduced complex space X consists of a normal complex space $\overline{X}$ and a morphism $\nu = \nu_X : \overline{X} \to X$ such that the following conditions are satisfied:

(1) ν is finite and surjective.

(2) The preimage of the non-normal locus, $\nu^{-1}(\mathrm{NNor}(X))$, is nowhere dense in $\overline{X}$, and the restriction

$$\nu : \overline{X} \setminus \nu^{-1}(\mathrm{NNor}(X)) \to X \setminus \mathrm{NNor}(X)$$

is biholomorphic.

Remark 1.94.1. It follows from the first Riemann removable singularity Theorem 1.97 that in Definition 1.94 (2) we may replace the non-normal locus $\mathrm{NNor}(X)$ by any nowhere dense analytic set $A \subset X$ (see [GrR2, Ch. 8, §4,2] and the definition of a normalization [GrR2, Ch. 8, §3,3]).

Theorem 1.95 (Normalization). *Let X be a reduced complex space. Then the following holds:*

(1) X admits a normalization.
(2) The normalization $\nu : \overline{X} \to X$ has the following characterizing universal property: every morphism $f : Z \to X$ with Z normal factors through $\nu : \overline{X} \to X$, that is, there exists a morphism $f : Z \to \overline{X}$ fitting in a commutative diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\nu"] \\ & X\,. & \end{tikzcd}$$

Property (2) implies that the normalization $\nu : \overline{X} \to X$ is uniquely determined up to a unique isomorphism. That is, if $\nu' : \overline{X}' \to X$ is another normalization of X, then there exists a unique isomorphism $\overline{X}' \to \overline{X}$ making the following diagram commute

$$\begin{tikzcd} \overline{X}' \arrow[rr, "\cong"] \arrow[dr, "\nu'"'] & & \overline{X} \arrow[dl, "\nu"] \\ & X\,. & \end{tikzcd}$$

Note that $\nu^{-1}(x)$ consists of as many points as the germ (X, x) has irreducible components and that, for each $z \in \nu^{-1}(x)$, the germ $(\overline{X}, z)$ is irreducible and is mapped by ν homeomorphically onto a unique irreducible component of (X, x).

Proof. For the existence of a normalization, consider the sheaf $\widetilde{\mathcal{O}}_X$ of weakly holomorphic functions on X. Here, a *weakly holomorphic function on X* is a holomorphic function $f : X \setminus \mathrm{Sing}(X) \to \mathbb{C}$ which is locally bounded on X (note that f is not defined on $\mathrm{Sing}(X)$). One can show that $\widetilde{\mathcal{O}}_X$ is a coherent $\mathcal{O}_X$-sheaf and that, for each $x \in X$, the stalk $\widetilde{\mathcal{O}}_{X,x}$ is the normalization of $\mathcal{O}_{X,x}$. If $(X_1, x), \ldots, (X_s, x)$ denote the irreducible components of the (reduced) germ (X, x), we thus get $\widetilde{\mathcal{O}}_{X,x} \cong \prod_{i=1}^{s} \widetilde{\mathcal{O}}_{X_i,x}$. Now, we may construct the normalization as follows: let $(\overline{X}, x_i)$ be the complex space germ defined by $\widetilde{\mathcal{O}}_{X_i,x}$ (see Remark 1.47.1 (2)), and let $\nu : (\overline{X}, x_i) \to (X, x)$ be the map induced by $\mathcal{O}_{X_i,x} \hookrightarrow \overline{\mathcal{O}_{X_i,x}} \cong \widetilde{\mathcal{O}}_{X_i,x}$, $i = 1, \dots, s$. Then, the complex space $\overline{X}$ and the morphism $\nu : \overline{X} \to X$ are obtained by glueing.

For details, see [GrR2, Ch. 8, §3, 3] or [Fis, Ch. 2, Appendix]. In Section 3.3, we give a different proof for the existence of a normalization in the special case of plane curve singularities. For (2), see the proofs in [GrR2, Ch. 8, §4, 2] or [Fis, Ch. 2, Appendix]. □

Remark 1.95.1. The embedding dimension $\operatorname{edim}(X, x)$ may behave in an unpredictable way under normalization. The normalization $(\overline{C}, \overline{x})$ of a curve singularity (C, x) is smooth (Theorem 1.96 (1)). In particular, the embedding dimension of the normalization, $\operatorname{edim}(\overline{C}, \overline{x}) = \dim(\overline{C}, \overline{x}) = 1$, is not related to the embedding dimension of (C, x). Moreover, by [GrR2, Ch. 8, §3], every normal complex germ of dimension d is the normalization of a hypersurface singularity in $(\mathbb{C}^{d+1}, \mathbf{0})$. Hence, for a fixed embedding dimension of (X, x), the embedding dimension of the normalization can become arbitrarily large. An important class of examples are *cyclic quotient singularities* $(X_n, x) = (\mathbb{C}^2, \mathbf{0})/C_n$, where the cyclic group C_n of n-th roots of unity acts on $(\mathbb{C}^2, \mathbf{0})$ via $\rho \cdot (z_1, z_2) = (\rho^n z_1, \rho^n z_2)$. It is known that (X_n, x) is a normal two-dimensional singularity with embedding dimension $n + 1$ (see [GrR, III, 3]).

In the following three theorems, we collect the most important properties of normal complex spaces:

Theorem 1.96. *Let X be a reduced complex space. Then the following holds:*

(1) If X is normal, then $\dim(\operatorname{Sing}(X)) \leq \dim(X) - 2$. *If X is Cohen-Macaulay, the inverse implication is also true.*

(2) The following are equivalent:

(a) X is normal.

(b) For every open set $U \subset X$, the restriction map

$$\Gamma(U, \mathcal{O}_X) \to \Gamma(U \setminus \operatorname{Sing}(X), \mathcal{O}_X)$$

is bijective.

(3) Let $f : X \to S$ be a morphism of reduced complex spaces such that $f^{-1}(\operatorname{NNor}(S))$ *is nowhere dense in X. Then there is a unique lifting of f to the normalization, that is, there is a commutative diagram*

$$\begin{tikzcd} \overline{X} \arrow[r, "\overline{f}"] \arrow[d, "\nu_X"'] & \overline{S} \arrow[d, "\nu_S"] \\ X \arrow[r, "f"'] & S\,. \end{tikzcd}$$

with $\overline{f} : \overline{X} \to \overline{S}$ being uniquely determined.

Proof. See [Fis, Ch. 2, Appendix]. □

Theorem 1.97 (First Riemann removable singularity theorem). *Let X be a reduced complex space. Then the following are equivalent:*

(1) X is normal.

(2) For each open set $U \subset X$, and each closed analytic subset $A \subset U$ which is nowhere dense in U, each holomorphic map $f : U \setminus A \to \mathbb{C}$ which is locally bounded on U has a unique holomorphic extension $\widetilde{f} : U \to \mathbb{C}$.

Proof. The implication (2) $\Rightarrow$ (1) follows from Corollary 1.111 and the construction of the normalization in the proof of Theorem 1.95. For (1) $\Rightarrow$ (2), we refer to [GrR2, Ch. 7]. □

Theorem 1.98 (Second Riemann removable singularity theorem). *Let X be a normal complex space, $U \subset X$ an open subset, and $A \subset U$ a closed analytic subset which is locally of codimension at least 2, that is, which satifies $\dim(A, x) \leq \dim(U, x) - 2$ at every $x \in A$. Then the restriction map $\Gamma(U, \mathcal{O}_X) \to \Gamma(U \setminus A, \mathcal{O}_X)$ is bijective.*

Proof. The statement basically follows from Theorem 1.97 and Theorem 1.96 (1), (2). For details, we refer to [Fis, Ch. 2, Appendix]. □

Note that Theorem 1.97 is, indeed, a generalization of the classical removable singularity theorem due to Riemann: A one-dimensional normal complex space is smooth and, for each open subset U of $\mathbb{C}^n$, each holomorphic function $f : U \setminus \{x\} \to \mathbb{C}$ which is bounded near x extends uniquely to a holomorphic function $\widetilde{f} : U \to \mathbb{C}$.

We turn now to morphisms having reduced, respectively normal, fibres.

Definition 1.99. Let $f : X \to S$ be a morphism of complex spaces. We call f *reduced at* $x \in X$ (resp. *normal at* $x \in X$) if f is flat at x and the fibre $f^{-1}(f(x))$ is reduced (resp. normal) at x. In this case, we also say that x is a *reduced* (resp. *normal*) *point of* f, and we call the induced morphism of germs $f : (X, x) \to (S, f(x))$ *reduced* (resp. *normal*).

We define f to be *reduced* (resp. *normal*) if it is reduced (resp. normal) at every $x \in X$.

Theorem 1.100 (Non-reduced and non-normal locus are closed). *Let $f : X \to S$ be a morphism of complex spaces. Then the sets*

$$\begin{aligned} \mathrm{NRed}(f) &:= \{x \in X \mid f \text{ is not reduced at } x\}, \\ \mathrm{NNor}(f) &:= \{x \in X \mid f \text{ is not normal at } x\}, \end{aligned}$$

are analytically closed in X.

Proof. We refer to [Fis, Prop. 3.22]. □

Note that Proposition 1.93 is a special case of Theorem 1.100 (for $S = \{\mathrm{pt}\}$).

Remark 1.100.1. If $f:(X,x)\to(S,s)$ is a flat morphism of complex space germs such that the special fibre $(f^{-1}(s),x)$ is reduced (resp. normal), then there is a representative $f:X\to S$ such that f is flat at every point of X, $f(X)=S$ and, for all $s'\in S$, the fibre $f^{-1}(s')$ is reduced (resp. normal).

Indeed, since the flat locus of a morphism $f:X\to S$ is analytically open by Frisch's Theorem 1.83, and since flat morphisms are open by Theorem 1.84, we may assume that f is everywhere flat and that $f:X\to S$ is surjective. After schrinking S and X (if necessary), the statement follows from Theorem 1.100.

Theorem 1.101. *Let $f:X\to S$ be a flat morphism of complex spaces, and let $x\in X$. Then the following holds:*

(1) If X is reduced (resp. normal) at x, then S is reduced (resp. normal) at $f(x)$.
(2) If the fibre $f^{-1}(f(x))$ is reduced (resp. normal) at x, and if S is reduced (resp. normal) at $f(x)$, then X is reduced (resp. normal) at x, and there is a neighbourhood $U\subset X$ of x such that all fibres $f^{-1}(f(x'))$, $x'\in U$, are reduced (resp. normal) at x'.

Proof. The statement follows immediately from Theorem B.8.19 and Theorem B.8.20. □

Theorem 1.102. *Let $f:X\to S$ be a morphism of reduced complex spaces. If f is a homeomorphism and S is normal, then f is an isomorphism.*

Proof. The proof is left as Exercise 1.9.3. □

Theorem 1.103 (Sard). *Let $f:X\to S$ be a morphism of complex manifolds with* $\mathrm{Sing}(f)\subsetneq X$. *Then the set of critical values, $f\big(\mathrm{Sing}(f)\big)$, has Lebesgue measure zero in S.*

For a proof, see [Nar, 1.4.6].

Exercises

Exercise 1.9.1. Let X be a complex space. Show that one can effectively compute an ideal sheaf defining the non-normal locus $\mathrm{NNor}(X)$ of X by using the Grauert-Remmert criterion (see also [GrP, Sect. 3.6]).

Exercise 1.9.2. Prove the following *theorem of Clements* ([Cle], [Nar1, Thm. 5.5]): Let $U\subset\mathbb{C}^n$ be open, and let $f:U\to\mathbb{C}^n$ be an injective holomorphic map. Then the image $\varphi(U)$ is open in $\mathbb{C}^n$ and

$$\varphi:U\to\varphi(U)$$

is an isomorphism of complex spaces.

HINT. Use the implicit function Theorem 1.18, the finite mapping theorem (Corollary 1.68) and Sard's Theorem 1.103.

Exercise 1.9.3. Prove Theorem 1.102 by using the theorem of Clements (Exercise 1.9.2) and the first Riemann removable singularity Theorem 1.97.

1.10 Singular Locus and Differential Forms

In this section we characterize singular points of complex spaces and of morphisms of complex spaces. One of the aims is to show that these sets are analytically closed.

Recall that x is a *regular* (or *smooth*) *point* of X, iff the local ring $\mathcal{O}_{X,x}$ is regular (cf. Definition 1.40); x is a *singular point* iff it is not regular. The set of singular points of X is referred to as the *singular locus of* X, denoted by $\mathrm{Sing}(X)$.

If X is *pure dimensional*, that is, if the dimension $\dim(X,x)$ is independent of $x \in X$, then we can easily give a local description of $\mathrm{Sing}(X)$. Since any isomorphism $X \to Y$ of complex spaces maps $\mathrm{Sing}(X)$ isomorphic to $\mathrm{Sing}(Y)$ (since $\dim(X,x)$ and $\mathrm{edim}(X,x)$ are preserved under isomorphisms), we may assume that X is a complex model space. In this situation we have

Proposition 1.104. *Let X be a pure n-dimensional complex subspace of $\mathbb{C}^m$ with ideal sheaf $\mathcal{I}$. If $x \in X$ and $\mathcal{I}_x = \langle f_1, \dots, f_k\rangle \cdot \mathcal{O}_{\mathbb{C}^m,x}$ with $f_1, \dots, f_k$ holomorphic functions in a neighbourhood U of x then*

$$\mathrm{Sing}(X) \cap U \;=\; \Bigl\{ \boldsymbol{y} \in X \cap U \;\Big|\; \mathrm{rank}\Bigl(\tfrac{\partial f_i}{\partial x_j}(\boldsymbol{y})\Bigr) < m-n \Bigr\}\,.$$

In particular, there is a canonical ideal sheaf $\mathcal{J}_{\mathrm{Sing}(X)}$ such that $\mathcal{J}_{\mathrm{Sing}(X)}|_U$ is generated by $f_1, \dots, f_k$ and all $(m-n)$-minors of the Jacobian matrix $\bigl(\frac{\partial f_i}{\partial x_j}\bigr)$ with $V(\mathcal{J}_{\mathrm{Sing}(X)}) = \mathrm{Sing}(X)$.

Proof. By Lemma 1.22, $\mathrm{rank}\bigl(\frac{\partial f_i}{\partial x_j}(\boldsymbol{y})\bigr) = \mathrm{jrk}(\mathcal{I}_{\boldsymbol{y}}) = m - \mathrm{edim}(X,\boldsymbol{y})$. Hence, $\boldsymbol{y} \in \mathrm{Sing}(X) \cap U$ iff $\mathrm{rank}\bigl(\frac{\partial f_i}{\partial x_j}(\boldsymbol{y})\bigr) < m - \dim(X,\boldsymbol{y})$. The result follows since X is purely n-dimensional. $\square$

If X has several irreducible components, $X = X_1 \cup \ldots \cup X_r$ then

$$\mathrm{Sing}(X) \;=\; \bigcup_{i=1}^{r} \mathrm{Sing}(X_i) \,\cup\, \bigcup_{i<j} (X_i \cap X_j)\,,$$

as will be shown in the exercises. As X_i is pure dimensional, $\mathrm{Sing}(X_i)$ is analytic in X, by Proposition 1.104. The intersection of two irreducible components is analytic, too. Hence, $\mathrm{Sing}(X)$ is analytic. We can use locally a primary decomposition of $\langle 0\rangle \subset \mathcal{O}_{X,x}$ to define an ideal for $\mathrm{Sing}(X)$ locally at x. But, since a primary decomposition is not unique it is not clear how to glue these locally defined sheaves to get a well-defined global ideal sheaf for $\mathrm{Sing}(X)$.

In the following, we shall give a different proof of the analyticity of $\mathrm{Sing}(X)$, which provides $\mathrm{Sing}(X)$ with a canonical structure, even if X is not pure dimensional. For this, we use differential forms.

Before we introduce differential forms, let us first recall the notion of derivations.

Definition 1.105. Let A be a B-algebra and M an A-module. Then a *B-derivation with values in M* is a B-linear map $\delta : A \to M$ satisfying the product rule, also called the *Leibniz rule*,

$$\delta(fg) = \delta(f)g + f\delta(g)\,, \quad f, g \in A\,.$$

The set

$$\mathrm{Der}_B(A, M) := \bigl\{\delta : A \to M \bigm| \delta \text{ is a } B\text{-derivation}\bigr\} \subset \mathrm{Hom}_B(A, M)$$

is via $(a \cdot \delta)(f) := a \cdot \delta(f)$ an A-module, the *module of B-derivations of A with values in M*.

We consider first the case $B = \mathbb{C}$. It is easy to see that for $A = \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$ the partial derivatives $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, are a basis of the free $\mathbb{C}\{\boldsymbol{x}\}$-module $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}\{\boldsymbol{x}\}, \mathbb{C}\{\boldsymbol{x}\})$.

For each local ring $(A, \mathfrak{m})$ and each A-derivation $\delta : A \to M$ we have $\delta(\mathfrak{m}^k) \subset \mathfrak{m}^{k-1}M$ for all $k > 0$. Note also that, by the Leibniz rule and Krull's intersection theorem, any derivation δ is already uniquely determined by the values $\delta(x_i)$ for $x_1, \dots, x_n$ a set of generators for $\mathfrak{m}$.

In particular, for $A = \mathbb{C}\{x_1, \dots, x_n\}$, each $\delta \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}\{\boldsymbol{x}\}, M)$ has a unique expression

$$\delta = \sum_{i=1}^{n} \delta(x_i) \cdot \frac{\partial}{\partial x_i}\,. \tag{1.10.1}$$

Now, let us define differential forms.

Theorem 1.106. *Let A be an analytic $\mathbb{C}$-algebra.*

(1) There exists a pair (Ω^1_A, d_A) consisting of a finitely generated A-module Ω^1_A and a derivation $d_A : A \to \Omega^1_A$ such that for each finitely generated A-module M the A-linear morphism

$$\theta_M : \mathrm{Hom}_A(\Omega^1_A, M) \longrightarrow \mathrm{Der}_{\mathbb{C}}(A, M)\,, \quad \varphi \longmapsto \varphi \circ d_A\,,$$

is an isomorphism of A-modules.

(2) The pair (Ω^1_A, d_A) is uniquely determined up to unique isomorphism.

(3) If $A = \mathbb{C}\{x_1, \dots, x_n\}$ then Ω^1_A is free of rank n with basis $dx_1, \dots, dx_n$ and $d = d_A : A \to \Omega^1_A$ is given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i\,.$$

(4) If $A = \mathbb{C}\{x_1, \dots, x_n\}/I$ then

$$\Omega^1_A = \Omega^1_{\mathbb{C}\{\boldsymbol{x}\}} \big/ \bigl(I \cdot \Omega^1_{\mathbb{C}\{\boldsymbol{x}\}} + \mathbb{C}\{\boldsymbol{x}\} \cdot dI\bigr)$$

with $d_A : A \to \Omega^1_A$ induced by $d : \mathbb{C}\{\boldsymbol{x}\} \to \Omega^1_{\mathbb{C}\{\boldsymbol{x}\}}$. In particular, Ω^1_A is generated, as A-module, by the classes of $dx_1, \dots, dx_n$.

The pair (Ω_A^1, d_A) is called the *module of (Kähler) differentials.* We usually write d instead of d_A.

Proof. Once we have shown the defining property of the modules constructed in (3) and (4), (1) is obviously satisfied. Moreover, (2) follows from (1), by the usual abstract argument.

(3) $\Omega_A^1 = Adx_1 \oplus \ldots \oplus Adx_n$ is finitely generated and $d : A \to \Omega_A^1$ is a derivation. We have to show that $\theta = \theta_M$ is bijective. If $\theta(\varphi) = 0$ then

$$\theta(\varphi)(x_i) = \varphi(dx_i) = 0, \qquad i = 1, \ldots, n,$$

hence $\varphi = 0$, and θ is injective.

Given a derivation $\delta \in \mathrm{Der}_{\mathbb{C}}(A, M)$ define $\varphi \in \mathrm{Hom}_A(\Omega_A^1, M)$ by $\varphi(dx_i) = \delta(x_i)$. Then

$$\theta(\varphi)(f) = \varphi(df) = \varphi\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \delta(x_i) = \delta(f),$$

by (1.10.1). That is, θ is surjective, too.

(4) One checks directly that $d : \mathbb{C}\{\boldsymbol{x}\}/I \to \Omega^1_{\mathbb{C}\{\boldsymbol{x}\}}/(I \cdot \Omega^1_{\mathbb{C}\{\boldsymbol{x}\}} + \mathbb{C}\{\boldsymbol{x}\} \cdot dI)$ is well-defined (by the Leibniz rule), and a derivation. If M is a finite A-module then it is also a finite $\mathbb{C}\{\boldsymbol{x}\}$-module. We set $N = \mathbb{C}\{\boldsymbol{x}\}dI + I\Omega$, $\Omega = \Omega^1_{\mathbb{C}\{\boldsymbol{x}\}}$. Induced by the exact sequences $0 \to N/I\Omega \to \Omega/I\Omega \to \Omega_A^1 \to 0$ and $0 \to I \to \mathbb{C}\{\boldsymbol{x}\} \to A \to 0$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathbb{C}\{\boldsymbol{x}\}}(N/I\Omega, M) & \longleftarrow & \mathrm{Hom}_{\mathbb{C}\{\boldsymbol{x}\}}(\Omega/I\Omega, M) & \longleftarrow & \mathrm{Hom}_{\mathbb{C}\{\boldsymbol{x}\}}(\Omega_A^1, M) & \longleftarrow & 0 \\
\downarrow & & \cong\downarrow & & \downarrow & & \\
\mathrm{Der}_{\mathbb{C}}(I, M) & \longleftarrow & \mathrm{Der}_{\mathbb{C}}(\mathbb{C}\{\boldsymbol{x}\}, M) & \longleftarrow & \mathrm{Der}_{\mathbb{C}}(A, M) & \longleftarrow & 0
\end{array}$$

where the vertical arrows are given by $\varphi \mapsto \varphi \circ d$. The middle arrow is bijective by (3) and the left one is injective by a direct check. It follows that the right-hand one is bijective, too. $\square$

Lemma 1.107. *For each analytic $\mathbb{C}$-algebra A there are canonical isomorphisms*

$$\Omega_A^1/\mathfrak{m}_A\Omega_A^1 \xrightarrow{\cong} \mathfrak{m}_A/\mathfrak{m}_A^2, \qquad \mathrm{Der}_{\mathbb{C}}(A, \mathbb{C}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_A/\mathfrak{m}_A^2, \mathbb{C}).$$

In particular, $\mathrm{edim}(A) = \mathrm{mng}(\Omega_A^1) = \dim_{\mathbb{C}}\big(\mathrm{Der}_{\mathbb{C}}(A, \mathbb{C})\big)$.

Proof. Consider the *point derivation* (putting $\mathfrak{m} = \mathfrak{m}_A$)

$$\delta_0 : A \longrightarrow \mathfrak{m}/\mathfrak{m}^2, \quad f \longmapsto (f - f(0))(\mathrm{mod}\ \mathfrak{m}^2),$$

which is an element of $\mathrm{Der}_{\mathbb{C}}(A, \mathfrak{m}/\mathfrak{m}^2)$.

Since $\mathrm{Hom}_A(\Omega^1_A, \mathfrak{m}/\mathfrak{m}^2) \to \mathrm{Der}_{\mathbb{C}}(A, \mathfrak{m}/\mathfrak{m}^2)$ is bijective, there is a unique homomorphism $\varphi_0 : \Omega^1_A \to \mathfrak{m}/\mathfrak{m}^2$ such that $\varphi_0 \circ d = \delta_0$. Since δ_0 is surjective, so is φ_0. From $\varphi_0(\mathfrak{m}\Omega^1_A) \subset \mathfrak{m}(\mathfrak{m}/\mathfrak{m}^2) = 0$ we get that φ_0 induces a surjective morphism $\varphi_0 : \Omega^1_A/\mathfrak{m}\Omega^1_A \to \mathfrak{m}/\mathfrak{m}^2$. On the other hand, Theorem 1.106 (4) implies $\mathrm{mng}(\Omega^1_A) = \dim_{\mathbb{C}}(\Omega^1_A/\mathfrak{m}\Omega^1_A) \leq \dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2)$. Hence, φ_0 is an isomorphism.

Dualizing this isomorphism and using $\mathrm{Hom}_{\mathbb{C}}(M/\mathfrak{m}M, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ and the universal property of Ω^1_A, we get the second isomorphism. □

Proposition 1.108. *For each morphism $\varphi : A \to B$ of analytic algebras there is a unique A-module homomorphism $d\varphi : \Omega^1_A \to \Omega^1_B$ making the following diagram commutative*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle d_A}\downarrow & & \downarrow{\scriptstyle d_B} \\ \Omega^1_A & \xrightarrow{d\varphi} & \Omega^1_B \end{array}$$

$d\varphi$ is called the *differential of* φ. It satisfies the *chain rule* $d(\psi \circ \varphi) = d\psi \circ d\varphi$.

Proof. Since $\Omega^1_A = A \cdot d_A A$, $d\varphi$ must satisfy

$$d\varphi\Big(\sum_i g_i \cdot d_A(f_i)\Big) = \sum_i g_i \cdot d\varphi(d_A(f_i)) = \sum_i g_i \cdot d_B(\varphi(f_i)),$$

and, hence, $d\varphi$ is uniquely defined if it exists. For the existence, let $A = \mathbb{C}\{\boldsymbol{x}\}/I$, $B = \mathbb{C}\{\boldsymbol{y}\}/J$ and $\widetilde{\varphi} : \mathbb{C}\{\boldsymbol{x}\} \to \mathbb{C}\{\boldsymbol{y}\}$ a lifting of φ (Lemma 1.14). We define

$$d\widetilde{\varphi} : \Omega^1_{\mathbb{C}\{\boldsymbol{x}\}} \to \Omega^1_{\mathbb{C}\{\boldsymbol{y}\}}, \quad d\widetilde{\varphi}(dx_i) := d_{\mathbb{C}\{\boldsymbol{y}\}}(\widetilde{\varphi}(x_i)),$$

which is well-defined, since $\Omega^1_{\mathbb{C}\{\boldsymbol{x}\}}$ is free and generated by the dx_i. It is now straightforward to check that $d\widetilde{\varphi}$ induces, via the surjections of Theorem 1.106, an A-linear map $\Omega^1_A \to \Omega^1_B$. □

Now, let X be a complex space and $x \in X$. Moreover, let $U \subset X$ be an open neighbourhood of x which is isomorphic to a local model space Y defined by a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_D$. Here, D is an open subset of $\mathbb{C}^n$, and we may assume that $\mathcal{I}$ is generated $f_1, \dots, f_k \in \Gamma(D, \mathcal{O}_D)$. The sheaf Ω^1_D is defined to be the free sheaf $\mathcal{O}_D dx_1 \oplus \ldots \oplus \mathcal{O}_D dx_n$ and the derivation $d : \mathcal{O}_D \to \Omega^1_D$ is defined by $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

Definition 1.109. Let $\mathcal{O}_Y = \mathcal{O}_D/\mathcal{I}$ and $\mathcal{I} = \langle f_1, \dots, f_k \rangle \mathcal{O}_D$. We define

$$\Omega^1_Y := \Omega^1_D / \big(\mathcal{I}\Omega^1_D + \mathcal{O}_D d\mathcal{I}\big)\big|_Y,$$

where $\mathcal{O}_D d\mathcal{I}$ is the subsheaf of Ω^1_D generated by $df_1, \dots, df_k$, and $\mathcal{I}\Omega^1_D$ is the subsheaf of Ω^1_D generated by $f_j dx_i$, $i = 1, \dots, n$, $j = 1, \dots, k$. The induced derivation is denoted by $d_Y : \mathcal{O}_Y \to \Omega^1_Y$.

Finally, if $\varphi : U \to Y$ is an isomorphism to the local model space Y, we define $\Omega^1_U := \varphi^*\Omega^1_Y$ where $\varphi^*\Omega^1_Y$ is the analytic preimage sheaf (A.6). Theorem 1.106 (2) implies that Ω^1_U is, up to a unique isomorphism, independent of the choice of φ.

It follows that we can glue the locally defined sheaves Ω^1_U to get a unique sheaf Ω^1_X on X, the *sheaf of holomorphic (Kähler) differentials* or *holomorphic 1-forms* on X, and a unique derivation $d_X : \mathcal{O}_X \to \Omega^1_X$ (A.2). Ω^1_X is a coherent $\mathcal{O}_X$-module (A.7), and it satisfies

$$\Omega^1_{X,x} = \Omega^1_{\mathcal{O}_{X,x}} \text{ for each } x \in X.$$

It is now easy to prove the important regularity criterion for complex spaces.

Theorem 1.110 (Regularity criterion for complex space germs). *Let X be a complex space and $x \in X$. Then X is regular at x iff $\Omega^1_{X,x}$ is a free $\mathcal{O}_{X,x}$-module (of rank* $\dim(X,x)$*).*

Proof. If X is regular at x, then $\mathcal{O}_{X,x} \cong \mathbb{C}\{x_1,\dots,x_n\}$, and $\Omega^1_{X,x}$ is free of rank n (Theorem 1.106).

On the other hand, if $\Omega^1_{X,x}$ is free of rank n then $\mathcal{O}_{X,x} \cong \mathbb{C}\{x_1,\dots,x_n\}/I$, where $n = \operatorname{edim}(X,x)$ (Lemma 1.107). Since $dx_1,\dots,dx_n \in \Omega^1_{\mathbb{C}^n,0}$ generate $\Omega^1_{\mathbb{C}^n,0}$, they induce a basis of $\Omega^1_{X,x}$. By definition, each $f \in I$ satisfies $[df] = 0$, where $[\ ]$ denotes the image in $\Omega^1_{X,x}$. Since $df = \sum_i \frac{\partial f}{\partial x_i}\,dx_i$, and since the images of $dx_1,\dots,dx_n$ are linearly independent in $\Omega^1_{X,x}$, we get $\left[\frac{\partial f}{\partial x_i}\right] = 0$, that is, $\frac{\partial f}{\partial x_i} \in I$ for $i = 1,\dots,n$. It follows that any partial derivative of f of any order is in I, hence vanishes at x. As a consequence, the Taylor series of f vanishes, that is, $f = 0$. Therefore, $I = 0$ and $\mathcal{O}_{X,x} \cong \mathbb{C}\{x_1,\dots,x_n\}$. □

Corollary 1.111 (Singular locus is closed). *Let X be a complex space and Ω^1_X the sheaf of Kähler differentials on X. Then*

$$\operatorname{Sing}(X) = X \setminus \operatorname{Free}(\Omega^1_X)$$

is a closed analytic set in X. Moreover, if X is reduced, the set of regular points of X, $X \setminus \operatorname{Sing}(X)$, is open and dense in X.

Proof. This is a consequence of Theorems 1.110 and 1.80. □

Remark 1.111.1. Note that the proof of Theorem 1.80 provides the topological space $\operatorname{Sing}(X)$ with a *natural structure* given by the 0-th Fitting ideal (see page 48) of $\mathcal{E}xt^1_{\mathcal{O}_X}\big(\Omega^1_X, \mathcal{S}yz_1(\Omega^1_X)\big)$ (see Exercises 1.7.7, 1.7.8).

The previous considerations can be generalized to morphisms between complex spaces:

Definition 1.112. A morphism $f : (X,x) \to (S,s)$ of complex germs is called *regular*, if there is a commutative diagram

$$\begin{tikzcd}
(X,x) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "f"'] & & (S,s)\times(T,t) \arrow[dl, "p"] \\
& (S,s) &
\end{tikzcd}$$

such that φ is an isomorphism, p the projection on the first factor, and (T,t) a regular germ. A morphism of complex spaces is called *regular at* $x \in X$, or x is called a *regular point of* f, if the induced morphism of germs, $(X,x) \to (S, f(x))$, is regular. f is called *regular* if this holds at every $x \in X$. x is called a *singular point of* f if it is not regular. Instead of regular, we say also *smooth* or *non-singular*.

Note that this definition coincides with Definition 1.41. Moreover, the complex space germ (X,x) is regular iff $f : X \to \{\mathrm{pt}\}$ is a regular morphism.

The regularity criterion for complex space germs (Theorem 1.110) generalizes to morphisms $f : X \to S$. For this, we need the concept of relative differential $\Omega^1_{X/S}$. These have the property that the analytic restriction to any fibre $f^{-1}(s)$, $s \in S$, coincides with $\Omega^1_{f^{-1}(s)}$.

We define relative differentials first for morphisms $\varphi : A \to B$ of analytic algebras.

Definition 1.113. Let $\varphi : A \to B$ be a morphism of analytic $\mathbb{C}$-algebras. Define

$$\alpha : \Omega^1_A \otimes_A B \to \Omega^1_B\,, \quad \omega \otimes b \mapsto b \cdot d\varphi(\omega)\,,$$

with $d\varphi$ as in Proposition 1.108, and call the B-module

$$\Omega^1_{B/A} := \mathrm{Coker}(\alpha) = \Omega^1_B / B \cdot d\varphi(\Omega^1_A),$$

together with the A-derivation $d_{B/A} : B \to \Omega^1_{B/A}$, $b \mapsto [d_B(b)]$, the *module of relative (Kähler) differentials* of B over A. We write d instead of $d_{B/A}$ if there is no ambiguity.

If $x_1, \dots x_n \in \mathfrak{m}_B$ generate $\mathfrak{m}_B$ as B-module, then Theorem 1.106 (4) implies that $\Omega^1_{B/A}$ is generated by the differentials $d_{B/A}(x_1), \dots, d_{B/A}(x_n)$. Moreover, the module of relative differentials $(\Omega^1_{B/A}, d_{B/A})$ satisfies the following *universal property*: For each finitely generated B-module M, the B-linear morphism

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \longrightarrow \mathrm{Der}_A(B, M)\,, \quad \varphi \longmapsto \varphi \circ d_{B/A}\,,$$

is an isomorphism of B-modules. Indeed, this is also an immediate consequence of Theorem 1.106.

It follows that $(\Omega^1_{B/A}, d_{B/A})$ is uniquely determined up to unique isomorphism. If $A = \mathbb{C}\{t_1, \dots, t_k\}/J$, then, by definition,

$$\Omega^1_{B/A} = \Omega^1_B / B\langle d\varphi(t_1), \dots, d\varphi(t_k)\rangle\,.$$

Proposition 1.114. *Let A be an analytic $\mathbb{C}$-algebra, and let B, C be analytic A-algebras. Then the following holds:*

(1) If $B = A\{x_1, \dots, x_n\}$ is a free power series algebra over A, then $\Omega^1_{B/A}$ is a free B-module of rank n, generated by $dx_1, \dots, dx_n$.
(2) $\Omega^1_{B/A} \otimes_A A/\mathfrak{m}_A \cong \Omega^1_{B/\mathfrak{m}_A B}$.
(3) If $\varphi : B \to C$ is an A-morphism, then there is an exact sequence of A-modules

$$\Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \xrightarrow{\beta} \Omega^1_{C/B} \to 0\,, \tag{1.10.2}$$

where $\alpha\big(d_{B/A}(b) \otimes c\big) = c \cdot d_{C/A}\big(\varphi(b)\big)$ and $\beta\big(d_{C/A}(c)\big) = d_{C/B}(c)$ for $b \in B$ and $c \in C$. If C is a free power series algebra over B then α is surjective and β is split surjective*, that is, admits a section $\Omega^1_{C/B} \to \Omega^1_{C/A}$.*
(4) If $\varphi : B \to C$ is a surjective A-morphism, then $\Omega^1_{C/B} = 0$, and we have an exact sequence of B-modules

$$I/I^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \to 0\,, \tag{1.10.3}$$

where $I := \mathrm{Ker}(\varphi)$, α is as in (3), and $\delta([b]) = d_{B/A}(b) \otimes 1$ for $b \in I$ and $[b]$ the class of b in I/I^2. If C is a free power series algebra over A, then δ is injective, and α is split surjective.

The proof of this proposition is straightforward and left as Exercise 1.10.1.

The exact sequences (1.10.2), (1.10.3) are called the *first,* respectively *second, fundamental exact sequence for relative differentials.*

For a morphism $f : X \to S$ of complex spaces, we define the *sheaf $\Omega^1_{X/S}$ of relative holomorphic (Kähler) differential forms* of X over S by the exact sequence

$$f^*\Omega^1_S \xrightarrow{\alpha} \Omega^1_X \to \Omega^1_{X/S} \to 0\,,$$

where α is the morphism of sheaves $f^{-1}\Omega^1_S \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X \to \Omega^1_X$ defined by $\alpha(\omega \otimes g) = g \cdot d\widehat{f}(\omega)$ on local sections. Note that, by Proposition 1.108, the morphism of sheaves $\widehat{f} : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ belonging to f induces a unique morphism of sheaves $d\widehat{f} : f^{-1}\Omega^1_S \to \Omega^1_X$ which commutes with the differentials $f^{-1}d_S$ and d_X. Then we have

$$\Omega^1_{X/S,x} = \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}$$

for all $x \in X$. Moreover,

$$\Omega^1_{X/S} = \Omega^1_D \big/ \big(\mathcal{I}\Omega^1_D + \mathcal{O}_D d\mathcal{I} + \mathcal{O}_D\langle df_1, \dots, df_k\rangle\big)\big|_X\,,$$

where X is the complex model space defined by the coherent ideal $\mathcal{I} \subset \mathcal{O}_D$ with $D \subset \mathbb{C}^n$ open, $S \subset \mathbb{C}^k$, and f is induced by $f = (f_1, \dots, f_k) : D \to \mathbb{C}^k$. From Proposition 1.114 (2), we get that the analytic restriction of $\Omega^1_{X/S}$ to a fibre $f^{-1}(s)$ is $\Omega^1_{f^{-1}(s)}$,

$$\left(\Omega^1_{X/S} \otimes_{\mathcal{O}_{S,s}} (\mathcal{O}_{S,s}/\mathfrak{m}_{S,s})\right)\Big|_{f^{-1}(s)} \cong \Omega^1_{f^{-1}(s)} . \tag{1.10.4}$$

We leave it as an exercise to formulate the other statements of Proposition 1.114 for morphisms of complex spaces.

Now, we are in the position to prove the following regularity criterion for morphisms:

Theorem 1.115 (Regularity criterion for morphisms). *Let $f : X \to S$ be a morphism of complex spaces, and let $x \in X$. Then the following are equivalent:*

(a) f is regular at x.
(b) $\mathcal{O}_{X,x}$ is a free power series algebra over $\mathcal{O}_{S,f(x)}$.
(c) f is flat at x and the fibre $(F,x) := \big(f^{-1}(f(x)),x\big)$ is regular.
(d) f is flat at x and $\Omega^1_{X/S,x}$ is a free $\mathcal{O}_{X,x}$-module (of rank $\dim(F,x)$*).*

Proof. Let $s = f(x)$ and $(F,x) = \big(f^{-1}(s),x\big)$.

The equivalence (a) $\Leftrightarrow$ (b) follows from the definition of the Cartesian product: $\mathcal{O}_{(S,s)\times(F,x)} = \mathcal{O}_{S,s} \mathbin{\widehat{\otimes}} \mathcal{O}_{F,x}$.

The implication (a) $\Rightarrow$ (c) follows from Corollary 1.88. Let us prove the inverse implication (a) $\Leftarrow$ (c): If $\dim(F,x) = k$, then $\mathcal{O}_{F,x} \cong \mathbb{C}\{t_1,\dots,t_k\}$, and the canonical surjection $\mathcal{O}_{X,x} \to \mathcal{O}_{F,x}$ has a section mapping t_i to some preimage $h_i \in \mathcal{O}_{X,x}$ (Remark 1.1.1 (5)). Mapping t_i to h_i induces also a unique morphism $\mathcal{O}_{S,s} \mathbin{\widehat{\otimes}} \mathcal{O}_{F,x} \cong \mathcal{O}_{S,s}\{\boldsymbol{t}\} \to \mathcal{O}_{X,x}$ of $\mathcal{O}_{S,s}$-algebras with $\boldsymbol{t} = (t_1,\dots,t_k)$. That is, we have a commutative diagram of germs,

$$\begin{tikzcd}[column sep=small] (X,x) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (S,s)\times(F,x) \arrow[dl, "p"] \\ & (S,s)\,, & \end{tikzcd} \tag{1.10.5}$$

where f is flat, and ϕ is an isomorphism on the special fibre. By Lemma 1.86, ϕ is an isomorphism. Hence, f is regular at x.

The implication (b) $\Rightarrow$ (d) follows directly from the definition of $\Omega^1_{X/S,x}$. To complete the proof, we show the implication (d) $\Rightarrow$ (a): By (1.10.4), we have $\Omega^1_{X/S,x} \otimes_{\mathcal{O}_{S,s}} \mathbb{C} \cong \Omega^1_{F,x}$. Hence, if $\Omega^1_{X/S,x}$ is $\mathcal{O}_{X,x}$-free of rank k, then $\Omega^1_{F,x}$ is $\mathcal{O}_{F,x}$-free of rank k. Theorem 1.110 implies that the germ (F,x) is regular of dimension k. $\square$

Corollary 1.116 (Singular locus of a morphism is closed). *The set of singular points of a morphism $f : X \to S$ of complex spaces satisfies*

$$\mathrm{Sing}(f) = \mathrm{NFree}(\Omega^1_{X/S}) \cup \mathrm{NFlat}(f)\,,$$

where $\mathrm{NFlat}(f)$ *denotes the non-flat locus of f. In particular,* $\mathrm{Sing}(f)$ *is analytically closed in X.*

If X is reduced and if f is flat, then the set of regular points of f is dense in X.

Proof. This follows from Theorems 1.115, 1.83 and 1.80 (4). □

Note that the set of flat points of a morphism $f : X \to S$ of complex spaces is not necessarily dense in X, even if X and S are reduced. Consider, for instance, a closed embedding $i : X \hookrightarrow S$ of a proper analytic subspace in an irreducible complex space S. Such a morphism i is nowhere flat.

Finally, we mention the following result, which follows from Theorem B.8.17 and Corollary 1.116:

Theorem 1.117. *Let $f : X \to S$ be a flat morphism of complex spaces, and let $x \in X$. Then the following holds:*

(1) If X is regular at x, then S is regular at $f(x)$.
(2) If the fibre $f^{-1}(f(x))$ is regular at x, and if S is regular at $f(x)$, then X is regular at x, and there is a neighbourhood $U \subset X$ of x such that all fibres $f^{-1}(f(x'))$, $x' \in U$, are regular at x'.

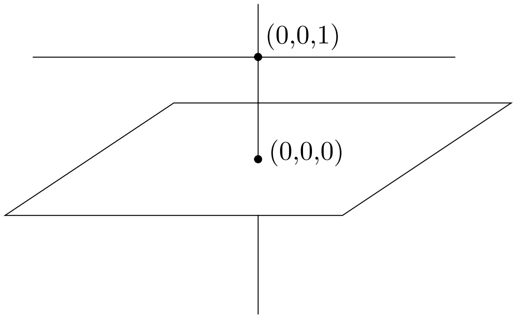
Remarks and Exercises

Proposition 1.104 and Theorem 1.110 provide two different ways to compute the singular locus of a complex space X. Let us assume that $I = \langle f_1, \dots, f_k \rangle$ is the ideal of $X \subset \mathbb{C}^n$ where the $f_i \in \mathbb{C}[x_1, \dots, x_n]$ are polynomials.

The first approach is to decompose X into pure-dimensional (e.g. irreducible) components $X_1, \dots, X_r$ by applying an equidimensional (e.g. primary) decomposition of I, $I = \bigcap_{i=1}^r Q_i$ such that $X_i = V(Q_i)$ is pure dimensional (see [GrP]). If $Q_i = \langle g_1^i, \dots, g_{k_i}^i \rangle$ then $\mathrm{Sing}(X_i)$ is given by the ideal J_i generated by Q_i and the $n - k_i$-minors of the Jacobian matrix of $(g_1^i, \dots, g_{k_i}^i)$. For $i < j$, let $J_{ij} = Q_i + Q_j$. Then the ideal $\bigcap_{j=1}^r J_j \cap \big(\bigcap_{i<j} J_{ij}\big)$ defines $\mathrm{Sing}(X)$.

Another way to show that $\mathrm{Sing}(X)$ is an analytic subset of X (and which does not use a primary decomposition) is based on Theorem 1.110 which states that
$$\mathrm{Sing}(X) = \mathrm{NFree}(\Omega_X^1)\,.$$
By Exercise 1.7.8, every $x \in X$ has an open neighbourhood U such that
$$\mathrm{NFree}(\Omega_X^1) \cap U = \mathrm{supp}\big(\mathcal{E}xt^1_{\mathcal{O}_X|_U}\big(\Omega_X^1|_U, \mathcal{S}yz(\Omega_X^1|U)\big)\big)\,,$$
where $\mathcal{O}_U^q \xrightarrow{A} \mathcal{O}_U^p \to \Omega_X^1|_U \to 0$ is a presentation of $\Omega_X^1|_U$ and $\mathcal{S}yz(\Omega_X^1|U) = \mathcal{I}m\,(A)$ is the first syzygy module of $\Omega_X^1|_U$. It follows that
$$\mathrm{Sing}(X) = \mathrm{supp}\big(\mathcal{E}xt^1_{\mathcal{O}_X}\big(\Omega_X^1, \mathcal{S}yz(\Omega_X^1)\big)\big)\,.$$
Hence, $\mathrm{Sing}(X)$ is defined by $\mathcal{J}$ where $\mathcal{J} \subset \mathcal{O}_X$ is either the annihilator ideal or the 0-th Fitting ideal of $\mathcal{E}xt^1_{\mathcal{O}_X}\big(\Omega_X^1, \mathcal{S}yz(\Omega_X^1)\big)$.

Next, we compute an example with Singular. Let
$$X = V(z) \cap V(x, y) \cap V(x, z-1) \subset \mathbb{C}^3\ :$$



Obviously, $(0,0,0)$ and $(0,0,1)$ are the only singular points. We compute an ideal of $\mathrm{Sing}(X)$ by using the first method, which is implemented in SINGULAR and can be accessed by the `slocus` command:

```
LIB "sing.lib";
ring R = 0,(x,y,z),dp;
ideal I = intersect(z,ideal(x,y),ideal(x,z-1));
interred(slocus(I));    // ideal of singular locus
//-> _[1]=y
//-> _[2]=x
//-> _[3]=z2-z
```

Now, let us compute $\mathrm{Sing}(X)$ via $\mathcal{E}xt^1_{\mathcal{O}_X}\big(\Omega^1_X, \mathcal{S}yz(\Omega^1_X)\big)$:

```
LIB "homolog.lib";
module Omega1X = transpose(jacob(I));
qring qr = std(I);       // pass to R/I
module Omega1X = imap(R,Omega1X);
module S = syz(Omega1X); // presentation matrix of syzygy module
module E = Ext(1,Omega1X,S);
Ann(E);                  // annihilator structure
//-> _[1]=y
//-> _[2]=x
//-> _[3]=z2-z
interred(minor(E,nrows(E))); // Fitting structure
//-> _[1]=x
//-> _[2]=yz-y
//-> _[3]=y2
//-> _[4]=z3-2z2+z
```

We see that `Ann(E)`, the annihilator ideal of $\mathcal{E}xt^1_{\mathcal{O}_X}\big(\Omega^1_X, \mathcal{S}yz(\Omega^1_X)\big)$ coincides with the structure computed via `slocus`, while the 0-th Fitting ideal provides $(0,0,1)$ with a non-reduced structure.

Exercise 1.10.1. Prove Proposition 1.114.

Exercise 1.10.2. Write SINGULAR procedures for computing the Fitting structure and the annihilator structure of $\mathcal{E}xt^1_{\mathcal{O}_X}\big(\Omega^1_X, \mathcal{S}yz(\Omega^1_X)\big)$ on $\mathrm{Sing}(X)$ if $X \subset \mathbb{C}^n$ is given by polynomials.

2 Hypersurface Singularities

This section is devoted to the study of isolated hypersurface singularities in $(\mathbb{C}^n, \mathbf{0})$. We introduce basic invariants like the Milnor and Tjurina number and show that they behave semicontinuously under deformations. This is an important application of the finite coherence theorem proved in Section 1.

We place some emphasis on (semi-)quasihomogeneous and Newton non-degenerate singularities. For these singularities many invariants have an easy combinatorial description and, more important, these singularities play a prominent role in the classification of singularities.

When dealing with hypersurface singularities given by a convergent power series f, $f(\mathbf{0}) = 0$, one can either consider the (germ of the) *function* f or, alternatively, the *zero set* of f, that is, the complex space germ $V(f) = f^{-1}(0)$ at $\mathbf{0}$. With respect to these different points of view we have different equivalence relations, different notions of deformation, etc. For example, we have two equivalence relations for hypersurface singularities: *right equivalence* (referring to functions) and *contact equivalence* (referring to zero sets of functions). We treat both cases in parallel, paying special attention to contact equivalence, since the latter is usually not considered in the literature. In most cases, statements about right equivalence turn out to be a special case of statements about contact equivalence.

We prove a finite determinacy theorem for isolated hypersurface singularities under right equivalence, as well as under contact equivalence. The finite determinacy reduces the consideration of power series to a consideration of polynomials. This allows us to apply the theory of algebraic groups to the classification of singularities. Using this and properties of invariants, we give a complete proof of the classification of the so-called *simple* or *ADE*-singularities, which turns out to be the same for right and contact equivalence.

2.1 Invariants of Hypersurface Singularities

We study the Milnor and Tjurina number and its behaviour under deformations.

Definition 2.1. Let $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$ be a convergent power series.

(1) The ideal

$$j(f) := \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathbb{C}\{\boldsymbol{x}\}}$$

is called the *Jacobian ideal*, or the *Milnor ideal* of f, and

$$\langle f, j(f) \rangle = \left\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathbb{C}\{\boldsymbol{x}\}}$$

is called the *Tjurina ideal* of f.

(2) The analytic algebras

$$M_f := \mathbb{C}\{\boldsymbol{x}\}/j(f)\,, \qquad T_f := \mathbb{C}\{\boldsymbol{x}\}/\langle f, j(f)\rangle$$

are called the *Milnor* and *Tjurina algebra* of f, respectively.

(3) The numbers

$$\mu(f) := \dim_{\mathbb{C}} M_f\,, \qquad \tau(f) := \dim_{\mathbb{C}} T_f$$

are called the *Milnor* and *Tjurina number* of f, respectively.

The Milnor and the Tjurina algebra and, in particular, their dimensions play an important role in the study of isolated hypersurface singularities.

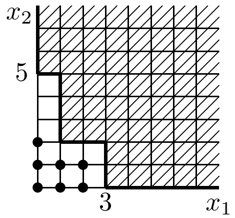
Let us consider some examples.

Example 2.1.1. (1) $f = x_1(x_1^2+x_2^3) + x_3^2 + \ldots + x_n^2$, $n \geq 2$, is called an E_7-singularity (see the classification in Section 2.4). Since

$$j(f) = \langle 3x_1^2+x_2^3, x_1x_2^2, x_3, \ldots, x_n\rangle$$

we see that $x_1^3, x_2^5 \in j(f)$, in particular, $f \in j(f)$.

As $\mathbb{C}\{x_1,\ldots,x_n\}/j(f) \cong \mathbb{C}\{x_1,x_2\}/\langle 3x_1^2+x_2^3, x_1x_2^2\rangle$ we can draw the monomial diagram of $j(f)$ in the 2-plane.



The monomials belonging to the shaded region are contained in $j(f)$ and it is easy to see that none of the monomials below the shaded region belongs to $j(f)$. The only relations between these monomials are $3x_1^2 \equiv -x_2^3 \bmod j(f)$ and, hence, $3x_1^2x_2 \equiv -x_2^4 \bmod j(f)$. It follows that $1, x_1, x_1^2, x_2, x_1x_2, x_1^2x_2, x_2^2$ is a $\mathbb{C}$-basis of both M_f and T_f and, thus, $\mu(f) = \tau(f) = 7$.

(2) $f = x^5 + y^5 + x^2y^2$ has $j(f) = \langle 5x^4 + 2xy^2, 5y^4 + 2x^2y\rangle$. We can compute a $\mathbb{C}$-basis of T_f as $1, x, \ldots, x^4, xy, y, \ldots, y^4$ and a $\mathbb{C}$-basis of M_f, which has an additional monomial y^5. Hence, $10 = \tau(f) < \mu(f) = 11$.

Such computations are quite tedious by hand, but can easily be done with a computer by using a computer algebra system which allows calculations in local rings. Here is the Singular code:

```
ring r=0,(x,y),ds; // a ring with a local ordering
poly f=x5+y5+x2y2;
ideal j=jacob(f);
```

```
vdim(std(j));      // the Milnor number
//-> 11

ideal fj=f,j;
vdim(std(fj));     // the Tjurina number
//-> 10
kbase(std(fj));
//-> _[1]=y4   _[2]=y3   _[3]=y2   _[4]=xy   _[5]=y
//-> _[6]=x4   _[7]=x3   _[8]=x2   _[9]=x   _[10]=1
```

Moreover, if f satisfies a certain non-degeneracy (NND) property then there is a much more handy way to compute the Milnor number. Indeed, it can be read from the Newton diagram of f (see Proposition 2.16 below).

Critical and Singular Points. Let $U \subset \mathbb{C}^n$ be an open subset, $f : U \to \mathbb{C}$ a holomorphic function and $x \in U$. We set

$$j(f) := \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \mathcal{O}(U) \subset \mathcal{O}(U)$$

and define

$$M_{f,x} := \mathcal{O}_{\mathbb{C}^n,x}/j(f)\mathcal{O}_{\mathbb{C}^n,x}\,, \qquad T_{f,x} := \mathcal{O}_{\mathbb{C}^n,x}/\langle f, j(f)\rangle \mathcal{O}_{\mathbb{C}^n,x}$$

to be the *Milnor* and *Tjurina algebra* of f at x. Furthermore, we introduce

$$\mu(f,x) := \dim_{\mathbb{C}} M_{f,x}\,, \qquad \tau(f,x) := \dim_{\mathbb{C}} T_{f,x}\,,$$

and call these numbers the *Milnor* and *Tjurina number* of f at x.

It is clear that $\mu(f,x) \neq 0$ iff $\frac{\partial f}{\partial x_i}(x) = 0$ for all i, and that $\tau(f,x) \neq 0$ iff additionally $f(x) = 0$. Hence, we see that μ counts the singular points of the *function* f, while τ counts the singular points of the *zero set* of f, each with multiplicity $\mu(f,x)$, respectively $\tau(f,x)$. The following definition takes care of this difference:

Definition 2.2. Let $U \subset \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ a holomorphic function, and $X = V(f) = f^{-1}(0)$ the hypersurface defined by f in U. We call

$$\operatorname{Crit}(f) := \operatorname{Sing}(f) := \left\{ x \in U \;\middle|\; \frac{\partial f}{\partial x_1}(x) = \ldots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}$$

the set of *critical*, or *singular, points of* f and

$$\operatorname{Sing}(X) := \left\{ x \in U \;\middle|\; f(x) = \frac{\partial f}{\partial x_1}(x) = \ldots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}$$

the set of *singular points of* X.

A point $x \in U$ is called an *isolated critical point* of f, if there exists a neighbourhood V of x such that $\operatorname{Crit}(f) \cap V \setminus \{x\} = \emptyset$. It is called an *isolated singular point* of X if $x \in X$ and $\operatorname{Sing}(X) \cap V \setminus \{x\} = \emptyset$. Then we say also that the germ $(X,x) \subset (\mathbb{C}^n, x)$ is an *isolated hypersurface singularity*.

Note that the definition of $\mathrm{Sing}(X)$, resp. $\mathrm{Sing}(f)$, is a special case of Definition 1.40, resp. 1.112.

Lemma 2.3. *Let $f : U \to \mathbb{C}$ be holomorphic and $x \in U$, then the following are equivalent.*

(a) x is an isolated critical point of f,
(b) $\mu(f, x) < \infty$,
(c) x is an isolated singularity of $f^{-1}(f(x)) = V(f - f(x))$,
(d) $\tau(f - f(x), x) < \infty$.

Proof. (a), respectively (c) says that x is an isolated point of the fibre over 0 (if it is contained in the fibre) of the morphisms

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) : U \longrightarrow \mathbb{C}^n, \qquad \left(f - f(x), \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) : U \longrightarrow \mathbb{C}^{n+1},$$

respectively. Hence, the equivalence of (a) and (b), respectively of (c) and (d), is a consequence of Proposition 1.70 or the Hilbert-Rückert Nullstellensatz 1.72.

Since $\mu(f, x) \le \tau(f - f(x), x)$, the implication (b) $\Rightarrow$ (d) is evident. Finally, (c) $\Rightarrow$ (a) follows from the following lemma, which holds also for non-isolated singularities. □

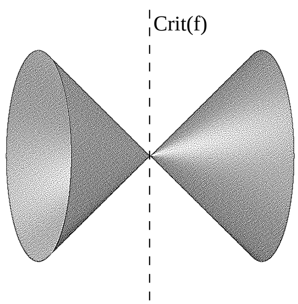
Lemma 2.4. *Let $U \subset \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ a holomorphic function, $x \in U$ and $f(x) = 0$. Then there is a neighbourhood V of x in U such that*

$$\mathrm{Crit}(f) \cap V \subset f^{-1}(0)\,.$$

In other words, the nearby fibres $f^{-1}(t) \cap V$, t sufficiently small, are smooth.

Proof. Consider $C = \mathrm{Crit}(f)$ with its reduced structure. As a reduced complex space, the regular points of C, $\mathrm{Reg}(C)$, are open and dense in C by Corollary 1.111. Since $\frac{\partial f}{\partial x_i}$ vanishes on C for $i = 1, \ldots, n$, f is locally constant on the complex manifold $\mathrm{Reg}(C)$. A sufficiently small neighbourhood V of x intersects only the connected components of $\mathrm{Reg}(C)$ having x in its closure. If $x \notin C$ the result is trivial. If $x \in C$ then $f|_{V \cap C} = 0$, since f is continuous and $f(x) = 0$. □

Hence, it cannot happen that the critical set of f (the dashed line) meets $f^{-1}(0)$ as in the following picture.



Semicontinuity of Milnor and Tjurina number. In the sequel we study the behaviour of μ and τ under deformations. Loosely speaking, a deformation of a power series $f \in \mathbb{C}\{\boldsymbol{x}\}$, usually called an *unfolding*, is given by a power series $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$ such that, setting $F_{\boldsymbol{t}}(\boldsymbol{x}) = F(\boldsymbol{x}, \boldsymbol{t})$, $F_0 = f$, while a *deformation* of the hypersurface germ $f^{-1}(0)$ is given by any power series $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$ satisfying $F_0^{-1}(0) = f^{-1}(0)$. So far, unfoldings and deformations are both given by a power series F, the difference appears later when we consider isomorphism classes of deformations. For the moment we only consider the power series F.

Definition 2.5. A power series $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\} = \mathbb{C}\{x_1, \dots, x_n, t_1, \dots, t_k\}$ is called an *unfolding* of $f \in \mathbb{C}\{x_1, \dots, x_n\}$ if $F(\boldsymbol{x}, \boldsymbol{0}) = f(\boldsymbol{x})$. We use the notation

$$F_{\boldsymbol{t}}(\boldsymbol{x}) \;=\; F(\boldsymbol{x}, \boldsymbol{t})\,, \quad \boldsymbol{t} \in T\,,$$

for the family of power series $F_{\boldsymbol{t}} \in \mathbb{C}\{\boldsymbol{x}\}$ or, after choosing a representative $F\colon U \times T \to \mathbb{C}$, for the family $F_{\boldsymbol{t}}\colon U \to \mathbb{C}$ of holomorphic functions parametrized by $t \in T$, where $U \subset \mathbb{C}^n$ and $T \subset \mathbb{C}^k$ are open neighbourhoods of the origin.

Theorem 2.6 (Semicontinuity of μ and τ).
Let $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$ be an unfolding of $f \in \mathbb{C}\{\boldsymbol{x}\}$, $f(\boldsymbol{0}) = 0$, and assume that $\boldsymbol{0}$ is an isolated critical point of f. Then there are neighbourhoods $U = U(\boldsymbol{0}) \subset \mathbb{C}^n$, $V = V(0) \subset \mathbb{C}$, $T = T(\boldsymbol{0}) \subset \mathbb{C}^k$, such that F converges on $U \times T$ and the following holds for each $\boldsymbol{t} \in T$:

(1) $\boldsymbol{0} \in U$ is the only critical point of $f = F_0\colon U \to V$, and $F_{\boldsymbol{t}}$ has only isolated critical points in U.
(2) For each $y \in V$,

$$\mu(f, \boldsymbol{0}) \geq \sum_{x \in \operatorname{Sing}(F_{\boldsymbol{t}}^{-1}(y))} \mu(F_{\boldsymbol{t}}, \boldsymbol{x}) \quad \textit{and}$$

$$\tau(f, \boldsymbol{0}) \geq \sum_{x \in \operatorname{Sing}(F_{\boldsymbol{t}}^{-1}(y))} \tau(F_{\boldsymbol{t}} - y, \boldsymbol{x}).$$

(3) Furthermore,

$$\mu(f, \boldsymbol{0}) = \sum_{x \in \operatorname{Crit}(F_{\boldsymbol{t}})} \mu(F_{\boldsymbol{t}}, \boldsymbol{x}).$$

Proof. (1) Choose U such that $\boldsymbol{0}$ is the only critical point of F_0 and consider the map

$$\varPhi\colon U \times T \to \mathbb{C}^n \times T\,, \quad (\boldsymbol{x}, \boldsymbol{t}) \mapsto \left(\frac{\partial F_{\boldsymbol{t}}}{\partial x_1}(\boldsymbol{x}), \dots, \frac{\partial F_{\boldsymbol{t}}}{\partial x_n}(\boldsymbol{x}), \boldsymbol{t}\right).$$

Then $\varPhi^{-1}(\boldsymbol{0}, \boldsymbol{0}) = \operatorname{Crit}(F_0) \times \{\boldsymbol{0}\} = \{(\boldsymbol{0}, \boldsymbol{0})\}$ by the choice of U. Hence, by the local finiteness Theorem 1.66, $\varPhi$ is a finite morphism if we choose U, T

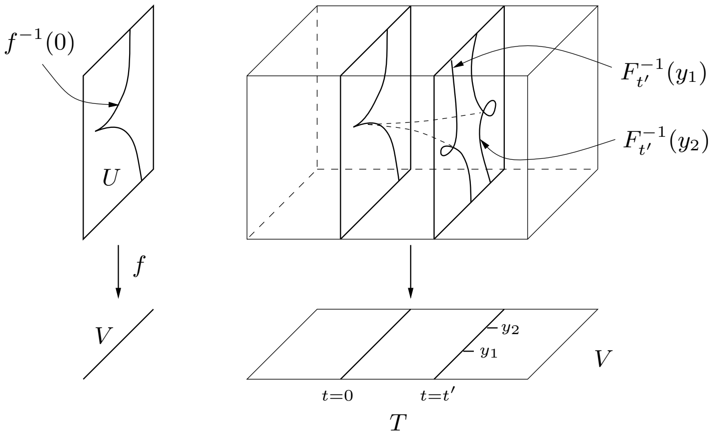


Fig. 2.5. Deformation of an isolated hypersurface singularity

to be sufficiently small. This implies that Φ has finite fibres, in particular, $\mathrm{Crit}(F_{\boldsymbol{t}}) \times \{\boldsymbol{t}\} = \Phi^{-1}(\mathbf{0}, \boldsymbol{t})$ is finite.

(2) The first inequality follows from (3). For the second consider the map

$$\Psi\colon U \times T \longrightarrow V \times \mathbb{C}^n \times T\,, \quad (\boldsymbol{x}, \boldsymbol{t}) \mapsto \left(F_{\boldsymbol{t}}(\boldsymbol{x}), \frac{\partial F_{\boldsymbol{t}}}{\partial x_1}(\boldsymbol{x}), \ldots, \frac{\partial F_{\boldsymbol{t}}}{\partial x_n}(\boldsymbol{x}), \boldsymbol{t}\right).$$

Then $\Psi^{-1}(0, \mathbf{0}, \mathbf{0}) = \mathrm{Sing}(f_0^{-1}(0)) \times \{\mathbf{0}\} = \{(\mathbf{0}, \mathbf{0})\}$ and, again by the local finiteness theorem, $\mathrm{Sing}\left(F_{\boldsymbol{t}}^{-1}(y)\right) \times \{\boldsymbol{t}\} = \Psi^{-1}(y, \mathbf{0}, \boldsymbol{t})$ is finite for U, V, T sufficiently small and $y \in V$, $\boldsymbol{t} \in T$. Moreover, the direct image sheaf $\Psi_* \mathcal{O}_{U \times T}$ is coherent on $V \times \mathbb{C}^n \times T$. The semicontinuity of fibre functions (Theorem 1.81) implies that the function

$$\begin{aligned} \nu(y, \boldsymbol{t}) &:= \nu\left(\Psi_* \mathcal{O}_{U \times T}, (y, \mathbf{0}, \boldsymbol{t})\right) \\ &= \sum_{(\boldsymbol{x}, \boldsymbol{t}) \in \Psi^{-1}(y, \mathbf{0}, \boldsymbol{t})} \dim_{\mathbb{C}} \mathcal{O}_{U \times T, (\boldsymbol{x}, \boldsymbol{t})} / \mathfrak{m}_{(y, \mathbf{0}, \boldsymbol{t})} \mathcal{O}_{U \times T, (\boldsymbol{x}, \boldsymbol{t})} \end{aligned}$$

is upper semicontinuous. Since

$$\mathcal{O}_{U \times T, (\boldsymbol{x}, \boldsymbol{t})} / \mathfrak{m}_{(y, \mathbf{0}, \boldsymbol{t})} \mathcal{O}_{U \times T, (\boldsymbol{x}, \boldsymbol{t})} \cong \mathcal{O}_{U, x} \Big/ \left\langle F_{\boldsymbol{t}} - y, \frac{\partial F_{\boldsymbol{t}}}{\partial x_1}, \ldots, \frac{\partial F_{\boldsymbol{t}}}{\partial x_n} \right\rangle$$

we have $\nu(0, \mathbf{0}) = \tau(f, \mathbf{0})$ and $\nu(y, \boldsymbol{t}) = \sum_{\boldsymbol{x} \in \mathrm{Sing}(F_{\boldsymbol{t}}^{-1}(y))} \tau(F_{\boldsymbol{t}}, \boldsymbol{x})$, and the result follows.

(3) We consider again the morphism Φ and have to show that the function

$$\nu(\boldsymbol{t}) := \nu\left(\Phi_*\mathcal{O}_{U\times T}, (\mathbf{0}, \boldsymbol{t})\right) = \sum_{\boldsymbol{x}\in \mathrm{Crit}(F_{\boldsymbol{t}})} \dim_{\mathbb{C}} \mathcal{O}_{U,\boldsymbol{x}} \Big/ \left\langle \frac{\partial F_{\boldsymbol{t}}}{\partial x_1}, \dots, \frac{\partial F_{\boldsymbol{t}}}{\partial x_n} \right\rangle$$

is locally constant on T. Thus, by Theorems 1.81 and 1.82 we have to show that Φ is flat at $(\mathbf{0}, \mathbf{0})$.

Since $\mathcal{O}_{U\times T,(\mathbf{0},\mathbf{0})} \cong \mathbb{C}\{x_1, \dots, x_n, t_1, \dots, t_k\}$ is a regular local ring, and since the $n+k$ component functions $\frac{\partial F_{\boldsymbol{t}}}{\partial x_1}, \dots, \frac{\partial F_{\boldsymbol{t}}}{\partial x_n}, t_1, \dots, t_k$ define a zero-dimensional, hence $(n+k)$-codimensional germ, the flatness follows from the following proposition. □

Proposition 2.7. *(1) Let $f = (f_1, \dots, f_k)\colon (X, x) \to (\mathbb{C}^k, \mathbf{0})$ be a holomorphic map germ and M a finitely generated $\mathcal{O}_{X,x}$-module. Then M is f-flat iff the sequence $f_1, \dots, f_k$ is M-regular*[11]*.*
In particular, f is flat iff $f_1, \dots, f_k$ is a regular sequence.
(2) If (X, x) is the germ of an n-dimensional complex manifold, then $f_1, \dots, f_k$ is $\mathcal{O}_{X,x}$-regular iff $\dim(f^{-1}(0), x) = n - k$.

The proof is given in Appendix B.8.

Remark 2.7.1. Let $(T, \mathbf{0}) \subset (\mathbb{C}^k, \mathbf{0})$ be an arbitrary reduced analytic subgerm, and let $F \in \mathcal{O}_{\mathbb{C}^n\times T,\mathbf{0}}$ map to $f \in \mathbb{C}\{\boldsymbol{x}\}$ (as in Theorem 2.6) under the canonical surjection $\mathcal{O}_{\mathbb{C}^n\times T,\mathbf{0}} \to \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} = \mathbb{C}\{\boldsymbol{x}\}$. Then we can lift F to $\widetilde{F} \in \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^k,\mathbf{0}}$ and apply Theorem 2.6 to obtain the semicontinuity of μ, resp. τ, for $\widetilde{F}$ and all t in a neighbourhood of $\mathbf{0} \in \mathbb{C}^k$. Since $F : (\mathbb{C}^n \times T, \mathbf{0}) \to (\mathbb{C}, 0)$ is the restriction of $\widetilde{F} : (\mathbb{C}^n \times \mathbb{C}^k, \mathbf{0}) \to (\mathbb{C}, 0)$, statements (1), (2) and (3) hold for F and a sufficiently small representative T of the germ $(T, \mathbf{0})$.

Alternatively, we may apply the proof of Theorem 2.6 directly to an arbitrary reduced germ $(T, \mathbf{0})$. The flatness of the maps ϕ and ψ follows from the flatness of the maps $\widetilde{\phi}$ and $\widetilde{\psi}$ (associated to $\widetilde{F}$) and the base change property for flatness (Propoisition 1.87 on page 89).

Example 2.7.2. (1) Consider the unfolding $F_t(x, y) = x^2 - y^2(t+y)$ of the cusp singularity $f(x, y) = x^2 - y^3$. We compute $\mathrm{Crit}(F_t) = \{(0, 0), (0, -\frac{2}{3}t)\}$ and $\mathrm{Sing}(F_t^{-1}(0)) = \{(0, 0)\}$. Moreover, $\mu(f) = \tau(f) = 2$, while for $t \neq 0$ we have $\mu(F_t, (0, 0)) = \tau(F_t, (0, 0)) = 1$ and $\mu(F_t, (0, -\frac{2}{3}t)) = 1$.

(2) For the unfolding $F_t(x, y) = x^5 + y^5 + tx^2y^2$ we compute the critical locus to be $\mathrm{Crit}(F_t) = V(5x^4 + 2txy^2, 5y^4 + 2tx^2y)$. The only critical point of F_0 is the origin $\mathbf{0} = (0, 0)$, and we have $\mu(F_0, \mathbf{0}) = \tau(F_0, \mathbf{0}) = 16$. Using Singular we compute that, for $t \neq 0$, F_t has a critical point at $\mathbf{0}$ with $\mu(F_t, \mathbf{0}) = 11$, $\tau(F_t, \mathbf{0}) = 10$, and five further critical points with $\mu = \tau = 1$ each. This shows that $\mu(F_0, \mathbf{0}) = \sum_{\boldsymbol{x}\in\mathrm{Crit}(F_t)} \mu(F_t, \boldsymbol{x})$ for each t as stated in Theorem 2.6.

[11] Recall that $f_1, \dots, f_k$ is an *M-regular sequence* or *M-regular* iff f_1 is a non-zerodivisor of M and f_i is a non-zerodivisor of $M/(f_1M + \ldots + f_{i-1}M)$ for $i = 2, \dots, k$.

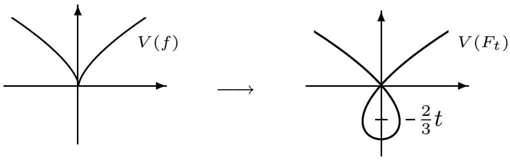


Fig. 2.6. Deformation of a cusp singularity

But $\tau(F_0, \mathbf{0}) = 16 > 15 = \sum_{\boldsymbol{x}\in\mathrm{Crit}(F_t)} \tau(F_t - F_t(\boldsymbol{x}), \boldsymbol{x})$, that is, even the "total" Tjurina number is not constant.

(3) The local, respectively total, Milnor number can be computed in Singular by the same formulas but with a different choice of monomial ordering. First, we work in the ring $\mathbb{Q}(t)[x,y]_{\langle x,y\rangle}$, by choosing the *local* monomial ordering `ds`:

```
ring r=(0,t),(x,y),ds;
poly f=x5+y5;
poly F=f+tx2y2;  // an unfolding of f
LIB "sing.lib";  // load library
milnor(f);       // (local) Milnor number of the germ (f,0)
//-> 16
tjurina(f);      // (local) Tjurina number of (f,0)
//-> 16
milnor(F);       // (local) Milnor number of F for generic t
//-> 11
tjurina(F);      // (local) Tjurina number of F for generic t
//-> 10
```

To obtain the total (affine) Milnor, respectively Tjurina, number, we repeat the same commands in a ring with the *global* monomial ordering `dp` (implementing $\mathbb{Q}(t)[x,y]$):

```
ring R=(0,t),(x,y),dp;
poly F=x5+y5+tx2y2;
milnor(F);       // global Milnor number of F_t for generic t
//-> 16
tjurina(F);      // global Milnor number of F_t for generic t
//-> 10
```

Since the local and the global Tjurina number for F_t coincide, the hypersurface $F_t^{-1}(\mathbf{0})$ has, for generic t, the origin as its only singularity.

If the first inequality in Theorem 2.6 (2) happens to be an equality (for some y suffciently close to 0) then the fibre $F_{\boldsymbol{t}}^{-1}(y)$ contains only one singular point:

Theorem 2.8. *Let $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$ be an unfolding of $f \in \langle\boldsymbol{x}\rangle^2 \subset \mathbb{C}\{\boldsymbol{x}\}$. Moreover, let $T \subset \mathbb{C}^k$ and $U \subset \mathbb{C}^n$ be open neighbourhoods of the origin, and let*

$F_{\boldsymbol{t}} \colon U \to \mathbb{C}$, $\boldsymbol{x} \mapsto F_{\boldsymbol{t}}(\boldsymbol{x}) = F(\boldsymbol{x}, \boldsymbol{t})$. *If* $\mathbf{0}$ *is the only singularity of the special fibre* $F_0^{-1}(0) = f^{-1}(0)$ *and, for all* $\boldsymbol{t} \in T$,

$$\sum_{x \in \operatorname{Sing}(F_{\boldsymbol{t}}^{-1}(0))} \mu(F_{\boldsymbol{t}}, \boldsymbol{x}) \;=\; \mu(f, \mathbf{0})$$

then all fibres $F_{\boldsymbol{t}}^{-1}(0)$, $\boldsymbol{t} \in T$, *have a unique singular point (with Milnor number* $\mu(f, \mathbf{0})$*).*

This was proven independently by Lazzeri [Laz] and Gabrièlov [Gab1].

Right and Contact Equivalence. Now let us consider the behaviour of μ and τ under coordinate transformation and multiplication with units.

Definition 2.9. Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$.

(1) f is called *right equivalent* to g, $f \overset{r}{\sim} g$, if there exists an automorphism φ of $\mathbb{C}\{\boldsymbol{x}\}$ such that $\varphi(f) = g$.
(2) f is called *contact equivalent* to g, $f \overset{c}{\sim} g$, if there exists an automorphism φ of $\mathbb{C}\{\boldsymbol{x}\}$ and a unit $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ such that $f = u \cdot \varphi(g)$

If $f, g \in \mathcal{O}_{\mathbb{C}^n, x}$ then we sometimes also write $(f, x) \overset{r}{\sim} (g, x)$, respectively $(f, x) \overset{c}{\sim} (g, x)$.

Remark 2.9.1. (1) Of course, $f \overset{r}{\sim} g$ implies $f \overset{c}{\sim} g$. The converse, however, is not true (see Exercise 2.1.3, below).

(2) Any $\varphi \in \operatorname{Aut} \mathbb{C}\{\boldsymbol{x}\}$ determines a biholomorphic local coordinate change $\Phi = (\Phi_1, \ldots, \Phi_n) \colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ by $\Phi_i = \varphi(x_i)$, and, vice versa, any isomorphism of germs Φ determines $\varphi \in \operatorname{Aut} \mathbb{C}\{\boldsymbol{x}\}$ by the same formula. We have $\varphi(g) = g \circ \Phi$ and, hence,

$$f \overset{r}{\sim} g \iff f = g \circ \Phi$$

for some biholomorphic map germ $\Phi \colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$, that is, the diagram

$$\begin{tikzcd} (\mathbb{C}^n, \mathbf{0}) \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "f"'] & & (\mathbb{C}^n, \mathbf{0}) \arrow[dl, "g"] \\ & (\mathbb{C}, 0) & \end{tikzcd}$$

commutes. The notion of right equivalence results from the fact that, on the level of germs, the group of local coordinate changes acts from the right.

(3) Since f and g generate the same ideal in $\mathbb{C}\{\boldsymbol{x}\}$ iff there is a unit $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ such that $f = u \cdot g$, we see that $f \overset{c}{\sim} g$ iff $\langle f \rangle = \langle \varphi(g) \rangle$ for some $\varphi \in \operatorname{Aut} \mathbb{C}\{\boldsymbol{x}\}$. Moreover, since any isomorphism of analytic algebras lifts to the power series ring by Lemma 1.14, we get

$$f \overset{c}{\sim} g \iff \mathbb{C}\{\boldsymbol{x}\}/\langle f \rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\langle g \rangle \text{ as analytic } \mathbb{C}\text{-algebras.}$$

Equivalently, $f \overset{c}{\sim} g$ iff the complex space germs $(f^{-1}(0), \mathbf{0})$ and $(g^{-1}(0), \mathbf{0})$ are isomorphic.

Hence, $f \overset{r}{\sim} g$ iff f and g define, up to a change of coordinates in $(\mathbb{C}^n, \mathbf{0})$, the same map germs $(\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$, while $f \overset{c}{\sim} g$ iff f and g have, up to coordinate change, the same zero-fibre.

Lemma 2.10. *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$. Then*

(1) $f \overset{r}{\sim} g$ implies that $M_f \cong M_g$ and $T_f \cong T_g$ as analytic algebras. In particular, $\mu(f) = \mu(g)$ and $\tau(f) = \tau(g)$.
(2) $f \overset{c}{\sim} g$ implies that $T_f \cong T_g$ and hence $\tau(f) = \tau(g)$.

Proof. (1) If $g = \varphi(f) = f \circ \Phi$, then

$$\left(\frac{\partial(f\circ\Phi)}{\partial x_1}(\boldsymbol{x}), \ldots, \frac{\partial(f\circ\Phi)}{\partial x_n}(\boldsymbol{x})\right) = \left(\frac{\partial f}{\partial x_1}(\Phi(\boldsymbol{x})), \ldots, \frac{\partial f}{\partial x_n}(\Phi(\boldsymbol{x}))\right) \cdot D\Phi(\boldsymbol{x}),$$

where $D\Phi$ is the Jacobian matrix of Φ, which is invertible in a neighbourhood of $\boldsymbol{x}$. It follows that $j(\varphi(f)) = \varphi(j(f))$ and $\langle \varphi(f), j(\varphi(f))\rangle = \varphi(\langle f, j(f)\rangle)$, which proves the claim.

(2) By the product rule we have $\langle u \cdot f, j(u \cdot f)\rangle = \langle f, j(f)\rangle$ for a unit u, which together with (1) implies $T_f \cong T_g$. □

In characteristic 0 it is even true that $f \overset{c}{\sim} g$ implies $\mu(f) = \mu(g)$, but this is more difficult. For an analytic proof we refer to [Gre] where the following formulas are shown (even for complete intersections):

$$\mu(f) = \begin{cases} \dim_{\mathbb{C}} \mathcal{O}_{X,\mathbf{0}} - 1\,, & \text{if } n = 1\,, \\ \dim_{\mathbb{C}} \Omega^{n-1}_{X,\mathbf{0}} / d\Omega^{n-2}_{X,\mathbf{0}}\,, & \text{if } n \geq 2\,, \end{cases}$$

with $(X, \mathbf{0}) = (f^{-1}(0), \mathbf{0})$. Even more, $\mu(f)$ *is a topological invariant* of $(f^{-1}(0), \mathbf{0})$ (cf. [Mil1] in general, respectively Section 3.4 for curves).

Example 2.10.1. (1) Consider the unfolding $F_t(x,y) = x^2 + y^2(t+y)$ with $\mathrm{Crit}(F_t) = \{(0,0), (0, -\frac{2}{3}t)\}$. The coordinate change $\varphi_t \colon x \mapsto x$, $y \mapsto y\sqrt{t+y}$, $(t \neq 0)$, satisfies $\varphi_t(x^2 + y^2) = x^2 + y^2(t+y) = F_t(x,y)$.

Hence, $(F_t, \mathbf{0}) \overset{r}{\sim} (x^2 + y^2, \mathbf{0})$ for $t \neq 0$. Thus, we have $\tau(x^2 + y^3, \mathbf{0}) = 2$, but for $t \neq 0$ we have $\tau(F_t, \mathbf{0}) = 1$, $\tau(F_t, (0, -\frac{2}{3}t)) = 1$. Hence $(F_t, \mathbf{0})$ and $(F_t, (0, -\frac{2}{3}t))$ are not contact equivalent to $(f, \mathbf{0})$.

(2) Consider the unfolding $F_t(x,y) = x^2 + y^2 + txy = x(x+ty) + y^2$. The coordinate change $\varphi \colon x \mapsto x - \frac{1}{2}ty$, $y \mapsto y$ satisfies $\varphi(F_t) = x^2 + y^2(1 - \frac{1}{4}t^2)$, which is right equivalent to $x^2 + y^2$ for $t \neq \pm 2$. In particular, $(F_t, \mathbf{0}) \overset{r}{\sim} (F_0, \mathbf{0})$ for all sufficiently small $t \neq 0$.

(3) The Milnor number is not an invariant of the contact class in positive characteristic: $f = x^p + y^{p+1}$ has $\mu(f) = \infty$, but $\mu((1+x)f) < \infty$ in $K[[x,y]]$ where K is a field of characteristic p.

Quasihomogeneous Singularities. The class of those isolated hypersurface singularities, for which the Milnor and Tjurina number coincide, attains a particular importance. Of course, an isolated hypersurface singularity $(X,x) \subset (\mathbb{C}^n, x)$ belongs to this class iff $f \in j(f)$ for some (hence, by the chain rule, all) local equation(s) $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$. In the following, we give a coordinate dependent description of this class:

Definition 2.11. A polynomial $f = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}[\boldsymbol{x}]$ is called *weighted homogeneous* or *quasihomogeneous*) of *type* $(\boldsymbol{w}; d) = (w_1, \ldots, w_n; d)$ if w_i, d are positive integers satisfying

$$\boldsymbol{w}\text{-deg}(\boldsymbol{x}^{\boldsymbol{\alpha}}) := \langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle = w_1\alpha_1 + \ldots + w_n\alpha_n = d$$

for each $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $a_{\boldsymbol{\alpha}} \neq 0$. The numbers w_i are called the *weights* and d the *weighted degree* or the $\boldsymbol{w}$*-degree* of f.

Note that this property is not invariant under coordinate changes (if the w_i are not all the same then it is not even invariant under linear coordinate changes).

In the above Example 2.1.1 (1), f is quasihomogeneous of type $(6, 4, 9; 18)$, while in Example 2.1.1 (2), f is not quasihomogeneous, not even after a change of coordinates.

Remark 2.11.1. A quasihomogeneous polynomial f of type $(\boldsymbol{w}; d)$ obviously satisfies the *Euler relation*[12]

$$d \cdot f = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} \quad \text{in } \mathbb{C}[\boldsymbol{x}],$$

and the relation

$$f(t^{w_1}x_1, \ldots, t^{w_n}x_n) = t^d \cdot f(x_1, \ldots, x_n) \quad \text{in } \mathbb{C}[\boldsymbol{x}, t].$$

The Euler relation implies that f is contained in $j(f)$, hence, $\mu(f) = \tau(f)$. The other relation implies that the hypersurface $V(f) \subset \mathbb{C}^n$ is invariant under the $\mathbb{C}^*$*-action* $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$, $(\lambda, \boldsymbol{x}) \mapsto \lambda \circ \boldsymbol{x} := (\lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n)$. In particular, the complex hypersurface $V(f) \subset \mathbb{C}^n$ is contractible.

Moreover, $\mathrm{Sing}(f)$ and $\mathrm{Crit}(f)$ are also invariant under $\mathbb{C}^*$ and, hence, the union of $\mathbb{C}^*$-orbits. It follows that if $V(f)$ has an isolated singularity at $\mathbf{0}$ then $\mathbf{0}$ is the only singular point of $V(f)$. Furthermore, $\boldsymbol{x} \mapsto \lambda \circ \boldsymbol{x}$ maps $V(f-t)$ isomorphically onto $V(f - \lambda^d t)$. Since $f \in j(f)$, $\mathrm{Sing}(f)$ and $\mathrm{Sing}(V(f))$ coincide in this situation.

Definition 2.12. An isolated hypersurface singularity $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ is called *quasihomogeneous* if there exists a quasihomogeneous polynomial $f \in \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$ such that $\mathcal{O}_{X,\mathbf{0}} \cong \mathbb{C}\{\boldsymbol{x}\}/\langle f \rangle$.

[12] The Euler relation generalizes the *Euler formula* for homogeneous polynomials $f \in \mathbb{C}[x_0, \ldots, x_n]$: $x_0 \frac{\partial f}{\partial x_0} + \ldots + x_n \frac{\partial f}{\partial x_n} = \deg(f) \cdot f$.

Lemma 2.13. *Let $f \in \mathbb{C}[\boldsymbol{x}]$ be quasihomogeneous and $g \in \mathbb{C}\{\boldsymbol{x}\}$ arbitrary. Then $f \overset{c}{\sim} g$ iff $f \overset{r}{\sim} g$.*

Proof. Let f be weighted homogeneous of type $(w_1, \ldots, w_n; d)$. If $f \overset{c}{\sim} g$ then there exists a unit $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ and an automorphism $\varphi \in \operatorname{Aut} \mathbb{C}\{\boldsymbol{x}\}$ such that $u \cdot f = \varphi(g)$. Choose a d-th root $u^{1/d} \in \mathbb{C}\{\boldsymbol{x}\}$. The automorphism

$$\psi \colon \mathbb{C}\{\boldsymbol{x}\} \longrightarrow \mathbb{C}\{\boldsymbol{x}\}, \quad x_i \mapsto u^{w_i/d} \cdot x_i$$

yields $\psi\big(f(\boldsymbol{x})\big) = f\big(u^{w_1/d}x_1, \ldots, u^{w_n/d}x_n\big) = u \cdot f(\boldsymbol{x})$ by Remark 2.11.1, implying the result. $\square$

It is clear that for quasihomogeneous isolated hypersurface singularities the Milnor and Tjurina number coincide (since $f \in j(f)$). It is a remarkable theorem of K. Saito [Sai] that for an isolated singularity the converse does also hold. Let $(X, x) \subset (\mathbb{C}^n, x)$ be an isolated hypersurface singularity and let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ be any local equation for (X, x), then

$$(X, x) \text{ quasihomogeneous} \iff \mu(f) = \tau(f)\,.$$

Since $\mu(f)$ and $\tau(f)$ are computable, the latter equivalence gives an effective characterization of isolated quasihomogeneous hypersurface singularities.

Newton Non-Degenerate and Semiquasihomogeneous Singularities. As mentioned before, for certain classes of singularities there is a much more handy way to compute the Milnor number. It can be read from the Newton diagram of an appropriate defining power series:

Definition 2.14. Let $f = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$, $a_{\boldsymbol{0}} = 0$. Then the convex hull in $\mathbb{R}^n$ of the support of f,

$$\Delta(f) := \operatorname{conv}\big\{\boldsymbol{\alpha} \in \mathbb{N}^n \,\big|\, a_{\boldsymbol{\alpha}} \neq 0\big\}\,,$$

is called the *Newton polytope of f*. We introduce $K(f) := \operatorname{conv}(\{\boldsymbol{0}\} \cup \Delta(f))$, and denote by $K_0(f)$ the closure of the set $(K(f) \setminus \Delta(f)) \cup \{\boldsymbol{0}\}$. Define the *Newton diagram*[13] *$\Gamma(f, \boldsymbol{0})$ of f at the origin* as the union of those faces of the polyhedral complex $K_0(f) \cap \Delta(f)$ through which one can draw a supporting hyperplane to $\Delta(f)$ with a normal vector having only positive coordinates. Moreover, we introduce for a face $\sigma \subset \Gamma(f, \boldsymbol{0})$ the *truncation*

$$f^\sigma := \sum_{\boldsymbol{\alpha} \in \sigma} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{i} \in \sigma \cap \mathbb{N}^n} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}\,,$$

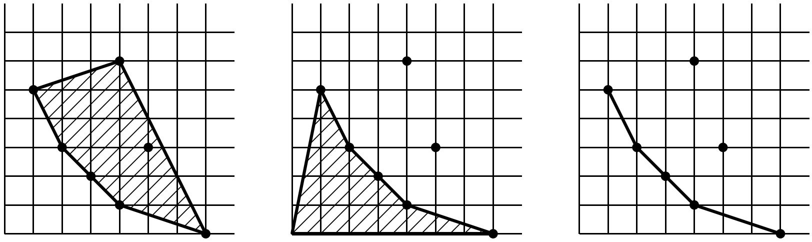
that is, the sum of the monomials in f corresponding to the integral points in σ.

[13] An equivalent definition is as follows: Define the *local Newton polytope* $N(f, \boldsymbol{0})$ as the convex hull of

$$\bigcup_{\boldsymbol{\alpha} \in \operatorname{supp}(f)} \boldsymbol{\alpha} + (\mathbb{R}_{\geq 0})^n\,.$$

Then $\Gamma(f, \boldsymbol{0})$ is the union of the compact faces of $N(f, \boldsymbol{0})$.

Example 2.14.1. Let $f = x \cdot (y^5 + xy^3 + x^2y^2 - x^2y^4 + x^3y - 10x^4y + x^6)$.



$\varDelta(f)$ $\qquad$ $K_0(f)$ $\qquad$ $\varGamma(f,\mathbf{0})$

In particular, the Newton diagram at $\mathbf{0}$ has three one-dimensional faces, with slopes $-2, -1, -\frac{1}{3}$.

Definition 2.15. A power series $f = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$ is called *convenient* if its Newton diagram $\varGamma(f,\mathbf{0})$ meets all the coordinate axes. A convenient power series f is called *Newton non-degenerate (NND)* at $\mathbf{0}$ if, for all faces $\sigma \subset \varGamma(f,\mathbf{0})$, the hypersurface $\{f^\sigma = 0\}$ has no singular point in the torus $(\mathbb{C}^*)^2$.

In the above example, we have 3 truncations on one-dimensional faces σ of $\varGamma(f,\mathbf{0})$, $f^\sigma = y^5 + xy^3$, $xy^3 + x^2y^2 + x^3y$ and $x^3y + x^6$, respectively. None of the corresponding hypersurfaces $\{f^\sigma = 0\}$ is singular in $(\mathbb{C}^*)^2$, and the truncations at the 0-dimensional faces are monomials, hence define hypersurfaces having no singular point in the torus $(\mathbb{C}^*)^2$. However f is not Newton non-degenerate, since it is not convenient. In turn, $x^{-1}f$ is NND. On the other hand, $x^{-1}f + xy^2$ is Newton degenerate, since its truncation at the face with slope -1, $y^3 + 2xy^2 + x^2y = y(x+y)^2$, is singular along the line $\{x + y = 0\}$.

Proposition 2.16. *Let $f \in \mathbb{C}\{x_1, \dots x_n\}$ be Newton non-degenerate. Then the Milnor number of f satisfies*

$$\mu(f) = n!\,\mathrm{Vol}_n\big(K_0(f)\big) + \sum_{i=1}^{n} (-1)^{n-i}(n-i)! \cdot \mathrm{Vol}_{n-i}\big(K_0(f) \cap H_{n-i}\big)\,,$$

where H_i denotes the union of all i-dimensional coordinate planes, and where Vol_i *denotes the i-dimensional Euclidean volume.*

For a proof, we refer to [Kou, Thm. I(ii)]. The right-hand side of the formula is called the *Minkowski mixed volume* of the polytope $K_0(f)$, or the *Newton number* of f.

In the above Example 2.14.1, we compute

$$\mu(x^{-1}f) = 2 \cdot \frac{19}{2} - 11 + 1 = 9\,.$$

Note that, in general, the Newton number of f gives a lower bound for $\mu(f)$ (cf. [Kou]).

Another important class of singularities is given by the class of semiquasihomogeneous singularities, which is characterized by means of the Newton diagram, too:

Definition 2.17. A power series $f \in \mathbb{C}\{x_1, \dots, x_n\}$ is called *semiquasihomogeneous (SQH)* at $\mathbf{0}$ (or, $\mathbf{0}$ is called a semiquasihomogeneous point of f) if there is a face $\sigma \subset \Gamma(f, \mathbf{0})$ of dimension $n-1$ (called the *main*, or *principal, face*) such that the truncation f^σ has no critical points in $\mathbb{C}^n \setminus \{\mathbf{0}\}$. f^σ is called the *main part*, or *principal part*, of f.

Note that f^σ is a quasihomogeneous polynomial, hence, it is contained in the ideal generated by its partials. It follows that f^σ has no critical point in $\mathbb{C}^n \setminus \{\mathbf{0}\}$ iff the hypersurface $\{f^\sigma = 0\} \subset \mathbb{C}^n$ has an isolated singularity at $\mathbf{0}$. In other words, due to Lemma 2.3, f is SQH iff we can write

$$f = f_0 + g\,, \quad \mu(f_0) < \infty$$

with $f_0 = f^\sigma$ a quasihomogeneous polynomial of *type* $(\boldsymbol{w}; d)$ and all monomials of g being of $\boldsymbol{w}$-degree at least $d+1$.

We should like to point out that we do not require that the Newton diagram $\Gamma(f, \mathbf{0})$ meets all coordinate axes (as for NND singularities). For instance, $xy + y^3 + x^2y^2 \in \mathbb{C}\{x, y\}$ is SQH with main part $xy + y^3$ (which is $\boldsymbol{w} = (2,1)$-weighted homogeneous of weighted degree 3); but it is not Newton non-degenerate, since the Newton diagram does not meet the x-axis. However, the results of the next section show that each SQH power series is right equivalent to a convenient one.

Note that each convenient SQH power series $f \in \mathbb{C}\{x, y\}$ is NND, while for higher dimensions this is not true. For instance $f = (x+y)^2 + xz + z^2$ is SQH (with $f = f_0$) and convenient, but Newton degenerate (the truncation $(x+y)^2$ at one of the one-dimensional faces has singular points in $(\mathbb{C}^*)^3$).

Corollary 2.18. *Let $f \in \mathbb{C}\{\boldsymbol{x}\}$ be SQH with principal part f_0. Then f has an isolated singularity at $\mathbf{0}$ and $\mu(f) = \mu(f_0)$.*

Proof. Let $f_0 \in \mathbb{C}[\boldsymbol{x}]$ be quasihomogeneous of type $(\boldsymbol{w}; d)$ and write

$$f = f_0 + \sum_{i \geq 1} f_i$$

with f_i quasihomogeneous of type $(\boldsymbol{w}; d+i)$. Clearly, f is singular at $\mathbf{0}$ iff f_0 is singular at $\mathbf{0}$. Consider for $t \in \mathbb{C}$ the unfolding

$$F_t(\boldsymbol{x}) := f_0(\boldsymbol{x}) + \sum_{i \geq 1} t^i f_i(\boldsymbol{x})\,,$$

which satisfies $F_0 = f_0$ and $F_1 = f$. Theorem 2.6 (1) implies that, for $t_0 \neq 0$ sufficiently small, F_{t_0} has an isolated critical point at $\mathbf{0}$. Since, for every $t \in \mathbb{C}^*$,

$$F_t(x_1,\dots,x_n) = \frac{1}{t^d}\cdot f(t^{w_1}x_1,\dots,t^{w_n}x_n)\,,$$

the $\mathbb{C}^*$-action $x \mapsto (t^{w_1}x_1,\dots,t^{w_n}x_n)$ maps

$$\mathrm{Crit}(F_t)\cap\left\{\boldsymbol{x}\;\middle|\;\forall\, i:\; |x_i| < \frac{\varepsilon}{|t|^{w_i}}\right\} \xrightarrow{\cong} \mathrm{Crit}(f)\cap\left\{\boldsymbol{x}\;\middle|\;\forall\, i:\; |x_i|<\varepsilon\right\}.$$

Hence, we can find some $\varepsilon > 0$, independent of t, such that, for all $|t| \leq 1$, $\mathrm{Crit}(F_t : B_\varepsilon(\mathbf{0}) \to \mathbb{C}) = \{\mathbf{0}\}$. Finally, the statement follows from Theorem 2.6 (3). □

Again, the SQH and NND property are both not preserved under analytic coordinate changes, for instance, $x^2 - y^3 \in \mathbb{C}\{x,y\}$ is SQH and NND, but $(x+y)^2 - y^3 \in \mathbb{C}\{x,y\}$ is neither SQH nor NND. Anyhow, we can make the following definition:

Definition 2.19. An isolated hypersurface singularity $(X,x) \subset (\mathbb{C}^n,x)$, is called *Newton non-degenerate* (respectively *semiquasihomogeneous*), if there exists a NND (respectively SQH) power series $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1,\dots,x_n\}$ such that $\mathcal{O}_{X,x} \cong \mathbb{C}\{\boldsymbol{x}\}/\langle f\rangle$.

Exercises

Exercise 2.1.1. Let $p_1,\dots,p_n \in \mathbb{Z}_{\geq 1}$, and let $f = x_1^{p_1} + \ldots + x_n^{p_n} \in \mathbb{C}[\boldsymbol{x}]$. Show that $\mu(f,\mathbf{0}) = (p_1 - 1)\cdot\ldots\cdot(p_n - 1)$.

More generally:

Exercise 2.1.2. Let $p_1,\dots,p_n \in \mathbb{Z}_{\geq 1}$, $\boldsymbol{w} = (w_1,\dots,w_n)$ with $w_i := \prod_{j\neq i} p_j$ and $d := \prod_{i=1}^n p_i$. Moreover, let $f \in \mathbb{C}[x_1,\dots,x_n]$ be a quasi-homogeneous polynomial of type $(\boldsymbol{w};d)$ which has an isolated critical point at the origin. Show that $\mu(f,\mathbf{0}) = (p_1 - 1)\cdot\ldots\cdot(p_n - 1)$.

Exercise 2.1.3. Consider the unfolding

$$f_t(x,y,z) = x^p + y^q + z^r + txyz, \qquad \frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1.$$

Show that for all $t,t' \neq 0$, $f_t \overset{c}{\sim} f_{t'}$ but $f_t \overset{r}{\not\sim} f_{t'}$.

Exercise 2.1.4. Show that $\mu - \tau$ is lower semicontinuous in the following sense: with the notations and under the assumptions of Theorem 2.6, we have $\mu(f,0) - \tau(f,0) \leq \mu(F_{\boldsymbol{t}},0) - \tau(F_{\boldsymbol{t}},0)$.

Exercise 2.1.5. Let $f = f_d + f_{d+1}$, where $f_d, f_{d+1} \in \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1,\dots,x_n]$ are homogeneous polynomials of degree $d, d+1$, respectively. Assume that the system

$$\frac{\partial f_d}{\partial x_1} = \ldots = \frac{\partial f_d}{\partial x_n} = f_{d+1} = 0$$

has the origin as only solution. Show that $\mu(f) = \mu(f, \mathbf{0}) < \infty$. Furthermore, if $n = 2$, show that $\mu(f) = d(d-1) - k$, where k is the number of distinct linear divisors of f_d.

Exercise 2.1.6. (1) Let $f \in \mathbb{C}\{x, y\}$ be of order $d \geq 2$ with a non-degenerate principal form[14] of degree d. Prove the following statements:

- If f is a polynomial of degree at most $d+1$, then

$$\mu(f) - \tau(f) \leq \begin{cases} (k-1)^2, & \text{if } d = 2k+1\,, \\ (k-1)(k-2), & \text{if } d = 2k\,. \end{cases}$$

  Furthermore, show that this bound is sharp for $d \leq 6$.
- In general, we have

$$\mu(f) - \tau(f) \leq \frac{(d-4)(d-3)}{2}\,.$$

(2) Improve the latter bound up to

$$\mu(f) - \tau(f) \leq \sum_{k=1}^{d-5} \min\left\{ \frac{(k+1)(k+2)}{2},\ d-5-k \right\}\,.$$

(3) Generalize the above bounds to semiquasihomogeneous plane curve singularities.

(4)* Generalize the above bounds to higher dimensions, for instance, prove that if $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ is a polynomial of degree at most $d+1$ with zero $(d-1)$-jet and a non-degenerate d-form, then

$$\mu(f) - \tau(f) \leq (k_0+1)^\ell k_0^{n-\ell}, \quad k_0 = \left[\frac{(n-1)d-1}{n}\right] - 2\,,$$

$$(n-1)d - 2n - 1 = nk_0 + \ell$$

(the upper bound is the maximum number of integral points in a parallelepiped with sides parallel to the coordinate axes and inscribed into the simplex

$$\left\{(i_1, \ldots, i_n) \in \mathbb{R}^n \;\middle|\; i_1 + \ldots + i_n \geq d+1,\ \max\{i_1, ..., i_n\} \leq d-2\right\}.$$

Exercise 2.1.7. Under the hypotheses of Corollary 2.18, is it true that $g \overset{\mathrm{r}}{\sim} f$, respectively $g \overset{\mathrm{c}}{\sim} f$?

[14] Let $f \in \mathbb{C}\{x, y\}$. Then we may write $f = f_d + f_{d+1} + \ldots$, where $d = \operatorname{ord}(f)$ and f_k is a homogeneous polynomial of degree k, $k \geq d$. The polynomial f_d is called the *principal form* (or *principal d-form*) of f. It is called *non-degenerate* if the hypersurface $\{f_d = 0\}$ has no critical points in $\mathbb{C}^n \setminus \{\mathbf{0}\}$.

2.2 Finite Determinacy

The aim of this section is to show that an isolated hypersurface singularity is already determined by its Taylor series expansion up to a sufficiently high order.

Definition 2.20. Let $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$. Then

$$\operatorname{jet}(f,k) := f^{(k)} := \text{ image of } f \text{ in } \mathbb{C}\{\boldsymbol{x}\}/\mathfrak{m}^{k+1}$$

denotes the *k-jet* of f and

$$J^{(k)} := \mathbb{C}\{\boldsymbol{x}\}/\mathfrak{m}^{k+1}$$

the complex vector space of all k-jets. We identify $f^{(k)} \in J^{(k)}$ with the power series expansion of f up to (and including) order k.

Definition 2.21. (1) $f \in \mathbb{C}\{\boldsymbol{x}\}$ is called *right k-determined*, respectively *contact k-determined* if for each $g \in \mathbb{C}\{\boldsymbol{x}\}$ with $f^{(k)} = g^{(k)}$ we have $f \overset{r}{\sim} g$, respectively $f \overset{c}{\sim} g$.

(2) The minimal such k is called the *right determinacy*, respectively the *contact determinacy* of f.

(3) A power series f is called *finitely right determined*, respectively *finitely contact determined*, if f is right k-determined, respectively contact k-determined for some k.

The finite determinacy theorem, saying that isolated singularities are finitely determined, will follow from the following theorem, which is fundamental in many respects.

Theorem 2.22 (Infinitesimal characterization of local triviality). *Let $F \in \mathbb{C}\{\boldsymbol{x}, t\} = \mathbb{C}\{x_1, \ldots, x_n, t\}$ and $b \geq 0$, $c \geq 0$ be integers.*

(1) The following are equivalent

(a) $\dfrac{\partial F}{\partial t} \in \langle x_1, \ldots, x_n \rangle^b \cdot \left\langle \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n} \right\rangle + \langle x_1, \ldots, x_n \rangle^c \cdot \langle F \rangle .$

(b) There exist $\phi = (\phi_1, \ldots, \phi_n) \in \mathbb{C}\{\boldsymbol{x}, t\}^n$, $u \in \mathbb{C}\{\boldsymbol{x}, t\}$ satisfying

(i) $u(\boldsymbol{x}, 0) = 1$,

(ii) $u(\boldsymbol{x}, t) - 1 \in \langle x_1, \ldots, x_n \rangle^c \cdot \mathbb{C}\{\boldsymbol{x}, t\}$,

(iii) $\phi_i(\boldsymbol{x}, 0) = x_i$, $i = 1, \ldots, n$,

(iv) $\phi_i(\boldsymbol{x}, t) - x_i \in \langle x_1, \ldots, x_n \rangle^b \cdot \mathbb{C}\{\boldsymbol{x}, t\}$, $i = 1, \ldots, n$,

(v) $u(\boldsymbol{x}, t) \cdot F\big(\phi(\boldsymbol{x}, t), t\big) = F(\boldsymbol{x}, 0)$.

(2) Moreover, the condition

$$\frac{\partial F}{\partial t} \in \langle x_1, \ldots, x_n \rangle^b \cdot \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle$$

is equivalent to (1)(b) with $u = 1$.

Remark 2.22.1. Set $\phi_t(\boldsymbol{x}) = \phi(\boldsymbol{x}, t)$ and $u_t(\boldsymbol{x}) = u(\boldsymbol{x}, t)$. Since $\phi_0 = \mathrm{id}$, the morphism $\phi_t \colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \phi_t(\mathbf{0}))$ is an isomorphism of germs for small t, and, similarly, u_t is a unit in $\mathbb{C}\{\boldsymbol{x}\}$ for small t. If $b > 0$, then $\phi_t(\mathbf{0}) = \mathbf{0}$ for all t, hence ϕ_t is an automorphism of $(\mathbb{C}^n, \mathbf{0})$. If $b = 0$, $\phi_t(\mathbf{0})$ is not necessarily $\mathbf{0}$ but, nevertheless, ϕ_t is biholomorphic for small t, with the origin getting displaced. Now, condition (1)(b) says that ϕ_t induces an isomorphism $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}} \xrightarrow{\cong} \mathcal{O}_{\mathbb{C}^n,\phi_t(\mathbf{0})}$ mapping the ideal $\langle F_0 \rangle$ to $\langle F_t \rangle$. Hence, we get an isomorphism of germs $(F_0^{-1}(0), \mathbf{0}) \cong (F_t^{-1}(0), \phi_t(\mathbf{0}))$ being the identity up to order b.

In the situation of statement (2) we get a commutative diagram of function germs.

$$\begin{tikzcd}
(\mathbb{C}^n, \mathbf{0}) \arrow[rr, "\phi_t", "\cong"'] \arrow[dr, "F_0"'] & & (\mathbb{C}^n, \phi_t(\mathbf{0})) \arrow[dl, "F_t"] \\
& \mathbb{C} &
\end{tikzcd}$$

Example 2.22.2. (1) The unfolding $F(x, y, t) = x^2 + y^3 + tx^\alpha y^\beta$ is right locally trivial if $\alpha + \beta \geq 4$. Namely, we have

$$\frac{\partial F}{\partial t} = x^\alpha y^\beta \in \langle x, y \rangle \cdot \langle 2x + \alpha t x^{\alpha-1} y^\beta, 3y^2 + \beta t x^\alpha y^{\beta-1} \rangle,$$

as the latter ideal is equal to $\langle x, y \rangle \cdot \langle x, y^2 \rangle$. Moreover, as $\mu(x^2 + y^3) = 2$, we shall show in Theorem 2.23 (respectively Corollary 2.24) that $x^2 + y^3$ is 3-determined, hence $F_t \overset{\mathrm{r}}{\sim} x^2 + y^3$ for all t.

(2) *Warning:* It is not sufficient to require

$$\frac{\partial F}{\partial t}\Big|_{t=0} \in \langle x_1, \ldots, x_n \rangle^b \cdot \Big\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \Big\rangle\Big|_{t=0}$$

for local triviality. As an example consider $F_t(x, y) = x^2 + y^3 + txy$. We have $\frac{\partial F}{\partial t} = xy \in \langle x, y \rangle \cdot \langle x, y^2 \rangle$ but $F_t \overset{\mathrm{r}}{\not\sim} F_0$ since $\mu(F_0) = 2$ and $\mu(F_t) = 1$ for $t \neq 0$.

Proof of Theorem 2.22. (1) *(a)* $\Rightarrow$ *(b)*: We write $\langle \boldsymbol{x} \rangle$ instead of $\langle x_1, \ldots, x_n \rangle$. By (a) there exist $Y_1, \ldots, Y_n \in \langle \boldsymbol{x} \rangle^b \cdot \mathbb{C}\{\boldsymbol{x}, t\}$ and $Z \in \langle \boldsymbol{x} \rangle^c \cdot \mathbb{C}\{\boldsymbol{x}, t\}$ such that

$$\frac{\partial F}{\partial t} = -\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \cdot Y_i - Z \cdot F. \tag{2.2.1}$$

Step 1. Set $Y = (Y_1, \ldots, Y_n)$ and let $\phi = (\phi_1, \ldots, \phi_n)$ be the (unique) solution, for t close to 0, of the ordinary differential equation[15]

[15] The theory of (analytic) ordinary differential equations guarantees, besides the existence and uniqueness of a solution, also the analytic dependence on the initial conditions (cf. [CoL]).

$$\frac{\partial\phi}{\partial t}(\boldsymbol{x},t) = Y(\phi(\boldsymbol{x},t),t)\,, \quad \text{initial condition: } \phi(\boldsymbol{x},0)=\boldsymbol{x}\,. \tag{2.2.2}$$

To see that the ϕ_i satisfy (iv) we assume $b \geq 1$ (since for $b=0$ there is nothing to show). Then $Y(\mathbf{0},t)=\mathbf{0}$ for t close to 0 and, hence, $\phi=\mathbf{0}$ is a solution of the ordinary differential equation

$$\frac{\partial\phi}{\partial t}(\mathbf{0},t) = Y(\phi(\mathbf{0},t),t)\,, \quad \text{initial condition: } \phi(\mathbf{0},0)=\mathbf{0}\,.$$

By uniqueness of the solution, $\phi(\mathbf{0},t)=\mathbf{0}$, that is, $\phi_i(\boldsymbol{x},t)\in\langle\boldsymbol{x}\rangle\cdot\mathbb{C}\{\boldsymbol{x},t\}$. Since $Y_i\in\langle\boldsymbol{x}\rangle^b\cdot\mathbb{C}\{\boldsymbol{x},t\}$ it follows that

$$\frac{\partial\phi_i}{\partial t}(\boldsymbol{x},t) = Y_i(\phi(\boldsymbol{x},t),t)\in\langle\boldsymbol{x}\rangle^b\cdot\mathbb{C}\{\boldsymbol{x},t\}\,,$$

and, hence, $\phi_i - x_i\in\langle\boldsymbol{x}\rangle^b\cdot\mathbb{C}\{\boldsymbol{x},t\}$.

Step 2. Set $\psi(\boldsymbol{x},t)=(\phi(\boldsymbol{x},t),t)$. Since the right-hand side of (v) is independent of t, differentiating (v) with respect to t yields

$$\frac{\partial}{\partial t}\Big(u\cdot\big(F\circ\psi\big)(\boldsymbol{x},t)\Big)=0\,.$$

Since (v) holds for $t=0$, the latter equation is in fact equivalent to (v).

Step 3. Let u be the unique solution of the ordinary differential equation

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) = u(\boldsymbol{x},t)\cdot(Z\circ\psi)(\boldsymbol{x},t)\,, \quad \text{initial condition: } u(\boldsymbol{x},0)=1\,. \tag{2.2.3}$$

$Z\in\langle\boldsymbol{x}\rangle^c$ implies $\frac{\partial u}{\partial t}\in\langle\boldsymbol{x}\rangle^c$ and, hence, $u-1\in\langle\boldsymbol{x}\rangle^c$. Using (2.2.1)–(2.2.3) we get

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(u\cdot\big(F\circ\psi\big)\Big) &= \frac{\partial u}{\partial t}\cdot\big(F\circ\psi\big)+u\cdot\frac{\partial(F\circ\psi)}{\partial t}\\
&= u\cdot(Z\circ\psi)\cdot(F\circ\psi)+u\cdot\left(\sum_{i=1}^n\frac{\partial F}{\partial x_i}\circ\psi\cdot\frac{\partial\phi_i}{\partial t}+\frac{\partial F}{\partial t}\circ\psi\right)\\
&= u\cdot\left((Z\cdot F)\circ\psi+\sum_{i=1}^n\frac{\partial F}{\partial x_i}\circ\psi\cdot\frac{\partial\phi_i}{\partial t}-\sum_{i=1}^n\Big(\frac{\partial F}{\partial x_i}\cdot Y_i\Big)\circ\psi-\big(Z\cdot F\big)\circ\psi\right)\\
&= 0\,,
\end{aligned}$$

which completes this part of the proof.

Now, let's prove the implication $(b)\Rightarrow(a)$. Note that $\psi(\boldsymbol{x},t)=\big(\phi(\boldsymbol{x},t),t\big)$ defines an isomorphism of $(\mathbb{C}^n\times\mathbb{C},(\mathbf{0},0))$ since $\phi(\boldsymbol{x},0)=\boldsymbol{x}$. Let $\chi=\psi^{-1}$ be the inverse.

If $b\geq 1$ then $\phi(\mathbf{0},t)=\mathbf{0}$ and, hence, $\chi(\boldsymbol{x},t)\in(\langle\boldsymbol{x}\rangle\cdot\mathbb{C}\{\boldsymbol{x},t\})^{n+1}$. Then (ii) implies $\frac{\partial u}{\partial t}\circ\chi\in\langle\boldsymbol{x}\rangle^c\cdot\mathbb{C}\{\boldsymbol{x},t\}$ and (iv) implies $\frac{\partial\phi_i}{\partial t}\circ\chi\in\langle\boldsymbol{x}\rangle^b\cdot\mathbb{C}\{\boldsymbol{x},t\}$. Differentiation of (v) gives

$$0 = \frac{\partial u}{\partial t} \cdot (F \circ \psi) + u \cdot \left(\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \circ \psi \cdot \frac{\partial \phi_i}{\partial t} \right) + u \cdot \left(\frac{\partial F}{\partial t} \circ \psi \right)$$

and, hence,

$$-\frac{\partial F}{\partial t} = \left(u^{-1} \cdot \frac{\partial u}{\partial t} \circ \chi \right) \cdot F + \sum_{i=1}^{n} \left(\frac{\partial \phi_i}{\partial t} \circ \chi \right) \cdot \frac{\partial F}{\partial x_i} ,$$

which implies (a).

The proof of (2) is a special case of (1): If $Z = 0$ then $u = 1$ is the unique solution of (2.2.3). □

As a corollary we obtain

Theorem 2.23 (Finite determinacy theorem). *Let $f \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$.*

(1) f is right k-determined if

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle . \tag{2.2.4}$$

(2) f is contact k-determined if

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle f \rangle . \tag{2.2.5}$$

Proof. Let k satisfy (2.2.4), respectively (2.2.5), and consider, for $h \in \mathfrak{m}^{k+1}$,

$$F(\boldsymbol{x}, t) = f(\boldsymbol{x}) + t \cdot h(\boldsymbol{x}) \in \mathbb{C}\{\boldsymbol{x}\}[t].$$

Obviously, it suffices to show that for every $t_0 \in \mathbb{C}$ the germ of F in $\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (\mathbf{0}, t_0)}$ satisfies the conditions of (1)(a), respectively (2), in Theorem 2.22 (since then $F_{t_0} \overset{c}{\sim} F_t$, respectively $F_{t_0} \overset{r}{\sim} F_t$, for $|t - t_0|$ small and therefore $f = F_0 \sim F_1 = f + h$). Thus, we have to show that, for contact equivalence,

$$h \in \left(\mathfrak{m}^2 \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle F \rangle \right) \cdot \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (\mathbf{0}, t_0)}$$

with $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. Since $h \in \mathfrak{m}^{k+1}$, $\mathfrak{m}^2 \frac{\partial h}{\partial x_i} + \mathfrak{m} \cdot h \subset \mathfrak{m}^{k+2}$ and, hence,

$$\begin{aligned} &\left(\mathfrak{m}^2 \cdot \left\langle \frac{\partial (f + th)}{\partial x_1}, \ldots, \frac{\partial (f + th)}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle f + th \rangle + \mathfrak{m}^{k+2} \right) \cdot \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (\mathbf{0}, t_0)} \\ &\qquad = \left(\mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle f \rangle + \mathfrak{m}^{k+2} \right) \cdot \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (\mathbf{0}, t_0)} . \end{aligned}$$

The latter module contains $\mathfrak{m}^{k+1}$ by assumption, in particular, it contains h. For right equivalence we just delete the terms $\mathfrak{m}\langle F \rangle$ and $\mathfrak{m}\langle f \rangle$. The claim now follows from Theorem 2.22 and Remark 2.23.1 (1). □

Remark 2.23.1. (1) Nakayama's lemma, applied to $\langle \mathfrak{m}^{k+1}, \mathfrak{m}^2 j(f)\rangle / \mathfrak{m}^2 j(f)$ gives that (2.2.4) is equivalent to

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 j(f) + \mathfrak{m}^{k+2}. \tag{2.2.6}$$

Hence, by passing to $\mathbb{C}\{\boldsymbol{x}\}/\mathfrak{m}^{k+2}$, condition (2.2.4) is a condition on finite dimensional vector spaces. The same applies to condition (2.2.5), which is equivalent to

$$\mathfrak{m}^{k+1} \subset \langle \mathfrak{m}^2 j(f), \mathfrak{m}f, \mathfrak{m}^{k+2}\rangle. \tag{2.2.7}$$

(2) If $f - g \in \mathfrak{m}^{k+1}$ then $\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i} \in \mathfrak{m}^k$ and $j(f) \subset j(g) + \mathfrak{m}^k$. Thus, (2.2.4) (resp. (2.2.5)) for f implies (2.2.6) (resp. (2.2.7)) for g. It follows that the conditions in the finite determinacy theorem depend only on the k-jet of f.

(3) Of course, (2.2.4) (respectively (2.2.5)) is implied by

$$\mathfrak{m}^k \subset \mathfrak{m} \cdot j(f) \quad \big(\text{respectively by } \mathfrak{m}^k \subset \langle f, \mathfrak{m} \cdot j(f)\rangle\big)\,.$$

(4) The theory of standard bases implies that the condition (2.2.4) (respectively (2.2.5)) is fulfilled if every monomial of degree $k+1$ is divisible by the leading monomial of some element of a standard basis of $\mathfrak{m}^2 j(f)$ (respectively of $\langle \mathfrak{m}^2 j(f), \mathfrak{m}f\rangle$) with respect to a local degree ordering (cf. [GrP]). Hence, these determinacy bounds can be computed effectively.

As an immediate consequence of Theorem 2.23, we obtain

Corollary 2.24. *If $f \in \mathbb{C}\{\boldsymbol{x}\}$, $f(\mathbf{0}) = 0$, has an isolated singularity with Milnor number μ and Tjurina number τ, then*

(1) f is right $(\mu+1)$-determined,
(2) f is contact $(\tau+1)$-determined.

Proof. If $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ then $\mu(f) = \tau(f) = 0$ and f is 1-determined by the implicit function theorem. Let $f \in \mathfrak{m}^2$. Then $\dim_{\mathbb{C}} \mathfrak{m}/\langle f, j(f)\rangle = \tau - 1$ and

$$\mathfrak{m}/\langle f, j(f)\rangle \supset \big(\mathfrak{m}^2 + \langle f, j(f)\rangle\big)/\langle f, j(f)\rangle \supset \ldots$$

is a strictly decreasing sequence of vector spaces, hence $\mathfrak{m}^\tau \subset \langle f, j(f)\rangle$. In particular, we obtain $\mathfrak{m}^{\tau+2} \subset \langle \mathfrak{m}^2 j(f), \mathfrak{m}f\rangle$, and (2) follows from Theorem 2.23. The argument for (1) is similar. $\square$

Example 2.24.1. For $f = x^5 + y^5 + x^2y^2$ we computed $\mu = 11$, $\tau = 10$. However, $\mathfrak{m}^6 \subset \mathfrak{m}^2 j(f)$, which can be seen, e.g., using Singular as explained in Remark 2.23.1 (4):

```
ring r=0,(x,y),ds;
poly f=x5+y5+x2y2;
size(reduce(maxideal(6),std(maxideal(2)*jacob(f))));
//-> 0
```

Hence, f is already 5-determined with respect to right and contact equivalence.

As Example 2.24.1 shows, the bounds in Corollary 2.24 are usually quite bad. Nevertheless, they are of great importance, since μ and τ are semicontinuous under deformations by Theorem 2.6.

The conditions for k-determinacy in Theorem 2.23 are sufficient but not necessary. However, they are close to necessary conditions as the following supplement (which follows directly from Theorem 2.22) shows.

Supplement to Theorem 2.23. *With the notation of Theorem 2.23 the following holds:*

(1) $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 j(f)$ *iff, for each* $g \in \mathfrak{m}^{k+1}$, *there is some* $\varphi \in \mathrm{Aut}(\mathbb{C}\{\boldsymbol{x}\})$ *with* $\varphi(\boldsymbol{x}) = \boldsymbol{x} + (\textit{higher order terms})$ *such that* $f \circ \varphi = f + g$.
(2) $\mathfrak{m}^{k+1} \subset \langle \mathfrak{m}^2 j(f), \mathfrak{m} f\rangle$ *iff, for each* $g \in \mathfrak{m}^{k+1}$, *there exists an automorphism* φ *of* $\mathbb{C}\{\boldsymbol{x}\}$ *with* $\varphi(\boldsymbol{x}) = \boldsymbol{x} + (\textit{higher order terms})$ *and a unit* $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ *with* $u(\boldsymbol{0}) = 1$ *such that* $u \cdot (f \circ \varphi) = f + g$.

Lemma 2.25. *Let* $f \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$, $I \subset \mathbb{C}\{\boldsymbol{x}\}$ *an ideal and* $h \in \mathfrak{m} I$ *satisfying*

(i) $\mathfrak{m} I \subset \mathfrak{m}^2 j(f) + \mathfrak{m}\langle f\rangle$, *respectively* $\mathfrak{m} I \subset \mathfrak{m}^2 j(f)$, *and*
(ii) $\left\langle \frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n} \right\rangle \subset I$.

Then f *and* $f + h$ *are contact, respectively right, equivalent.*

The proof is actually identical to that of Theorem 2.23, which is a special case of Lemma 2.25 with $I = \mathfrak{m}^k$.

The following example shows that the conditions in the finite determinacy theorem are in general not necessary for k-determinacy.

Example 2.25.1. Consider the singularity E_7 given by $f = x^3 + xy^3$. We have $\mathfrak{m}^6 \subset \mathfrak{m}^2 j(f) + \mathfrak{m}\langle f\rangle$ but $\mathfrak{m}^5 \not\subset \mathfrak{m}^2 j(f) + \mathfrak{m}\langle f\rangle$ (the element y^5 is missing). The finite determinacy theorem gives that E_7 is 5-determined. However, the determinacy is, indeed, 4. To see this, consider the 5-jet of any unfolding F of $x^3 + xy^3$ with terms of order at least 5:

$$F^{(5)} = x^3 + xy^3 + ty^5 + \beta xy^4 + \gamma x^2 y^3 + ax^3 y^2 + bx^4 y + cx^5 .$$

Substituting y by $y\sqrt[3]{1 + \beta y + \gamma x}$ yields

$$F^{(5)} = \underbrace{x^3 + xy^3 + ty^5}_{=:\, g} + \underbrace{ax^3 y^2 + bx^4 y + cx^5}_{=:\, h} .$$

Using SINGULAR, we compute $y^5 = (1 + \frac{25}{3} t^2 y)^{-1} \cdot \left(y^2 \cdot \frac{\partial g}{\partial x} - (x - \frac{5}{3} t y^2) \cdot \frac{\partial g}{\partial y}\right)$:

```
ring r=(0,t),(x,y),ds;
poly g=x3+xy3+t*y5;
division(y5,jacob(g));
```

```
//-> [1]:
//->    _[1,1]=y2
//->    _[2,1]=-x+(5/3t)*y2
//-> [2]:
//->    _[1]=0
//-> [3]:
//->    _[1,1]=1+(25/3t2)*y
```

In particular, $y^5 \in \mathfrak{m} \cdot j(g)$, and Theorem 2.22 implies $g \overset{\mathrm{r}}{\sim} f$. Now, we introduce the ideal $I = \langle x^4, x^3y, x^2y^2, xy^3\rangle$ and check that $\mathfrak{m}I \subset \mathfrak{m}^2 j(g)$:

```
ideal I=x4,x3y,x2y2,xy3;
size(reduce(maxideal(1)*I,std(maxideal(2)*jacob(g))));
//-> 0
```

Since $h \in \mathfrak{m}I$ and $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \in I$, we get $F^{(5)} = g + h \overset{\mathrm{r}}{\sim} g$ by Lemma 2.25, that is, $F^{(5)} \overset{\mathrm{c}}{\sim} f = F^{(4)}$. We conclude that E_7 is right 4-determined.

We are now going to prove the well-known theorem of Mather and Yau [MaY] stating that the contact class of an isolated hypersurface singularity is already determined by its Tjurina algebra.

Theorem 2.26 (Mather-Yau). *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{x_1, \ldots, x_n\}$. The following are equivalent:*

(a) $f \overset{\mathrm{c}}{\sim} g$.
(b) For all $b \geq 0$, $\mathbb{C}\{\boldsymbol{x}\}/\langle f, \mathfrak{m}^b j(f)\rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\langle g, \mathfrak{m}^b j(g)\rangle$ as $\mathbb{C}$-algebras.
(c) There is some $b \geq 0$ such that $\mathbb{C}\{\boldsymbol{x}\}/\langle f, \mathfrak{m}^b j(f)\rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\langle g, \mathfrak{m}^b j(g)\rangle$ as $\mathbb{C}$-algebras.

In particular, $f \overset{\mathrm{c}}{\sim} g$ iff $T_f \cong T_g$, where $T_f = \mathbb{C}\{\boldsymbol{x}\}/\langle f, j(f)\rangle$ is the Tjurina algebra of f.

Note that the original proof in [MaY] was for $b = 0, 1$ and required f to be an isolated singularity.

Proof. (a) $\Rightarrow$ (b) is just an application of the chain rule, as performed in the proof of Lemma 2.10 (for $b = 0$). The implication (b) $\Rightarrow$ (c) is trivial. Finally, we are left with (c) $\Rightarrow$ (a).

If, for some $b \geq 0$, φ is an isomorphism of the $\mathbb{C}$-algebras in (c), then φ lifts to an isomorphism $\widetilde{\varphi}\colon \mathbb{C}\{\boldsymbol{x}\} \to \mathbb{C}\{\boldsymbol{x}\}$ with $\widetilde{\varphi}\big(\langle f, \mathfrak{m}^b j(f)\rangle\big) = \langle g, \mathfrak{m}^b j(g)\rangle$ (cf. Lemma 1.23). Since $\widetilde{\varphi}\langle f, \mathfrak{m}^b j(f)\rangle = \langle \widetilde{\varphi}(f), \mathfrak{m}^b j(\widetilde{\varphi}(f))\rangle$, we may actually assume that

$$\langle f, \mathfrak{m}^b j(f)\rangle = \langle g, \mathfrak{m}^b j(g)\rangle. \tag{2.2.8}$$

Put $h := g - f$ and consider the family of ideals

$$I_t := \left\langle f + th,\, \mathfrak{m}^b \cdot \left\langle \frac{\partial(f+th)}{\partial x_1}, \ldots, \frac{\partial(f+th)}{\partial x_n} \right\rangle \right\rangle \subset \mathbb{C}\{\boldsymbol{x}, t\}\,, \quad t \in \mathbb{C}\,.$$

Due to (2.2.8), $I_t \subset I_0 \cdot \mathbb{C}\{\boldsymbol{x}, t\} = \big\langle f, \mathfrak{m}^b j(f)\big\rangle \cdot \mathbb{C}\{\boldsymbol{x}, t\}$ and $I_1 = I_0$.

Now, represent f, g in a neighbourhood $V = V(\mathbf{0}) \subset \mathbb{C}^n$ by holomorphic functions and consider the coherent $\mathcal{O}_{V\times\mathbb{C}}$-module

$$\mathcal{F} := \Big\langle f, \mathfrak{m}^b \cdot \Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big\rangle\Big\rangle \Big/ \Big\langle f+th, \mathfrak{m}^b \cdot \Big\langle \frac{\partial(f+th)}{\partial x_1}, \dots, \frac{\partial(f+th)}{\partial x_n}\Big\rangle\Big\rangle,$$

whose support is a closed analytic set in $V\times\mathbb{C}$ (A.7). Moreover, note that

$$\operatorname{supp}(\mathcal{F}) \cap (\{\mathbf{0}\}\times\mathbb{C}) = \big\{t \in \mathbb{C} \,\big|\, \mathcal{F}_{(\mathbf{0},t)} \neq 0\big\} = \big\{t \in \mathbb{C} \,\big|\, I_0 \neq I_t\big\},$$

which is a closed analytic, hence a discrete, set of points in $\mathbb{C} = \{\mathbf{0}\}\times\mathbb{C}$. It follows that the set $U = \big\{t \in \mathbb{C} \,\big|\, I_t = I_0\big\}$ is open and connected and contains 0 and 1. Hence,

$$\frac{\partial(f+th)}{\partial t} = h \in I_0 = I_t = \big\langle f+th, \mathfrak{m}^b \cdot j(f+th)\big\rangle$$

for all $t \in U$, and, by Theorem 2.22, we get that $f+th \overset{c}{\sim} f+t'h$ for $t, t' \in U$ such that $|t-t'|$ is sufficiently small. Hence, $f+th \overset{c}{\sim} f$ for all t in U, in particular, $f \overset{c}{\sim} g$. $\square$

Corollary 2.27. *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{x_1, \dots, x_n\}$ with f defining an isolated singularity.*

(1) If $\langle g, j(g)\rangle \subset \langle f, j(f)\rangle$ then $f+tg \overset{c}{\sim} f$ for almost all $t \in \mathbb{C}$.
(2) If $\langle g, j(g)\rangle \subset \mathfrak{m}\cdot\langle f, j(f)\rangle$ then $f+tg \overset{c}{\sim} f$ for all $t \in \mathbb{C}$.

Proof. By assumption, there exists a matrix $A(\boldsymbol{x}) = (a_{ij})_{i,j=0\dots n}$ such that

$$\Big(f+tg, \frac{\partial(f+tg)}{\partial x_1}, \dots, \frac{\partial(f+tg)}{\partial x_n}\Big) = \Big(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big) \cdot \big(\mathbf{1} + tA(\boldsymbol{x})\big).$$

In Case (1) $\det(\mathbf{1}+tA(\mathbf{0}))$ vanishes for at most $n+1$ values of t, while in Case (2) we have $\det(\mathbf{1}+tA(\mathbf{0})) = 1$ for all t (since $a_{ij} \in \mathfrak{m}$). Since the Tjurina ideals $\langle f, j(f)\rangle$ and $\langle f+tg, j(f+tg)\rangle$ coincide if $\det(\mathbf{1}+tA(\mathbf{0})) \neq 0$, (1) and (2) follow from Theorem 2.26.

Remark 2.27.1. It is in general not true that f is right equivalent to g if the Milnor algebras M_f and M_g are isomorphic.

For example, $F_t(x,y) = x^4 + y^5 + t\cdot x^2y^3$ satisfies $F_t \overset{r}{\sim} F_1$ for only finitely many t. However, for $t \neq 0$, the assignment $\varphi_t(x) = x/\sqrt{t^5}$, $\varphi_t(y) = y/t^2$ defines isomorphisms $\varphi_t\colon \mathbb{C}\{x,y\} \to \mathbb{C}\{x,y\}$ satisfying $\varphi_t(j(F_t)) = j(F_1)$. Hence, all Milnor algebras M_{F_t}, $t \neq 0$, are isomorphic.

However, if we impose more structure on M_f than just the $\mathbb{C}$-algebra structure, we obtain an analogue of the Mather-Yau theorem for right equivalence: we equip the Milnor algebra $M_{f,b} = \mathbb{C}\{\boldsymbol{x}\}/\mathfrak{m}^b j(f)$ with a $\mathbb{C}\{t\}$-algebra structure via $\mathbb{C}\{t\} \to M_{f,b}$, $t \mapsto f \bmod \mathfrak{m}^b j(f)$, then the following holds true:

Theorem 2.28. *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{x_1, \ldots, x_n\}$ be hypersurface singularities. Then the following are equivalent:*

(a) $f \overset{\mathrm{r}}{\sim} g$.
(b) For all $b \geq 0$, $\mathbb{C}\{\boldsymbol{x}\}/\langle \mathfrak{m}^b j(f)\rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\langle \mathfrak{m}^b j(g)\rangle$ *as* $\mathbb{C}\{t\}$*-algebras.*
(c) For some $b \geq 0$, $\mathbb{C}\{\boldsymbol{x}\}/\langle \mathfrak{m}^b j(f)\rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\langle \mathfrak{m}^b j(g)\rangle$ *as* $\mathbb{C}\{t\}$*-algebras.*

In particular, $f \overset{\mathrm{r}}{\sim} g \Longleftrightarrow M_f \cong M_g$ *are isomorphic as* $\mathbb{C}\{t\}$*-algebras.*

Proof. The proof is an easy adaptation of Theorem 2.26 and left as Exercise 2.2.5. □

Note that in the above example, we have $\varphi_t(F_t) = t^{-10} \cdot F_1$, hence, φ_t is not a $\mathbb{C}\{t\}$-algebra morphism.

There is another theorem, due to Shoshitaishvili [Sho], which says that the Milnor algebra, as $\mathbb{C}$-algebra, determines f up to right equivalence if f is quasihomogeneous.

Theorem 2.29. *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ have isolated singularities. Then:*

(1) If f is quasihomogeneous, then, for all $g \in \mathfrak{m}$,

$$f \overset{\mathrm{r}}{\sim} g \iff M_f \cong M_g \text{ as } \mathbb{C}\text{-algebras}. \tag{2.2.9}$$

(2) Conversely if "$\Leftarrow$" of (2.2.9) holds for all $g \in \mathfrak{m}$, then f is quasihomogeneous.

All definitions in this section also make sense if we work over fields K of any characteristic. However, Theorem 2.23 does not hold for $\mathrm{char}(K) > 0$, not even the statement about contact equivalence. Instead we have (cf. [GrK1]):

Remark 2.29.1. If $f \in K\langle \boldsymbol{x}\rangle$, $\mathrm{char}(K) > 0$, then f is right $2\mu(f)$-determined. and contact $2\tau(f)$-determined.

Exercises

Exercise 2.2.1. Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$, and assume that f has an isolated singularity. Moreover, assume that $g \in \mathfrak{m}I$, where $I \subset \mathbb{C}\{\boldsymbol{x}\}$ denotes the ideal of all power series h satisfying $\langle h, j(h)\rangle \subset \langle f, j(f)\rangle$. Prove that $f + tg \overset{\mathrm{c}}{\sim} f$ for all $t \in \mathbb{C}$.

Exercise 2.2.2. Prove the claims of Remark 2.27.1.

Exercise 2.2.3. Show that the degree of the contact (resp., right) determinacy of isolated hypersurface singularities is a contact (resp., right) equivalence invariant.

Exercise 2.2.4. Show that the degree of contact or right determinacy is not upper semicontinuous (see Theorem 2.6 for μ and τ).

HINT. Show that $x^{2p} - y^{2p}$ is $4p$-determined, while $(ty - x^p)^2 - y^{2p}$ is not $4p$-determined for sufficiently large p.

Exercise 2.2.5. Prove Theorem 2.28.

HINT. Show first that if $\mathfrak{m}^b j(f) = \mathfrak{m}^b j(g)$ and $f - g \in \mathfrak{m}^b j(f)$ then $f \overset{r}{\sim} g$ as in the proof of Theorem 2.26. Then show that the assumptions in (c) allow to reduce to this situation.

Exercise 2.2.6. Prove Theorem 2.29.

Exercise 2.2.7. (1) Show that any germ $f \in \mathfrak{m}^d \subset \mathbb{C}\{x, y\}$ with a non-degenerate principal d-form is right $(2d-2)$-determined.

(2)* Show that any germ $f \in \mathfrak{m}^d \subset \mathbb{C}\{x_1, \ldots, x_n\}$, $n \geq 2$, with a non-degenerate principal d-form is right $(nd - 2n + 2)$-determined.

HINT. Use the fact that the Jacobian ideal of a non-degenerate d-form in n variables contains $\mathfrak{m}^{nd-2n+1}$.

2.3 Algebraic Group Actions

The classification with respect to right, respectively contact, equivalence may be considered in terms of algebraic group actions.

Definition 2.30. The group $\mathcal{R} := \mathrm{Aut}(\mathbb{C}\{\boldsymbol{x}\})$ of automorphisms of the analytic algebra $\mathbb{C}\{\boldsymbol{x}\}$ is called the *right group*. The *contact group* is the semidirect product $\mathcal{K} := \mathbb{C}\{\boldsymbol{x}\}^* \ltimes \mathcal{R}$ of $\mathcal{R}$ with the group of units of $\mathbb{C}\{\boldsymbol{x}\}$, where the product in $\mathcal{K}$ is defined by

$$(u', \varphi')(u, \varphi) = (u'\varphi'(u), \varphi'\varphi).$$

These groups act on $\mathbb{C}\{\boldsymbol{x}\}$ by

$$\begin{array}{rcl} \mathcal{R} \times \mathbb{C}\{\boldsymbol{x}\} & \longrightarrow & \mathbb{C}\{\boldsymbol{x}\}\,, \\ (\varphi, f) & \longmapsto & \varphi(f)\,, \end{array} \qquad \begin{array}{rcl} \mathcal{K} \times \mathbb{C}\{\boldsymbol{x}\} & \longrightarrow & \mathbb{C}\{\boldsymbol{x}\}\,, \\ ((u, \varphi), f) & \longmapsto & u \cdot \varphi(f). \end{array}$$

We have

$$f \overset{r}{\sim} g \iff f \in \mathcal{R} \cdot g\,, \qquad f \overset{c}{\sim} g \iff f \in \mathcal{K} \cdot g,$$

where $\mathcal{R} \cdot g$ (respectively $\mathcal{K} \cdot g$) denotes the *orbit* of g under $\mathcal{R}$ (respectively $\mathcal{K}$), that is, the image of $\mathcal{R} \times \{f\}$, respectively $\mathcal{K} \times \{f\}$, in $\mathbb{C}\{\boldsymbol{x}\}$ under the maps defined above.

Neither $\mathcal{R}$ nor $\mathcal{K}$ are algebraic groups or Lie groups, since they are infinite dimensional. Therefore we pass to the k-jets of these groups

$$\mathcal{R}^{(k)} := \big\{ \mathrm{jet}(\varphi, k) \,\big|\, \varphi \in \mathcal{R} \big\}, \;\; \mathcal{K}^{(k)} := \big\{ \big(\mathrm{jet}(u, k), \mathrm{jet}(\varphi, k)\big) \,\big|\, (u, \varphi) \in \mathcal{K} \big\},$$

where $\mathrm{jet}(\varphi,k)(x_i)=\mathrm{jet}(\varphi(x_i),k)$ is the truncation of the power series of the component functions of φ.

As we shall show below, $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ are algebraic groups acting algebraically on the jet space $J^{(k)}$, which is a finite dimensional complex vector space. The action is given by

$$\varphi\cdot f=\mathrm{jet}(\varphi(f),k)\,,\qquad (u,\varphi)\cdot f=\mathrm{jet}(u\cdot\varphi(f),k)\,,$$

for $\varphi\in\mathcal{R}^{(k)}$, $(u,\varphi)\in\mathcal{K}^{(k)}$. Hence, we can apply the theory of algebraic groups to the action of $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$. If k is bigger or equal to the determinacy of g (see Definition 2.21), then $g\overset{r}{\sim}f$ (respectively $g\overset{c}{\sim}f$) iff $g\in\mathcal{R}^{(k)}f$ (respectively iff $g\in\mathcal{K}^{(k)}f$). Hence, the orbits of these algebraic groups are in one-to-one correspondence with the corresponding equivalence classes.

Before we make use of this point of view, we recall some basic facts about algebraic group actions. For a detailed study we refer to [Bor, Spr, Kra].

Definition 2.31. (1) An (affine) *algebraic group* G (over an algebraically closed field K) is a reduced (affine) algebraic variety over K, which is also a group such that the group operations are morphisms of varieties. That is, there exists an element $e\in G$ (the unit element) and morphisms of varieties over K

$$\begin{aligned} G\times G\longrightarrow G\,,&\quad (g,h)\mapsto g\cdot h\quad\text{(the multiplication)},\\ G\longrightarrow G\,,&\quad g\mapsto g^{-1}\quad\text{(the inverse)}\end{aligned}$$

satisfying the usual group axioms.

(2) A *morphism of algebraic groups* is a group homomorphism, which is also a morphism of algebraic varieties over K.

Example 2.31.1. (1) $\mathrm{GL}(n,K)$ and $\mathrm{SL}(n,K)$ are affine algebraic groups.

(2) For any field K, the additive group $(K,+)$ and the multiplicative group $(K^*,\cdot)$ of K are affine algebraic groups.

(3) The groups $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ are algebraic groups for any $k\geq 1$. This can be seen as follows: an element φ of $\mathcal{R}^{(k)}$ is uniquely determined by

$$\varphi^{(i)}:=\varphi(x_i)=\sum_{j=1}^{n}a_j^{(i)}x_j+\sum_{|\alpha|=2}^{k}a_\alpha^{(i)}\boldsymbol{x}^\alpha\,,\quad i=1,\dots,n,$$

such that $\det\big(a_j^{(i)}\big)\neq 0$. Hence, $\mathcal{R}^{(k)}$ is an open subset of a finite dimensional K-vectorspace (with coordinates the coefficients $a_j^{(i)}$ and $a_\alpha^{(i)}$). It is affine, since it is the complement of the hypersurface defined by the determinant.

The elements of the contact group $\mathcal{K}^{(k)}$ are given by pairs (u,φ), $\varphi\in\mathcal{R}^{(k)}$, $u=u_0+\sum_{|\alpha|=1}^{k}u_\alpha\boldsymbol{x}^\alpha$ with $u_0\neq 0$, hence $\mathcal{K}^{(k)}$ is also open in some finite dimensional vectorspace and an affine variety.

The group operations are morphisms of affine varieties, since the component functions are rational functions. Indeed the coefficients of $\varphi \cdot \psi$ are polynomials in the coefficients of φ, ψ, while the coefficients of φ^{-1} are determined by solving linear equations and involve $\det\left(a_j^{(i)}\right)$ (respectively $\det\left(a_j^{(i)}\right)$ and u_0) in the denominator.

Proposition 2.32. *Every algebraic group G is a smooth variety.*

Proof. Since G is a reduced variety, it contains smooth points by Corollary 1.111. For any $g \in G$ the translation $h \mapsto gh$ is an automorphism of G and in this way G acts transitively on G. Hence, a smooth point can be moved to any other point of G by some automorphism of G. $\square$

Definition 2.33. (1) An (*algebraic*) *action* of G on an algebraic variety X is given by a morphism of varieties

$$G \times X \longrightarrow X\,, \quad (g, x) \mapsto g \cdot x,$$

satisfying $ex = x$ and $(gh)x = g(hx)$ for all $g, h \in G$, $x \in X$.
(2) The *orbit* of $x \in X$ under the action of G on X is the subset

$$Gx := G \cdot x := \{g \cdot x \in X \mid g \in G\} \subset X\,,$$

that is, the image of $G \times \{x\}$ in X under the orbit map $G \times X \to X$.
(3) G acts *transitively* on X if $Gx = X$ for some (and then for any) $x \in X$.
(4) The *stabilizer* of $x \in X$ is the subgroup $G_x := \{g \in G \mid gx = x\}$ of G, that is, the preimage of x under the induced map $G \times \{x\} \to X$.

In this sense, $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ act algebraically on $J^{(k)}$. Note that the somehow unexpected multiplication on $\mathcal{K}^{(k)}$ as a semidirect product (and not just as direct product) was introduced in order to guarantee $(gh)x = g(hx)$ (check this!).

For the classification of singularities we need the following important properties of orbits.

Theorem 2.34. *Let G be an affine algebraic group acting on an algebraic variety X, and $x \in X$ an arbitrary point. Then*

(1) Gx is open in its (Zariski-) closure $\overline{Gx}$.
(2) Gx is a smooth subvariety of X.
(3) $\overline{Gx} \setminus Gx$ is a union of orbits of smaller dimension.
(4) G_x is a closed subvariety of G.
(5) If G is connected, then $\dim(Gx) = \dim(G) - \dim(G_x)$.

Proof. (1) By Theorem 2.35, below, Gx contains an open dense subset of $\overline{Gx}$; in particular, it contains interior points of $\overline{Gx}$. For any $g \in G$, $g \cdot \overline{Gx}$ is closed and contains Gx. Hence, $\overline{Gx} \subset g \cdot \overline{Gx}$. Replacing g with g^{-1} and then multiplying with g we also obtain $g \cdot \overline{Gx} \subset \overline{Gx}$. It follows that $\overline{Gx} = g \cdot \overline{Gx}$, that is, $\overline{Gx}$ is stable under the action of G.

Now, consider the induced action of G on $\overline{Gx}$. Since G acts transitively on Gx and Gx contains an interior point of its closure, every point of Gx is an interior point of $\overline{Gx}$, that is, Gx is open in its closure.

(2) Gx with its reduced structure contains a smooth point and, hence, it is smooth everywhere by homogeneity (see the proof of Proposition 2.32).

(3) $\overline{Gx} \setminus Gx$ is closed, of dimension strictly smaller than $\dim Gx$ and G-stable, hence a union of orbits.

(4) follows, since G_x is the fibre, that is, the preimage of a closed point, of a morphism, and since morphisms are continuous maps.

(5) Consider the map $f\colon G \times \{x\} \to \overline{Gx}$ induced by $G \times \{x\} \to X$. Then f is dominant and $G_x = f^{-1}(x)$. Since G is connected, $G \times \{x\}$ and $\overline{Gx}$ are both irreducible. Since for $y = gx \in Gx$ we have $Gx = Gy$ and $G_y = gG_xg^{-1}$, the statement is independent of the choice of $y \in Gx$. Hence, the result follows from (2) of the following theorem. □

We recall that a morphism $f\colon X \to Y$ of algebraic varieties is called *dominant* if for any open dense set $U \subset Y$, $f^{-1}(U)$ is dense in X. When we study the (non-empty) fibres $f^{-1}(y)$ of any morphism $f\colon X \to Y$ we may replace Y by $\overline{f(X)}$, that is, we may assume that $f(X)$ is dense in Y. If X and Y are irreducible, then f is dominant iff $\overline{f(X)} = Y$.

The following theorem concerning the dimension of the fibres of a morphism of algebraic varieties has many applications.

Theorem 2.35. *Let $f\colon X \to Y$ be a dominant morphism of irreducible varieties, $W \subset Y$ an irreducible, closed subvariety and Z an irreducible component of $f^{-1}(W)$. Put $r = \dim X - \dim Y$.*

(1) If Z dominates W then $\dim Z \geq \dim W + r$. In particular, for $y \in f(X)$, any irreducible component of $f^{-1}(y)$ has dimension $\geq r$.
(2) There is an open dense subset $U \subset Y$ (depending only on f) such that $U \subset f(X)$ and $\dim Z = \dim W + r$ or $Z \cap f^{-1}(U) = \emptyset$. In particular, for $y \in U$, any irreducible component of $f^{-1}(y)$ has dimension equal to r.
(3) If X and Y are affine, then the open set U in (2) may be chosen such that $f\colon f^{-1}(U) \to U$ factors as follows

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "f"'] & & U \times \mathbb{A}^r \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

with π finite and pr_1 the projection onto the first factor.

Proof. See [Mum1, Ch. I, §8] and [Spr, Thm. 4.1.6]. □

Observe that the theorem implies that for dominant morphisms $f\colon X \to Y$ there is an open dense subset U of Y such that $U \subset f(X) \subset Y$.

Recall that a morphism $f\colon X \to Y$ of algebraic varieties with algebraic structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ is *finite* if there exists a covering of Y by open, affine varieties U_i such that for each i, $f^{-1}(U_i)$ is affine and such that $\mathcal{O}_X\left(f^{-1}(U_i)\right)$ is a finitely generated $\mathcal{O}_Y(U_i)$-module.

If $f\colon X \to Y$ is finite, then the following holds:

(1) f is a closed map,
(2) for each $y \in Y$ the fibre $f^{-1}(y)$ is a finite set,
(3) for *every* open affine set $U \subset Y$, $f^{-1}(U)$ is affine and $\mathcal{O}_X\left(f^{-1}(U)\right)$ is a finitely generated $\mathcal{O}_Y(U)$-module.
(4) If X and Y are affine, then f is surjective if and only if the induced map of coordinate rings $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is injective.
(5) Moreover, if $f\colon X \to Y$ is a dominant morphism of irreducible varieties and $y \in f(X)$ such that $f^{-1}(y)$ is a finite set, then there exists an open, affine neighbourhood U of y in Y such that $f^{-1}(U)$ is affine and $f\colon f^{-1}(U) \to U$ is finite.

For proofs see [Mum1, Ch. I, §7], [Spr, Ch. 4.2] and [Har, Ch. II, Exe. 3.4–3.7].

Now, let $f\colon X \to Y$ be a morphism of complex algebraic varieties and let $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the induced morphism of complex spaces. It follows that f finite implies that f^{an} is finite. The converse, however, is not true (see [Har, Ch. II, Exe. 3.5(c)]).

Let $f\colon X \to Y$ be a morphism of algebraic varieties, $x \in X$ a point and $y = f(x)$. Then the induced map of local rings $f^{\#}\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces a K-linear map $\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2 \to \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ of the cotangent spaces and, hence, its dual is a K-linear map of tangent spaces

$$T_x f\colon T_x X \longrightarrow T_y Y$$

where $T_x X = \mathrm{Hom}_K(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, K)$ is the Zariski tangent space of X at x.

Observe that the cotangent and, hence, the tangent spaces coincide, independently of whether we consider X as an algebraic variety or as a complex space. Hence, if $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is the induced map of complex spaces, then the induced map $(f^{\mathrm{an}})^{\#}\colon \mathcal{O}_{Y^{\mathrm{an}},y} \to \mathcal{O}_{X^{\mathrm{an}},x}$ induces the same map as $f^{\#}$ on the cotangent spaces and, hence, on the Zariski tangent spaces.

Proposition 2.36. *Let $f\colon X \to Y$ be a dominant morphism of reduced, irreducible complex algebraic varieties. Then there is an open dense subset $V \subset X$ such that for each $x \in V$ the map $T_x f\colon T_x X \longrightarrow T_{f(x)} Y$ is surjective.*

Proof. By Theorem 2.35 there is an open dense subset $U \subset Y$ such that the restriction $f\colon f^{-1}(U) \to U$ is surjective.

By deleting the proper closed set $A = \mathrm{Sing}(f^{-1}(U)) \cup f^{-1}(\mathrm{Sing}(U))$ and considering $f\colon f^{-1}(U) \setminus A \to U \setminus \mathrm{Sing}(U)$, we obtain a map f between complex manifolds. The tangent map of f is just given by the (transpose of the)

Jacobian matrix of f with respect to local analytic coordinates, which is surjective on the complement of the vanishing locus of all maximal minors. $\square$

Another corollary of Theorem 2.35 is the theorem of Chevalley. For this recall that a subset Y of a topological space X is called *constructible* if it is a finite union of locally closed subsets of X. We leave it as an exercise to show that a constructible set Y contains an open dense subset of $\overline{Y}$. Moreover, the system of constructible subsets is closed under the Boolean operations of taking finite unions, intersections and differences.

If X is an algebraic variety (with Zariski topology) and $Y \subset X$ is constructible, then $Y = \bigcup_{i=1}^{s} L_i$ with L_i locally closed, and we can define the *dimension* of Y as the maximum of $\dim L_i$, $i = 1, \ldots, s$. The following theorem is a particular property of algebraic varieties. In general, it does not hold for complex analytic varieties.

Theorem 2.37 (Chevalley). *Let $f\colon X \to Y$ be any morphism of algebraic varieties. Then the image of any constructible set is constructible. In particular, $f(X)$ contains an open dense subset of $\overline{f(X)}$.*

Proof. It is clear that the general case follows if we show that $f(X)$ is constructible. Since X is a finite union of irreducible varieties, we may assume that X is irreducible. Moreover, replacing Y by $\overline{f(X)}$ we may assume that Y is irreducible and that f is dominant.

We prove the theorem now by induction on $\dim Y$, the case $\dim Y = 0$ being trivial. Let the open set $U \subset Y$ be as in Theorem 2.35, then $Y \setminus U$ is closed of strictly smaller dimension. By induction hypothesis, $f(f^{-1}(Y \setminus U))$ is constructible in $Y \setminus U$ and hence in Y. Then $f(X) = U \cup f(f^{-1}(Y \setminus U))$ is constructible. $\square$

We return to the action of $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ on $J^{(k)} = \mathbb{C}\{x_1, \ldots, x_n\}/\mathfrak{m}^{k+1}$, the affine space of k-jets. Note that $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ are both connected as they are complements of hypersurfaces in some $\mathbb{C}^N$.

Proposition 2.38. *Let G be either $\mathcal{R}^{(k)}$, or $\mathcal{K}^{(k)}$, and for $f \in J^{(k)}$ let Gf be the orbit of f under the action of G on $J^{(k)}$. We denote by $T_f(Gf)$ the tangent space to Gf at f, considered as a linear subspace of $J^{(k)}$. Then, for $k \geq 1$,*

$$\begin{aligned} T_f(\mathcal{R}^{(k)}f) &= \big(\mathfrak{m} \cdot j(f) + \mathfrak{m}^{k+1}\big)/\mathfrak{m}^{k+1}, \\ T_f(\mathcal{K}^{(k)}f) &= \big(\mathfrak{m} \cdot j(f) + \langle f \rangle + \mathfrak{m}^{k+1}\big)/\mathfrak{m}^{k+1}. \end{aligned}$$

Proof. Note that the orbit map and translation by $g \in G$ induce a commutative diagram

$$\begin{tikzcd} T_eG \arrow[r] \arrow[d, "\cong"'] & T_f(Gf) \arrow[d, "\cong"] \\ T_gG \arrow[r] & T_{gf}(Gf)\,. \end{tikzcd}$$

Since the orbit map $G \times \{f\} \to Gf$ satisfies the assumptions of Proposition 2.36, $T_gG \to T_{gf}(Gf)$ and, hence, $T_eG \to T_f(Gf)$ are surjective. Hence, the tangent space to the orbit at f is the image of the tangent map at $e \in G$ of the map $\mathcal{R}^{(k)} \to J^{(k)}$, $\Phi \mapsto f \circ \Phi$, respectively $\mathcal{K}^{(k)} \to J^{(k)}$, $(u, \Phi) \mapsto u \cdot (f \circ \Phi)$.

Let us treat only the contact group (the statement for the right group follows with $u \equiv 1$): consider a curve $t \mapsto (u_t, \Phi_t) \in \mathcal{K}^{(k)}$ such that $u_0 = 1$, $\Phi_0 = \mathrm{id}$, that is,

$$\begin{aligned}\Phi(\boldsymbol{x}, t) &= \boldsymbol{x} + \varepsilon(\boldsymbol{x}, t) \colon\ (\mathbb{C}^n \times \mathbb{C}, (\mathbf{0}, 0)) \longrightarrow (\mathbb{C}^n, \mathbf{0})\\ u(\boldsymbol{x}, t) &= 1 + \delta(\boldsymbol{x}, t) \colon\ (\mathbb{C}^n \times \mathbb{C}, (\mathbf{0}, 0)) \longrightarrow\ \mathbb{C}\,,\end{aligned}$$

with $\varepsilon(\boldsymbol{x}, t) = \varepsilon^1(\boldsymbol{x})\, t + \varepsilon^2(\boldsymbol{x})\, t^2 + \dots$, $\varepsilon^i = (\varepsilon^i_1, \dots, \varepsilon^i_n)$ such that $\varepsilon^i_j \in \mathfrak{m}$, and $\delta(\boldsymbol{x}, t) = \delta_1(\boldsymbol{x})\, t + \delta_2(\boldsymbol{x})\, t^2 + \dots$, $\delta_i \in \mathbb{C}\{\boldsymbol{x}\}$. The image of the tangent map are all vectors of the form

$$\begin{aligned}&\frac{\partial}{\partial t}\Big((1 + \delta(\boldsymbol{x}, t)) \cdot f(\boldsymbol{x} + \varepsilon(\boldsymbol{x}, t)\Big)\Big|_{t=0} \mod \mathfrak{m}^{k+1}\\ &\qquad = \delta_1(\boldsymbol{x}) \cdot f(\boldsymbol{x}) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\boldsymbol{x}) \cdot \varepsilon^1_j(\boldsymbol{x}) \mod \mathfrak{m}^{k+1},\end{aligned}$$

which proves the claim. $\square$

Of course, instead of using analytic curves, we could have used the interpretation of the Zariski tangent space T_xX as morphisms $T_\varepsilon \to X$, where $T_\varepsilon = \mathrm{Spec}\,\big(\mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle\big)$.

In view of Proposition 2.38 we call $\mathfrak{m} \cdot j(f)$, respectively $\mathfrak{m} \cdot j(f) + \langle f\rangle$ the *tangent space at f to the orbit* of f under the right action $\mathcal{R} \times \mathbb{C}\{\boldsymbol{x}\} \to \mathbb{C}\{\boldsymbol{x}\}$, respectively the contact action $\mathcal{K} \times \mathbb{C}\{\boldsymbol{x}\} \to \mathbb{C}\{\boldsymbol{x}\}$.

Corollary 2.39. *For $f \in \mathbb{C}\{x_1, \dots, x_n\}$, $f(\mathbf{0}) = 0$, the following are equivalent.*

(a) f has an isolated critical point.
(b) f is right finitely-determined.
(c) f is contact finitely-determined.

Proof. (a) $\Rightarrow$ (b). By Corollary 2.24, f is $\mu(f)+1$-determined. On the other hand, $\mu(f) < \infty$ due to Lemma 2.3. Since the implication (b) $\Rightarrow$ (c) is trivial, we are left with (c) $\Rightarrow$ (a). Let f be contact k-determined and $g \in \mathfrak{m}^{k+1}$. Then $f_t = f + tg \in \mathcal{K}^{(k+1)}f \mod \mathfrak{m}^{k+2}$ and, hence,

$$g = \frac{\partial f_t}{\partial t}\Big|_{t=0} \in \mathfrak{m} \cdot j(f) + \langle f\rangle \mod \mathfrak{m}^{k+2}\,,$$

by Proposition 2.38. By Nakayama's lemma $\mathfrak{m}^{k+1} \subset \mathfrak{m} \cdot j(f) + \langle f\rangle$, the latter being contained in $j(f) + \langle f\rangle$. Hence, $\tau(f) < \infty$ and f has an isolated critical point by Lemma 2.3. $\square$

Lemma 2.40. *Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1,\dots,x_n\}$ be an isolated singularity. Let k satisfy $\mathfrak{m}^{k+1} \subset \mathfrak{m}\cdot j(f)$, respectively $\mathfrak{m}^{k+1} \subset \mathfrak{m}\cdot j(f) + \langle f\rangle$, and call*

$$\begin{aligned} r\text{-}\operatorname{codim}(f) &:= \text{ codimension of } \mathcal{R}^{(k)}f \text{ in } J^{(k)}\,, \text{ respectively} \\ c\text{-}\operatorname{codim}(f) &:= \text{ codimension of } \mathcal{K}^{(k)}f \text{ in } J^{(k)} \end{aligned}$$

the codimension of the orbit *of f in $J^{(k)}$ under the action of $\mathcal{R}^{(k)}$, respectively $\mathcal{K}^{(k)}$. Then*

$$r\text{-}\operatorname{codim}(f) = \mu(f) + n\,, \qquad c\text{-}\operatorname{codim}(f) = \tau(f) + n.$$

Proof. In view of Proposition 2.38 and the definition of $\mu(f)$ and $\tau(f)$, one has to show that

$$\dim_{\mathbb{C}}\big(j(f)/\mathfrak{m}j(f)\big) = \dim_{\mathbb{C}}(j(f)+\langle f\rangle)/(\mathfrak{m}j(f)+\langle f\rangle) = n\,. \tag{2.3.1}$$

Both linear spaces in question are generated by the partials $\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}$, and it is sufficient to prove that none of these derivatives belongs to the ideal $\mathfrak{m}j(f)+\langle f\rangle$. Arguing to the contrary, assume that $\frac{\partial f}{\partial x_1} \in \mathfrak{m}j(f)+\langle f\rangle$. This implies

$$\frac{\partial f}{\partial x_1} = \sum_{i=2}^{n} \alpha_i(\boldsymbol{x})\frac{\partial f}{\partial x_i} + \beta(\boldsymbol{x})f$$

for some $\alpha_2,\dots,\alpha_n \in \mathfrak{m}$, $\beta \in \mathbb{C}\{\boldsymbol{x}\}$.

The system of differential equations

$$\frac{dx_i}{dx_1} = -\alpha_i(x_1,\dots,x_n), \quad x_i(0) = y_i\,, \quad i = 2,\dots,n\,,$$

has an analytic solution

$$x_i = \varphi_i(x_1,y_2,\dots,y_n) \in \mathbb{C}\{y_2,\dots,y_n\}\{x_1\}, \quad i = 2,\dots,n\,,$$

convergent in a neighbourhood of zero. In particular, we can define an isomorphism $\mathbb{C}\{x_1,x_2,\dots,x_n\} \cong \mathbb{C}\{x_1,y_2,\dots,y_n\}$ which sends $f(\boldsymbol{x})$ to

$$\widetilde{f}(x_1,y_2,\dots,y_n) = f(x_1,\varphi_2(x_1,y_2,\dots,y_n),\dots,\varphi_n(x_1,y_2,\dots,y_n))$$

such that

$$\frac{\partial\widetilde{f}}{\partial x_1} = \left(\frac{\partial f}{\partial x_1} - \sum_{i=2}^{n}\alpha_i(\boldsymbol{x})\frac{\partial f}{\partial x_i}\right)_{\substack{x_i=\varphi_i(x_1,y_2,\dots,y_n)\\ i=2,\dots,n}} = \widetilde{\beta}(x_1,y_2,\dots,y_n)\cdot\widetilde{f}\,.$$

This equality can only hold if $\widetilde{f}$ does not depend on x_1. But then $\widetilde{f}$ and $\frac{\partial\widetilde{f}}{\partial x_i}$, $i = 1,\dots,n$, all vanish along the line $\{(t,0,\dots,0) \mid t \in \mathbb{C}\}$, contradicting the assumption that f (and, hence, $\widetilde{f}$) has an isolated singularity at the origin. $\square$

Remark 2.40.1. In Section 3.4, we will study another important classification of (plane curve) singularities: the classification with respect to (embedded) topological equivalence. Unlike the classifications studied above, the topological classification has no description in terms of an algebraic group action.

Exercises

Exercise 2.3.1. (1) Let $f = y^2 + x^{2p^2} \in \mathbb{C}\{x,y\}$. Show that the $\mathcal{R}^{(k)}$- and $\mathcal{K}^{(k)}$-orbits of f contain an element of degree $2p$, where $k \geq 2p^2$.

(2)* Let $f = x_1^{2p^n} + x_2^2 + \ldots + x_n^2$, $n \geq 3$. Show that the $\mathcal{R}^{(k)}$- and $\mathcal{K}^{(k)}$-orbits of f contain an element of degree $2p$, where $k \geq 2p^n$.

HINT. See [Wes].

(3)* Show that the $\mathcal{R}^{(k)}$-orbit (resp. $\mathcal{K}^{(k)}$-orbit) of a germ $f \in \mathbb{C}\{x,y\}$ with Milnor number $\mu(f) = \mu < \infty$ contains an element of degree less than $4\sqrt{\mu}$ (resp. less than $3\sqrt{\mu}$).

HINT. See [Shu].

(4)** *(Unsolved problem).* Is it true that the $\mathcal{R}^{(k)}$- and $\mathcal{K}^{(k)}$-orbits of a germ $f \in \mathbb{C}\{x_1,\ldots,x_n\}$, $n \geq 3$, with Milnor number $\mu(f) = \mu < \infty$ contain an element of degree less than $\alpha_n \sqrt[n]{\mu}$, where $\alpha_n > 0$ depends only on n?

(5)** *(Unsolved problem).* Given an integer $p \geq 10$ such that $\sqrt{p} \notin \mathbb{Z}$, does there exist a series of semiquasihomogeneous $f_m \in \mathbb{C}\{x,y\}$, $m \geq 1$, of type $(p,1;2mp)$ whose $\mathcal{K}^{(k)}$-orbits contain elements of degree less than $m\sqrt{p}(1+o(m))$?

Exercise 2.3.2. Introduce the *right-left group*

$$\mathcal{RL} = \operatorname{Aut}(\mathbb{C},0) \times \operatorname{Aut}(\mathbb{C}^n,\mathbf{0})$$

with the product

$$(\psi',\varphi')\cdot(\psi,\varphi) = (\psi'\circ\psi, \varphi\circ\varphi')\,,$$

acting on $\mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$ by $(\psi,\varphi)(f) = \psi(f(\varphi))$, and define the *right-left equivalence*

$$f \overset{rl}{\sim} g \quad :\Longleftrightarrow \quad g = \Phi(f) \text{ for some } \Phi \in \mathcal{RL}\,.$$

(1) Show the implications

$$f \overset{\mathrm{r}}{\sim} g \implies f \overset{rl}{\sim} g \implies f \overset{\mathrm{c}}{\sim} g\,,$$

and that the right-left equivalence neither coincides with the right, nor with the contact equivalence.

(2) Determine $T_f(\mathcal{RL}^{(k)}f)$ for $f \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$ and k sufficiently large.

Exercise 2.3.3. Show that the right classification of the germs $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1,\ldots,x_n\}$ of order d with a non-degenerate d-form depends on $N-n$ parameters (*moduli*), where

$$N = \#\big\{(i_1,\ldots,i_n) \in \mathbb{Z}^n \;\big|\; i_1+\ldots+i_n \geq d,\ \max\{i_1,\ldots,i_n\} \leq d-2\big\}\,.$$

Exercise 2.3.4. Let $\mathbb{C}_{(\boldsymbol{w},d)}(\boldsymbol{x})$ be the space of semiquasihomogeneous germs $f \in \mathbb{C}\{\boldsymbol{x}\}$ with a non-degenerate quasihomogeneous part of type $(\boldsymbol{w},d)$, and let $\mathcal{R}_{(\boldsymbol{w},d)} \subset \mathcal{R}$ be the subgroup leaving $\mathbb{C}_{(\boldsymbol{w},d)}(\boldsymbol{x})$ invariant. Determine $T_f(\mathcal{R}_{(\boldsymbol{w},d)}f)$ and compute the number of moduli in the right classification of the above germs.

2.4 Classification of Simple Singularities

We want to classify singularities having no "moduli" up to contact equivalence. No moduli means that, in a sufficiently high jet space, there exists a neighbourhood of f, which meets only finitely many orbits of the contact group. A singularity having no moduli is also called 0-*modal*, while k-*modal* means, loosely speaking, that any small neighbourhood of f meets k- (and no higher) dimensional families of orbits.

The same notion makes sense for right equivalence and, indeed, these notions were introduced by Arnol'd for right equivalence in a series of papers, which was of utmost importance for the development of singularity theory (cf. [AGV]).

Here we treat simultaneously right and contact equivalence, since it means almost no additional work.

We recall that the space of k-jets $J^{(k)} = \mathbb{C}\{x_1, \dots, x_n\}/\mathfrak{m}^{k+1}$ is a finite dimensional complex vector space with a natural topology: for a power series $f = \sum_{|\boldsymbol{\nu}|=0}^{\infty} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \in \mathbb{C}\{\boldsymbol{x}\}$, we identify $f^{(k)} = \mathrm{jet}(f,k) \in J^{(k)}$ with the truncated power series $f^{(k)} = \sum_{|\boldsymbol{\nu}|=0}^{k} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$. Then an open neighbourhood of $f^{(k)}$ in $J^{(k)}$ consists of all truncated power series $\sum_{|\boldsymbol{\nu}|=0}^{k} b_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$ such that $b_{\boldsymbol{\nu}}$ is contained in some open neighbourhood of $a_{\boldsymbol{\nu}}$ in $\mathbb{C}$, for all $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| \leq k$.

Consider the projections

$$\mathbb{C}\{\boldsymbol{x}\} \longrightarrow J^{(k)}, \quad k \geq 0.$$

The preimages of open sets in $J^{(k)}$ generate a topology on $\mathbb{C}\{\boldsymbol{x}\}$, the coarsest topology such that all projections are continuous. Hence, a neighbourhood of f in $\mathbb{C}\{\boldsymbol{x}\}$ consists of all those $g \in \mathbb{C}\{\boldsymbol{x}\}$ for which the coefficients up to some degree k are in a neighbourhood of the coefficients of f but with no restrictions on the coefficients of higher order terms. The neighbourhood becomes smaller if the coefficients up to order k get closer to the coefficients of f and if k gets bigger.

Definition 2.41. Consider the action of the right group $\mathcal{R}$, respectively of the contact group $\mathcal{K}$, on $\mathbb{C}\{\boldsymbol{x}\}$. Call $f \in \mathbb{C}\{\boldsymbol{x}\}$ *right simple*, respectively *contact simple*, if there exists a neighbourhood U of f in $\mathbb{C}\{\boldsymbol{x}\}$ such that U intersects only finitely many orbits of $\mathcal{R}$, respectively of $\mathcal{K}$.

This means that there exists some k and a neighbourhood U_k of $f^{(k)}$ in $J^{(k)}$ such that the set of all g with $g^{(k)} \in U_k$ decomposes into only finitely many right classes, respectively contact classes. It is clear that right simple implies contact simple. However, as the classification will show, the converse is also true.

We show now that for an isolated singularity f a sufficiently high jet is not only sufficient for f but also for all g in a neighbourhood of f.

Proposition 2.42. *Let $f \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ have an isolated singularity. Then there exists a neighbourhood U of f in $\mathbb{C}\{\boldsymbol{x}\}$ such that each $g \in U$ is right $(\mu(f)+1)$-determined, respectively contact $(\tau(f)+1)$-determined.*

Proof. We consider contact equivalence, the proof for right equivalence is analogous. Let $\tau = \tau(f)$, $k = \tau + 1$, and consider

$$f^{(k)} = \sum_{|\boldsymbol{\nu}|=0}^{k} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \in J^{(k)} .$$

By Corollary 2.24, $f \overset{c}{\sim} f^{(k)}$, and any element $h = \sum_{|\boldsymbol{\nu}|=0}^{k} b_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \in J^{(k)}$ can be written as

$$h(\boldsymbol{x}) \;=\; f^{(k)}(\boldsymbol{x}) + \sum_{|\boldsymbol{\nu}|=0}^{k} t_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \tag{2.4.1}$$

with $t_{\boldsymbol{\nu}} = b_{\boldsymbol{\nu}} - a_{\boldsymbol{\nu}}$. Considering $t_{\boldsymbol{\nu}}$, $|\boldsymbol{\nu}| \leq k$, as variables, then (2.4.1) defines an unfolding of $f^{(k)}$, and the semicontinuity theorem 2.6 says that there is a neighbourhood $U_k \subset J^{(k)}$ of $f^{(k)}$ such that $\tau(h) \leq \tau(f^{(k)}) = \tau(f)$ for each $h \in \mathfrak{m} \cap U_k$. Hence, h is k-determined and, therefore, also every $g \in \mathbb{C}\{\boldsymbol{x}\}$ with $g^{(k)} \in U_k$. If $U \subset \mathbb{C}\{\boldsymbol{x}\}$ is the preimage of U_k under $\mathbb{C}\{\boldsymbol{x}\} \twoheadrightarrow J^{(k)}$ then this says that every $g \in U \cap \mathfrak{m}$ is contact $(\tau+1)$-determined. □

Remark 2.42.1. We had to use μ, respectively τ, as a bound for the determinacy, since the determinacy itself is not semicontinuous. For example the singularity E_7 is 4-determined (cf. Example 2.25.1) and deforms into A_6, which is 6-determined. This will follow from the classification in below.

Corollary 2.43. *Let $f \in \mathfrak{m}$ have an isolated singularity, and suppose that $k \geq \mu(f)+1$, respectively $k \geq \tau(f)+1$. Then f is right simple, respectively contact simple, iff there is a neighbourhood of $f^{(k)}$ in $J^{(k)}$, which meets only finitely many $\mathcal{R}^{(k)}$-orbits, respectively $\mathcal{K}^{(k)}$-orbits.*

Proof. The necessity is clear, the sufficiency is an immediate consequence of Proposition 2.42. □

Now let us start with the classification. The aim is to show that the right simple as well as the contact simple singularities $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ are exactly the so-called *ADE-singularities*:

$$\begin{array}{ll}
A_k\colon\ x_1^{k+1} + x_2^2 + \ldots + x_n^2 \,, & k \geq 1\,, \\
D_k\colon\ x_1(x_2^2 + x_1^{k-2}) + x_3^2 + \ldots + x_n^2 \,, & k \geq 4\,, \\
E_6\colon\ x_1^3 + x_2^4 + x_3^2 + \ldots + x_n^2 \,, & \\
E_7\colon\ x_1(x_1^2 + x_2^3) + x_3^2 + \ldots + x_n^2 \,, & \\
E_8\colon\ x_1^3 + x_2^5 + x_3^2 + \ldots + x_n^2 \,. &
\end{array}$$

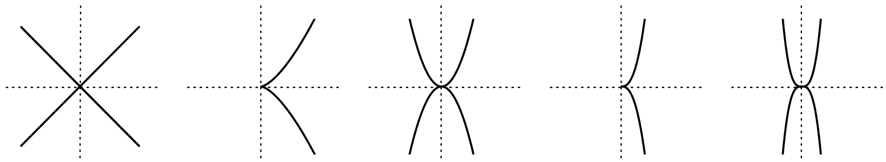


Fig. 2.7. Real pictures of one-dimensional A_k-singularities

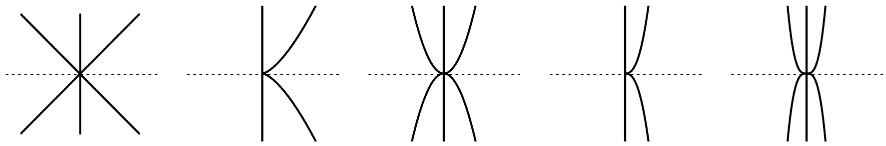


Fig. 2.8. Real pictures of one-dimensional D_k-singularities

Note that A_0 is usually not included in the list of simple singularities, since it is non-singular. It is however simple in the sense of Definition 2.41, since in a neighbourhood of A_0 in $\mathbb{C}\{\boldsymbol{x}\}$ there are only smooth germs or units. A_1-singularities are also called *(ordinary) nodes*, and A_2-singularities *(ordinary) cusps*.

Classification of Smooth Germs.

Lemma 2.44. *For $f \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$ the following are equivalent.*

(a) $\mu(f) = 0$,
(b) $\tau(f) = 0$,
(c) f *is non-singular,*
(d) $f \overset{\mathrm{r}}{\sim} f^{(1)}$,
(e) $f \overset{\mathrm{r}}{\sim} x_1$.

Proof. $\mu(f) = 0 \Leftrightarrow \tau(f) = 0 \Leftrightarrow \frac{\partial f}{\partial x_i}(\mathbf{0}) \neq 0$ for some i $\Leftrightarrow f$ is non-singular. The remaining equivalences follow from the implicit function theorem. □

Classification of Non-Degenerate Singularities. Let $U \subset \mathbb{C}^n$ be open, and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Then we denote by

$$H(f) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n} \in \operatorname{Mat}(n \times n, \mathbb{C}\{\boldsymbol{x}\})$$

the *Hessian (matrix)* of f.

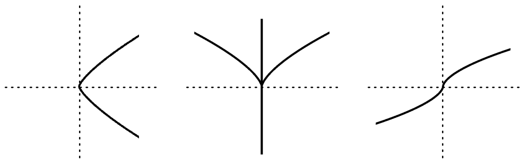


Fig. 2.9. Real pictures of one-dimensional E_6-, E_7-, E_8-singularities

Definition 2.45. A critical point p of f is called a *non-degenerate*, or *Morse singularity* if the rank of the Hessian matrix at p, $\operatorname{rank} H(f)(p)$, is equal to n. The number $\operatorname{crk}(f,p) := n - \operatorname{rank} H(f)(p)$ is called the *corank* of f at p. We write $\operatorname{crk}(f)$ instead of $\operatorname{crk}(f,\mathbf{0})$.

The notion of non-degenerate critical points is independent of the choice of local analytic coordinates. Namely, if $\phi\colon (\mathbb{C}^n,p) \to (\mathbb{C}^n,p)$ is biholomorphic, then

$$\begin{aligned}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\big(f\circ\phi(\boldsymbol{x})\big) &= \frac{\partial}{\partial x_i}\left(\sum_\nu \frac{\partial f}{\partial x_\nu}\big(\phi(\boldsymbol{x})\big)\cdot\frac{\partial\phi_\nu}{\partial x_j}(\boldsymbol{x})\right)\\ &= \sum_{\mu,\nu}\frac{\partial^2 f}{\partial x_\mu\partial x_\nu}\big(\phi(\boldsymbol{x})\big)\cdot\frac{\partial\phi_\mu}{\partial x_i}(\boldsymbol{x})\cdot\frac{\partial\phi_\nu}{\partial x_j}(\boldsymbol{x}) + \sum_\nu\frac{\partial f}{\partial x_\nu}\big(\phi(\boldsymbol{x})\big)\cdot\frac{\partial^2\phi_\nu}{\partial x_i\partial x_j}(\boldsymbol{x}).\end{aligned}$$

Since p is a critical point of f and $\phi(p)=p$ we have $\frac{\partial f}{\partial x_\nu}\big(\phi(p)\big)=0$, hence

$$H(f\circ\phi)(p) = J(\phi)(p)^t\cdot H(f)(p)\cdot J(\phi)(p), \tag{2.4.2}$$

where $J(\phi)$ is the Jacobian matrix of ϕ, which has rank n.

Similarly, if p is a singular point of the hypersurface $f^{-1}(0)$, that is, if $\frac{\partial f}{\partial x_i}(p) = f(p) = 0$, then $\operatorname{rank} H(f)(p) = \operatorname{rank} H(uf)(p)$ for any unit u.

Hence, $\operatorname{crk}(f,p)$ is an invariant of the right equivalence class of f at a critical point and an invariant of the contact class at a singular point of $f^{-1}(0)$. However, if p is non-singular, then $\operatorname{rank} H(f)(p)$ may depend on the choice of coordinates.

Note that for a critical point p, the rank of the Hessian matrix $H(f)(p)$ depends only on the 2-jet of f.

Theorem 2.46 (Morse lemma). *For $f\in\mathfrak{m}^2\subset\mathbb{C}\{x_1\dots,x_n\}$ the following are equivalent:*

(a) $\operatorname{crk}(f,\mathbf{0})=0$, *that is,* $\mathbf{0}$ *is a non-degenerate singularity of* f,
(b) $\mu(f)=1$,
(c) $\tau(f)=1$,
(d) $f\overset{\mathrm{r}}{\sim} f^{(2)}$ *and* $f^{(2)}$ *is non-degenerate,*
(e) $f\overset{\mathrm{r}}{\sim} x_1^2+\ldots+x_n^2$,

(f) $f \overset{c}{\sim} x_1^2 + \ldots + x_n^2$.

Proof. The apparently simple proof makes use of the finite determinacy theorem (Theorem 2.23, which required some work). Since $f \in \mathfrak{m}^2$, we can write

$$f(x) = \sum_{1 \le i,j \le n} h_{i,j}(\boldsymbol{x}) x_i x_j \,, \quad h_{i,j} \in \mathbb{C}\{\boldsymbol{x}\} \,,$$

with $\big(h_{i,j}(\mathbf{0})\big) = \frac{1}{2} \cdot H(f)(\mathbf{0})$ where $H(f)(\mathbf{0})$ is the Hessian of f at $\mathbf{0}$.

$(a) \Rightarrow (b)$. Since $h_{i,j}(\mathbf{0}) = h_{j,i}(\mathbf{0})$, we have

$$\frac{\partial f}{\partial x_\nu} = \sum_{i,j} \frac{\partial h_{i,j}}{\partial x_\nu} x_i x_j + \sum_j h_{\nu,j} x_j + \sum_i h_{i,\nu} x_i \equiv 2 \cdot \sum_{j=1}^n h_{\nu,j}(\mathbf{0}) x_j \mod \mathfrak{m}^2 .$$

Since $H(f)(\mathbf{0})$ is invertible by assumption, we get

$$\Big\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \Big\rangle = \big\langle x_1, \ldots, x_n \big\rangle \mod \mathfrak{m}^2 .$$

Nakayama's lemma implies $j(f) = \mathfrak{m}$ and, hence, $\mu(f) = 1$.

$(b) \Rightarrow (c)$ is obvious, since $\tau \le \mu$ and $\tau = 0$ can only happen if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

$(c) \Rightarrow (b) \Rightarrow (d)$. If $\tau = 1$ then $\mathfrak{m} = \langle f, j(f) \rangle$ and, hence, by Nakayama's lemma, $\mathfrak{m} = j(f)$, since $f \in \mathfrak{m}^2$. Then $\mu(f) = 1$, and by Corollary 2.24 f is right 2-determined, whence (d).

$(d) \Rightarrow (e)$. By the theory of quadratic forms over $\mathbb{C}$ there is a non-singular matrix T such that

$$T^t \cdot \frac{1}{2} H(f)(\mathbf{0}) \cdot T = \mathbf{1}_n \,,$$

where $\mathbf{1}_n$ is the $n \times n$ unit matrix. The linear coordinate change $\boldsymbol{x} \mapsto T \cdot \boldsymbol{x}$ provides, for $f = f^{(2)}$,

$$f(T \cdot \boldsymbol{x}) = \boldsymbol{x} \cdot T^t \cdot \frac{1}{2} H(f)(\mathbf{0}) \cdot T \cdot \boldsymbol{x}^t = x_1^2 + \ldots + x_n^2 \,.$$

The implication $(e) \Rightarrow (f)$ is trivial. Finally, (f) implies $\tau(f) = 1$ and, hence, (e) as shown above. The implication $(e) \Rightarrow (a)$ is again obvious. □

The Morse lemma gives a complete classification of non-degenerate singularities in a satisfying form: they are classified by one numerical invariant, the Milnor number, respectively the Tjurina number, and we have a very simple normal form.

In general, we cannot hope for such a simple answer. There might not be a finite set of complete invariants (that is, completely determining the singularity), and there might not be just one normal form but a whole family of normal forms. However, as we shall see, the simple singularities have a similar nice classification.

Splitting Lemma and Classification of Corank 1 Singularities. The following theorem, called *generalized Morse lemma* or *splitting lemma*, allows us to reduce the classification to germs of corank n or, equivalently, to germs in $\mathfrak{m}^3$.

Theorem 2.47 (Splitting lemma). *If* $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$ *has* $\operatorname{rank} H(f)(\mathbf{0}) = k$*, then*

$$f \overset{\mathrm{r}}{\sim} x_1^2 + \ldots + x_k^2 + g(x_{k+1}, \dots, x_n)$$

with $g \in \mathfrak{m}^3$*.* g *is called the* residual part *of* f*. It is uniquely determined up to right equivalence.*

Proof. As the Hessian matrix of f at $\mathbf{0}$ has rank k, the 2-jet of f can be transformed into $x_1^2 + \ldots + x_k^2$ by a linear change of coordinates (cf. the proof of Theorem 2.46). Hence, we can assume that

$$f(\boldsymbol{x}) = x_1^2 + \ldots + x_k^2 + f_3(x_{k+1}, \dots, x_n) + \sum_{i=1}^{k} x_i \cdot g_i(x_1, \dots, x_n)\,,$$

with $g_i \in \mathfrak{m}^2$, $f_3 \in \mathfrak{m}^3$. The coordinate change $x_i \mapsto x_i - \frac{1}{2} g_i$ for $i = 1, \dots, k$, and $x_i \mapsto x_i$ for $i > k$, yields

$$f(\boldsymbol{x}) = x_1^2 + \ldots + x_k^2 + f_3(x_{k+1}, \dots, x_n) + f_4(x_{k+1}, \dots, x_n) + \sum_{i=1}^{k} x_i \cdot h_i(\boldsymbol{x})\,,$$

with $h_i \in \mathfrak{m}^3$, $f_4 \in \mathfrak{m}^4$. Continuing with h_i instead of g_i in the same manner, the last sum will be of arbitrary high order, hence 0 in the limit.

In case f has an isolated singularity, the result follows from the finite determinacy theorem 2.23. In general, we get at least a formal coordinate change such that $g(x_{k+1}, \dots, x_n)$ in the theorem is a formal power series. We omit the proof of convergence.

To prove the uniqueness of g, let $\boldsymbol{x}' = (x_{k+1}, \dots, x_n)$ and assume

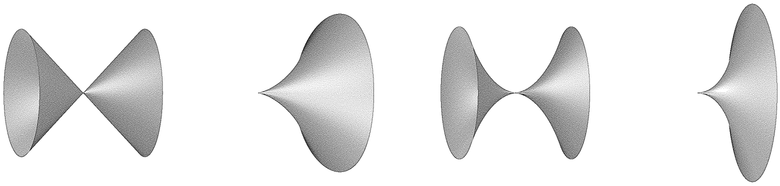
$$f_0(\boldsymbol{x}) := x_1^2 + \ldots + x_k^2 + g_0(\boldsymbol{x}') \overset{\mathrm{r}}{\sim} x_1^2 + \ldots + x_k^2 + g_1(\boldsymbol{x}') =: f_1(\boldsymbol{x})\,.$$

Then, by Theorem 2.28, we obtain isomorphisms of $\mathbb{C}\{t\}$-algebras,

$$\mathbb{C}\{\boldsymbol{x}'\}\Big/\Big\langle \frac{\partial g_0}{\partial x_{k+1}}, \dots, \frac{\partial g_0}{\partial x_n} \Big\rangle \cong M_{f_0} \cong M_{f_1} \cong \mathbb{C}\{\boldsymbol{x}'\}\Big/\Big\langle \frac{\partial g_1}{\partial x_{k+1}}, \dots, \frac{\partial g_1}{\partial x_n} \Big\rangle\,,$$

t acting on M_{f_0}, respectively on M_{f_1}, via multiplication with f_0, respectively with f_1. It follows that M_{g_0} and M_{g_1} are isomorphic as $\mathbb{C}\{t\}$-algebras. Hence, $g_0 \overset{\mathrm{r}}{\sim} g_1$, again by Theorem 2.28. □

We use the splitting lemma to classify the singularities of corank ≤ 1.



A_1 A_2 A_3 A_4

Fig. 2.10. Real pictures of two-dimensional A_k-singularities

Theorem 2.48. *Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\boldsymbol{x}\}$ and $k \geq 1$, then the following are equivalent:*

(a) $\operatorname{crk}(f) \leq 1$ *and* $\mu(f) = k$,
(b) $f \overset{\mathrm{r}}{\sim} x_1^{k+1} + x_2^2 + \ldots + x_n^2$, *that is, f is of type A_k,*
(c) $f \overset{\mathrm{c}}{\sim} x_1^{k+1} + x_2^2 + \ldots + x_n^2$.

Moreover, f is of type A_1 iff $\operatorname{crk}(f) = 0$. *It is of type A_k for some $k \geq 2$ iff* $\operatorname{crk}(f) = 1$.

Proof. The implications $(b) \Rightarrow (c) \Rightarrow (a)$ are obvious. Hence, it is only left to prove $(a) \Rightarrow (b)$. By the splitting lemma, we may assume that

$$f = g(x_1) + x_2^2 + \ldots + x_n^2 = u \cdot x_1^{k+1} + x_2^2 + \ldots + x_n^2$$

with $u \in \mathbb{C}\{x_1\}$ a unit and $k \geq 1$, since $\operatorname{crk}(f) \leq 1$. The coordinate change $x_1' = \sqrt[k+1]{u} \cdot x_1$, $x_i' = x_i$ for $i \geq 2$ transforms f into A_k. □

Corollary 2.49. *A_k-singularities are right (and, hence, contact) simple. More precisely, there is a neighbourhood of f in $\mathfrak{m}^2$, which meets only orbits of singularities of type A_ℓ with $\ell \leq k$.*

Proof. Since $\operatorname{crk}(f)$ is semicontinuous on $\mathfrak{m}^2$, a neighbourhood of A_k contains only A_ℓ-singularities. Since $\mu(f)$ is semicontinuous on $\mathfrak{m}^2$, too, we obtain $\ell = \mu(A_\ell) \leq \mu(A_k) = k$. □

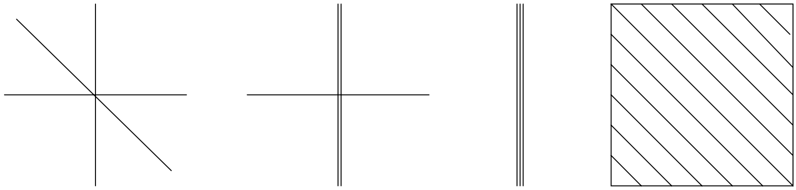
On the Classification of Corank 2 Singularities. If $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\boldsymbol{x}\}$ has corank 2 then the splitting lemma implies that $f \overset{\mathrm{r}}{\sim} g(x_1, x_2) + x_3^2 + \ldots + x_n^2$ with a uniquely determined $g \in \mathfrak{m}^3$. Hence, we may assume $f \in \mathbb{C}\{x, y\}$ and $f \in \mathfrak{m}^3$.

Proposition 2.50. *Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x, y\}$. Then there exists a linear automorphism $\varphi \in \mathbb{C}\{x, y\}$ such that $f^{(3)}$, the 3-jet of $\varphi(f)$, is of one of the following forms*

(1) $xy(x + y)$ or, equivalently, $f^{(3)}$ factors into 3 different linear factors,
(2) $x^2 y$ or, equivalently, $f^{(3)}$ factors into 2 different linear factors,

(3) x^3 *or, equivalently,* $f^{(3)}$ *has a unique linear factor (of multiplicity* 3*),*
(4) 0.

We may draw the zero-sets:



(1) 3 different lines (2) a line and a double line (3) a triple line (4) a plane

Proof. Let $f^{(3)} = ax^3 + bx^2y + cxy^2 + dy^3 \neq 0$. After a linear change of coordinates we get a homogenous polynomial g of the same type, but with $a \neq 0$. Dehomogenizing g by setting $y = 1$, we get a univariate polynomial of degree 3, which decomposes into linear factors. Homogenizing the factors, we see that g factorizes into 3 homogeneous factors of degree 1, either 3 simple factors or a double factor and a simple factor or a triple factor. This corresponds to the cases (1) – (3).

To obtain the exact normal forms in (1) – (3) we may first assume $a = 1$ (replacing x by $\frac{1}{\sqrt[3]{a}}x$). Then g factors as

$$g = (x - \lambda_1 y) \cdot (x - \lambda_2 y) \cdot (x - \lambda_3 y)\,.$$

Having a triple factor would mean $\lambda_1 = \lambda_2 = \lambda_3$, and, replacing $x - \lambda_1 y$ by x, we end up with the normal form (3). One double plus one simple factor can be transformed similarly to the normal form in (2).

Three different factors can always be transformed to $xy(x - \lambda y)$ with $\lambda \neq 0$. Replacing $-\lambda y$ by y, we get $\alpha xy(x + y)$, $\alpha \neq 0$. Finally, replacing x by $\alpha^{-\frac{1}{3}}x$ and y by $\alpha^{-\frac{1}{3}}y$ yields $xy(x + y)$. □

Remark 2.50.1. If $f \in \mathbb{C}\{\boldsymbol{x}\}$ then we can always write

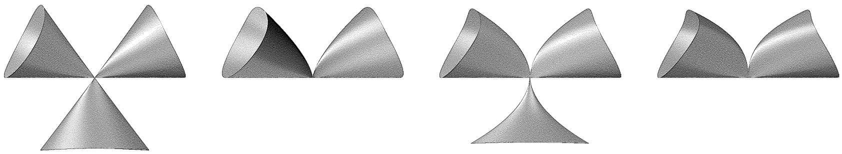
$$f = \sum_{i \geq d} f_i\,, \quad f_d \neq 0\,,$$

where f_i are homogeneous polynomials of degree i. The lowest non-vanishing term f_d is called the *tangent cone* of f, where $d = \operatorname{ord}(f)$ is the order of f. If f is contact equivalent to g with $u \cdot \varphi(f) = g$, $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ and $\varphi \in \operatorname{Aut}\mathbb{C}\{\boldsymbol{x}\}$, then $\operatorname{ord}(f) = \operatorname{ord}(g) = d$ and

$$u^{(0)} \cdot \varphi^{(1)}(f_d) = g_d\,,$$

where $u^{(0)} = u(\mathbf{0})$ is the 0-jet of u and $\varphi^{(1)}$ the 1-jet of φ. In particular we have $f_d \overset{c}{\sim} g_d$, and Lemma 2.13 implies $f_d \overset{r}{\sim} g_d$.

In other words, if f is contact equivalent to g then the tangent cones are right equivalent by some linear change of coordinates, that is, they are in the same $\mathrm{GL}(n, \mathbb{C})$-orbit acting on $\mathfrak{m}^d/\mathfrak{m}^{d+1}$.



D_4 D_5 D_6 D_7

Fig. 2.11. Real pictures of two-dimensional D_k-singularities

Remark 2.50.2. During the following classification we shall make several times use of the so-called *Tschirnhaus transformation*: let A be a ring and

$$f = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \ldots + \alpha_0 \in A[x]$$

a polynomial of degree d with coefficients in A. Assume that the quotient $\beta := \alpha_{d-1}/(d\alpha_d)$ exists in A. Then, substituting x by $x-\beta$ yields a polynomial of degree d with no term of degree $d-1$. In other words, the isomorphism $\varphi\colon A[x] \to A[y]$, $\varphi(x) = y-\beta$, maps f to

$$\varphi(f) = \alpha_d y^d + \beta_{d-2} y^{d-2} + \ldots + \beta_0 \in A[y]$$

for some $\beta_i \in A$.

Let us now analyse the four cases of Proposition 2.50, starting with the cases (1) and (2).

Theorem 2.51. *Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x,y\}$ and $k \geq 4$. Then the following are equivalent:*

(a) $f^{(3)}$ factors into at least two different factors and $\mu(f) = k$,
(b) $f \overset{\mathrm{r}}{\sim} x(y^2 + x^{k-2})$, that is, f is of type D_k,
(c) $f \overset{\mathrm{c}}{\sim} x(y^2 + x^{k-2})$.

Moreover, $f^{(3)}$ factors into three different factors iff f is of type D_4.

The proof will also show that D_k is $(k-1)$-determined.

Proof. The implications (b) $\Rightarrow$ (c),(a) being trivial, and (c) $\Rightarrow$ (b) being implied by Lemma 2.13, we can restrict ourselves on proving (a) $\Rightarrow$ (b).

Assume that $f^{(3)}$ factors into three different factors. Then, due to Proposition 2.50 (1), $f^{(3)} \overset{\mathrm{r}}{\sim} g := xy(x+y)$. But now it is easy to see that $\mathfrak{m}^4 \subset \mathfrak{m}^2 \cdot j(g)$, hence g is right 3-determined due to the finite determinacy theorem. In particular, $g \overset{\mathrm{r}}{\sim} f$.

If $f^{(3)}$ factors into exactly two different factors then, due to Proposition 2.50 (2), we can assume $f^{(3)} = x^2 y$. Note that $f \neq f^{(3)}$ (otherwise $\mu(f) = \infty$). Hence, we can define $m := \mathrm{ord}(f - f^{(3)})$ and consider the m-jet of f,

$$f^{(m)} = x^2 y + \alpha y^m + \beta x y^{m-1} + x^2 \cdot h(x,y) \tag{2.4.3}$$

with $\alpha, \beta \in \mathbb{C}$, $h \in \mathfrak{m}^{m-2}$, $m \geq 4$. Applying the Tschirnhaus transformations $x = x - \frac{1}{2}\beta \cdot y^{m-2}$, $y = y - h(x,y)$ turns $f^{(m)}$ into

$$f^{(m)}(x,y) = x^2 y + \alpha y^m. \tag{2.4.4}$$

Case A. If $\alpha = 0$ consider $f^{(m+1)}$, which has the form (2.4.3), hence can be transformed to (2.4.4) with m replaced by $m+1$ and, if still $\alpha = 0$, we continue. This procedure stops, since $\alpha = 0$ implies that

$$\begin{aligned}\mu(f) &\geq \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\big(j(f) + \mathfrak{m}^{m-1}\big) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\big(j(f^{(m)}) + \mathfrak{m}^{m-1}\big) \\ &= \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle x^2, xy, y^{m-1}\rangle = m\,.\end{aligned}$$

Hence, we have only to consider

Case B. If $\alpha \neq 0$, then, replacing y by $\alpha^{-1/m}y$ and x by $\alpha^{2/m}x$, we obtain

$$f^{(m)}(x,y) = x^2 y + y^m\,,$$

which is m-determined by Theorem 2.23. In particular, $f \overset{r}{\sim} y(x^2 + y^{m-1})$, which is a D_{m+1}-singularity. □

Corollary 2.52. *D_k-singularities are right (and, hence, contact) simple. More precisely, there is a neighbourhood of f in $\mathfrak{m}^2$, which meets only orbits of singularities of type A_ℓ for $\ell < k$ or D_k for $\ell \leq k$.*

Proof. For any $g \in \mathfrak{m}^2$ in a neighbourhood of D_k we have either $\operatorname{crk}(g) \leq 1$, which implies $g \overset{r}{\sim} A_\ell$ and $\ell \leq k$ by Theorem 2.48, respectively the semicontinuity theorem 2.6 (for the strict inequality we refer to Exercise 2.4.2, below), or we have $\operatorname{crk}(g) = 2$. In the latter case, for any power series g close to f, the 3-jet $g^{(3)}$ must factor into 2 or 3 different linear forms, since this is an open property (by continuity of the roots of a polynomial, cf. the proof of Proposition 2.50). Hence, $g \overset{r}{\sim} D_\ell$ for some $\ell \leq k$. □

Remark 2.52.1. Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x,y\}$ and $g = f^{(3)}$. Then g factors into

- three different linear factors iff the ring $\mathbb{C}\{x,y\}/j(g)$ is Artinian, that is, has dimension 0,
- two different linear factors iff $\mathbb{C}\{x,y\}/j(g)$ has dimension 1, and the ring $\mathbb{C}\{x,y\}/\big\langle \frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial x \partial y}, \frac{\partial^2 g}{\partial y^2}\big\rangle$ has dimension 0,
- one (triple) linear factor iff $\mathbb{C}\{x,y\}/j(g)$ has dimension 1, and the ring $\mathbb{C}\{x,y\}/\big\langle \frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial x \partial y}, \frac{\partial^2 g}{\partial y^2}\big\rangle$ has dimension 1.

This can be seen by considering the singular locus of g, respectively the singular locus of the singular locus, and it gives in fact an effective characterization of the D_k-singularities by using standard bases in local rings (as implemented in Singular).

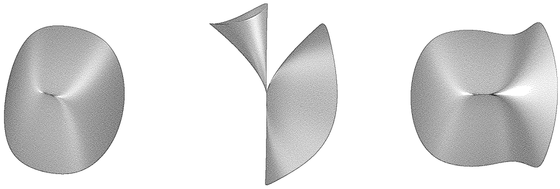


Fig. 2.12. Real pictures of two-dimensional E_k-singularities

Theorem 2.53. *Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x, y\}$. The following are equivalent:*

(a) $f^{(3)}$ has a unique linear factor (of multiplicity 3) and $\mu(f) \le 8$,
(b) $f^{(3)} \overset{\mathrm{r}}{\sim} x^3$ and if $f^{(3)} = x^3$ then $f \notin \langle x, y^2\rangle^3 = \langle x^3, x^2y^2, xy^4, y^6\rangle$.
(c) $f \overset{\mathrm{r}}{\sim} g$ with $g \in \{x^3 + y^4, x^3 + xy^3, x^3 + y^5\}$, that is, f is of type E_6, E_7 or E_8.
(d) $f \overset{\mathrm{c}}{\sim} g$ with $g \in \{x^3 + y^4, x^3 + xy^3, x^3 + y^5\}$.

Moreover, $\mu(E_k) = k$ for $k = 6, 7, 8$.

Proof. Let us prove the implication (b) $\Rightarrow$ (c). The 4-jet $f^{(4)}$ can be written as

$$f^{(4)}(x, y) = x^3 + \alpha y^4 + \beta xy^3 + x^2 \cdot h(x, y)$$

with $\alpha, \beta \in \mathbb{C}$, $h \in \mathfrak{m}^2$. After substituting $x = x - \frac{1}{3}h$, we may assume

$$f^{(4)}(x, y) = x^3 + \alpha y^4 + \beta xy^3 \,. \tag{2.4.5}$$

Case E_6: $\alpha \neq 0$ in (2.4.5). Applying a Tschirnhaus transformation (with respect to y), we obtain

$$f^{(4)}(x, y) = x^3 + y^4 + x^2 \cdot h \,, \quad h \in \mathfrak{m}^2 \,,$$

and by applying another Tschirnhaus transformation (with respect to x) we obtain $f^{(4)} = x^3 + y^4$, which is 4-determined due to the finite determinacy theorem. Hence, $f \overset{\mathrm{r}}{\sim} f^{(4)}$.

Case E_7: $\alpha = 0$, $\beta \neq 0$ in (2.4.5). Replacing y by $\beta^{-1/3}y$, we obtain the 4-jet $f^{(4)} = x^3 + xy^3$, which is 4-determined by Example 2.25.1, hence $f \overset{\mathrm{r}}{\sim} f^{(4)}$.

Case E_8: $\alpha = 0$, $\beta = 0$ in (2.4.6). Then $f^{(4)} = x^3$, and we consider the 5-jet of f.

$$f^{(5)}(x, y) = x^3 + \alpha y^5 + \beta xy^4 + x^2 \cdot h(x, y) \,, \quad h \in \mathfrak{m}^3.$$

Replacing x by $x - \frac{1}{3}h(x, y)$ we obtain

$$f^{(5)} = x^3 + \alpha y^5 + \beta xy^4. \tag{2.4.6}$$

If $\alpha \neq 0$ then, replacing y by $\alpha^{-1/5}y$ and renaming β, we obtain

$$f^{(5)}(x,y) = x^3 + y^5 + \beta xy^4 .$$

Applying a Tschirnhaus transformation (with respect to y) gives

$$f^{(5)}(x,y) = x^3 + y^5 + x^2 \cdot h(x,y)\,, \quad h \in \mathfrak{m}^3\,,$$

and, again replacing x by $x - \frac{1}{3}h$ yields $f^{(5)} = x^3 + y^5$, which is 5-determined due to the finite determinacy theorem.

If $\alpha = 0$ in (2.4.6) then $f^{(5)} = x^3 + \beta xy^4$ and, hence,

$$f \in \langle x^3, xy^4\rangle_{\mathbb{C}} + \mathfrak{m}^6 \subset \langle x, y^2\rangle^3.$$

This proves (c).

By Exercise 2.4.3 it follows that $\mu(f) > 8$ if $f \in \langle x, y^2\rangle^3$. Since $\mu(E_k) = k$ for $k = 6, 7, 8$, we get the equivalence of (a) and (b) and the implication (c) $\Rightarrow$ (a). Finally, since E_6, E_7, E_8 are quasihomogeneous, (c) and (d) are equivalent, by Lemma 2.13. $\square$

Corollary 2.54. *E_6, E_7, E_8 are right (hence, contact) simple. More precisely, there is a neighbourhood of f in $\mathfrak{m}^2$, which meets only orbits of singularities of type A_k or D_k or E_k for k at most* 8.

Proof. Let $g \in \mathfrak{m}^2$ be in a (sufficiently small) neighbourhood of f. Then either $\mathrm{crk}(g) \leq 1$, or $\mathrm{crk}(g) = 2$.

If $\mathrm{crk}(g) \leq 1$ then $g \overset{\mathrm{r}}{\sim} A_k$ for some k by 2.48. If $\mathrm{crk}(g) = 2$ and $g^{(3)}$ factors into three or two factors, then $g \overset{\mathrm{r}}{\sim} D_k$ for some k by 2.51. If $g^{(3)} \overset{\mathrm{r}}{\sim} x^3$, then g is right equivalent to E_6, E_7 or E_8 since the condition $f \notin \langle x, y^2\rangle^3$ is open. $\square$

Remark 2.54.1. We have shown that the singularities of type A_k ($k \geq 1$), D_k ($k \geq 4$), and E_6, E_7, E_8 are right simple (and, hence, contact simple). Moreover, we have also shown that if $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ is not contact equivalent to one of the ADE classes, then either

(1) $\mathrm{crk}(f) \geq 3$, or

(2) $\mathrm{crk}(f) = 2$, $f \overset{\mathrm{r}}{\sim} g(x_1, x_2) + x_3^2 + \ldots + x_n^2$ with
 (i) $g \in \mathfrak{m}^4$, or
 (ii) $g \in \langle x_1, x_2^2\rangle^3$.

We still have to show that all singularities belonging to one of these latter classes are, indeed, not contact simple. In particular, if f has a non-isolated singularity, then it must belong to class (1) or (2). An alternative way to prove that non-isolated singularities are not simple is given in the exercises below.

Theorem 2.55. *If $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ belongs to one of the classes (1),(2) above, then f is not contact simple and hence not right simple.*

Proof. (1) We may assume that $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x_1, x_2, x_3\}$, and we may consider its 3-jet $f^{(3)}$ as an element in $\mathfrak{m}^3/\mathfrak{m}^4$, which is a 10-dimensional vector space. If $f \overset{c}{\sim} g$, then $f^{(3)}$ and $g^{(3)}$ are in the same $GL(3,\mathbb{C})$-orbit. Since $\dim GL(3,\mathbb{C}) = 9$, this orbit has dimension ≤ 9 by Theorem 2.34. Since the orbits are locally closed by 2.34, and since a finite union of at most 9-dimensional locally closed subvarieties is a constructible set of dimension ≤ 9, a neighbourhood of $f^{(3)}$ in $\mathfrak{m}^3/\mathfrak{m}^4$ must meet infinitely many $GL(3,\mathbb{C})$-orbits. Hence, any neighbourhood of f in $\mathfrak{m}^3$ must meet infinitely many $\mathcal{K}$-orbits, that is, f is not contact simple.

(2) We may assume $f \in \mathbb{C}\{x, y\}$. The argument for (i) is the same as in (1) except that we consider $f^{(4)}$ in the 5-dimensional vector space $\mathfrak{m}^4/\mathfrak{m}^5$ and the action of $GL(2,\mathbb{C})$, which has dimension 4.

In case (ii) it is not sufficient to consider the tangent cone. Instead we use the *weighted* tangent cone: first notice that an arbitrary element f can be written as

$$f(x,y) = \sum_{d\geq 6} f_d(x,y)\,, \quad f_d(x,y) = \sum_{2i+j=d} \alpha_{i,j} x^i y^j\,,$$

that is, f_d is weighted homogeneous of type $(2,1;d)$. The weighted tangent cone f_6 has the form

$$f_6(x,y) = \alpha x^3 + \beta x^2y^2 + \gamma xy^4 + \delta y^6.$$

Applying the coordinate change φ given by

$$\begin{aligned}\varphi(x) &= a_1x + b_1y + c_1x^2 + d_1xy + e_1y^2 + \dots\,,\\ \varphi(y) &= a_2x + b_2y + c_2x^2 + d_2xy + e_2y^2 + \dots\,,\end{aligned}$$

we see that $\varphi(f) \in \langle x, y^2\rangle^3$ forces $b_1 = 0$. Then the weighted order of $\varphi(x)$ is at least 2, while the weighted order of $\varphi(y)$ is at least 1. This implies that, for all $d \geq 6$, $\varphi(f_d)$ has weighted order at least d. Therefore, only f_6 is mapped to the space of weighted 6-jets of $\langle x, y^2\rangle^3$. However, the weighted 6-jet of $\varphi(f_6)$ involves only the coefficients a_1, e_1 and b_2 of φ as a simple calculation shows. Therefore, the orbit of f_6 under the right group intersects the space of weighted 6-jets of $\langle x, y^2\rangle^3$ in a locally closed variety of dimension at most 3. Since f_6 is quasihomogeneous, the right orbit coincides with the contact orbit. As the space of weighted 6-jets of $\langle x, y^2\rangle^3$ is 4-dimensional, generated by x^3, x^2y^2, xy^4 and y^6, it must intersect infinitely many contact orbits of elements of $\langle x, y^2\rangle^3$. Hence, f is not contact simple. □

We now give explicit examples of non-simple singularities belonging to the classes (1) and (2) (i),(ii).

Example 2.55.1. (1) Consider the family of surface singularities given by

$$E = y^2z - 4x^3 + g_2xz^2 + g_3z^3\,, \quad g_1, g_2 \in \mathbb{C}\,,$$

of corank 3. This equation $E = 0$ defines the cone over an *elliptic curve*, defined by $E = 0$ in $\mathbb{P}^2$, in *Weierstraß normal form*. The *J-invariant* of this equation is

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2}\,.$$

The number J varies continuously in $\mathbb{C}$ if the coefficients g_2, g_3 vary, and two isomorphic elliptic curves in Weierstraß form have the same J-invariant (cf. [BrK, Sil]). Therefore the family $E = E(g_2, g_3)$ meets infinitely many right (and, hence, contact) orbits.

Another normal form is the *Hesse normal form* of an elliptic curve,

$$x^3 + y^3 + z^3 + \lambda xyz = 0\,.$$

The singularity in $(\mathbb{C}^3, \mathbf{0})$ defined by this equation is denoted by $\widetilde{E}_6$, or by P_8, or by $T_{3,3,3}$ (see [AGV], [Sai1]).

(2) Given four lines in $\mathbb{C}^2$ through $\mathbf{0}$, defined by $a_i x + b_i y = 0$, then

$$f = \prod_{i=1}^{4}(a_i x + b_i y) \ \in \mathfrak{m}^4\,,$$

defines the union of these lines. Similar to the J-invariant for elliptic curves, there is an invariant of 4 lines (equivalently, 4 points in $\mathbb{P}^1$), the *cross-ratio*

$$r = \frac{(a_1b_3 - a_3b_1)\cdot(a_2b_4 - a_4b_2)}{(a_1b_4 - a_4b_1)\cdot(a_2b_3 - a_3b_2)}\,.$$

A direct computation shows that r is an invariant under linear coordinate changes. Since this is quite tedeous to do by hand, we provide the SINGULAR code for checking this.

```
ring R = (0,A,B,C,D,a1,a2,a3,a4,b1,b2,b3,b4),(x,y),dp;
ideal i= Ax+By, Cx+Dy;          // the coordinate transformation
ideal i1 = subst(i,x,a1,y,b1);
ideal i2 = subst(i,x,a2,y,b2);
ideal i3 = subst(i,x,a3,y,b3);
ideal i4 = subst(i,x,a4,y,b4);
poly r1 = (a1b3-a3b1)*(a2b4-a4b2);
poly r2 = (a1b4-a4b1)*(a2b3-a3b2);
// cross-ratio = r1/r2

poly s1 = (i1[1]*i3[2]-i3[1]*i1[2])*(i2[1]*i4[2]-i4[1]*i2[2]);
poly s2 = (i1[1]*i4[2]-i4[1]*i1[2])*(i2[1]*i3[2]-i3[1]*i2[2]);
// cross-ratio of transformed lines = s1/s2

// The difference of the cross-ratios:
r1/r2-s1/s2;
//-> 0
```

(3) Consider three parabolas which are tangent to each other,

$$f(x,y) = (x - t_1 y^2) \cdot (x - t_2 y^2) \cdot (x - t_3 y^2) \in \langle x, y^2 \rangle^3 .$$

Two such polynomials for different (t_1, t_2, t_3) are, in general, not contact equivalent. We show this for the family

$$f_t(x,y) = x(x - y^2)(x - ty^2) .$$

As in the proof of Theorem 2.55 we make a coordinate change φ and then consider the weighted 6-jet of $\varphi(f_t) - f_s$. The relation between t and s can be computed explicitly by eliminating the coefficients of the coordinate change. For this computation, the use of a computer is necessary. Here is the SINGULAR code:

```
ring r = 0,(a,b,c,d,e,f,g,h,i,j,s,t,x,y),dp;
poly ft = x*(x-y2)*(x-sy2);
poly fs = x*(x-y2)*(x-ty2);
ideal i = maxideal(1);
i[13] = ax+by+cx2+dxy+ey2;  // phi(x)
i[14] = fx+gy+hx2+ixy+jy2;  // phi(y)
map phi = r,i;
poly dd = phi(ft)-fs;
intvec w;
w[13],w[14]=2,1;            // weights for the variables x,y
coef(jet(dd,3,w),xy);       // weighted 3-jet (must be 0)
//-> _[1,1]=y3
//-> _[2,1]=b3             // hence, we must have b=0
dd=subst(dd,b,0);           // set b=0
// Now consider the weighted 6-jet:
matrix C = coef(jet(dd,6,w),xy);
ideal cc=C[2,1..ncols(C)];  // note: cc=0 iff the weighted
                            // 6-jets of phi(ft) and fs coincide
cc;
//-> cc[1]=eg4s-e2g2s-e2g2+e3
//-> cc[2]=ag4s-2aeg2s-2aeg2+3ae2-t
//-> cc[3]=-a2g2s-a2g2+3a2e+t+1
//-> cc[4]=a3-1
// We eliminate a,e,g in cc to get the relation between t and s:
eliminate(cc,aeg);
//-> _[1]=s6t4-s4t6-2s6t3-3s5t4+3s4t5+2s3t6+s6t2+6s5t3
//->      -6s3t5-s2t6-3s5t2-5s4t3+5s3t4+3s2t5+3s4t+5s3t2-5s2t3
//->      -3st4-s4-6s3t+6st3+t4+2s3+3s2t-3st2-2t3-s2+t2
```

For fixed t, the vanishing of this polynomial in s is necessary for $f_t \overset{c}{\sim} f_s$. Hence, there are at most 6 values of s such that f_t and f_s are contact equivalent.

Algorithmic Classification of ADE-Singularities. The proof of the classification of the simple singularities is effective and provides a concrete algorithm for deciding whether a given polynomial $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$, $n > 1$, is simple or not, and if it is simple to determine the type of f.

Step 1. Compute $\mu := \mu(f)$. If $\mu = \infty$ then f has a non-isolated singularity and, hence, is not simple.

The Milnor number can be computed as follows: compute a standard basis $sj(f)$ of $j(f)$ with respect to a local monomial ordering and let $L(j(f))$ be the ideal generated by the leading monomials of the generators of $sj(f)$. Then $\mu = \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/L(f)$, which can be determined combinatorially (cf. [GrP]). The Singular library `sing.lib` contains the command `milnor` (see also Example 2.7.2 (3)).

Step 2. Assume $\mu < \infty$. Let $f^{(2)}$ be the 2-jet of f and compute

$$r := \operatorname{rank}\Big(\frac{\partial^2 f^{(2)}}{\partial x_i \partial x_j}(\mathbf{0})\Big).$$

Then $n-r = \operatorname{crk}(f)$ and, if $n-r \geq 3$, then f is not simple. On the other hand if $n-r \leq 1$, then $f \overset{r}{\sim} A_\mu$. If $n-r = 2$ goto Step 3.

Step 3. Assume $n-r = 2$. Note that, in order to decide whether f is of type D or E, we need only to consider the 3-jet of $f^{(3)}$. That is, by a linear change of coordinates we get

$$f^{(3)} = x_3^2 + \ldots + x_n^2 + f_3(x_1, x_2) + \sum_{i=3}^{n} x_i g_i(\boldsymbol{x}), \quad f_3 \in \mathfrak{m}^3,\ g_i \in \mathfrak{m}^2.$$

The coordinate change $x_i \mapsto x_i - \frac{1}{2} g_i$, $i = 3, \ldots, n$ transforms $f^{(3)}$ into

$$g(x_1, x_2) + x_3^2 + \ldots + x_n^2 + h(\boldsymbol{x}), \quad g \in \mathfrak{m}^3,\ h \in \mathfrak{m}^4.$$

Assume $g \neq 0$. If g factors over $\mathbb{C}$ into two or three different linear factors, then $f \overset{r}{\sim} D_\mu$. If g has only one factor and $\mu \in \{6, 7, 8\}$, then $f \overset{r}{\sim} E_\mu$. If $g = 0$ or $\mu \notin \{6, 7, 8\}$, then f is not simple (and necessarily $\mu > 8$).

The splitting lemma uses linear algebra to adjust the 2-jet of f and then applies Tschirnhaus transformations in order to adjust higher and higher order terms. In order to check the number of different linear factors of g, one can apply for example the method discussed in Remark 2.52.1.

Let us treat an example with Singular, using some procedures from the library `classify.lib`.

```
LIB "classify.lib";
ring R = 0,(x,y,z,t),ds;
poly f = x4+3x3y+3x2y2+xy3+y4+4y3t+6y2t2+4yt3+t4+x3+z2+zt;
milnor(f);
//-> 6
corank(f);                      // the corank
//-> 2
poly g = morsesplit(f);
g;                              // the residual part
```

```
//-> x3+x4+3x3y+3x2y2+xy3+y4+16y6
poly h = jet(g,3);                  // the 3-jet of g
ideal jh = jacob(h);
nvars(R) - dim(std(jh));            // codim of Sing(h)
//-> 1
```

Hence, $\dim \mathbb{C}\{x,y\}/j(g^{(3)}) = 1$ and $f \overset{r}{\sim} E_6$.

SINGULAR is also able to classify many other classes of singularities. Some of them can be identified by computing invariants without applying the splitting lemma. The procedure `quickclass` uses this method. Arnol'd's original method [AGV] is implemented in the procedure `classify` of the library `classify.lib`.

```
poly nf = quickclass(f);
//-> Singularity R-equivalent to :  E[6k]=E[6]
//-> normal form : z2+t2+x3+xy3+y4
nf;
//-> z2+t2+x3+xy3+y4
```

Exercises

Exercise 2.4.1. Show that the *modality* (that is, the number of moduli) of isolated singularities is upper semicontinuous under deformations.

Exercise 2.4.2. Show that for $k \geq 4$ there exists a neighbourhood of D_k in $\mathfrak{m}^2$ which does not contain an A_k-singularity.

Exercise 2.4.3. Show that $\mu(f) > 8$ if $f \in \langle x, y^2\rangle^3$.

HINT: Choose a generic element from $\langle x, y^2\rangle^3$ and use the semicontinuity of μ.

Exercise 2.4.4. (1) Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\boldsymbol{x}\}$ have an isolated singularity, and let $g \in \mathbb{C}\{\boldsymbol{x}\}$ satisfy $g \notin \mathfrak{m} \cdot j(f)$, respectively $g \notin \mathfrak{m} \cdot j(f) + \langle f\rangle$.

Show that $f \overset{r}{\sim} f + tg$, respectively $f \overset{c}{\sim} f + tg$, for only finitely many $t \in \mathbb{C}$.

(2) Use this to show that if f has a non-isolated singularity, then, for each $k > 0$, there is some $g_k \in \mathfrak{m}^k \setminus (\mathfrak{m} \cdot j(f) + \langle f\rangle + \mathfrak{m}^{k+1})$ such that $f + tg_k \overset{c}{\not\sim} f$ for arbitrary small t. Hence, f is not contact simple and, therefore, also not right simple.

Exercise 2.4.5. Give a contact classification of

(1) the plane curve singularities of order 4 with a non-degenerate principal 4-form;
(2) the surface singularities in $(\mathbb{C}^3, \mathbf{0})$ of order 3 with a non-degenerate principal 3-form.

Show that, in both cases, one obtains a one-parametric space of normal forms. More precisely, show that the first problem reduces to the projective classification of 4-tuples on the projective line, and the parameter is the cross-ratio. Similarly, show that the second problem reduces to the projective classification of nonsingular plane cubics, and the parameter is the J-invariant.

Exercise 2.4.6. Describe all semiquasihomogeneous curve singularities with a one-parametric contact classification.

3 Plane Curve Singularities

This section is devoted to the study of reduced plane curve singularities, that is, isolated one-dimensional hypersurface singularities, given by a reduced power series $f \in \mathfrak{m} \subset \mathbb{C}\{x,y\}$. Here, we have an additional very powerful technique, the parametrization, which is not available in higher dimensions. Indeed, giving a reduced plane curve singularity either by an equation $f=0$ or by a parametrization is mathematically equivalent. However, since the data structures are quite different, the different points of view have quite different advantages. Hence, the combination of both gives very powerful tools for the investigation of plane curve singularities (this will be even more significant in Section II.2). We treat in detail the parametrization and the resolution by successive blowing ups which, besides its general importance, is a concrete way to compute the parametrization. The main emphasis of this section, which is rather classical, is on numerical analytic and topological invariants. Our presentation is in part influenced by the book of Casas-Alvero [Cas1], where many more aspects of plane curve singularities, like polar invariants, linear families of germs and complete ideals, are treated.

Starting with a reduced power series $f \in \mathfrak{m} \subset \mathbb{C}\{x,y\}$, we concentrate on the investigation of the zero set of f, that is, of the complex space germ $(C,\mathbf{0}) := V(f) \subset (\mathbb{C}^2,\mathbf{0})$, or, equivalently, of the analytic $\mathbb{C}$-algebra $\mathbb{C}\{x,y\}/\langle f\rangle$ (which is the same as studying f up to contact equivalence, see Remark 2.9.1 (3)).

We call $f=0$, or, by abuse of notation, also $f \in \mathbb{C}\{x,y\}$, a *local equation* for the plane curve germ $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$. Moreover, if $f = f_1^{n_1} \cdot \ldots \cdot f_r^{n_r}$ is the irreducible decomposition of $f \in \mathbb{C}\{x,y\}$ then

$$V(f) = V(f_1) \cup \ldots \cup V(f_r)\,,$$

and we call $(C_i,\mathbf{0}) = V(f_i)$ a *branch* of $(C,\mathbf{0})$, which is reduced if $n_i = 1$. The germ $(C,\mathbf{0})$ is reduced iff all n_i are 1. Since f is irreducible iff $r=1$ and $n_1 = 1$, an irreducible power series f defines an irreducible and reduced germ $(C,\mathbf{0})$. In order to be consistent in notation, $(C,\mathbf{0})$ irreducible means reduced and irreducible in this section.

3.1 Parametrization

Definition 3.1. Let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be an irreducible plane curve singularity. Then by a *parametrization of* $(C,\mathbf{0})$, we denote a holomorphic map germ

$$\varphi : (\mathbb{C},0) \longrightarrow (\mathbb{C}^2,\mathbf{0})\,, \quad t \longmapsto \big(x(t),y(t)\big)$$

with $\varphi(\mathbb{C},0) \subset (C,\mathbf{0})$ and satisfying the following *universal factorization property*: each holomorphic map germ $\psi : (\mathbb{C},0) \to (\mathbb{C}^2,\mathbf{0})$, $\psi(\mathbb{C},0) \subset (C,\mathbf{0})$, factors in a unique way through φ, that is, there exists a unique holomorphic map germ $\psi' : (\mathbb{C},0) \to (\mathbb{C},0)$ making the following diagram commute:

$$\begin{tikzcd} (\mathbb{C},0) \arrow[r, "\psi"] \arrow[d, "\psi'"'] & (\mathbb{C}^2,\mathbf{0})\,. \\ (\mathbb{C},0) \arrow[ur, "\varphi"'] & \end{tikzcd} \tag{3.1.1}$$

If $(C,\mathbf{0})$ decomposes into several branches then a *parametrization of* $(C,\mathbf{0})$ is a system of parametrizations of the branches. If $(C,\mathbf{0}) = V(f)$ then we call a parametrization of $(C,\mathbf{0})$ also a *parametrization of* f.

Example 3.1.1. Let $(C,\mathbf{0}) = V(f) \subset (\mathbb{C}^2,\mathbf{0})$, $f = y^2 - x^3$. Then the map germ $\varphi : (\mathbb{C},0) \to (\mathbb{C}^2,\mathbf{0})$, $t \mapsto \big(t^2,t^3\big)$, defines a parametrization of $(C,\mathbf{0})$, while $\psi : t \mapsto \big(t^4,t^6\big)$ maps $(\mathbb{C},0)$ onto $(C,\mathbf{0})$, but does not satisfy the universal factorization property (3.1.1).

Lemma 3.2. *Let $f \in \mathbb{C}\{x,y\}$ be irreducible, and let*

$$\varphi : (\mathbb{C},0) \to (\mathbb{C}^2,\mathbf{0}), \quad t \mapsto \big(x(t),y(t)\big)\,,$$

be a parametrization of $V(f)$. Then $\psi = (\psi_1,\psi_2) : (\mathbb{C},0) \to (\mathbb{C}^2,\mathbf{0})$ defines a parametrization of $V(f)$ iff there exists a unit $u \in \mathbb{C}\{t\}$ such that

$$\psi_1(t) = x\big(u\cdot t\big)\,, \quad \psi_2(t) = y\big(u\cdot t\big)\,.$$

Proof. The “if”-statement being obvious, it suffices to consider the case that ψ is a parametrization, too.

Then the universal factorization property of φ, respectively ψ, gives the existence of (unique) holomorphic map germs ψ', $\varphi' : (\mathbb{C},0) \to (\mathbb{C},0)$ such that $\varphi = \psi \circ \varphi' = (\varphi \circ \psi') \circ \varphi'$. By uniqueness, we have necessarily $\psi' \circ \varphi' = \mathrm{id}$, and, in the same manner we obtain $\varphi' \circ \psi' = \mathrm{id}$. In particular, ψ' is an isomorphism, that is, given by $t \mapsto u \cdot t$, $u \in \mathbb{C}\{t\}$ a unit. □

Remark 3.2.1. A parametrization of an irreducible power series $f \in \mathbb{C}\{x,y\}$ is given by power series $x(t), y(t) \in \mathbb{C}\{t\}$ satisfying

- $f\big(x(t),y(t)\big) = 0$ in $\mathbb{C}\{t\}$,

- if $\widetilde{x}(t), \widetilde{y}(t) \in \mathbb{C}\{t\}$ satisfy $f\big(\widetilde{x}(t), \widetilde{y}(t)\big) = 0$ in $\mathbb{C}\{t\}$ then there is a unique unit $u \in \mathbb{C}\{t\}$ such that $\widetilde{x}(t) = x(u \cdot t)$, $\widetilde{y}(t) = y(u \cdot t)$.

Replacing $\mathbb{C}\{t\}$ by $\mathbb{C}[[t]]$ we obtain the definition of a *parametrization of a formal* (irreducible) *power series* $f \in \mathbb{C}[[x, y]]$. A parametrization of a reduced, but possibly reducible $f = f_1 \cdot \ldots \cdot f_r \in \mathbb{C}[[x, y]]$ is given by a system of parametrizations for the factors f_i. In the same way, we define a parametrization of $f \in K\langle x, y\rangle$, K any algebraically closed field.

The main result of this section is the following generalization of the implicit function theorem for convergent, respectively formal, power series. As before, we write $\mathbb{C}\langle x, y\rangle$ to denote either $\mathbb{C}\{x, y\}$ or $\mathbb{C}[[x, y]]$.

Theorem 3.3 (Puiseux expansion). *Let $f \in \mathfrak{m} \subset \mathbb{C}\langle x, y\rangle$ be irreducible and y-general of order b. Then there exists $y(t) \in \langle t\rangle \cdot \mathbb{C}\langle t\rangle$ such that*

$$f\big(t^b, y(t)\big) = 0\,.$$

Moreover, $t \mapsto \big(t^b, y(t)\big)$ is a parametrization of f.

For $(x, y) \in V(f)$ we have $x = x(t) = t^b$ and $y = y(t) = y(x^{1/b})$. The fractional power series $y\big(x^{1/b}\big) \in \mathbb{C}\langle x^{1/b}\rangle$, $y(0) = 0$, is called a *Puiseux expansion* for f. The ring $\mathbb{C}\langle x^{1/b}\rangle$ is equal to the ring $\mathbb{C}\langle y\rangle$ ($x^{1/b}$ is just a symbol as t).We have natural inclusions

$$C\langle x\rangle \subset \mathbb{C}\langle t\rangle = \mathbb{C}\langle x^{1/b}\rangle \subset \mathbb{C}\langle s\rangle = \mathbb{C}\langle x^{1/ab}\rangle\,,$$

given by $x \mapsto t^b$, $t \mapsto s^a$, for $b, a \geq 1$.

For the proof pf Theorem 3.3, we follow Newton's constructive method as presented in Algorithm 3.6 (see also the historical considerations in [BrK, pp. 372ff]).

Before going into details, we give an important application, showing that each y-general Weierstraß polynomial $f \in \mathbb{C}\langle x\rangle[y]$ of order b decomposes over $\mathbb{C}\langle x^{1/b}\rangle[y]$ into (conjugated) linear factors. This implies that factorization over the ring $\mathbb{C}\{x, y\}$ is equivalent to factorization over $\mathbb{C}[[x, y]]$ (cf. Corollary 3.5, below).

Proposition 3.4. *Let $f \in \mathfrak{m} \subset \mathbb{C}\langle x, y\rangle$ be irreducible and y-general of order b.*

(1) Let $y(t) = \sum c_k t^k \in \langle t\rangle \cdot \mathbb{C}\langle t\rangle$ satisfy $f\,(t^m, y(t)) = 0$, m chosen minimally, that is, $\gcd\big(m, \{k \mid c_k \neq 0\}\big) = 1$. Then for ξ a primitive m-th root of unity the power series $y\big(\xi^j t\big) \in \langle t\rangle \cdot \mathbb{C}\langle t\rangle$, $j = 1, \ldots, m$, are pairwise different, and there is a unit $u \in \mathbb{C}\langle x, y\rangle$ such that

$$f = u \cdot \prod_{j=1}^{m} \Big(y - y\big(\xi^j x^{1/m}\big)\Big)\ .$$

In particular, $m = b$.

(2) If $f = y^b + a_1 y^{b-1} + \ldots + a_b \in \mathbb{C}\langle x\rangle[y]$ is a Weierstraß polynomial then there exists a power series $y(t) \in \langle t\rangle \cdot \mathbb{C}\langle t\rangle$ such that

$$f = \prod_{j=1}^{b} \Big(y - y\big(\xi^j x^{1/b}\big) \Big) \,, \tag{3.1.2}$$

ξ a primitive b-th root of unity. Moreover, the decomposition (3.1.2) is unique.

Proof. (1) By the Weierstraß preparation theorem 1.6, we obtain a decomposition $f = ug$ with $u \in \mathbb{C}\langle x, y\rangle$ a unit and $g \in \mathbb{C}\langle x\rangle[y]$ a Weierstraß polynomial of degree b. By our assumption, $g\,(t^m, y(t)) = 0 \in \mathbb{C}\langle t\rangle$, which, due to the Weierstraß division theorem, implies that $y - y(t)$ divides $g(t^m, y)$ as elements of $\mathbb{C}\langle t, y\rangle$.

Let $\xi := e^{2\pi i/m}$. Since no divisor of m divides all k with $c_k \neq 0$, the power series $y\big(\xi^j t\big)$, $j = 1, \ldots, m$, are pairwise different. On the other hand,

$$0 \;=\; g\big((\xi^j t)^m,\, y(\xi^j t)\big) \;=\; g\big(t^m,\, y(\xi^j t)\big) \,,$$

and, as before, $y - y(\xi^j t)$ divides $g(t^m, y)$ in $\mathbb{C}\langle t, y\rangle$. It follows that

$$\varPi \;:=\; \prod_{j=1}^{m} \big(y - y\big(\xi^j t\big)\big)$$

divides g as an element of $\mathbb{C}\langle t, y\rangle$. But $\varPi$ is invariant under the conjugation $t \mapsto \xi^j t$, hence, $\varPi \in \mathbb{C}\langle x\rangle[y]$. Indeed, the Galois group of the field extension $K = \mathrm{Quot}(\mathbb{C}\langle x\rangle) \hookrightarrow \mathrm{Quot}(\mathbb{C}\langle t\rangle) = L$, $x \mapsto t^m$, consists of the m-th roots of unity, and $\varPi \in L[y]$ is invariant under this group.

Since g is irreducible, we obtain $g = u'\varPi$, $u' \in \mathbb{C}\langle x, y\rangle$ a unit. The uniqueness statement of the Weierstraß preparation theorem implies even $g = \varPi$. Finally, (2) follows from (1) and Theorem 3.3, the uniqueness follows, since $\mathbb{C}\langle t, y\rangle$ is factorial (Theorem 1.16). □

Corollary 3.5. *Let $f \in \mathbb{C}\{x, y\}$. Then f is irreducible as an element of $\mathbb{C}\{x, y\}$ iff it is irreducible in $\mathbb{C}[[x, y]]$.*

Proof. We need only to show that an irreducible element $f \in \mathbb{C}\{x, y\}$ is also irreducible in $\mathbb{C}[[x, y]]$. By Lemma 1.5 and the Weierstraß preparation theorem, we can assume that $f \in \mathbb{C}\{x\}[y]$ is a Weierstraß polynomial of degree $b > 0$.

In this case, Proposition 3.4 gives a decomposition of $f(t^b, y)$ in b *linear* factors $g_i \in \mathbb{C}\{t\}[y] \subset \mathbb{C}[[t, y]]$, $g_i(0, 0) = 0$. Since $\mathbb{C}[[t, y]]$ is factorial, (Theorem 1.16), $f(t^b, y) = g_1 \cdot \ldots \cdot g_b$ is the unique prime decomposition in $\mathbb{C}[[t, y]]$, and there is a partition $S_1 \cup \ldots \cup S_r$ of $\{1, \ldots, b\}$ such that $\prod_{j \in S_i} g_j$, $i = 1, \ldots, r$, are irreducible elements of $\mathbb{C}[[t^b, y]]$. Since each product $\prod_{j \in S_i} g_j$ is convergent, our assumption implies $r = 1$. □

If $f \in \mathbb{C}\{x,y\}$ decomposes as $f = f_1 \cdot f_2$ in $\mathbb{C}[[x,y]]$, then the factors f_1, f_2 need not be convergent, but there exists a unit $u \in \mathbb{C}[[x,y]]$ such that uf_1 and $u^{-1}f_2$ are convergent.

Remark 3.5.1. Artin's approximation theorem [Art1] gives a generalization of the latter statement: a convergent power series $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$ is irreducible as element of $\mathbb{C}\{\boldsymbol{x}\}$ iff it is irreducible in $\mathbb{C}[[\boldsymbol{x}]]$. To see this, consider the analytic equation $XY - f = 0$ (see Remarks and Exercises (B) on page 32).

The main tool for Newton's algorithm to compute a parametrization for a branch of a plane curve singularity is the Newton diagram $\Gamma(f) := \Gamma(f, \mathbf{0})$ of f at $\mathbf{0}$ (cf. Definition 2.14). Recall that for any *facet* (= one-dimensional face) $\sigma \subset \Gamma(f)$ we denote by f^σ the truncation of f at σ. Moreover, f is called *convenient* if the Newton diagram meets the coordinate axes, that is, there exist positive integers k, ℓ such that $(k,0), (0,\ell) \in \operatorname{supp}(f)$.

Note that any $f \in \mathbb{C}\langle x,y\rangle$ can be written as $f = x^k y^\ell f_1$ with $f_1 \in \mathbb{C}\langle x,y\rangle$ convenient.

Algorithm 3.6 (Newton-Puiseux). Let $f \in \langle x,y\rangle \subset \mathbb{C}\langle x,y\rangle$ be a convenient power series. Then the following algorithm computes a Puiseux expansion $s^{(0)}$ for some irreducible factor of f.

STEP 0. Set $i=0$, $f^{(0)} = f$, $x_0 = x$, $y_0 = y$.

STEP 1. Let $i \geq 0$, $f^{(i)} \in \mathbb{C}\langle x_i, y_i\rangle$ and $\sigma \subset \Gamma\big(f^{(i)}\big)$ be the steepest facet, that is, the facet with minimal slope $-\frac{p_i}{q_i}$, p_i, q_i coprime positive integers. With respect to the weights (p_i, q_i), the truncation $f^{(i),\sigma}$ is a quasihomogeneous polynomial of some degree $d_i > 0$. Let a_i be an arbitrary root of the univariate polynomial $f^{(i),\sigma}(1,y) \in \mathbb{C}[y]$. Then we substitute in $f^{(i)}$

$$x_i = x_{i+1}^{p_i}, \quad y_i = x_{i+1}^{q_i}(a_0 + y_{i+1}),$$

where x_{i+1}, y_{i+1} are new variables. Set

$$f^{(i+1)}(x_{i+1}, y_{i+1}) := \frac{1}{x_{i+1}^{d_i}} f^{(i)}\big(x_{i+1}^{p_i}, x_{i+1}^{q_i}(a_i + y_{i+1})\big) \in \mathbb{C}\langle x_{i+1}, y_{i+1}\rangle,$$
$$s^{(i)} := x_i^{q_i/p_i}\big(a_i + s^{(i+1)}\big),$$

where $s^{(i+1)}$ is a fractional power series in x to be determined in the subsequent steps.

STEP 2A. If the Newton diagram $\Gamma\big(f^{(i+1)}\big)$ does not reach the x_{i+1}-axis, that is, $(k,0) \not\in \operatorname{supp}\big(f^{(i+1)}\big)$ for any positive integer k, then set

$$s^{(i+1)} := 0$$

and go to Step 3.

Step 2b. If the Newton diagram $\Gamma\big(f^{(i+1)}\big)$ reaches the x-axis then raise i by 1 and return to Step 1.

Step 3. Replace successively $s^{(j+1)}$ in the definition of $s^{(j)}$, $j = 0, \ldots, i$, and obtain

$$\begin{aligned} s^{(0)} &= x^{q_0/p_0}\Big(a_0 + x_1^{q_1/p_1}\Big(a_1 + \ldots + x_i^{q_i/p_i}\Big(a_i + s^{(i+1)}\Big)\Big)\Big) \\ &= x^{q_0/p_0}\Big(a_0 + x^{q_1/(p_0p_1)}\Big(a_1 + \ldots + x^{q_i/(p_0\cdots p_i)}\Big(a_i + s^{(i+1)}\Big)\Big)\Big)\,. \end{aligned}$$

Note that, in general, this algorithm does not terminate (that is, it does not reach Step 3). What we claim is, that

$$s^{(0)} = \sum_{k=0}^{\infty} a_k x^{q_0/p_0 + q_1/(p_0p_1) + \ldots + q_k/(p_0\cdots p_k)}$$

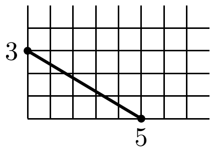
is, indeed, an element of $\mathbb{C}\langle x^{1/N}\rangle$ for some positive integer N and satisfies $f\big(x, s^{(0)}\big) = 0$. Before giving a proof, let's consider two simple (irreducible) examples.

Example 3.6.1. (1) Let $f = y^3 - x^5$. Then, of course, a parametrization is given by $t \mapsto (t^3, t^5)$, and we have a decomposition

$$f = \big(y - x^{5/3}\big)\cdot\big(y - \xi x^{5/3}\big)\cdot\big(y - \xi^2 x^{5/3}\big)\,, \qquad \xi = e^{2\pi i/3}$$

(Proposition 3.4).

Let's check what happens when we apply the Newton-Puiseux algorithm. The Newton diagram $\Gamma(f)$ has only one facet σ which has slope $-\frac{3}{5}$.



Since the support of f is contained in σ, $f = f^\sigma$ which is a (3,5)-weighted homogeneous polynomial of degree $d_0 = 15$. $f^\sigma(1, y) = y^3 - 1$ has the three roots $1, \xi, \xi^2$. Let's choose $a_0 = 1$. Then

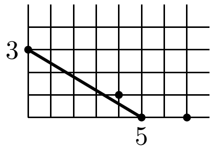
$$\begin{aligned} f^{(1)}(x_1, y_1) &= \frac{1}{x_1^{15}}\, f\big(x_1^3, x_1^5(1 + y_1)\big) = \frac{1}{x_1^{15}}\big(x_1^{15}(1 + y_1)^3 - x_1^{15}\big) \\ &= 3y_1 + 3y_1^2 + y_1^3 \in \mathbb{C}\{x_1, y_1\} \end{aligned}$$

In particular, the Newton diagram $\Gamma\big(f^{(1)}\big)$ does not reach the x_1-axis, and we obtain

$$s^{(0)} = x^{5/3}\big(1 + s^{(1)}\big) = x^{5/3}.$$

Note that we would have obtained the other two solutions, $\xi x^{5/3}, \xi^2 x^{5/3}$, when choosing $a_0 = \xi, \xi^2$.

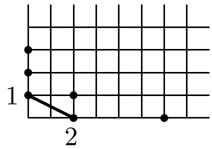
(2) Let $f = y^3 - x^5 - 3x^4y - x^7$. Now, it is not so obvious, what a possible parametrization could be.



The Newton diagram of f at $\mathbf{0}$ is the same as in example (1). The only difference is that the support of f is no longer contained in $\Gamma(f)$. As before, we obtain

$$\begin{aligned}
f^{(1)}(x_1,y_1) &= \frac{1}{x_1^{15}}\, f\big(x_1^3, x_1^5(1+y_1)\big)\\
&= \frac{1}{x_1^{15}}\,\big(x_1^{15}(1+y_1)^3 - x_1^{15} - 3x_1^{17}(1+y_1) - x_1^{21}\big)\\
&= 3y_1 + 3y_1^2 + y_1^3 - 3x_1^2 - 3x_1^2y_1 - x_1^6\,.
\end{aligned}$$

The Newton diagram of $f^{(1)}$ at $\mathbf{0}$ looks like



and we can apply the implicit function theorem to obtain the existence of a solution $Y(x_1)$ for $f^{(1)}\big(x_1, Y(x_1)\big) = 0$. However, to compute $Y(x_1)$ (up to an arbitrary precision) we can go on with the algorithm. We obtain $f^{(1),\sigma}(1,y_1) = 3y_1 - 3$, which has $a_1 = 1$ as only root. We set

$$\begin{aligned}
f^{(2)}(x_2,y_2) &= \frac{1}{x_2^2}\, f^{(1)}\big(x_2, x_2^2(1+y_2)\big)\\
&= 3y_2 + 3x_2^2(1+y_2)^2 + x_2^4(1+y_2)^3 - 3x_2^2(1+y_2) - x_2^4\\
&= 3y_2 + 3x_2^2y_2 + 3x_2^2y_2^2 + 3x_2^4y_2 + 3x_2^4y_2^2 + x_2^4y_2^3\,.
\end{aligned}$$

$\Gamma\big(f^{(2)}\big)$ does not reach the x_2-axis, whence $s^{(2)} := 0$, and we conclude

$$s^{(0)} = x^{5/3}\big(1 + x_1^2\big(1 + s^{(2)}\big)\big) = x^{5/3}\big(1 + x^{2/3}\big) = x^{5/3} + x^{7/3}\,.$$

Finally, due to Proposition 3.4, the Weierstraß polynomial f decomposes

$$f = \big(y - x^{5/3} - x^{7/3}\big)\big(y - \xi^2x^{5/3} - \xi x^{7/3}\big)\big(y - \xi x^{5/3} - \xi^2x^{7/3}\big)$$

with $\xi = e^{2\pi i/3}$.

Proof of Theorem 3.3. It suffices to show that the (infinite) Newton-Puiseux algorithm 3.6 is well-defined and returns a power series

$$s^{(0)} = \sum_{k=0}^{\infty} a_k x^{q_0/p_0+q_1/(p_0p_1)+\ldots+q_k/(p_0\cdots p_k)} \in \mathbb{C}\langle x^{1/N}\rangle$$

for some positive integer N, satisfying $f\bigl(x, s^{(0)}\bigr) = 0$. More precisely, using the above notations, we prove for any $i \geq 0$:

(1) $f^{(i)}\bigl(x_{i+1}^{p_i}, x_{i+1}^{q_i}(a_i + y_{i+1})\bigr)$ contains $x_{i+1}^{d_i}$ as a factor.
(2) $f^{(i)} \in \mathbb{C}\langle x_i, y_i\rangle$ is y_i-general of order b_i with $b_0 \geq b_1 \geq \ldots \geq b_i > 0$. Moreover, if $b_i = b_{i-1}$ then $p_{i-1} = 1$.

In particular, there exists a positive integer i_0 such that $b_i = b_{i-1}$ for any $i \geq i_0$, and $s^{(0)} \in \mathbb{C}[[x^{1/N}]]$ where $N = p_0 \cdot \ldots \cdot p_{i_0}$. Finally, we show:

(3) $s^{(0)}$ satisfies $f\bigl(x, s^{(0)}\bigr) = 0$.
(4) $s^{(0)} \in \mathbb{C}\langle x^{1/N}\rangle$.
(5) If f is irreducible and y-general of order b then $b = N$.

Proof of (1). We can write

$$f^{(i)} = \sum_{d \geq d_i} f_d^{(i)}$$

with $f_d^{(i)} \in \mathbb{C}\langle x_i, y_i\rangle$ (p_i, q_i)-weighted homogeneous of degree d. Hence, we obtain

$$f^{(i)}\bigl(x_{i+1}^{p_i}, x_{i+1}^{q_i}(a_i + y_{i+1})\bigr) \;=\; \sum_{d \geq d_i} x_{i+1}^{d} \cdot f_d^{(i)}\bigl(1,\, a_i + y_{i+1}\bigr)\,.$$

Proof of (2). We proceed by induction on i, $f^{(0)} = f$ being y-regular of order $b_0 = b$, by assumption. Due to the above, we can write

$$f^{(i+1)}(x_{i+1}, y_{i+1}) \;=\; \sum_{d \geq d_i} x_{i+1}^{d-d_i} \cdot f_d^{(i)}\bigl(1,\, a_i + y_{i+1}\bigr)$$

and have to show that $f_{d_i}^{(i)}\bigl(1,\, a_i + y_{i+1}\bigr)$ is y_{i+1}-regular of order $0 < b_{i+1} \leq b_i$. The univariate polynomial $f_{d_i}^{(i)}(1, y_i) = c \cdot y_i^{b_i} +$ lower terms in y_i, $c \neq 0$, factorizes

$$f_{d_i}^{(i)}(1, y_i) \;=\; c \cdot \bigl(y_i - a_i\bigr) \cdot \bigl(y_i - a_i^{(1)}\bigr) \cdot \ldots \cdot \bigl(y_i - a_i^{(b_i-1)}\bigr)\,,$$

$a_i,\, a_i^{(1)}, \ldots,\, a_i^{(b_i-1)} \in \mathbb{C}$. It follows that

$$f_{d_i}^{(i)}\bigl(1,\, a_i + y_{i+1}\bigr) \;=\; c \cdot y_{i+1} \cdot \bigl(y_{i+1} + a_i - a_i^{(1)}\bigr) \cdot \ldots \cdot \bigl(y_{i+1} + a_i - a_i^{(b_i-1)}\bigr)\,,$$

is y_{i+1}-regular of order $0 < b_{i+1} \leq b_i$.

Moreover, $b_{i+1} = b_i$ implies that $a_i^{(k)} = a_i$ for any $k = 1, \ldots, b_i - 1$, that is, $f_{d_i}^{(i)}(1, y_i) = c \cdot (y_i - a_i)^{b_i}$. Since $f_{d_i}^{(i)}$ is (p_i, q_i)-weighted homogeneous, we obtain $f^{(i)}_{d_i}(x_i, y_i) = c \cdot (y_i - a_i x_i^m)^{b_i}$ with $m \in \mathbb{N}$ satisfying $mp_i = q_i$. Recall that, by assumption, $\gcd(p_i, q_i) = 1$, which implies $p_i = 1$.

Proof of (3). By construction, we have for any $i \geq 0$ either

$$\begin{aligned} f^{(i)}\big(x_i, s^{(i)}\big) = f^{(i)}\big(x_i, x_i^{q_i/p_i}(a_i + s^{(i+1)})\big) \; &= \; f^{(i)}\big(x_{i+1}^{p_i}, x_{i+1}^{q_i}(a_i + s^{(i+1)})\big) \\ = x_{i+1}^{d_i} \cdot f^{(i+1)}\big(x_{i+1}, s^{(i+1)}\big)\,, & \end{aligned}$$

or the Newton diagram $\Gamma\big(f^{(i)}\big)$ does not reach the x_i-axis, that is, $f^{(i)}$ is divisible by y_i, and we have set $f^{(i)}\big(x_i, s^{(i)}\big) = f^{(i)}\big(x_i, 0\big) = 0$. In the first case the degrees of the lowest non-vanishing terms satisfy

$$\operatorname{ord}_{x_i}\big(f^{(i)}(x_i, s^{(i)})\big) \; = \; \frac{d_i}{p_i} + \frac{1}{p_i} \cdot \operatorname{ord}_{x_{i+1}}\big(f^{(i+1)}(x_{i+1}, s^{(i+1)})\big)\,,$$

and, by induction,

$$\operatorname{ord}_x f\big(x, s^{(0)}\big) \; > \; \sum_{j=0}^{i} \underbrace{\frac{d_j}{p_0 p_1 \cdots p_j}}_{\geq 1/N} \; \geq \; \frac{i+1}{N}$$

for any $i \geq 0$. Hence, the vanishing order of $f\big(x, s^{(0)}\big)$ is infinite, that is, $f\big(x, s^{(0)}\big) = 0$.

Proof of (4). If the algorithm terminated, that is, reached Step 3, then there is nothing to show ($s^{(0)}$ is even a polynomial in $x^{1/N}$). Moreover, by construction, the power series $s^{(i)} \in \mathbb{C}[[x^{1/N}]]$ converges exactly if $s^{(i+1)}$ does. Hence, by (2), we can assume without restriction that all $f^{(i)}$, $i \geq 0$, are y_i-general of the same order $b > 0$. In particular, as we have seen above, $N = 1$ and the Newton diagram of $f^{(i)}$ at 0 has a unique facet $\sigma = \overline{(0, b), (q_i b, 0)}$.

Case 1. $b = 1$.

Then $f = f^{(0)}$ is y-regular of order 1, that is, $f(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) \neq 0$. The implicit function theorem implies the existence of a *convergent* power series $Y(x) \in \langle x \rangle \cdot \mathbb{C}\langle x \rangle$ such that $f\big(x, Y(x)\big) = 0$. By the Weierstraß division theorem, both $y - s^{(0)}(x) \in \mathbb{C}[[x, y]]$ and $y - Y(x) \in \mathbb{C}\langle x, y \rangle$ divide f as *formal* power series. Finally, the uniqueness of the Weierstraß polynomial in the Weierstraß preparation theorem implies $s^{(0)} = Y(x)$.

Case 2. $b > 1$.

We show that this can only occur if $f\big(x, s(x)\big) = 0$ has a (unique) root $s = s^{(0)} \in \mathbb{C}[[x]]$ of order b. Clearly, this is then a solution of

$$\frac{\partial^{b-1} f}{\partial y^{b-1}}\big(x, s(x)\big) \; = \; 0\,,$$

and the statement follows as in Case 1, since $\frac{\partial^{b-1} f}{\partial y^{b-1}} \in \mathbb{C}\langle x, y \rangle$ is y-regular of order 1.

Let $s=\sum_{k\geq k_0} s_k x^k$ satisfy $f\bigl(x,s\bigr)=0$. Then, in particular, the terms of lowest degree in x cancel. Hence, there are at least two monomials of $f=\sum c_{k\ell}x^k y^\ell$ of minimal $(1,k_0)$-weighted degree d and $\sum_{k+k_0\ell=d} c_{k\ell}s_{k_0}^\ell=0$. In other words, there is a facet $\sigma'\subset\Gamma(f)$ of slope $-\frac{1}{k_0}$ such that $f^{\sigma'}(1,s_{k_0})$ vanishes.

On the other hand, we have seen above that our assumptions imply that $\Gamma(f)$ has a unique facet σ and $f^\sigma(1,y)=c\cdot(y-a_0)^b$, $c\neq 0$. It follows that $\sigma'=\sigma$, $k_0=q_0$ and $s_{k_0}=a_0$. Moreover,

$$s' := x^{-q_0}(s-s_{q_0}x^{q_0}) = \sum_{k>q_0} s_k x^{k-q_0}$$

satisfies $f^{(1)}\bigl(x,s'\bigr)=0$. By induction, we obtain $s=s^{(0)}\in\mathbb{C}[[x]]$.

Finally, we can apply inductively the Newton-Puiseux algorithm to $\bigl(y-s^{(0)}\bigr)^{-i}f\in\mathbb{C}[[x,y]]$, $i=1,\dots,b-1$, to show that $s^{(0)}\in\mathbb{C}[[x]]$ is a root of order b.

Proof of (5). In the proof of Proposition 3.4, we have already shown that

$$\Pi := \prod_{j=1}^{N}\Bigl(y-s^{(0)}\bigl(\xi^j x^{1/N}\bigr)\Bigr)\,,\quad \xi=e^{2\pi i/N}\,,$$

divides f as an element of $\mathbb{C}\langle x,y\rangle$. The irreducibility of f implies that it is y-general of order N as Π is. □

Proposition 3.7. *Let $f\in\mathbb{C}\langle x\rangle[y]$ be an irreducible Weierstraß polynomial of degree b, and let $y(x^{1/b})\in\mathbb{C}\langle x^{1/b}\rangle$ be any Puiseux expansion of f. Moreover, let $w_1(t),w_2(t)\in\mathbb{C}\langle t\rangle$ satisfy $f\bigl(w_1(t),w_2(t)\bigr)=0$. Then there exists a unique power series $h(t)\in\mathbb{C}\langle t\rangle$ such that*

$$\bigl(w_1(t),w_2(t)\bigr) = \bigl(h(t)^b,y(h(t))\bigr)\,. \tag{3.1.3}$$

Proof. Case 1. $w_1=0$.

By our assumptions, this implies $0=f\bigl(0,w_2(t)\bigr)=w_2(t)^b$, hence, $w_2=0$, and we can set $h(t):=0\in\mathbb{C}\langle t\rangle$.

Case 2. $w_1\neq 0$.

Then we can write $w_1(t)=t^m w_1'(t)$, $w_1'(0)\neq 0$. In particular, there exists a unit $u\in\mathbb{C}\langle t\rangle$ such that $u^m=w_1'$. Setting $h':=tu\in\langle t\rangle\cdot\mathbb{C}\langle t\rangle$, we obtain $w_1(t)=h'(t)^m$.

Let $w_2(t)=\sum_{k=0}^\infty c_k t^k$, and denote by M the greatest common divisor of m with all indices k in the support of w_2. Setting $h''(t):=h'(t)^M$, we have

$$\bigl(w_1(t),w_2(t)\bigr) = \left(h''(t)^{m'},\sum_{k=0}^\infty c_{Mk}\frac{h''(t)^k}{u^{Mk}}\right),\qquad m'=\frac{m}{M}\,,$$

and, by Proposition 3.4 (1) and the factoriality of $\mathbb{C}[[h'', y]]$, it follows that $m' = b$ and

$$\sum_{k=0}^{\infty} c_{Mk} \frac{h''(t)^k}{u^{Mk}} = y\big(\xi^j h''(t)\big)$$

for some $1 \leq j \leq b$, ξ a primitive b-th root of unity. Finally, $h(t) := \xi^j h''(t)$ satisfies (3.1.3).

The proof of uniqueness is standard and left as an exercise. □

Corollary 3.8. *Let $f \in \mathbb{C}\{x, y\}$ be irreducible. Then there exists a parametrization $\varphi : (\mathbb{C}, 0) \to V(f) \subset (\mathbb{C}^2, \mathbf{0})$, $t \mapsto \big(x(t), y(t)\big)$. Moreover, after a linear coordinate change, we may assume that $\big(x(t), y(t)\big) = \big(t^b, y(t)\big)$, with $b = \operatorname{ord}(f)$ and $\operatorname{ord}(y(t)) > b$.*

Proof. Applying a linear coordinate change (Exercise 1.1.6) and the Weierstraß preparation theorem, we can assume that $f \in \mathbb{C}\{x\}[y]$ is a Weierstraß polynomial of degree $b = \operatorname{ord}(f)$. In this case, Proposition 3.7 shows that any Puiseux expansion $y(x^{1/b})$ defines a parametrization $t \mapsto \big(t^b, y(t)\big)$ of $V(f)$. Remark 3.8.1 gives that, indeed, $\operatorname{ord}(y(t)) > b$. □

Remark 3.8.1. We shall often use the following simple fact which follows from the comparison of the terms of lowest degree. Let $f = y^m + f_{m+1} + \ldots$ with $f_i \in \mathbb{C}[x, y]$ homogeneous of degree i, and let $f\big(t^b, y(t)\big) = 0$. Then $\operatorname{ord}\big(y(t)\big) > b$.

If $f = f_m + f_{m+1} + \ldots \in \mathbb{C}\{x, y\}$ is irreducible then Lemma 3.19 implies that, indeed, (up to a linear coordinate change) we may assume that $f_m = y^m$.

Remark 3.8.2. There exists an analogue of Puiseux expansions when working over an algebraically closed field K of positive characteristic, the so-called *Hamburger-Noether expansions* (HNE) for the branches $V(f_\nu)$ of a plane curve germ $V(f) \subset (\mathbb{C}^2, \mathbf{0})$:

$$\begin{aligned}
z_{-1} &= a_{0,1} z_0 + a_{0,2} z_0^2 + \ldots + a_{0,h_0} z_0^{h_0} + z_0^{h_0} z_1 \\
z_0 &= \qquad\qquad a_{1,2} z_1^2 + \ldots + a_{1,h_1} z_1^{h_1} + z_1^{h_1} z_2 \\
&\vdots \qquad\qquad\quad \vdots \\
z_{i-1} &= \qquad\qquad a_{i,2} z_i^2 + \ldots + a_{i,h_i} z_i^{h_i} + z_i^{h_i} z_{i+1} \\
&\vdots \qquad\qquad\quad \vdots \\
z_{s-2} &= \qquad\qquad a_{s-1,2} z_{s-1}^2 + \ldots + a_{s-1,h_{s-1}} z_{s-1}^{h_{s-1}} + z_{s-1}^{h_{s-1}} z_s \\
z_{s-1} &= \qquad\qquad a_{s,2} z_s^2 + a_{s,3} z_s^3 + \ldots\ldots\ldots\ldots
\end{aligned}$$

where s is a non-negative integer, $a_{j,i} \in K$, and the h_j, $j = 1, \ldots, s-1$, are positive integers, such that $f_\nu\big(z_0(z_s), z_{-1}(z_s)\big) = 0$ in $K[[z_s]]$ (here, we assume that x is not in the tangent cone of f_ν).

Note that any HNE leads to a parametrization $\varphi : K[[x,y]] \to K[[t]]$ of the branch (setting $t := z_s$ and mapping $x \mapsto z_0(z_s)$, $y \mapsto z_{-1}(z_s)$), but in general we cannot achieve the parametrization with $\varphi(x) = t^b$.

There exist constructive algorithms to compute a system of HNE's (up to a given degree) for the branches of a reduced plane curve singularity (cf. [Cam] and [Ryb] for details in the reducible case). A modification of the latter algorithm is implemented in SINGULAR. We can use it, for instance, to compute a parametrization for the (reducible) plane curve singularity in Example 2.14.1:

```
LIB "hnoether.lib";
ring r = 0,(x,y),ds;
poly f = y5+xy3+2x2y2-x2y4+x3y-10x4y+x6;

list L = hnexpansion(f);      // result is a list of rings
def R  = L[1];
setring R;                    // contains list hne = HNE of f
```

Let us look at the (computed jets of the) parametrization of the first branch:

```
parametrisation(hne[1],0); // with optional second parameter 0
                           // the exactness is returned, too
//-> [1]:
//->    [1]:
//->       _[1]=1/9x2-4/81x3
//->       _[2]=-1/9x2+13/81x3-4/81x4
//->    [2]:
//->       3,3
```

We read the parametrization: $t \mapsto \left(\frac{1}{9}t^2 - \frac{4}{81}t^3 + \ldots\,,\, -\frac{1}{9}t^2 + \frac{13}{81}t^3 + \ldots\right)$. To compute the terms up to order 10 we can extend the computation, by typing

```
parametrisation(extdevelop(hne[1],10));
//-> // Warning: result is exact up to order 10 in x and 10 in y !
//-> _[1]=1/9x2-4/81x3+70/729x4-856/6561x5+9679/59049x6
//          -118906/531441x7+1438831/4782969x8-17658157/43046721x9
//          +216843244/387420489x10
//-> _[2]=-1/9x2+13/81x3-106/729x4+1486/6561x5-17383/59049x6
//          +206017/531441x7-2508985/4782969x8+30607636/43046721x9
//          -375766657/387420489x10+216843244/387420489x11
```

Finally, we have a short look at the parametrizations of the other (smooth) branches of f:

```
parametrisation(hne[2]);
//-> _[1]=x
//-> _[2]=-x3-10x4
parametrisation(hne[3]);
//-> _[1]=-x2
//-> _[2]=x
```

If the field K is not algebraically closed then there exists a parametrization of $f \in K\langle x,y\rangle$ with $x(t), y(t) \in L\langle t\rangle$, where $L \supset K$ is some finite field extension of K.

Exercises

Exercise 3.1.1. Let K be a field and $t \mapsto \big(x(t), y(t)\big)$ a parametrization of $f \in \langle x,y\rangle \subset K\langle x,y\rangle$ with $x(t), y(t) \in K\langle t\rangle$, $x(t)$ monic of order b.

(1) Show that, if $\mathrm{char}(K) = 0$, then f has a parametrization $t \mapsto \big(t^b, \widetilde{y}(t)\big)$ with $\widetilde{y}(t) \in K\langle t\rangle$, too. Moreover, show that, for any $m \geq 0$, the m-jet of $\widetilde{y}(t)$ can be computed from sufficiently high jets of $x(t), y(t)$.

(2) Give an example that (1) does not hold for $\mathrm{char}(K) > 0$.

(3) Write a Singular procedure taking as input an integer m and polynomials $x(t), y(t)$ and returning t^b and the m-jet of $\widetilde{y}(t)$. Test your procedure for $x(t) = t^2 + t^3 + \ldots + t^{10}$ and $y(t) = t^5 + t^7 + t^9$.

Hint: You need subprocedures to compute the b-th root of a unit in $K\langle t\rangle$ and the inverse of an isomorphism $K\langle t\rangle \xrightarrow{\cong} K\langle t\rangle$, each up to a given order.

Exercise 3.1.2. Let $2 \leq b < a_1 < a_2 < a_3 < a_4 < \ldots$ be integers, and let $x(t) = t^b$, $y(t) = \sum_{i=1}^{\infty} \alpha_i t^{a_i}$ define a parametrization of an irreducible plane curve germ $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ with isolated singularity at the origin. Put $D_0 = b$, $D_1 = \gcd(b, a_1)$, $D_2 = \gcd(b, a_1, a_2)$,... . Show that

$$\mu(C, \mathbf{0}) = \sum_{i \geq 1} (a_i - 1)(D_{i-1} - D_i)\,.$$

Exercise 3.1.3. (1) Show that the ring of locally convergent Puiseux series $\bigcup_{m \geq 1} \mathbb{C}\langle x^{1/m}\rangle$ is Henselian.

(2) Show that the set of locally convergent series $\sum_{k \in I} a_k x^k$, $a_k \in \mathbb{C}$, $k \in I$, $I \subset [0, \infty)$ some set such that each subset of I has a minimal element, is a Henselian ring.

Exercise 3.1.4. Show that the field of power series $\sum_{k \in I} a_k x^k$, $a_k \in \mathbb{C}$, $k \in I$, where $I \subset \mathbb{R}$ is any set bounded from below whose subsets all have minimal elements, is algebraically closed.

Exercise 3.1.5. Show that any irreducible one-dimensional complex space germ $X \subset (\mathbb{C}^n, \mathbf{0})$, $n \geq 2$, which is not contained in the hyperplane $\{x_1 = 0\}$, possesses a parametrization

$$x_1 = t^m, \qquad x_i = \sum_{j=1}^{\infty} a_{ij} t^j, \quad i = 2, \ldots, n\,,$$

for some $a_{ij} \in \mathbb{C}$ such that $\gcd\big(m, \bigcup_{i \geq 2} \{j \mid a_{ij} \neq 0\}\big) = 1$.

Hint. Use projections to 2-planes.

3.2 Intersection Multiplicity

In this section, we introduce the intersection multiplicity of two plane curve germs. It is a numerical invariant which measures in some sense the (higher order) tangency of the germs.

Definition 3.9. (1) Let $g \in \mathbb{C}\{x,y\}$ be irreducible. Then the *intersection multiplicity* of any $f \in \mathbb{C}\{x,y\}$ with g is given by

$$\begin{aligned} i(f,g) \;:=\; i_{\mathbf{0}}(f,g) &:= \mathrm{ord}_t\, f\big(x(t),y(t)\big) \\ &= \sup\big\{m \in \mathbb{N} \;\big|\; t^m \text{ divides } f\big(x(t),y(t)\big)\big\}\,, \end{aligned}$$

where $t \mapsto \big(x(t),y(t)\big)$ is a parametrization for the plane curve germ defined by g. If u is a unit then we define $i(f,u) := 0$.

(2) The intersection multiplicity of f with a reducible power series $g_1 \cdot \ldots \cdot g_s$ is defined to be the sum

$$i(f, g_1 \cdot \ldots \cdot g_s) \;:=\; i(f,g_1) + \ldots + i(f,g_s)\,.$$

Lemma 3.2 implies that $i(f,g)$ is well-defined, that is, independent of the chosen parametrization. Moreover, if $\phi : (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0})$ is an analytic isomorphism and $u,u' \in \mathbb{C}\{x,y\}$ units then $i(f,g) = i(uf\circ\phi, u'g\circ\phi)$. Hence, we can define the intersection multiplicity of two plane curve germs $(C,\mathbf{0})$, $(D,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ as

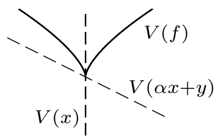
$$i_{\mathbf{0}}(C,D) \;:=\; i(f,g)\,,$$

where $f,g \in \mathbb{C}\{x,y\}$ are local equations for $(C,\mathbf{0})$, respectively $(D,\mathbf{0})$.

Note that $i_{\mathbf{0}}(C,D) > 0$ iff the germs $(C,\mathbf{0})$ and $(D,\mathbf{0})$ are non-empty, that is, iff $f,g \in \mathfrak{m}$. We say that $(C,\mathbf{0})$ and $(D,\mathbf{0})$ intersect *transversally (at* $\mathbf{0}$*)*, if $i_{\mathbf{0}}(C,D) = 1$. If $C,D \subset U$ are representatives of the germs $(C,\mathbf{0})$, $(D,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$, then we say that C,D intersect *transversally in* U, if, for each $z \in U$, the germs (C,z) and (D,z) intersect transversally.

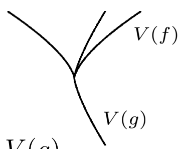
Example 3.9.1. (1) Consider the intersection multiplicity of the ordinary cusp (local equation $f = x^2 - y^3$) with a line ($g = \alpha x - \beta y$). The line being parametrized by $t \mapsto (\beta t, \alpha t)$, we obtain

$$\begin{aligned} i(f, \alpha x - \beta y) &= \mathrm{ord}_t(\beta^2 t^2 - \alpha^3 t^3) \\ &= \begin{cases} 2\,, & \text{if } \beta \neq 0\,, \\ 3\,, & \text{if } \beta = 0\,. \end{cases} \end{aligned}$$



(2) The intersection of the ordinary cusp ($f = x^2 - y^3$) with a tangential E_7-singularity ($g = x^3 - y^4$) can be computed as

$$\begin{aligned} i(f,g) &= \mathrm{ord}_t\big(x(t)^2 - y(t)^3\big) \\ &= \mathrm{ord}_t\big(t^8 - t^9\big) \;=\; 8\,, \end{aligned}$$



$t \mapsto \big(x(t),y(t)\big) = \big(t^4, t^3\big)$ being a parametrization of $V(g)$.

Proposition 3.10 (Halphen's formula[16]). *Let $f, g \in \mathbb{C}\{x,y\}$ and assume*

$$f = \prod_{i=1}^{m} \big(y - y_i(x^{1/N})\big), \quad g = \prod_{j=1}^{m'} \big(y - y'_j(x^{1/N})\big)$$

with $y_i(t), y'_j(t) \in \langle t\rangle \cdot \mathbb{C}\{t\}$, N some positive integer. Then the intersection multiplicity of f and g is

$$i(f,g) = \sum_{i=1}^{m}\sum_{j=1}^{m'} \frac{\operatorname{ord}_t\big(y_i(t) - y'_j(t)\big)}{N}. \tag{3.2.1}$$

Proof. Since both sides are additive, it suffices to prove (3.2.1) in the case f being irreducible.

Note that f is y-general of order m. Hence, due to Proposition 3.4 there exists a convergent power series $y(t) \in \langle t\rangle \cdot \mathbb{C}\{t\}$ and a unit $u \in \mathbb{C}\{x,y\}$ such that, in $\mathbb{C}\{x^{1/mN}, y\}$,

$$\prod_{i=1}^{m} \big(y - y_i(x^{1/N})\big) = f = u \cdot \prod_{i=1}^{m} \Big(y - y\big(\xi^i x^{1/m}\big)\Big),$$

ξ a primitive m-th root of unity. Since the power series ring $\mathbb{C}\{x^{1/mN}, y\}$ is factorial, we may assume

$$y_i\big(x^{1/N}\big) = y\big(\xi^i x^{1/m}\big), \quad i = 1, \dots, m.$$

In particular, $s \mapsto \big(s^m, y(\xi^i s)\big)$ is a parametrization for $V(f)$, and we obtain ($t^N = x = s^m$)

$$\begin{aligned} i(g,f) = \operatorname{ord}_s g\big(s^m, y(\xi^i s)\big) &= \frac{m}{N} \cdot \operatorname{ord}_t g\big(t^N, y(\xi^i t^{N/m})\big) \\ = \frac{m}{N} \cdot \operatorname{ord}_t g\big(t^N, y_i(t)\big) &= m \cdot \sum_{j=1}^{m'} \frac{\operatorname{ord}_t\big(y_i(t) - y'_j(t)\big)}{N}. \end{aligned}$$

Since this holds for any $i = 1, \dots, m$, we derive the equality (3.2.1). □

As an immediate corollary, we obtain

Corollary 3.11. *Let $f, g \in \mathbb{C}\{x,y\}$. Then*

(1) $i(f,g) = i(g,f)$.
(2) $i(f,g) < \infty \iff f$ and g have no common non-trivial factor.

[16] This kind of formula was already used by Zeuthen [ZeP] to determine the multiplicities of fixed points of one-dimensional algebraic correspondences. Hence, sometimes, it is also called *Zeuthen's formula*.

Using the finite coherence theorem, we can give a completely different formula for the intersection multiplicity which does not involve a parametrization[17]:

Proposition 3.12. *Let $f, g \in \mathbb{C}\{x,y\}$. Then*

$$i(f,g) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f,g\rangle\,, \tag{3.2.2}$$

in the sense, that if one of the two sides is finite then so is the other and they are equal. In particular,

$$i(f,g) < \infty \iff V(f) \cap V(g) \subset \{\mathbf{0}\}\,,$$

with $V(f), V(g) \subset (\mathbb{C}^2, \mathbf{0})$ denoting the plane curve germs defined by f, g.

For the proof of Proposition 3.12 we need the following

Lemma 3.13. *Let $f \in \mathbb{C}\{x,y\}$ be irreducible, let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be the plane curve germ defined by f, and let $\varphi : (\mathbb{C},\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0})$ be a parametrization of $(C,\mathbf{0})$. Then there exist open neighbourhoods of the origin, $D \subset \mathbb{C}$ and $B \subset \mathbb{C}^2$, and a holomorphic representative $\varphi : D \to B$ for the parametrization such that $\big(\varphi(D),\mathbf{0}\big) = (C,\mathbf{0})$ and*

(1) $\varphi : D \to B$ is finite;
(2) $\varphi : D \to \varphi(D) =: C$ is bijective;
(3) $\varphi : D \setminus \{\mathbf{0}\} \to C \setminus \{\mathbf{0}\}$ is biholomorphic.

Proof. Let $\varphi(t) = \big(x(t), y(t)\big)$ and assume that $x(t) \neq 0 \in \mathbb{C}\{t\}$ (otherwise the statement is obvious). After a reparametrization (Lemma 3.2), we may assume that $x(t) = t^b$. Then φ is quasifinite and, by the local finiteness theorem (Theorem 1.66), we can find D and B such that $\varphi : D \to B$ is finite.

By Corollary 1.68, the image $\varphi(D) \subset B$ is a closed analytic subset which we endow with its reduced structure. Since $f \circ \varphi = 0$, the latter is contained in the plane curve (germ) C. Now, let $(x_0, y_0) \in C$, $x_0 \neq 0$, be sufficiently close to $\mathbf{0}$, and let $t_0 \in \mathbb{C}$ be a fixed b-th root of x_0. By Proposition 3.4,

$$f(x_0, y_0) = c_0 \cdot \prod_{j=1}^{b} \big(y_0 - y(\xi^j t_0)\big)\,,$$

with ξ a primitive b-th root of unity and $c_0 \in \mathbb{C} \setminus \{0\}$. Moreover, due to the identity theorem for univariate holomorphic functions, we may assume that $y\big(\xi^j t_0\big) \neq y\big(\xi^i t_0\big)$ for $i \neq j$. It follows that there exists a unique j_0 such that $y_0 = y\big(\xi^{j_0} t_0\big)$, hence (2).

Since the choice of the b-th root can be made holomorphically along a fixed branch near $x_0 \in \mathbb{C} \setminus \{0\}$, the map $t \mapsto \big(t^b, \xi^{j_0} t_0\big)$ has a holomorphic inverse in a neighbourhood of $(x_0, y_0) \in C \setminus \{\mathbf{0}\}$, which implies (3). $\square$

[17] Alternatively to the use of the finite coherence theorem, we could use the resolution of plane curve singularities via blowing up (as introduced in Section 3.3) and the recursive formula (3.3.2) for the intersection multiplicities of the strict transforms, cf. Remark 3.29.1.

Proof of Proposition 3.12. First, let f be irreducible. If f divides g, both sides of (3.2.2) are infinite. Hence, assume that this is not the case, and choose a representative $\varphi : D \to B$ of a parametrization of f as in Lemma 3.13. Since $\varphi : D \to C$ is surjective and biholomorphic outside the origin, the induced map $\mathcal{O}_C \to \varphi_*\mathcal{O}_D$ is injective and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \varphi_*\mathcal{O}_D \longrightarrow \varphi_*\mathcal{O}_D\big/\mathcal{O}_C \longrightarrow 0\,,$$

where the quotient sheaf $\varphi_*\mathcal{O}_D\big/\mathcal{O}_C$ is supported at $\{\mathbf{0}\}$. By the finite coherence theorem (Theorem 1.67), $\varphi_*\mathcal{O}_D$ is coherent, hence the quotient $\varphi_*\mathcal{O}_D\big/\mathcal{O}_C$ is coherent, too (A.7, Fact 2). By Corollary 1.74, $\big(\varphi_*\mathcal{O}_D\big/\mathcal{O}_C\big)_{\mathbf{0}}$ is a finite dimensional complex vector space. Since $\mathcal{O}_{C,\mathbf{0}} \cong \mathbb{C}\{x,y\}/\langle f\rangle$ and $(\varphi_*\mathcal{O}_D)_{\mathbf{0}} \cong \mathbb{C}\{t\}$, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & x,y & \longmapsto & x(t),y(t) & & & & \\
0 & \longrightarrow & \mathbb{C}\{x,y\}/\langle f\rangle & \longrightarrow & \mathbb{C}\{t\} & \longrightarrow & \big(\varphi_*\mathcal{O}_D\big/\mathcal{O}_C\big)_{\mathbf{0}} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \cdot g} & & \Big\downarrow{\scriptstyle \cdot g\left(x(t),y(t)\right)} & & \Big\downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & \mathbb{C}\{x,y\}/\langle f\rangle & \longrightarrow & \mathbb{C}\{t\} & \longrightarrow & \big(\varphi_*\mathcal{O}_D\big/\mathcal{O}_C\big)_{\mathbf{0}} & \longrightarrow & 0\,.
\end{array}$$

Since f does not divide g, multiplication by g is injective on $\mathbb{C}\{x,y\}/\langle f\rangle$, and the snake lemma gives an exact sequence

$$0 \to \operatorname{Ker}(\pi) \to \mathbb{C}\{x,y\}\big/\big\langle f,g\big\rangle \to \mathbb{C}\{t\}\big/\big\langle g\big(x(t),y(t)\big)\big\rangle \to \operatorname{Coker}(\pi) \to 0\,.$$

Since $\dim_{\mathbb{C}}\big(\varphi_*\mathcal{O}_D\big/\mathcal{O}_C\big)_{\mathbf{0}} < \infty$, the $\mathbb{C}$-vector spaces $\operatorname{Ker}(\pi)$ and $\operatorname{Coker}(\pi)$ have the same dimension. Hence,

$$\dim_{\mathbb{C}} \mathbb{C}\{x,y\}\big/\big\langle f,g\big\rangle = \dim_{\mathbb{C}} \mathbb{C}\{t\}\big/\big\langle g\big(x(t),y(t)\big)\big\rangle\,. \tag{3.2.3}$$

If $m = \operatorname{ord} g\big(x(t),y(t)\big)$ then $g\big(x(t),y(t)\big) = t^m \cdot u(t)$ for a unit $u \in \mathbb{C}\{t\}$. But this just means that the dimension on the right-hand side of (3.2.3) equals $m = i(f,g)$.

If f is reducible, and if $\big(x_i(t_i),y_i(t_i)\big)$, $i = 1,\dots,r$, are parametrizations of the irreducible factors of f, then the same argument as before works, noting that then $D = \coprod_{i=1}^r D_i$ and $(\varphi_*\mathcal{O}_D)_{\mathbf{0}} \cong \bigoplus_{i=1}^r \mathbb{C}\{t_i\}$.

Finally, as the quotient sheaf $\mathcal{O}_B/\langle f,g\rangle$ is coherent, Corollary 1.74 gives that the stalk at $\mathbf{0}$, $\mathbb{C}\{x,y\}\big/\big\langle f,g\big\rangle$, is a finite dimensional $\mathbb{C}$-vector space iff the germ of the support of $\mathcal{O}_B/\langle f,g\rangle$ at $\mathbf{0}$ is contained in $\{\mathbf{0}\}$. As the support of $\mathcal{O}_B/\langle f,g\rangle$ equals $V(f,g) = V(f)\cap V(g)$, this implies the second statement of the proposition. □

Using the equality (3.2.2) and the principle of conservation of numbers (Section 1.6), we can give a beautiful geometric description of the intersection multiplicity of two (not necessarily reduced) plane curve germs $(C,\mathbf{0})$, $(D,\mathbf{0})$ as number of intersection points of two neighbouring curves (obtained by small deformations) in a small neighbourhood $U \subset \mathbb{C}^2$ of z:

Proposition 3.14. *Let $f, g \in \mathbb{C}\{x, y\}$ have no common factor, and let $F, G \in \mathbb{C}\{x, y, t\}$ be unfoldings of f, respectively g.*

Then, for all sufficiently small neighbourhoods $U = U(\mathbf{0}) \subset \mathbb{C}^2$, we can choose an open neighbourhood $W = W(0) \subset \mathbb{C}$ such that

- *F and G converge on $U \times W$,*
- *the curves $C = V(f) = V(F_0)$ and $D = V(g) = V(G_0)$ have the unique intersection point $\mathbf{0}$ in U,*
- *for all $t \in W$, we have*

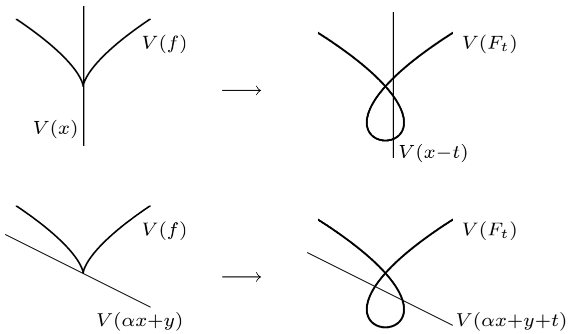
$$i(f, g) = i_U(C_t, D_t) := \sum_{z \in U} i_z(C_t, D_t)\,, \tag{3.2.4}$$

where $C_t = V(F_t)$ and $D_t = V(G_t)$ in U.

In particular, if the curves C_t and D_t are reduced and intersect transversally in U then $i(f, g)$ is just the number of points in $C_t \cap D_t$.

We call $i_U(C_t, D_t)$ the *total intersection multiplicity* of the plane curves C_t and D_t in U.

Example 3.14.1. We reconsider the intersection multiplicity of the ordinary cusp ($f = x^2 - y^3$) with the smooth curve germs given by x, respectively $\alpha x + y$. The unfolding $F_t := x^2 - y^3 - ty^2$ turns the cusp into an ordinary node. Now, for $t \neq 0$ small, we can compute $i(f, x)$, respectively $i(f, \alpha x + y)$ as the number of (simple) intersection points of the curves $V(F_t)$ and $V(x - t)$, respectively $V(\alpha x + y + t)$:



Indeed, there are three, respectively two, simple intersection points appearing on the scene.

Proof of Proposition 3.14. Choose open neighbourhoods $U = U(\mathbf{0}) \subset \mathbb{C}^2$ and $W = W(0) \subset \mathbb{C}$ such that F and G converge on $U \times W$. Since $\mathbf{0}$ is an isolated point of the intersection $V(f) \cap V(g)$ in U, we can assume (after shrinking U if necessary) that it is, indeed, the unique intersection point of the curves C and D in U. It remains to deduce (3.2.4), maybe again after shrinking U and

W. To do so, we apply the principle of conservation of numbers (Theorem 1.81).

Let $X \subset U \times W$ be given by the ideal sheaf $\mathcal{J} := \langle F, G\rangle$, and consider the map $\pi : X \to W$ induced by the natural projection $U \times W \to W$. The structure sheaf $\mathcal{O}_X = \mathcal{O}_{U\times W}/\mathcal{J}$ is coherent (Corollary 1.64) and satisfies

$$\sum_{z\in\pi^{-1}(t)} \dim_{\mathbb{C}} \mathcal{O}_{X,z}/\mathfrak{m}_{W,t}\mathcal{O}_{X,z} = \sum_{z\in U} \dim_{\mathbb{C}} \mathcal{O}_{U,z}/\langle F_t, G_t\rangle = \sum_{z\in U} i_z(F_t, G_t)\,.$$

Hence, it only remains to show that π is a flat morphism.

Since $(C, \mathbf{0})$ and $(D, \mathbf{0})$ have no common component, we can assume by Proposition 1.70 that π is finite. Hence, due to Theorem 1.78, the flatness of π is equivalent to the local freeness of $\pi_*\mathcal{O}_X$, and for our needs it is even sufficient to show that $(\pi_*\mathcal{O}_X)_0 \cong \mathbb{C}\{x, y, t\}/\langle F, G\rangle$ is a free $\mathbb{C}\{t\}$-module (using Theorem 1.80 (1)). But $\mathbb{C}\{x, y, t\}/\langle F, G\rangle$ is a complete intersection, hence Cohen-Macaulay (Corollary B.8.10) and, hence, free (Corollary B.8.12).

A direct proof of the freeness goes as follows: since $\mathbb{C}\{t\}$ is a principal ideal domain, and since $\mathbb{C}\{x, y, t\}/\langle F, G\rangle$ is a finitely generated $\mathbb{C}\{t\}$-module, it suffices to show that $\mathbb{C}\{x, y, t\}/\langle F, G\rangle$ is torsion free (cf. [Lan, Thm. III.7.3]), or, equivalently, that for any $H \in \mathbb{C}\{x, y, t\}$ and $k \geq 1$ we have the implication

$$t^k \cdot H \in \langle F, G\rangle \iff H \in \langle F, G\rangle\,.$$

Assume that $t^kH = AF + BG$ with $A, B \in \mathbb{C}\{x, y, t\}$. Setting $t = 0$ gives $A(x, y, 0) \cdot f + B(x, y, 0) \cdot g = 0$, which implies

$$A(x, y, 0) = h \cdot g\,, \quad B(x, y, 0) = -h \cdot f\,, \tag{3.2.5}$$

for some $h \in \mathbb{C}\{x, y\}$ (since $\mathbb{C}\{x, y\}$ is a UFD and $\gcd(f, g) = 1$). Moreover, obviously,

$$t^kH \;=\; AF + BG \;=\; (A - hG)F + (B + hF)G\,,$$

and (3.2.5) implies that the power series $A - hG$ and $B + hF$ are both divisible by t. It follows that $t^{k-1}H \in \langle F, G\rangle$, and, by induction, $H \in \langle F, G\rangle$. □

There exists another, purely topological, characterization of the (local) total intersection number of two plane curves germs without common components:

Proposition 3.15. *Let $f, g \in \mathbb{C}\{x, y\}$ be reduced and without common factors, and let B be a closed ball centred at the origin such that f, g converge on B and $(f(z), g(z)) \neq (0, 0)$ as $z \in \partial B$. Then $i_B(f, g)$ is equal to the (topological) degree of the map*

$$\Phi : \partial B \longrightarrow S^3\,, \quad z \longmapsto \frac{(f(z), g(z))}{\sqrt{|f(z)|^2 + |g(z)|^2}}\,.$$

Proof. See [Mil1, Lemma B.2]. □

We close this section by a computational remark. If we want to compute the intersection multiplicity of two polynomials (or, power series) f and g in a computer algebra system as SINGULAR, we may either use a parametrization as in the definition of the intersection multiplicity or use the formula (3.2.2) expressing the intersection multiplicity as codimension of an ideal (which can be computed then by using standard bases, see, e.g., [GrP, Cor. 7.5.6]).

Example 3.15.1. To compute the intersection of $f = (x^3 - y^4)(x^2 - y^2 - y^3)$ with $g = (x^3 + y^4)(x^2 - y^2 - y^3 + y^{10})$ in SINGULAR, we may either start by computing a (sufficiently high[18] jet of a) parametrization for each irreducible factor of f,

```
LIB "hnoether.lib";
ring r = 0,(x,y),ls;          // we have to use a local ordering
poly f = (x3-y4)*(x2-y2-y3);
poly g = (x3+y4)*(x2-y2-y3+y10);
list L = hnexpansion(f);      // result is a list of rings
def R  = L[1];
setring R;                    // contains list hne = HNE of f

// computing higher jets of the HNE (where needed):
for (int i=1; i<=size(hne); i++) {
   if (hne[i][4]<>0) { hne[i]=extdevelop(hne[i],10) };
};
// deducing a parametrization:
list P = parametrisation(hne);
// substituting the parametrization for x,y
for (i=1; i<=size(P); i++) { map phi(i) = r,P[i]; };
ord(phi(1)(g))+ord(phi(2)(g))+ord(phi(3)(g));
//-> 44
```

or we may compute the codimension of the (complete intersection) ideal generated by f, g:

```
setring r;       // we have to change from R back to r
ideal I=f,g;
vdim(std(I));
//-> 44
```

[18] A priori, it is clear, that for N sufficiently large, the N-jet of the parametrization is sufficient for the computation of the intersection number. The problem is to have a good lower bound for N. To get such a bound with SINGULAR, one may compute a system of Hamburger-Noether expansions `hne` for the product $f \cdot g$, and compute `list P = parametrisation(hne,0);`. Then the maximal integer in the entries `P[i][2]`, $i = 1, \ldots,$ `size(P)`, gives an appropriate lower bound for N. In the example, this maximal entry is 10.

Exercises

Exercise 3.2.1. Let $f, g \in \mathbb{C}\{x,y\}$. Show that the intersection multiplicity of f and g is at least the product of the respective multiplicities, that is,

$$i(f,g) \;\geq\; \mathrm{mt}(f)\cdot \mathrm{mt}(g)\,.$$

Moreover, show that the multiplicity of f can be expressed in terms of intersection multiplicities

$$\mathrm{mt}(f) \;=\; \min\bigl\{ i(f,g) \bigm| g \in \langle x,y\rangle \subset \mathbb{C}\{x,y\}\bigr\}\,,$$

and that the minimum is attained for $g = \alpha x + \beta y$ a general linear form[19]. In particular, if f is irreducible, then

$$\mathrm{mt}(f) \;=\; \min\bigl\{\mathrm{ord}\, x(t), \mathrm{ord}\, y(t)\bigr\}\,,$$

where $t \mapsto \bigl(x(t), y(t)\bigr)$ is a parametrization of the germ $\bigl(V(f), \mathbf{0}\bigr)$.

Exercise 3.2.2. For any $n \geq 3$ and $f_1, \dots, f_n \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$, introduce the *intersection multiplicity*

$$i(f_1, \dots, f_n) = \dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}/\langle f_1, \dots, f_n\rangle\,.$$

Show that

(1) $i(f_1, \dots, f_n) < \infty \iff V(f_1) \cap \ldots \cap V(f_n) \subset \{\mathbf{0}\}$,
(2) $i(f_1, \dots, f_n) \geq \mathrm{mt}(f_1)\cdot \ldots \cdot \mathrm{mt}(f_n)$.

Exercise 3.2.3. Let $f \in \mathbb{C}\{x,y\}$ split into s nonsingular irreducible components $f_1, \dots, f_s$ which pairwise intersect transversally, and let $g \in \mathbb{C}\{x,y\}$ satisfy $i(f_j, g) \geq s$ for all $j = 1, \dots, s$. Show that $f + tg$ splits in $\mathbb{C}\{x,y\}$ into s irreducible components for almost all $t \in \mathbb{C}$.

3.3 Resolution of Plane Curve Singularities

In the following we introduce our main tool in the study of plane curve singularities, the "blowing up" of a point z in a smooth surface M. It is a purely local process, which can be thought of as a "mighty microscope" replacing the point z by a projective line $\mathbb{P}^1$ and attaching to a point $(a:b) \in \mathbb{P}^1$ the "view" of M from z in direction $(a:b)$. As a result, curves which previously met at z get separated (e.g., smooth curves with different tangents), or, at least, their intersection multiplicity decreases. In any case, the singularities of curves at z become simpler after blowing up.

It turns out that by successively blowing up points, we can *resolve* a reduced plane curve singularity, that is, transform it to a smooth (multi)germ.

[19] More precisely, $\mathrm{mt}(f) = i(f,\, \alpha x + \beta y)$ iff $\alpha x + \beta y$ is not a tangent of f (cf. Definition 3.18).

Blowing Up $\mathbf{0} \in \mathbb{C}^2$. We identify the projective line $\mathbb{P}^1$ with the set of lines $L \subset \mathbb{C}^2$ through the origin $\mathbf{0}$ and define $B\ell_{\mathbf{0}}\mathbb{C}^2$ to be the closed complex subspace

$$\begin{aligned} B\ell_{\mathbf{0}}\mathbb{C}^2 &:= \big\{(z, L) \in \mathbb{C}^2 \times \mathbb{P}^1 \,\big|\, z \in L\big\} \\ &= \big\{(x, y; s : t) \in \mathbb{C}^2 \times \mathbb{P}^1 \,\big|\, tx - sy = 0\big\} \subset \mathbb{C}^2 \times \mathbb{P}^1 . \end{aligned}$$

Definition 3.16. The projection $\pi : B\ell_{\mathbf{0}}\mathbb{C}^2 \to \mathbb{C}^2$, $(p, L) \mapsto p$, is called a *σ-process with centre* $\mathbf{0} \in \mathbb{C}^2$, or the *blowing up of* $\mathbf{0} \in \mathbb{C}^2$.

We write $E := \pi^{-1}(\mathbf{0}) \subset B\ell_{\mathbf{0}}\mathbb{C}^2$ and call it the *exceptional divisor*[20] of π. Frequently, $E = \{\mathbf{0}\} \times \mathbb{P}^1$ (which we identify with $\mathbb{P}^1$) is also called the *first infinitely near neighbourhood* of the point $\mathbf{0} \in \mathbb{C}^2$.

Note that each point of E corresponds to a unique line through the origin in $\mathbb{C}^2$. Each fibre $\pi^{-1}(z)$, $z \neq \mathbf{0}$, consists of exactly one point (z, L) where $L \subset \mathbb{C}^2$ is the unique line through $\mathbf{0}$ and $z = (x, y)$, that is, $\pi^{-1}(z) = \big\{(x, y; x : y)\big\}$. In particular, the preimages of any two lines $L \neq L' \subset \mathbb{C}^2$ through $\mathbf{0}$ do not have any point in common. In other words, blowing up $\mathbf{0} \in \mathbb{C}^2$ *"separates lines through the origin"*.

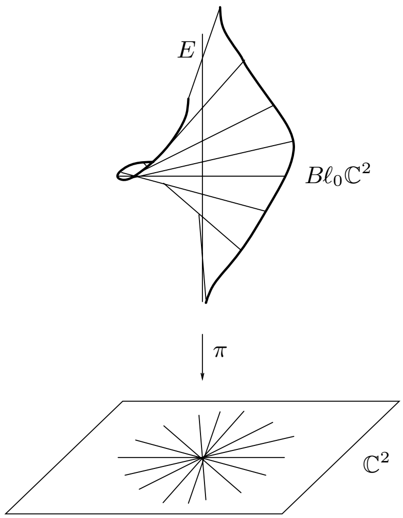


Fig. 3.13. The blowing up of $\mathbf{0} \in \mathbb{C}^2$

Remark 3.16.1. In the same manner, we define the blowing up $\pi : B\ell_{z_0}U \to U$ of z_0 in an open neighbourhood U of $z_0 \in \mathbb{C}^2$. Let $\varphi = (\varphi_1, \varphi_2) : U \to \mathbb{C}^2$ be

[20] A *(Cartier) divisor* D in a complex space is a subspace which is locally defined by one equation $f = 0$. We denote by mD the divisor given by $f^m = 0$. If D' is another divisor, given by $g = 0$, then $D + D'$ denotes the divisor given by $fg = 0$.

biholomorphic onto some open neighbourhood of $\mathbf{0} \in \mathbb{C}^2$ with $\varphi(z_0) = \mathbf{0}$, and let $s:t$ be homogeneous coordinates on $\mathbb{P}^1$. Then we can describe $B\ell_{z_0}U$ in coordinates:

$$B\ell_{z_0}U := B\ell^{\varphi}_{z_0}U := \big\{(z;s:t) \in U \times \mathbb{P}^1 \,\big|\, \varphi_1(z)t - \varphi_2(z)s = 0\big\} \subset U \times \mathbb{P}^1 .$$

As usual, we write x, y instead of φ_1, φ_2 and call them *local coordinates of* U *with centre* z_0

Note that if $\psi : U \to \mathbb{C}^2$ provides other local coordinates on U with centre z_0 then we get a canonical isomorphism

$$B\ell^{\varphi}_{z_0}U \xrightarrow{\cong} B\ell^{\psi}_{z_0}U , \quad (z;s:t) \longmapsto \big(\psi^{-1} \circ \varphi(z); s:t\big) .$$

In particular, the notation $B\ell_{z_0}U$ is justified. We cover $U \times \mathbb{P}^1$ by two charts, induced by the canonical charts $V_0 := \{s \neq 0\}$, $V_1 := \{t \neq 0\}$ for $\mathbb{P}^1$:

Chart 1. $U \times V_0 \subset U \times \mathbb{P}^1$ (with coordinates x, y; $v = t/s$).

In this chart $B\ell_{z_0}U$ is the zero-set of $xv - y$. In particular, it is smooth, and we can introduce coordinates $u = x$, v on $B\ell_{z_0}U \cap (U \times V_0)$. With respect to these coordinates, the morphism π can be described as

$$\pi :\ B\ell_{z_0}U \cap (U \times V_0) \longrightarrow U , \quad (u,v) \longmapsto (u, uv) .$$

The exceptional divisor in this chart is $E \cap (U \times V_0) = \big\{(u,v) \mid u = 0\big\} = \{0\} \times \mathbb{C}$ with coordinate v.

Chart 2. $U \times V_1 \subset U \times \mathbb{P}^1$ (with coordinates x, y; $\overline{u} = s/t$).

$B\ell_{z_0}U \cap (U \times V_1)$ is the zero-set of $x - y\overline{u}$. Hence, it is smooth, and we can introduce local coordinates $\bar{u}, \bar{v} = y$ such that

$$\pi :\ B\ell_{z_0}U \cap (U \times V_1) \longrightarrow U , \quad (\bar{u}, \bar{v}) \longmapsto (\bar{u}\bar{v}, \bar{v}) ,$$

with exceptional divisor $E \cap (U \times V_1) = \big\{(\overline{u}, \overline{v}) \mid \overline{v} = 0\big\} = \mathbb{C} \times \{0\}$.

Note that the coordinate of E (v in Chart 1, resp. $\overline{u}$ in Chart 2) is not only local but affine. That is, if $U = U_1 \times U_2 \subset \mathbb{C}^2$, then $(U \times V_1) \cap B\ell_{z_0}U = \big\{(u,v) \in U_1 \times \mathbb{C} \,\big|\, vu \in U_2\big\}$ is an open neighbourhood of $\{0\} \times \mathbb{C}$, and $(U \times U_2) \cap B\ell_{z_0}U = \big\{(\overline{u}, \overline{v}) \in \mathbb{C} \times U_2 \,\big|\, \overline{uv} \in U_1\big\}$ is an open neighbourhood of $\mathbb{C} \times \{0\}$.

Sometimes, we want to make a point $p = (\beta : \alpha) \in \mathbb{P}^1 = \pi^{-1}(\mathbf{0})$ in the exceptional divisor the centre of the coordinate system (u,v), resp. $(\overline{u}, \overline{v})$. Since $\mathbb{P}^1 = V_0 \cup \{(0:1)\}$, we have $p = (1:\alpha)$, $\alpha \in \mathbb{C}$, or $p = (0:1)$. In Chart 1, a point $p = (1:\alpha)$ has coordinates $(0, \alpha)$; in Chart 2, the point $p = (0:1)$ has coordinates $(0,0)$.

If $(u', v') = (u, v - \alpha)$ are new coordinates in Chart 1 then $p = (1:\alpha)$ has coordinates $(u', v') = (0,0)$ and in these coordinates we have

$$\pi :\ B\ell_{z_0}U \cap (U \times V_0) \longrightarrow U , \quad (u', v') \longmapsto \big(u', u'(v' + \alpha)\big) .$$

Lemma 3.17. *Let U be an open neighbourhood of $z \in \mathbb{C}^2$. Then $B\ell_z U$ is a 2-dimensional complex manifold, and the restriction $\pi : B\ell_z U \setminus E \to U \setminus \{z\}$ is an analytic isomorphism.*

Proof. The transition function between $U \times V_0$ and $U \times V_1$ is given by $(x, y, t) \mapsto (x, y, \frac{1}{t})$, hence, analytic, which implies that $B\ell_z U$ is a complex manifold. In local coordinates $x, y; s : t$, the inverse morphism is given by

$$\pi^{-1} : U \setminus \{z\} \longrightarrow B\ell_z U \setminus \pi^{-1}(z)\,, \quad (x, y) \longmapsto (x, y;\, x : y)\,. \qquad \square$$

Blowing Up a Point on a Smooth Surface. Blowing up is a purely local process. Hence, we can generalize the blowing up of $\mathbf{0} \in \mathbb{C}^2$ to define the blowing up of a point in an arbitrary smooth complex surface (i.e., 2-dimensional complex manifold) M.

Let $z \in M$ be a point. Then there exists an open neighbourhood $U \subset M$ of z being isomorphic to an open neighbourhood of $\mathbf{0} \in \mathbb{C}^2$. Choosing local coordinates with centre $\mathbf{0}$, we can apply the above construction and define the blowing up of $z \in U$, $\pi : B\ell_z U \to U \subset M$.

Since the graph of the restriction $\pi : B\ell_z U \setminus \pi^{-1}(z) \xrightarrow{\cong} U \setminus \{z\}$ is obviously closed in $B\ell_z U \times (U \setminus \{z\})$, the glueing lemma [GuR, Prop. V.5] allows to define the blowing up of $z \in M$,

$$B\ell_z M \;:=\; B\ell_z U \cup_\pi \big(M \setminus \{z\}\big) \longrightarrow M\,,$$

by glueing $B\ell_z U$ and $M \setminus \{z\}$. Again, for different choice of local coordinates the result will be canonically isomorphic.

Remark 3.17.1. The following statements follow easily from the definition and are left as exercises.

(1) Let $\pi : B\ell_z M \to M$ be the blowing up of $z \in M$. Then the exceptional divisor $E := \pi^{-1}(z) \subset B\ell_z$ satisfies

- $E \cong \mathbb{P}^1$;
- its complement $B\ell_z M \setminus E$ is dense in $B\ell_z M$;
- the restriction $B\ell_z M \setminus E \to M \setminus \{z\}$ is an analytic isomorphism.

(2) The blowing up $\pi : B\ell_z M \to M$ of a point $z \in M$ is *well-defined up to isomorphism (over M)*, that is, if $\pi' : B\ell'_z M \to M$ is another blowing up of $z \in M$ (obtained from local coordinates x', y' on $U' \subset M$) then there exists a unique isomorphism $\varphi : B\ell_z M \to B\ell'_z M$ making the following diagram commute

$$\begin{tikzcd} B\ell_z M \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & & B\ell'_z M \arrow[dl, "\pi'"] \\ & M. & \end{tikzcd}$$

Moreover, φ induces a linear projectivity $\pi^{-1}(z) = E \xrightarrow{\cong} E' = \pi'^{-1}(z)$.

We call $B\ell_z M \to M$ a *monoidal transformation blowing up $z \in M$*, or simply *the blowing up of the point $z \in M$.*

(3) In particular, we can define the *blowing up of the germ* (M,z) at z as the germ of $\pi : B\ell_z M \to M$ along $E = \pi^{-1}(z)$ (that is, an equivalence class of morphisms[21] defined on a neighbourhood of $E \subset B\ell_z M$). We write

$$\pi : B\ell_z M \longrightarrow (M,z), \text{ or } \pi : (B\ell_z M, E) \longrightarrow (M,z).$$

More generally, if (M,V) is the germ of M along the subvariety $V \subset M$ containing z, then we define the blowing up of (M,V) at z as the germ of $\pi : B\ell_z M \to M$ along $\pi^{-1}(V)$.

(4) *Analytic isomorphisms lift* to the blown-up surfaces. More precisely, let $\varphi : M \to M'$ be an analytic isomorphism of smooth complex surfaces. Then there exists a unique isomorphism $\widetilde{\varphi} : B\ell_z M \to B\ell_{\varphi(z)} M'$ making the following diagram commute

$$\begin{array}{ccc} B\ell_z M & \xrightarrow[\cong]{\widetilde{\varphi}} & B\ell_{\varphi(z)} M' \\ \downarrow & & \downarrow \\ M & \xrightarrow[\varphi]{\cong} & M'. \end{array}$$

(5) Let $z \neq w \in M$. Then the surfaces $B\ell_{z,w} M$ (obtained by blowing up $z \in M$ first and then blowing up the point $\pi^{-1}(w) \in B\ell_z M$) and $B\ell_{w,z} M$ (obtained by blowing up in opposite order) are isomorphic over M.

Blowing up Curves and Germs. In the following we study the effect of the blowing up $\pi : B\ell_z M \to M$ on a curve $C \subset M$. We define the *total transform* of C to be the pull-back

$$\widehat{C} := \pi^{-1}(C) \subset B\ell_z M.$$

As we shall see below, as a divisor we have

$$\widehat{C} = \widetilde{C} + mE, \quad m = \mathrm{mt}(C,z),$$

where E is the exceptional divisor and $\widetilde{C}$ is the *strict transform* of C,

$$\widetilde{C} := \overline{\pi^{-1}(C) \setminus E} \subset B\ell_z M$$

provided with the induced, reduced structure. Here $\overline{}$ denotes the closure[22] in $B\ell_z M$. E being an irreducible component of $\widehat{C}$, it follows that the strict

[21] The *category of germs of complex spaces along a subspace* consists of pairs (X,E) of complex spaces with E a subspace of X. Morphisms $(X,E) \to (Y,F)$ are equivalence classes of morphisms $X \to Y$ mapping E to F, where two such morphisms are called equivalent if they coincide on some common neighbourhood of E.

[22] It follows from the equations of $\widehat{C}$ in Chart 1, respectively in Chart 2, that the topological closure is a complex curve, hence an analytic subvariety of $B\ell_z M$.

transform $\widetilde{C}$ consists precisely of the remaining irreducible components of the total transform $\widehat{C}$.

Since the blowing up map is an isomorphism outside the exceptional divisor $E = \pi^{-1}(z)$, it suffices to study the induced total (respectively strict) transform of the germ (C, z). Note that the total transform of the curve germ (C, z) is not a curve germ but the *germ* of $\widehat{C}$ *along* E, while the strict transform of (C, z) is a multi-germ of plane curve singularities.

Remark 3.17.2. Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a plane curve singularity with local equation $f \in \mathbb{C}\{x, y\}$. Then we can describe the total (respectively strict) transform of $(C, \mathbf{0})$ w.r.t. the local coordinates introduced in Remark 3.16.1: let

$$f = f_m + f_{m+1} + \ldots\,, \qquad f_j \text{ homogeneous of degree } j,$$

$f_m \neq 0$, that is, $m = \operatorname{mt} f$. Then the total transform of $(C, \mathbf{0})$ is the germ of the total transform of a representative C along E with (local) equation:

- in Chart 1: $\widehat{f}(u, v) = f(u, uv) = u^m \big(\underbrace{f_m(1, v) + u f_{m+1}(1, v) + \ldots}_{=:\, \widetilde{f}(u,v)}\big)\,,$
- in Chart 2: $\widehat{f}(\bar{u}, \bar{v}) = f(\bar{u}\bar{v}, \bar{v}) = \bar{v}^m \big(\underbrace{f_m(\bar{u}, 1) + \bar{v} f_{m+1}(\bar{u}, 1) + \ldots}_{=:\, \widetilde{f}(\bar{u},\bar{v})}\big)\,.$

Then u, respectively $\bar{v}$, are the local equation of the exceptional divisor, $\widehat{f}(u, v)$, respectively $\widetilde{f}(u, v)$, are the local equations of $\widehat{C}$, respectively $\widetilde{C}$ in Chart 1, while $\widehat{f}(\bar{u}, \bar{v})$, respectively $\widetilde{f}(\bar{u}, \bar{v})$, are the local equations of $\widehat{C}$, respectively $\widetilde{C}$ in Chart 2. It follows that, as a divisor, $\widehat{C} = mE + \widetilde{C}$. In particular, the total transform is non-reduced whenever $m > 1$.

The intersection of the strict transform $\widetilde{C}$ with E consists of at most m points given by the local equations

$$u = 0 = f_m(1, v)\,, \quad \text{respectively } \bar{v} = 0 = f_m(\bar{u}, 1)\,. \tag{3.3.1}$$

Recall that the points of $E \subset B\ell_{\mathbf{0}}\mathbb{C}^2$ correspond to lines in $\mathbb{C}^2$ through the origin. The points of intersection of E with the strict transform of a plane curve correspond precisely to those lines being tangent to the curve at the origin, as we shall see in the following.

Definition 3.18. Let $f \in \mathbb{C}\{x, y\}$, $m := \operatorname{mt}(f)$, and let $f_m \in \mathbb{C}[x, y]$ denote the *tangent cone*, that is, the homogeneous part of lowest degree. Then f_m decomposes into (possibly multiple) linear factors,

$$f_m = \prod_{i=1}^{s} (\alpha_i x - \beta_i y)^{m_i}\,,$$

with $(\beta_i : \alpha_i) \in \mathbb{P}^1$ pairwise distinct, $m = m_1 + \ldots + m_s$. We call the factors $(\alpha_i x - \beta_i y)$, $i = 1, \ldots, s$, the *tangents* of f, the m_i are called *multiplicities of the tangent.* We also refer to $(\beta_i : \alpha_i) \in \mathbb{P}^1$, $i = 1, \ldots, s$, as the *tangent directions* of the plane curve germ $(C, \mathbf{0}) = V(f) \subset (\mathbb{C}^2, \mathbf{0})$ (with respect to the chosen local coordinates).

The tangents of f are in 1–1 correspondence with the points of intersection of the strict transform $\widetilde{C}$ of $(C, \mathbf{0})$ with the exceptional divisor E (cf. (3.3.1)). Moreover, the multiplicity of the tangent coincides with the intersection multiplicity of $\widetilde{C}$ and E at the respective point.

Let C be a representative of the curve germ at $\mathbf{0}$ defined by $f = 0$. We leave it as an exercise to show that the tangents of f correspond uniquely to the limits of secant lines $\overline{0s}$ with $s \in C$, $s \to 0$.

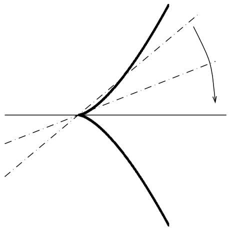


Fig. 3.14. Tangents are limits of secant lines.

Lemma 3.19. *Each irreducible factor of $f \in \mathbb{C}\{x, y\}$ has a unique tangent.*

Proof. After a linear coordinate change, f is y-general of order b (cf. Exercise 1.1.6), hence, by the WPT we can assume that f is, indeed, a Weierstraß polynomial

$$f \;=\; y^b + a_1(x) y^{b-1} + \ldots + a_b\,, \qquad a_i(0) = 0\,.$$

Let $f = f_m + f_{m+1} + f_{m+2} + \ldots$ and consider the strict transform

$$\widetilde{f}(u, v) \;:=\; \frac{f(u, uv)}{u^m} \;\equiv\; f_m(1, v) \bmod \langle u \rangle \cdot \mathbb{C}\{u, v\}\,.$$

It follows that $\widetilde{f}(u, v) \in \mathbb{C}\{u\}[v]$ is monic, and $\widetilde{f}(0, v) = f_m(1, v)$ is a complex polynomial in v of degree m. In particular, it decomposes into linear factors,

$$\widetilde{f}(0, v) = (v - c_1)^{m_1} \cdot \ldots \cdot (v - c_n)^{m_n}\,, \qquad \sum_{i=1}^{n} m_i = m\,.$$

Hensel's lemma 1.17 implies the existence of polynomials $\widetilde{f}_i \in \mathbb{C}\{u\}[v]$ of degree m_i such that $\widetilde{f} = \widetilde{f}_1 \cdot \ldots \cdot \widetilde{f}_n$ and $\widetilde{f}_i(0, v) = (v - c_i)^{m_i}$, $i = 1, \ldots n$. Hence, we can write

$$f(x,y) \;=\; x^m \widetilde{f}\,(x,\, y/x) \;=\; \underbrace{x^{m_1}\widetilde{f}_1\,(x,\, y/x)}_{\in\, \mathfrak{m}\, \subset\, \mathbb{C}\{x,y\}} \cdot \ldots \cdot \underbrace{x^{m_n}\widetilde{f}_n\,(x,\, y/x)}_{\in\, \mathfrak{m}\, \subset\, \mathbb{C}\{x,y\}} \,.$$

In particular, each tangent corresponds to a unique (not necessarily irreducible) factor $\widetilde{f}_i$. □

Example 3.19.1. (1) $f = x \in \mathbb{C}\{x,y\}$. The (local) equations of the total, respectively strict, transform are

$$\text{in CHART 1:}\quad \widehat{f} = u\,,\quad \widetilde{f} = 1\,, \qquad \text{in CHART 2:}\quad \widehat{f} = \bar{u}\bar{v}\,,\quad \widetilde{f} = \bar{u}\,,$$

u, respectively $\bar{v}$, being the local equation of the exceptional divisor.

It follows that the strict transform of a smooth germ is, again, smooth and intersects the exceptional divisor transversally.

(2) $f = x^m - y^m \in \mathbb{C}\{x,y\}$. Then we obtain

$$\text{in CHART 1:}\quad \widehat{f} = u^m(1-v^m)\,,\qquad \widetilde{f} = 1 - v^m = \prod_{k=0}^{m}\,(1 - e^{2\pi i/k}v)\,,$$

$$\text{in CHART 2:}\quad \widehat{f} = \bar{v}^m(\bar{u}^m - 1)\,,\qquad \widetilde{f} = \bar{u}^m - 1 = \prod_{k=0}^{m}\,(\bar{u} - e^{2\pi i/k})\,.$$

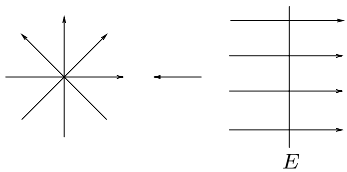
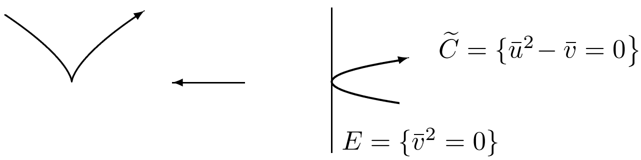


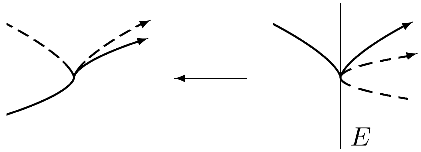
Fig. 3.15. Blowing up the curve germ defined by $x^4 - y^4$.

The strict transform intersects the exceptional divisor in m different points (corresponding to the m tangents of f), the germ at each of these points being smooth (see Fig. 3.15 on page 188).

(3) $f = x^2 - y^3 \in \mathbb{C}\{x,y\}$. Here, the strict transform is smooth (local equation $\widetilde{f} = \bar{u}^2 - \bar{v}$), but intersects the exceptional divisor (in the point corresponding to the unique tangent x) with multiplicity 2, that is, not transversally:



(4) $f = (x^2 - y^3)(x^3 - y^5) \in \mathbb{C}\{x,y\}$. The strict transform meets the exceptional divisor in a unique point (corresponding to the unique tangent x) and has local equation $\widetilde{f} = (\bar{u}^2 - \bar{v})(\bar{u}^3 - \bar{v}^2)$. The corresponding curve germ is singular and decomposes in one smooth and one singular branch:



Remark 3.19.2. Let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be a plane curve germ, and $f \in \mathbb{C}\{x,y\}$ a local equation. Since $\mathrm{mt}(f)$ is invariant under the action of the contact group (Remark 2.50.1), we can introduce the *multiplicity of* $(C,\mathbf{0})$,

$$\mathrm{mt}(C,\mathbf{0}) := \mathrm{mt}(f) = \mathrm{ord}(f)\,.$$

Moreover, since any element of the contact group induces a *linear* isomorphism of the tangent cone, we can define the *number of tangents of* $(C,\mathbf{0})$ as the number of tangents of f. We also speak about the *tangents of the plane curve germ* $(C,\mathbf{0})$.

As we have seen above, each tangent of f correspond to a unique point of the exceptional divisor E, hence,

$$\bigl\{\text{tangents of } (C,\mathbf{0})\bigr\} \stackrel{1:1}{\longleftrightarrow} \bigl\{\text{points of } E \cap \widetilde{C}\bigr\}\,.$$

Moreover, taking the strict transform gives a correspondence

$$\left\{\begin{array}{c}\text{plane curve germs}\\ (C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})\\ \text{with unique tangent}\\ \text{corresponding to } q \in E\end{array}\right\} \stackrel{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{plane curve germs}\\ (D,q) \subset (B\ell_{\mathbf{0}}\mathbb{C}^2,q)\\ \text{with}\\ (D,q) \cap (E,q) = \{q\}\end{array}\right\}.$$

This is the content of the following lemma.

Lemma 3.20. *Let q be a point in the first neighbourhood E of the blowing-up map $\pi : B\ell_z(M) \to M$ and let $\bigl(D,q\bigr) \subset \bigl(B\ell_z M,q\bigr)$ be a reduced plane curve germ such that $(D,q) \cap (E,q) = \{q\}$. Then there exists a unique unitangential plane curve germ $(C,z) \subset (M,z)$ such that $(D,q) = \bigl(\widetilde{C},q\bigr)$ is the germ of the strict transform of (C,z) at q.*

Proof. By Proposition 1.70, the restriction $\pi : (D,q) \to (M,z)$, is a finite morphism of complex space germs. We can choose a finite representative of the latter and define (C,z) as the germ of its image at z with its reduced structure. The uniqueness is obvious. $\square$

We have seen that blowing up a point separates curve germs with different tangents. For germs with the same tangents, at least the intersection multiplicity decreases. More precisely, we have the following

Proposition 3.21. *Let* $(C,\mathbf{0}),(D,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ *be two plane curve singularities and* $\widetilde{C},\widetilde{D}$ *the corresponding strict transforms after blowing up the origin. Then*

$$i_{\mathbf{0}}(C,D) \;=\; \mathrm{mt}(C,\mathbf{0})\cdot\mathrm{mt}(D,\mathbf{0})+\sum_{p\in E} i_p\big(\widetilde{C},\widetilde{D}\big)\,, \tag{3.3.2}$$

where E *is the exceptional divisor.*

In particular, $i_{\mathbf{0}}(C,D)=\mathrm{mt}(C,\mathbf{0})\cdot\mathrm{mt}(D,\mathbf{0})$ *iff* $(C,\mathbf{0})$ *and* $(D,\mathbf{0})$ *have no common tangent.*

Proof. Let $f,g\in\mathfrak{m}\subset\mathbb{C}\{x,y\}$ be local equations for $(C,\mathbf{0})$, $(D,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$. Since both sides of (3.3.2) are additive, we can assume that g is irreducible.

Hence, by Lemma 3.19, it has a unique tangent, which we can assume to be y, that is,

$$g \;=\; y^m+g_{m+1}+g_{m+2}+\dots\,, \qquad m=\mathrm{mt}(g)\,.$$

In particular, g is y-general of order m, and, due to Proposition 3.4 and the WPT, there is a power series $y(t)\in\mathbb{C}\{t\}$ and a unit $u\in\mathbb{C}\{x,y\}$ such that

$$g=u\cdot\prod_{j=1}^{m}\Big(y-y\big(\xi^j x^{1/m}\big)\Big)\,, \qquad \xi=e^{2\pi i/m}\,.$$

Moreover, comparing coefficients shows that $\mathrm{ord}(y(t))>m$.

The germ of the strict transform $\widetilde{D}$ at the unique intersection point $p\in\widetilde{D}\cap E$ (the origin in Chart 1, corresponding to the tangent direction $(0\,{:}\,1)$) has local equation

$$\widetilde{g}(u,v) \;=\; \frac{g(u,uv)}{u^m} \;=\; \prod_{j=1}^{m}\frac{\big(uv-y\big(\xi^j u^{1/m}\big)\big)}{u}\,,$$

and, therefore, it is parametrized by $t\mapsto\big(t^m,v(t)\big)$,

$$v(t):=\frac{y(t)}{t^m}\in\langle t\rangle\cdot\mathbb{C}\{t\}\,.$$

Let $n:=\mathrm{mt}(f)$. Then we obtain

$$\begin{aligned} i_p(\widetilde{C},\widetilde{D}) &= \mathrm{ord}\,\widetilde{f}\big(t^m,v(t)\big) \;=\; \mathrm{ord}\,\frac{f\big(t^m,t^m v(t)\big)}{t^{mn}}\\ &= \mathrm{ord}\,f\big(t^m,y(t)\big)-mn \;=\; i_{\mathbf{0}}(C,D)-mn\,. \end{aligned}$$

□

Since blowing up a point on a smooth surface leads, again, to a smooth surface, we can repeat this process. Let's consider what happens in the above examples (3), (4) when successively blowing up the non-nodal singular points of the respective total transform:

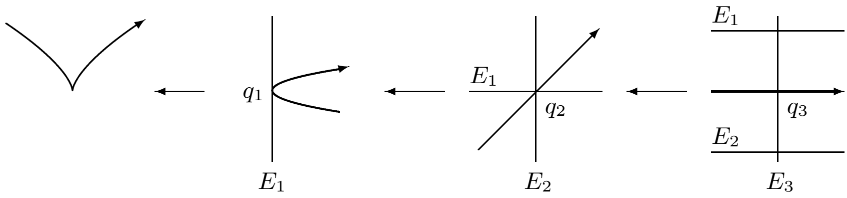


Fig. 3.16. Blowing up the cusp.

Example 3.21.1. $f = x^2 - y^3 \in \mathbb{C}\{x, y\}$ (cf. Figure 3.16).

Step 1. Let $\pi_1 : M^{(1)} \to (\mathbb{C}^2, \mathbf{0})$ be the blowing up of $\mathbf{0} \in (\mathbb{C}^2, \mathbf{0})$. By the above, the reduction of the total transform of $(C, \mathbf{0}) = V(f) \subset (\mathbb{C}^2, \mathbf{0})$ has local equation $f^{(1)} = \bar{v}(\bar{u}^2 - \bar{v})$ at its unique singular point q_1 (corresponding to the tangent x).

Step 2. Let $\pi_2 : M^{(2)} \to M^{(1)}$ be the blowing up of $q_1 \in M^{(1)}$. Then the total transform $(\pi_1 \circ \pi_2)^{-1}(C, \mathbf{0})$ has the (local) equation

$$\begin{aligned} &\text{in CHART 1:} \quad (xy)^2(x^2 - xy) = x^3y^2(x - y)\,, \ \text{respectively} \\ &\text{in CHART 2:} \quad \bar{y}^2((\bar{x}\bar{y})^2 - \bar{y}) = \bar{y}^3(\bar{x}^2\bar{y} - 1)\,, \end{aligned}$$

x, respectively $\bar{y}$, being the local equation of $E_2 = \pi_2^{-1}(q_1) \subset M^{(2)}$. In particular, the reduced total transform has local equation $f^{(2)} = xy(x - y)$ at its unique singular point q_2 (corresponding to the unique tangent $\bar{v}$ of $f^{(1)}$). Note that y is the local equation of the (reduced) strict transform of $E_1 = \pi_1^{-1}(\mathbf{0})$ at $q_2 \in M^{(2)}$.

Step 3. Let $\pi_3 : M^{(3)} \to M^{(2)}$ be the blowing up of $q_2 \in M^{(2)}$. The total transform $(\pi_1 \circ \pi_2 \circ \pi_3)^{-1}(C, \mathbf{0})$ is given by the (local) equation

$$\begin{aligned} &\text{in CHART 1:} \quad u^3(uv)^2(u - uv) = u^6v^2(1 - v)\,, \ \text{respectively} \\ &\text{in CHART 2:} \quad (\bar{u}\bar{v})^3\bar{v}^2(\bar{u}\bar{v} - \bar{v}) = \bar{v}^6\bar{u}^3(\bar{u} - 1)\,, \end{aligned}$$

u, respectively $\bar{v}$, being the local equation of $E_3 = \pi_3^{-1}(q_2) \subset M^{(3)}$. In particular, the reduced total transform has exactly the three (nodal) singular points given by

- $u = v = 0$, that is, the intersection point of (the strict transform of) E_1 with E_3,
- $\bar{v} = \bar{u} = 0$, that is, the intersection point of (the strict transform of) E_2 with E_3, respectively
- $u = 0$, $v = 1$ (respectively $\bar{v} = 0\,, \bar{u} = 1$), that is, the intersection point of the strict transform of $(C, \mathbf{0})$ with E_3.

Example 3.21.2. $f = (x^2 - y^3)(x^3 - y^5)$ (cf. Figure 3.17).

We proceed as in Example 3.21.1. The total transform of $(C, \mathbf{0}) = V(f)$ under the composition of blowing ups $\pi_3 \circ \pi_2 \circ \pi_1$ has the (local) equation

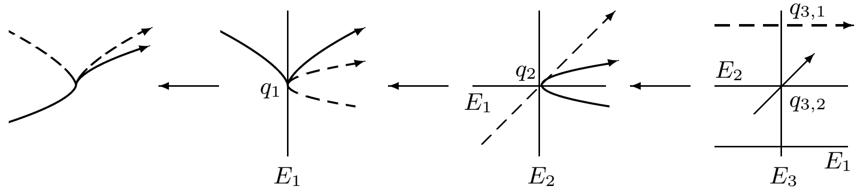


Fig. 3.17. Blowing up the curve germ defined by $(x^2-y^3)(x^3-y^5)$.

$$\begin{aligned}
&\text{in Chart 1:} \quad \big(u^6v^2(1-v)\big)\big(u^9v^3(1-uv^2)\big) = u^{15}v^5(1-v)(1-uv^2)\,,\\
&\text{in Chart 2:} \quad \big(\bar{v}^6\bar{u}^3(\bar{u}-1)\big)\big(\bar{v}^9\bar{u}^5(\bar{u}-\bar{v})\big) = \bar{v}^{15}\bar{u}^8(\bar{u}-1)(\bar{u}-\bar{v})\,,
\end{aligned}$$

u, respectively $\bar{v}$, being the local equation of E_3. In particular, the two branches of the strict transform of $(C,\mathbf{0})$ are separated and the reduced total transform has exactly two nodal singular points given by

- $u = v = 0$, that is, the intersection point of (the strict transform of) E_1 with E_3, respectively
- $u = 0$, $v = 1$ (respectively $\bar{v} = 0\,, \bar{u} = 1$), that is, the intersection point $q_{3,1}$ of the first (smooth) branch of the strict transform with E_3,

and one non-nodal singular point $q_{3,2}$ given by

- $\bar{v} = \bar{u} = 0$, that is, the intersection point of E_2, E_3 and the second (smooth) branch of the strict transform.

The latter being an ordinary singularity[23], blowing up $q_{3,2} \in M^{(3)}$ leads to a reduced total transform with only nodal singularities.

In both examples, we end up with a map $\pi : M^{(N)} \to (\mathbb{C}^2,\mathbf{0})$, satisfying the property

(EmbRes) π is the composition

$$\pi :\, M^{(N)} \xrightarrow{\pi_N} M^{(N-1)} \longrightarrow \,\dots\, \longrightarrow M^{(1)} \xrightarrow{\pi_1} (\mathbb{C}^2,\mathbf{0})\,,$$

of σ-processes π_{i+1} with centre $p_i \in E^{(i)} := (\pi_i \circ \ldots \circ \pi_1)^{-1}(\mathbf{0}) \subset M^{(i)}$, respectively $p_0 = \mathbf{0} \in (\mathbb{C}^2,\mathbf{0})$, such that the strict transform

$$C^{(N)} \;=\; \overline{\pi^{-1}(C,\mathbf{0}) \setminus E^{(N)}}$$

is smooth and intersects the reduced exceptional divisor $E^{(N)}$ transversally in smooth points, that is, for each point $p \in C^{(N)} \cap E^{(N)}$ we have $i_p\big(C^{(N)}, E^{(N)}\big) = 1$.

[23] A reduced plane curve singularity is called *ordinary* if all its local branches are smooth and intersect pairwise transversally. If an ordinary singularity has k branches, we call it an *ordinary k-multiple point*. We thus have *ordinary double points*, also called *nodes*, *ordinary triple points*, etc.

Definition 3.22. A commutative diagram

$$\begin{tikzcd} M^{(N)} \arrow[r, "\pi"] & (\mathbb{C}^2,\mathbf{0}) \\ C^{(N)} \arrow[u, hook] \arrow[r, "\pi"'] & (C,\mathbf{0}) \arrow[u, hook] \end{tikzcd}$$

with $\pi : M^{(N)} \to (\mathbb{C}^2,\mathbf{0})$ satisfying the property (EmbRes) is called an *embedded resolution (of singularities)* of $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$.

It is called a *minimal embedded resolution* of $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ if $(C,\mathbf{0})$ is singular and if the π_{i+1}, $i > 0$, blow up only non-nodal singular points p_i of the reduced total transform of $(C,\mathbf{0})$ in $M^{(i)}$.

If $\pi : \widetilde{M} \to (\mathbb{C}^2,\mathbf{0})$ is a composition of finitely many blowing ups of points ($\widetilde{M}$ being a germ of a two-dimensional complex manifold along $E = \pi^{-1}(\mathbf{0})$) and if $q \in \widetilde{M}$ belongs to the strict transform of $(C,\mathbf{0})$, then we call $\widetilde{M}$ together with π an *infinitely near neighbourhood of* $(C,\mathbf{0})$ and p a *point infinitely near to* $\mathbf{0}$ *and belonging to* $(C,\mathbf{0})$.

Theorem 3.23 (Desingularization Theorem).

(1) Let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be a singular and reduced plane curve singularity. Then there exists a (minimal) embedded resolution

$$\begin{tikzcd} M^{(N)} \arrow[r, "\pi_N"] & M^{(N-1)} \arrow[r, "\pi_{N-1}"] & \cdots \arrow[r, "\pi_2"] & M^{(1)} \arrow[r, "\pi_1"] & (\mathbb{C}^2,\mathbf{0}) \\ C^{(N)} \arrow[u, hook] \arrow[r] & C^{(N-1)} \arrow[u, hook] \arrow[r] & \cdots \arrow[r] & C^{(1)} \arrow[u, hook] \arrow[r] & (C,\mathbf{0}) \arrow[u, hook] \end{tikzcd}\,.$$

(2) The minimal embedded resolution is unique up to isomorphism.

Proof. (1) We define $\pi_1 : M^{(1)} \to (\mathbb{C}^2,\mathbf{0})$ to be the blowing up of $\mathbf{0} \in (\mathbb{C}^2,\mathbf{0})$, and $\pi_i : M^{(i)} \to M^{(i-1)}$, $i \geq 2$, to be induced by (successively) blowing up of all those intersection points of the strict transform $C^{(i)}$ and the exceptional divisor $E^{(i)}$, where the reduced germ of $C^{(i)} \cup E^{(i)}$ is not a node. Our claim is that after finitely many steps there are no such points left to be blown up.

Before starting with the proof of this claim, we should like to point out that, by construction, the reduced exceptional divisors $E^{(i)}$ are nodal curves, with all irreducible components being $\mathbb{P}^1$'s, which pairwise intersect transversally and in at most one point.

Case 1. Assume that $(C,\mathbf{0})$ is irreducible.

We show that either $\mathrm{mt}(C,\mathbf{0}) = 1$, or the multiplicity of the strict transform $C^{(i)}$ drops after finitely many blowing ups: choosing suitable local coordinates on $(\mathbb{C}^2,\mathbf{0})$, we can assume that $(C,\mathbf{0})$ is given by a Weierstraß polynomial

$$f(x,y) = \prod_{j=1}^{m} \Big(y - \varphi\big(\xi^j x^{1/m}\big)\Big)\ , \quad m = \mathrm{mt}(C,\mathbf{0})\,,$$

with $\varphi \in \langle t\rangle \cdot \mathbb{C}\{t\}$ and ξ a primitive m-th root of unity (Proposition 3.4 (2)). Note that, in particular, $\operatorname{ord}(\varphi) \geq m$. Blowing up $\mathbf{0}$ leads to the strict transform (in Chart 1)

$$\widetilde{f}(u,v) = \prod_{j=1}^{m} \left(v - \frac{\varphi\big(\xi^j u^{1/m}\big)}{u} \right) =: \prod_{j=1}^{m} \Big(v - \varphi^{(1)}\big(\xi^j u^{1/m}\big)\Big) ,$$

which has multiplicity equal to $\min\big\{m, \operatorname{ord}(\varphi) - m\big\}$.

Since $\operatorname{ord}(\varphi^{(1)}) = \operatorname{ord}(\varphi) - m$, we can proceed by induction to conclude that the multiplicity will drop after finitely many steps.

Hence, after finitely many blowing ups, we end up with a strict transform $C^{(k)}$ which is smooth at the (unique) intersection point q with the exceptional divisor $E^{(k)}$. It still might be that $i_q(C^{(k)}, E^{(k)}) > 1$. But after blowing up q the intersection number (of the respective strict transforms) has dropped, due to Proposition 3.21. Moreover, the new components of the exceptional divisor (not belonging to the strict transform of $E^{(k)}$) are intersected transversally by $C^{(k+1)}$. It follows that after finitely many further blowing ups the only non-nodal singularity of $C^{(k)} \cup E^{(k)}$ might be an ordinary triple point, that is, two components of the exceptional divisor and the strict transform of $(C, \mathbf{0})$ intersecting transversally at a point q'. But the latter is resolved by blowing up q'.

Case 2. If $(C, \mathbf{0})$ is reducible then, by the above, after finitely many blowing ups the strict transform of each branch $(C_j, \mathbf{0}) \subset (C, \mathbf{0})$ intersects the exceptional divisor transversally in smooth points. Now the statement follows, since after finitely many further blowing ups the strict transforms of the branches are separated (that is, don't intersect each other), applying Proposition 3.21 again and proceeding by induction.

(2) The proof of the uniqueness is left as Exercise 3.3.1 □

Transforming Rings. In the following, we study the algebraic counterpart of the geometric resolution process (by means of successive blowing ups) described before. The following statements about rings and ring maps hold in the same way for arbitrary algebraically closed fields of characteristic 0.

Let $\mathcal{O}$ be the local ring of a plane curve singularity. By Lemma 1.5 and the Weierstraß preparation theorem, we can assume that $\mathcal{O} = \mathbb{C}\{x\}[y]/\langle f\rangle$ where f is a Weierstraß polynomial (of order $m > 0$), that is,

$$\mathcal{O} \;=\; \mathbb{C}\{x\}[y]/\langle f\rangle \;\cong\; \mathbb{C}\{x\} \oplus \mathbb{C}\{x\} \cdot y \oplus \ldots \oplus \mathbb{C}\{x\} \cdot y^{m-1} ,$$

the latter being an isomorphism of $\mathbb{C}\{x\}$-modules. Additionally, we assume that x is not a tangent of f, that is,

$$f \;\equiv\; \prod_{i=1}^{s} (y - \alpha_i x)^{m_i} \bmod \mathfrak{m}^{m+1} ,$$

with $\alpha_i \in \mathbb{C}$ pairwise distinct, $m = m_1 + \ldots + m_s$. In other words, we assume that $\alpha_1, \ldots, \alpha_s$ are the different zeros of $f_m(1, y)$ (f_m the tangent cone of f).

Now, let $U = U(\mathbf{0}) \subset \mathbb{C}^2$ be a (small) open neighbourhood of the origin, and let

$$\pi : \widetilde{U} := \big\{(x, y; s : t) \in U \times \mathbb{P}^1 \,\big|\, xt - ys = 0\big\} \longrightarrow U$$

be the blowing up map. Note that the above assumptions allow to restrict ourselves on the chart $V_0 = \{s \neq 0\} \subset \mathbb{P}^1$ when considering the strict transform of $V(f)$. As before, we introduce the coordinates $u = x$, $v = t/s$ (cf. Remark 3.16.1) on $\widetilde{U} \cap (U \times V_0)$, and set $(u_i, v_i) := (u, v - \alpha_i)$, $i = 1, \ldots, s$, the latter being local coordinates at the intersection point $q_i := (0, 0; \alpha_i : 1)$ of the strict transform of $V(f)$ and the exceptional divisor. Then π induces injective morphisms

$$\pi^\sharp : \mathbb{C}\{x, y\} \to \mathbb{C}\{u_i, v_i\}, \quad (x, y) \mapsto \big(u_i, u_i(v_i + \alpha_i)\big)$$

of local $\mathbb{C}$-algebras, mapping f to its total transform $\widehat{f}$ at the point q_i. In particular, it induces a morphism

$$\phi : \mathcal{O} \longrightarrow \mathcal{O}^{(1)} := \bigoplus_{i=1}^{s} \mathbb{C}\{u_i, v_i\} \Big/ \Big\langle \widetilde{f}\big(u_i, u_i(v_i + \alpha_i)\big) \Big\rangle, \tag{3.3.3}$$

where $\widetilde{f}$ is the strict transform of f. $\mathcal{O}^{(1)}$ is classically called the *first neighbourhood ring of* $\mathcal{O}$ (cf. [Nor]).

Lemma 3.24. *With the above notations, the morphism $\phi : \mathcal{O} \to \mathcal{O}^{(1)}$ is injective. Moreover,*

$$\mathcal{O}^{(1)} \cong \mathcal{O}\left[\frac{y}{x}\right] \subset \mathrm{Quot}(\mathcal{O})$$

as ring extensions of $\mathcal{O}$, and $1, y/x, \ldots, (y/x)^{m-1}$ is a minimal set of generators for $\mathcal{O}[y/x]$ as $\mathcal{O}$-module.

In particular, $\mathcal{O} \hookrightarrow \mathcal{O}^{(1)}$ is an integral extension of $\mathcal{O}$ in the total quotient ring $\mathrm{Quot}(\mathcal{O})$, and we have the equivalence

$$\mathcal{O} \cong \mathcal{O}^{(1)} \iff V(f) \text{ is a smooth germ}. \tag{3.3.4}$$

Proof of Lemma 3.24. We proceed in three steps:

Step 1. We show that y/x is integral over $\mathcal{O}$ and that $1, y/x, \ldots, (y/x)^{m-1}$ is a minimal set of generators for $\mathcal{O}[y/x]$ as $\mathcal{O}$-module.

By our assumptions, x is not a zero-divisor in $\mathcal{O}$ (since it is not a tangent of f) and

$$f = y^m + \sum_{i=0}^{m-1} a_i(x) \cdot y^i, \qquad a_i \in \big\langle x^{m-i} \big\rangle \cdot \mathbb{C}\{x\}.$$

Hence, in $\mathrm{Quot}(\mathcal{O})$ we have the equality

$$0 = \left(\frac{y}{x}\right)^m + \sum_{i=0}^{m-1} a'_i(x) \cdot \left(\frac{y}{x}\right)^i, \qquad a'_i := \frac{a_i}{x^{m-i}} \in \mathbb{C}\{x\}\,,$$

which shows the integral dependence of y/x over $\mathcal{O}$. On the other hand, it is not difficult to see that this is an integral equation of minimal degree: let $b_i \in \mathbb{C}\{x\}[y]$, $i = 0, \ldots, N-1$ satisfy the equation

$$0 = \left(\frac{y}{x}\right)^N + \sum_{i=0}^{N-1} b_i(x,y) \cdot \left(\frac{y}{x}\right)^i = \frac{y^N + \sum_i b_i(x,y) \cdot x^{N-i} y^i}{x^N} \in \operatorname{Quot}(\mathcal{O})\,.$$

Then there exists some $h \in \mathbb{C}\{x\}[y]$ which is not a divisor of f and satisfies

$$h(x,y) \cdot \left(y^N + \sum_{i=0}^{N-1} b_i(x,y) \cdot x^{N-i} y^i \right) \in \langle f \rangle \subset \mathbb{C}\{x\}[y]\,.$$

In particular, f divides $y^N + \sum_i b_i(x,y) \cdot x^{N-i} y^i$, which implies $N \geq m$.

Step 2. Let $\widetilde{f}(u,v) = f(u,uv)/u^m$ be the strict transform of f. Then there exists an isomorphism

$$\mathcal{O}\left[\frac{y}{x}\right] \xrightarrow{\cong} \mathbb{C}\{u\}[v] \big/ \langle \widetilde{f}\, \rangle\,,$$

such that the composition with the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}[y/x]$ is induced by mapping $x \mapsto u$, $y \mapsto uv$.

First note that, due to the considerations in Step 1, mapping

$$\sum_{i=0}^{m-1} b_i(x,y) \cdot \left(\frac{y}{x}\right)^i \longmapsto \sum_{i=0}^{m-1} b_i(u,uv) \cdot v^i$$

induces a well-defined, surjective morphism $\psi : \mathcal{O}\,[y/x] \twoheadrightarrow \mathbb{C}\{u\}[v]/\big\langle \widetilde{f}\,\big\rangle$. It remains to show that ψ is injective, too. To do so, let

$$\sum_{i=0}^{m-1} b_i(u,uv) \cdot v^i \;=\; h(u,v) \cdot \widetilde{f}(u,v)$$

with $h \in \mathbb{C}\{u\}[v]$ a polynomial in v of degree N. Then $u^N \cdot h(u,v) = h'(u,uv)$ for some polynomial $h' \in \mathbb{C}\{x\}[y]$, and, by resubstituting x for u and y for uv, we obtain

$$x^N \cdot \sum_{i=0}^{m-1} b_i(x,y) \cdot x^{m-i} y^i \;=\; h'(x,y) \cdot f(x,y)\,.$$

Since x is not a factor of f, this implies $\sum_i b_i(x,y) \cdot x^{m-i} y^i = 0 \in \mathcal{O}[y/x]$.

Step 3. There exists an isomorphism

$$\mathbb{C}\{u\}[v]/\langle\widetilde{f}\,\rangle \xrightarrow{\cong} \mathcal{O}^{(1)} = \bigoplus_{i=1}^{s} \mathbb{C}\{u_i, v_i\}\Big/\Big\langle \widetilde{f}\big(u_i, u_i(v_i+\alpha_i)\big)\Big\rangle\,.$$

Due to Lemma 3.19 we can decompose

$$f = f^{(1)} \cdot \ldots \cdot f^{(s)}\,, \qquad f^{(j)} \equiv (y - \alpha_j x)^{m_j} \mod \mathfrak{m}^{m_j+1}\,.$$

Moreover, by the Weierstraß preparation theorem, we can assume that the $f^{(j)}$, $j = 1, \ldots, s$, are, indeed, Weierstraß polynomials in $\mathbb{C}\{x\}[y]$. Now,

$$\begin{aligned}\widetilde{f}\big(u_i, u_i(v_i+\alpha_i)\big) &= \frac{f\big(u_i, u_i(v_i+\alpha_i)\big)}{u_i^m} = \frac{\prod_j f^{(j)}\big(u_i, u_i(v_i+\alpha_i)\big)}{u_i^m} \\ &= \text{unit} \cdot \frac{f^{(i)}\big(u_i, u_i(v_i+\alpha_i)\big)}{u_i^{m_i}} = \text{unit} \cdot \widetilde{f}^{(i)}\big(u_i, u_i(v_i+\alpha_i)\big)\end{aligned}$$

in $\mathbb{C}\{u_i, v_i\}$, $\widetilde{f}^{(i)} \in \mathbb{C}\{u\}[v]$ denoting the strict transform of $f^{(i)}$. Finally, the statement follows from the chinese remainder theorem. For this, we have to show that the polynomials $\widetilde{f}^{(i)}$ are pairwise coprime in $\mathbb{C}\{u\}[v]$, that is, if $\big\langle \widetilde{f}^{(i)}, \widetilde{f}^{(j)}\big\rangle = \mathbb{C}\{u\}[v]$ for all $i \neq j$.

We compute $\widetilde{f}^{(i)}(0, v) = (v - \alpha_i)^{m_i}$, which implies $\gcd\big(\widetilde{f}^{(i)}, \widetilde{f}^{(j)}\big) = 1$ in $\mathbb{C}\{u\}[v]$. Hence, there are $A_{ij}, B_{ij} \in \big(\operatorname{Quot}\mathbb{C}\{u\}\big)[v]$, $\deg_v A_{ij} < m_j$ such that $A_{ij}\widetilde{f}^{(i)} + B_{ij}\widetilde{f}^{(j)} = 1$. Equivalently, there are polynomials $a_{ij}, b_{ij} \in \mathbb{C}\{u\}[v]$ and a non-negative integer $N \geq 0$, such that

$$a_{ij}\widetilde{f}^{(i)} + b_{ij}\widetilde{f}^{(j)} = u^N, \qquad \deg_v(a_{ij}) < m_j\,.$$

We assume N to be chosen minimally, that is, $(a_{ij}, b_{ij})(0, v) \neq (0, 0)$. If $N > 0$ then $a_{ij}(0, v)\widetilde{f}^{(i)}(0, v) + b_{ij}(0, v)\widetilde{f}^{(j)}(0, v) = 0 \in \mathbb{C}[v]$. In particular, $\widetilde{f}^{(j)}(0, v)$ would divide $a_{ij}(0, v)$, contradicting the assumption $\deg_v(a_{ij}) < m_j$. Hence, $N = 0$, which yields $1 \in \big\langle \widetilde{f}^{(i)}, \widetilde{f}^{(j)}\big\rangle$. □

Proceeding by induction, we introduce the *k-th neighbourhood ring $\mathcal{O}^{(k)}$ of $\mathcal{O}$*, $k \geq 2$: let $\mathcal{O}^{(k-1)}$ be the direct sum of local rings

$$\mathcal{O}^{(k-1)} = \bigoplus_{i=1}^{s_{k-1}} \mathcal{O}_i^{(k-1)}\,.$$

Then we define $\mathcal{O}^{(k)}$ to be the direct sum of the respective first neighbourhood rings,

$$\mathcal{O}^{(k)} := \bigoplus_{i=1}^{s_{k-1}} \big(\mathcal{O}_i^{(k-1)}\big)^{(1)}\,. \tag{3.3.5}$$

Lemma 3.25. *The k-th neighbourhood rings $\mathcal{O}^{(k)}$, $k \geq 1$, of $\mathcal{O}$ are integral extensions of $\mathcal{O}$, contained in the total quotient ring* $\mathrm{Quot}(\mathcal{O})$.

Proof. This follows from Lemma 3.24, applying induction and using the following (easy) fact: if $R_i \subset S_i$, $i = 1, \dots, N$, are (finite) ring extensions in the respective full quotient ring, then $R_1 \oplus \dots \oplus R_N \subset S_1 \oplus \dots \oplus S_N$ is a (finite) ring extension in $\mathrm{Quot}\big(R_1 \oplus \dots \oplus R_N\big)$. □

Since we know already that after finitely many blowing ups the strict transforms of f at the respective points become non-singular (Theorem 3.23), and since regular local rings are normal (that is, integrally closed in its full quotient ring), the above equivalence (3.3.4) allows the following conclusion:

Proposition 3.26. *Let $\mathcal{O} = \mathcal{O}_{C,\mathbf{0}}$ be the local ring of a reduced plane curve singularity, and let $\mathcal{O}^{(k)}$ denote the k-th neighbourhood ring of $\mathcal{O}$, $k \geq 1$. Then the following conditions are equivalent and hold for k sufficiently large:*

(a) $\mathcal{O}^{(k)}$ is a direct sum of regular local rings.
(b) $\mathcal{O}^{(k+1)} = \mathcal{O}^{(k)}$.
(c) $\mathcal{O}^{(j)} = \mathcal{O}^{(k)}$ for all $j \geq k$.
(d) $\mathcal{O}^{(k)}$ is integrally closed in its full quotient ring.
(e) $\mathcal{O}^{(k)}$ is the integral closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (in its full quotient ring).

Hence, we have a sequence of inclusions

$$\mathcal{O} \hookrightarrow \mathcal{O}^{(1)} \hookrightarrow \mathcal{O}^{(2)} \hookrightarrow \dots \hookrightarrow \mathcal{O}^{(k)} = \overline{\mathcal{O}}\,.$$

We call the map $\mathcal{O} \hookrightarrow \overline{\mathcal{O}}$ the *normalization* of $\mathcal{O}$. As we shall see in the following corollary, normalization and parametrization of reduced plane curve singularities are closely related.

Corollary 3.27. *Let $\mathcal{O} = \mathbb{C}\{x, y\}/\langle f \rangle$ be the local ring of a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$, and let $\overline{\mathcal{O}}$ be the integral closure of $\mathcal{O}$. Then the following holds true:*

(a) $\overline{\mathcal{O}}$ is a finitely generated $\mathcal{O}$-module.
(b) $\overline{\mathcal{O}} \cong \bigoplus\limits_{i=1}^{r} \mathbb{C}\{t_i\}$, where r is the number of branches of $(C, \mathbf{0})$.

Moreover, the induced map

$$\phi = (\phi_1, \dots, \phi_r) : \mathbb{C}\{x, y\} \twoheadrightarrow \mathbb{C}\{x, y\}/\langle f \rangle \hookrightarrow \mathbb{C}\{t_1\} \oplus \dots \oplus \mathbb{C}\{t_r\}$$

defines a parametrization $\varphi : \bigoplus_i (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0})$, $t_i \mapsto \big(\phi_i(x)(t_i), \phi_i(y)(t_i)\big)$ of $(C, \mathbf{0})$. In particular,

(c) $i(f, g) = \sum\limits_{i=1}^{r} \mathrm{ord}_{t_i}\, \phi_i(g)$ for any $g \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$.

Proof. (a) follows immediately from Proposition 3.26 and Lemma 3.25. To see (b), note that, by the above, $\overline{\mathcal{O}} = \mathcal{O}^{(k)}$ for some $k \gg 0$, and the latter is the direct product of regular local rings. Moreover, the number of direct factors is easily seen to coincide with the number of intersection points of the strict transform with the exceptional divisor when having resolved the singularity (Theorem 3.23). But the latter is just r, the number of branches of $(C, \mathbf{0})$.

(c) It suffices to show that $\phi : \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \to \mathbb{C}\{t_1\} \oplus \ldots \oplus \mathbb{C}\{t_r\}$ defines, indeed, a parametrization of $(C, \mathbf{0})$. To do so, we can restrict ourselves to the case that $(C, \mathbf{0})$ is irreducible. Moreover, we can assume that $(C, \mathbf{0})$ is given by a Weierstraß polynomial $f \in \mathbb{C}\{x\}[y]$ with the (unique) tangent y.

Let $\psi : (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0})$, $t \mapsto \big(x(t), y(t)\big)$, be a holomorphic map such that $\psi(\mathbb{C}, 0) \subset (C, \mathbf{0})$, that is, $f\big(x(t), y(t)\big) = 0$. Comparing coefficients, we obtain $\operatorname{ord}(x(t)) < \operatorname{ord}(y(t))$. Hence, we can consider the holomorphic map of complex space germs $\eta^{(1)} : (\mathbb{C}, 0) \to (\widetilde{C}, \mathbf{0})$ induced by

$$t \longmapsto \big(x(t), \widetilde{y}(t)\big)\,, \qquad \widetilde{y}(t) := \frac{y(t)}{x(t)} \in \langle t \rangle \cdot \mathbb{C}\{t\}\,.$$

Obviously, we obtain a splitting $\psi = \phi^{(1)} \circ \eta^{(1)}$, where $\phi^{(1)} : (\widetilde{C}, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ denotes the holomorphic map induced by the composition $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \to \mathcal{O} \hookrightarrow \mathcal{O}^{(1)}$.

Finally, proceeding by induction, we can deduce the existence of a holomorphic map $\eta^{(k)} : (\mathbb{C}, 0) \to (C^{(k)}, \mathbf{0}) \cong (\mathbb{C}, 0)$ satisfying $\psi = \phi \circ \eta^{(k)}$. The uniqueness of $\eta^{(k)}$ is obvious, since ϕ is a bijection (of germs of sets). $\square$

Remark 3.27.1. The latter corollary states that the normalization $\mathcal{O} \hookrightarrow \overline{\mathcal{O}}$ of the local ring of a reduced plane curve singularity induces parametrizations $\varphi_i : (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0})$ of the branches $(C_i, \mathbf{0})$, $i = 1, \ldots, r$, of the singularity. Vice versa, let $\varphi'_i : (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0})$, $i = 1, \ldots, r$, be parametrizations of the branches. Then the corresponding morphisms of local rings ϕ'_i factor through $\mathcal{O}$ and we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \arrow[rr, "\phi'_i"] \arrow[dr, "\mathrm{pr}"'] & & \mathcal{O}_{\mathbb{C},0} \arrow[r, "\cong"] & \mathbb{C}\{t_i\}\,. \\ & \mathcal{O} \arrow[ur] \arrow[urr, "\eta_i"'] & & \end{tikzcd}$$

Now the universal factorization property (3.1.1) of the parametrizations shows that $\eta = (\eta_1, \ldots, \eta_r)$ coincides with the normalization (up to an isomorphism $\mathbb{C}\{t_1\} \oplus \ldots \oplus \mathbb{C}\{t_r\} \xrightarrow{\cong} \overline{\mathcal{O}}$).

Proposition 3.28. *Let $\mathcal{O}$ be the local ring of a reduced plane curve singularity, and let $\mathcal{O}^{(k)}$ denote the k-th neighbourhood ring of $\mathcal{O}$, $k \geq 1$.*

Then the ideal $\mathcal{O} : \mathcal{O}^{(k)} := \big\{ g \in \mathcal{O} \,\big|\, g \cdot \mathcal{O}^{(k)} \in \mathcal{O} \big\}$ is either $\mathcal{O}$ or an $\mathfrak{m}$-primary ideal ($\mathfrak{m} \subset \mathcal{O}$ denoting the maximal ideal).

Proof. Since $\mathcal{O} \hookrightarrow \mathcal{O}^{(k)}$ is a finite ring extension in the full quotient ring $\operatorname{Quot}(\mathcal{O})$, there exists a non-zerodivisor $h \in \mathcal{O}$ such that $h \cdot \mathcal{O}^{(k)} \subset \mathcal{O}$, that is, $h \in \mathcal{O} : \mathcal{O}^{(k)}$.

On the other hand, it is not difficult to see that the prime ideals of $\mathcal{O}$ are just the maximal ideal $\mathfrak{m}$ and the ideals generated by the classes of the irreducible factors of f. In particular, the maximal ideal is the unique prime ideal containing a non-zerodivisor of $\mathcal{O}$. Hence, if $\mathcal{O} : \mathcal{O}^{(k)}$ is contained in a prime ideal, then it is necessarily $\mathfrak{m}$-primary. □

Corollary 3.29. *Let $\mathcal{O}$ be the local ring of a reduced plane curve singularity and $\overline{\mathcal{O}}$ its integral closure. Then* $\dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O} < \infty$.

The dimension of $\overline{\mathcal{O}}/\mathcal{O}$, the so-called *$\delta$-invariant* (or, *order*) *of a singularity* is one of the most important invariants when studying plane curve singularities. It will be treated in detail in Section 3.4, below.

Proof of Corollary 3.29. Let $\mathfrak{m} \subset \mathcal{O}$ be the maximal ideal. Then, due to Proposition 3.28, there exists some $k \geq 0$ such that $\mathfrak{m}^k\overline{\mathcal{O}} \subset \mathcal{O}$. Hence, it suffices to show that $\dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathfrak{m}^k\overline{\mathcal{O}}$ is finite. But,

$$\dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathfrak{m}^k\overline{\mathcal{O}} = \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathfrak{m}\overline{\mathcal{O}} + \dim_{\mathbb{C}} \mathfrak{m}\overline{\mathcal{O}}/\mathfrak{m}^2\overline{\mathcal{O}} + \ldots + \dim_{\mathbb{C}} \mathfrak{m}^{k-1}\overline{\mathcal{O}}/\mathfrak{m}^k\overline{\mathcal{O}},$$

where all summands on the right-hand side are finite due to Corollary 3.27 (a) and Nakayama's lemma. □

Remark 3.29.1. Knowing that $\overline{\mathcal{O}}/\mathcal{O}$ is a finite dimensional complex vector space implies another proof of proposition 3.12: Let $\mathcal{O} = \mathbb{C}\{x,y\}/\langle f\rangle$ and $\mathcal{O} \hookrightarrow \overline{\mathcal{O}}$ the normalization. Applying the snake lemma to the commutative diagram of $\mathcal{O}$-modules

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\cdot g"] \arrow[d, hook] & \mathcal{O} \arrow[r, two heads] \arrow[d, hook] & \mathcal{O}/g\mathcal{O} \arrow[d, "\rho"] \\ \overline{\mathcal{O}} \arrow[r, "\cdot g"] & \overline{\mathcal{O}} \arrow[r, two heads] & \overline{\mathcal{O}}/g\overline{\mathcal{O}}, \end{tikzcd}$$

we can argue as in the proof of Proposition 3.12 to deduce the equality $i(f,g) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f,g\rangle$.

Exercises

Exercise 3.3.1. Show that the minimal embedded resolution of a reduced plane curve singularity $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ is unique. That is, if

$$\begin{tikzcd} M'^{(N')} \arrow[r, "\pi'_{N'}"] & M'^{(N'-1)} \arrow[r, "\pi'_{N'-1}"] & \cdots \arrow[r, "\pi'_2"] & M'^{(1)} \arrow[r, "\pi'_1"] & (\mathbb{C}^2,\mathbf{0}). \\ C'^{(N')} \arrow[r] \arrow[u, hook] & C'^{(N'-1)} \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & C'^{(1)} \arrow[r] \arrow[u, hook] & (C,\mathbf{0}). \arrow[u, hook] \end{tikzcd}$$

is a second minimal embedded resolution of $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ then $N = N'$ and there exists an isomorphism $M^{(N)} \xrightarrow{\cong} M'^{(N)}$ making the following diagram commute

$$
\begin{tikzcd}
& M^{(N)} \arrow[dl, "\pi_N"'] \arrow[rr, "\cong"] & & M'^{(N)} \arrow[dr, "\pi'_N"] & \\
M^{(N-1)} \arrow[d, "\pi_{N-1}"'] & & & & M'^{(N-1)} \arrow[d, "\pi'_{N-1}"] \\
\vdots \arrow[d, "\pi_2"'] & & & & \vdots \arrow[d, "\pi'_2"] \\
M^{(1)} \arrow[drr, "\pi_1"'] & & & & M'^{(1)} \arrow[dll, "\pi'_1"] \\
& & (\mathbb{C}^2, \mathbf{0})\,. & &
\end{tikzcd}
$$

Exercise 3.3.2. Denote by $N(C, \mathbf{0})$ the number of blowing ups needed to obtain a minimal embedded resolution of a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$. Show that $N(C, \mathbf{0}) \leq 1$ iff $(C, \mathbf{0})$ is an ordinary singularity where $N(C, \mathbf{0}) = 0$ iff $(C, \mathbf{0})$ is smooth.

Exercise 3.3.3. (1) Show that each reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ has a smooth strict transform after at most $\delta = \delta(C, \mathbf{0})$ blowing ups.

(2) Is it true that the minimal embedded resolution of any isolated curve singularity contains at most δ blowing ups (that is, $N(C, \mathbf{0}) \leq \delta(C, \mathbf{0})$ using the notation of Exercise 3.3.2)?

Exercise 3.3.4. Let $\mathcal{O}$ be the local ring of a reduced plane curve singularity of multiplicity $m \geq 1$. Prove that $\mathcal{O} : \mathcal{O}^{(1)} = \mathfrak{m}^{m-1}$.

Exercise 3.3.5. Show that the local ring of the isolated surface singularity $\{x^2 + y^2 + z^2 = 0\}$ is integrally closed (in particular, in higher dimensions the normalization does not resolve an isolated hypersurface singularity).

3.4 Classical Topological and Analytic Invariants

In Section 2, we have already introduced (and studied in some detail) two of the most important numerical invariants of isolated hypersurface singularities, the Milnor number μ and the Tjurina number τ. In the following, we shall discuss two further (classical) invariants of reduced (that is, isolated) plane curve singularities, the δ- and the *κ-invariant*. In particular, we study the interrelations of these four invariants.

Moreover, we introduce the *semigroup of values* associated to a plane curve singularity, and the *conductor*.

In Section 2 we studied for isolated hypersurface singularities the action of the group of analytic isomorphisms on $(\mathbb{C}^2, \mathbf{0})$, which leads to the notion of *analytic types*. For plane curves we also study the action of the group of homeomorphisms, leading to a weaker equivalence relation (the topological equivalence, the equivalence classes being called *topological types*). Note that, in contrast to the action leading to the notion of analytic types the group action leading to the notion of topological types is *not an algebraic group action*.

In the final part of this section we shall introduce the so-called system of multiplicity sequences of a reduced plane curve singularity which completely determines its topological type. In particular, this will enable us to show that the above analytic invariants μ, δ, κ and the conductor of the semigroup of values are actually *topological* invariants, while the Tjurina number τ is not a topological invariant.

Analytic and Topological Types. Even if we mainly deal with (invariants) of plane curve singularities, we should like to introduce the notions of analytic, respectively topological, types in the more general context of isolated hypersurface singularities.

Definition 3.30. Let $(X,z) \subset (\mathbb{C}^n, z)$ and $(Y,w) \subset (\mathbb{C}^n, w)$ be two germs of isolated hypersurface singularities. Then (X,z) and (Y,w) (or any defining power series) are said to be *analytically equivalent* (or *contact equivalent*) if there exists a local analytic isomorphism $(\mathbb{C}^n, z) \to (\mathbb{C}^n, w)$ mapping (X,z) to (Y,w). The corresponding equivalence classes are called *analytic types.*

(X,z) and (Y,w) (or any defining power series) are said to be *topologically equivalent* if there exists a homeomorphism $(\mathbb{C}^n, z) \to (\mathbb{C}^n, w)$ mapping (X,z) to (Y,w). The corresponding equivalence classes are called *topological types* (or sometimes "complex" topological types as opposed to "real" topological types).

A number (or a set, or a group, ...) associated to a singularity is called an *analytic*, respectively *topological*, *invariant* if it does not change its value within an analytic, respectively topological, equivalence class.

Example 3.30.1. Any two ordinary k-multiple points (consisting of smooth branches with different tangents) have the same topological type. However, if $k \geq 4$ then there are infinitely many analytic types of ordinary k-multiple points. For instance, if $k = 4$ then the analytic type depends precisely on the cross-ratios of the 4 tangents (see also Example 3.43.2). Thus, the cross-ratio is an analytic but not a topological invariant of four lines through $\mathbf{0}$ in $\mathbb{C}^2$.

With respect to the topological type we just like to mention *Milnor's cone theorem.* For further information, we refer to the literature, e.g., [BrK, EiN, MiW, Mil1, Pha, Loo].

Let $U \subset \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ holomorphic and z an isolated singularity of the hypersurface $f^{-1}(0)$. Milnor [Mil1] considered a small $2n$-dimensional ball B_ε, a $(2n-1)$-dimensional sphere ∂B_ε of radius $\varepsilon > 0$ centred at z, and its intersection with the singular fibre $f^{-1}(0)$ for ε sufficiently small. Denote

- $B_\varepsilon := \big\{ x \in \mathbb{C}^n \bigm| |x - z| \leq \varepsilon \big\}$, the *Milnor ball*,
- $\partial B_\varepsilon := \big\{ x \in \mathbb{C}^n \bigm| |x - z| = \varepsilon \big\}$, the *Milnor sphere*,
- $X_\varepsilon := f^{-1}(0) \cap B_\varepsilon$,

- $\partial X_\varepsilon := f^{-1}(0) \cap \partial B_\varepsilon$, the *neighbourhood boundary*, or the *link* of the singularity $\bigl(f^{-1}(0), z\bigr)$.

Then Milnor showed that, for a given f as above, there exists some $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds:

(1) $B_\varepsilon \subset U$ and z is the only singular point of $f^{-1}(0)$ in X_ε.
(2) ∂B_ε and $f^{-1}(0)$ intersect transversally, in particular, ∂X_ε is a compact, real analytic submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$ of real dimension $2n-3$.
(3) The pair $\bigl(B_\varepsilon, X_\varepsilon\bigr)$ is homeomorphic to the pair $\bigl(B_\varepsilon, \mathrm{cone}(\partial X_\varepsilon)\bigr)$. More precisely, there exists a homeomorphism $h : B_\varepsilon \to B_\varepsilon$, $h|_{\partial B_\varepsilon} = \mathrm{id}$, such that $h(X_\varepsilon) = \mathrm{cone}(\partial X_\varepsilon)$, $h(z) = z$.

Recall that as a topological space the *cone* over a space M, $\mathrm{cone}(M)$, is obtained from $M \times [0,1]$ by collapsing $M \times \{0\}$ to a point. As subspace of B_ε, the cone over ∂X_ε is the union of segments in B_ε joining points of ∂X_ε with the centre z.

Since $B_\varepsilon = \mathrm{cone}(\partial B_\varepsilon)$, the topological type of the pair $\bigl(B_\varepsilon, X_\varepsilon\bigr)$ is completely determined by the pair $\bigl(\partial B_\varepsilon, \partial X_\varepsilon\bigr)$, that is, by the link ∂X_ε and its embedding in the (Milnor) sphere ∂B_ε.

In $n > 2$ then ∂X_ε is connected. If $n = 2$ then the number of connected components of ∂X_ε is equal to the number of branches of the curve singularity (X_ε, z). Furthermore, each connected component of ∂X_ε is homeomorphic to S^1 embedded in $\partial B_\varepsilon \approx S^3$, that is, a *knot*. Different connected components are linked with each other (see Figure 3.18).

In general, for $n = 2$, and f irreducible, ∂X_ε is an iterated torus knot (cable knot), characterized by the Puiseux pairs of f. For several branches, the linking numbers of the different knots are the intersection number of the corresponding branches. Thus, the Puiseux pairs of branches and the pairwise intersection numbers determine (and are determined by) the topological type of a reduced plane curve singularity (cf., e.g., [BrK]).

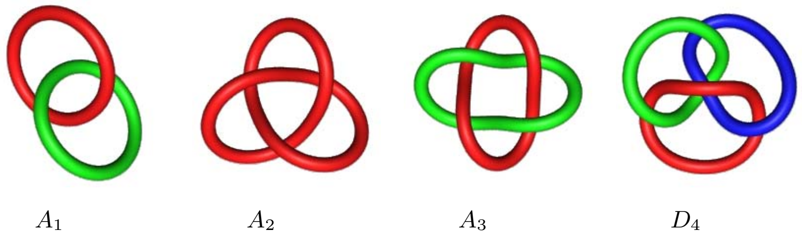


Fig. 3.18. The links, respectively knots, of some simple plane curve singularities.

Though we do not make essential use of Puiseux pairs later in this book, for completeness we shortly present this notion and the notion of characteristic exponents as well as their relation to the topology of the link of a singularity.

Our short discussion follows the lines of [BrK, Chapter III], to which we refer for details and proofs.

Puiseux Pairs and Characteristic Exponents. Let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be a reduced irreducible plane curve germ with isolated singularity of multiplicity m, given by a local equation $f \in \mathbb{C}\{x,y\}$. If $m=1$, the germ $(C,\mathbf{0})$ is non-singular, and no Puiseux pairs and characteristic exponents are defined. So, suppose that $m \geq 2$. Assume that f is y-general of order m and consider its Puiseux expansion $y = \sum_{r\in R} a_r x^r \in \mathbb{C}\{x^{1/m}\}$, $R \subset \frac{1}{m}\cdot\mathbb{Z}_{>0}$, with $a_r \neq 0$, $r \in R$.

Since $m \geq 2$, and since m is the least common multiple of the denominators of the Puiseux exponents $r \in R$, the set $R \setminus \mathbb{Z}$ is non-empty. Choose

$$r_1 = \min\bigl(R \setminus \mathbb{Z}\bigr) = \frac{q_1}{p_1}$$

with coprime integers $q_1 > p_1 > 1$. The pair (p_1,q_1) is called the *first Puiseux pair* of $(C,\mathbf{0})$ (or of f). If $p_1 = m$, we end up with only one Puiseux pair, otherwise we take

$$r_2 = \min\left(R \setminus \frac{1}{p_1}\cdot\mathbb{Z}_{>0}\right) = \frac{q_2}{p_1p_2}$$

with coprime integers $q_2 > p_2 > 1$. The pair (q_2,p_2) is called the *second Puiseux pair* of $(C,\mathbf{0})$.

In general, having defined Puiseux pairs $(p_1,q_1),\dots,(p_k,q_k)$, $k \geq 1$, we look for

$$r_{k+1} = \min\left(R \setminus \frac{1}{p_1\cdots p_k}\cdot\mathbb{Z}_{>0}\right) = \frac{q_{k+1}}{p_1\cdots p_k p_{k+1}}$$

with coprime integers $q_{k+1} > p_{k+1} > 1$, and define the $(k+1)$*-st Puiseux pair* (p_{k+1},q_{k+1}). The process terminates when we come to the common denominator $p_1\cdots p_s = m$ of all the Puiseux exponents $r \in R$.

The Puiseux pairs satisfy the conditions

$$1 < p_1 < q_1, \quad q_k p_{k+1} < q_{k+1}, \quad \gcd(p_k,q_k) = 1, \quad k \geq 1\,. \tag{3.4.1}$$

Conversely, each sequence of pairs of positive integers $(p_1,q_1),\dots,(p_s,q_s)$ obeying (3.4.1), is the sequence of Puiseux pairs for some irreducible curve germ. For example, we may choose the germ with Puiseux expansion

$$y(x) = x^{q_1/p_1} + x^{q_2/(p_1p_2)} + \ldots + x^{q_s/(p_1\cdots p_s)}\,. \tag{3.4.2}$$

Having the Puiseux parameterization $x = t^m$, $y = b_1t^{\alpha_1} + b_2t^{\alpha_2} + \ldots$ of f, where $m \leq \alpha_1 < \alpha_2 < \ldots \in \mathbb{Z}$ and $b_k \neq 0$ for all k, we define the *Puiseux characteristic exponents* of f as follows. Put

$$\Delta_0 = m, \quad \Delta_k = \gcd(m,\alpha_1,\ldots,\alpha_k), \quad k \geq 1\,.$$

This is a non-increasing sequence of positive integers, which stabilizes for some k_0 with $\Delta_k = 1$ for all $k \geq k_0$. We define the sequence of characteristic exponents $\beta_0 < \ldots < \beta_\ell$, setting $\beta_0 = m$ and choosing the other β_i's precisely as the α_k for k satisfying $\Delta_{k-1} > \Delta_k$.

Lemma 3.31. *If $(p_1, q_1), \ldots, (p_s, q_s)$ are the Puiseux pairs and $\beta_0, \ldots, \beta_\ell$ are the characteristic exponents of an irreducible plane curve singularity, then $s = \ell$ and the following relations hold:*

$$\beta_0 = m = p_1 \cdots p_s, \quad \beta_k = \frac{mq_k}{p_1 \cdots p_k}, \ k = 1, \ldots, \ell$$

and, conversely,

$$p_k = \frac{D_{k-1}}{D_k}, \quad q_k = \frac{\beta_k}{D_k}, \ k = 1, \ldots, s\,,$$

where $D_0 = \beta_0$, $D_1 = \gcd(\beta_0, \beta_1)$, $\ldots$, $D_s = \gcd(\beta_0, \ldots, \beta_s) = 1$.

The proof is straightforward and we leave it to the reader.

The Puiseux pairs determine the topology of the knot $\partial X_\varepsilon \subset \partial B_\varepsilon \approx S^3_\varepsilon$ in the following way: Consider for simplicity example (3.4.2). Take the first approximation $y_1 = x^{q_1/p_1}$ to the expansion (3.4.2). The corresponding (oriented) knot $\partial X^{(1)}_\varepsilon \subset S^3_\varepsilon$ is parameterized by

$$x = \left(\frac{\varepsilon}{2}\right)^{1/q_1} e^{p_1\theta\sqrt{-1}}, \quad y_1 = \left(\frac{\varepsilon}{2}\right)^{1/p_1} e^{q_1\theta\sqrt{-1}}, \quad 0 \leq \theta \leq 2\pi\,.$$

This is a torus knot of type (p_1, q_1): it sits in the torus

$$T^{(1)} := \left\{|x| = \left(\frac{\varepsilon}{2}\right)^{1/q_1}\right\} \times \left\{|y| = \left(\frac{\varepsilon}{2}\right)^{1/p_1}\right\} \subset S^3_\varepsilon$$

and makes p_1 (resp. q_1) positive rotations in the direction of the cycle $T^{(1)} \cap \{y = (\varepsilon/2)^{1/p_1}\}$ (resp. $T^{(1)} \cap \{x = (\varepsilon/2)^{1/q_1}\}$). Then we proceed inductively as follows: Suppose that the knot $\partial X^{(k)}_\varepsilon$, defined by the approximation $y_k = x^{q_1/p_1} + \ldots + x^{q_k/(p_1 \cdots p_k)}$ to (3.4.2), is parametrized by

$$x = \varphi_k(\theta_k) e^{p_1 \cdots p_k \theta_k \sqrt{-1}}, \quad y_k = \psi_k(\theta_k), \quad 0 \leq \theta \leq 2\pi\,, \tag{3.4.3}$$

where φ_k is a positive function with image close to $(\varepsilon/2)^{1/q_1}$, and $|\psi_k|$ is close to $(\varepsilon/2)^{1/p_1}$.

The deviation of the next approximation $y_{k+1} = y_k + x^{q_{k+1}/(p_1 \cdots p_{k+1})}$ of y_k satisfies

$$\Delta y := y_{k+1} - y_k = x^{q_{k+1}/(p_1 \cdots p_{k+1})}\,. \tag{3.4.4}$$

Since

$$r_{k+1} = \frac{q_{k+1}}{p_1 \cdots p_{k+1}} > r_k = \frac{q_k}{p_1 \cdots p_k}\,,$$

one can show that the respective knot $\partial X_\varepsilon^{(k+1)}$ lies on the boundary of a small tubular neighbourhood of $\partial X_\varepsilon^{(k)}$ in S^3_ε. This boundary is a torus $T^{(k+1)}$ with axis $\partial X_\varepsilon^{(k)}$. Substituting the expression for x in (3.4.3) into (3.4.4), we resolve the equation (3.4.4) in the form

$$x = \varphi_{k+1}(\theta_{k+1})e^{p_1\cdots p_{k+1}\theta_{k+1}\sqrt{-1}}, \quad \Delta y = \widetilde{\psi}(\theta_{k+1})e^{q_{k+1}\theta_{k+1}\sqrt{-1}} ,$$

where $0 \le \theta_{k+1} \le 2\pi$, that is, $\theta_k = p_{k+1}\theta_k$. Geometrically, this means that $\partial X_\varepsilon^{(k+1)}$ is a torus knot in $T^{(k+1)}$ which makes p_{k+1} positive rotations in the direction of the axis $\partial X_\varepsilon^{(k)}$ and q_{k+1} positive rotations in the orthogonal direction.

Finally, we obtain ∂X_ε as an iterated torus knot. It can be also regarded as a closed positive braid with $m = p_1 \cdots p_s$ strings over the circle $\{|x| = (\varepsilon/2)^{1/q_1},\ y = 0\}$ (cf. [BrK, Section 8.3]).

Remark 3.31.1. Letting $p_{k+1} = 1$ in the above procedure, we obtain a knot $\partial X_\varepsilon^{(k+1)}$ isotopic to $\partial X_\varepsilon^{(k)}$, that is, the non-characteristic exponents of the Puiseux expansion do not contribute to the topology of the link ∂X_ε. In turn, for $p_{k+1} > 1$, the knot $\partial X_\varepsilon^{(k+1)}$ is not equivalent to $\partial X_\varepsilon^{(k)}$ (see [Zar]).

We continue studying further invariants of plane curve singularities.

δ-Invariant. Let $f \in \mathbb{C}\{x, y\}$ be a reduced power series, and let

$$\mathcal{O} = \mathbb{C}\{x, y\}/\langle f\rangle \hookrightarrow \mathbb{C}\{t_1\} \oplus \ldots \oplus \mathbb{C}\{t_r\} = \overline{\mathcal{O}}$$

denote the normalization (cf. p. 198). Then we call

$$\delta(f) := \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O}$$

(identifying $\mathcal{O}$ with its image in $\overline{\mathcal{O}}$) the *δ-invariant* of f.

Example 3.31.2. (a) Let $f = y^2 - x^{2k+1}$ be an A_{2k}-singularity. Then we compute $\delta(f) = \dim_{\mathbb{C}} \mathbb{C}\{t\}/\mathbb{C}\{t^2, t^{2k+1}\} = k$.

(b) Let $f = y^2 - x^{2k}$ be an A_{2k-1}-singularity. It has two irreducible factors, and the normalization is induced by $x \mapsto (t_1, t_2)$, $y \mapsto (t_1^k, -t_2^k)$. A monomial basis of $\overline{\mathcal{O}}/\mathcal{O}$ is given by, for instance, $(1, 0)$, $(t_1, 0), \ldots, (t_1^{k-1}, 0)$. In particular, $\delta(f) = k$.

The following lemma is due to Hironaka [Hir].

Lemma 3.32. *Let $f, g \in \mathbb{C}\{x, y\}$ be two reduced power series which have no factor in common. Then*

$$\delta(fg) = \delta(f) + \delta(g) + i(f, g) .$$

Proof. Since f and g have no common factor, $\langle f\rangle \cap \langle g\rangle = \langle fg\rangle$. Hence, there is an obvious exact sequence

$$0 \to \mathbb{C}\{x,y\}/\langle fg\rangle \to \underbrace{\mathbb{C}\{x,y\}/\langle f\rangle}_{=:\,\mathcal{O}_1} \oplus \underbrace{\mathbb{C}\{x,y\}/\langle g\rangle}_{=:\,\mathcal{O}_2} \to \mathbb{C}\{x,y\}/\langle f,g\rangle \to 0\,,$$

the second map given by $(\varphi,\psi) \to [\varphi-\psi]$. It follows that

$$\begin{aligned}
\delta(fg) &= \dim_{\mathbb{C}}\Big(\bigoplus_{i=1}^{r}\mathbb{C}\{t_i\}\Big)\Big/\big(\mathbb{C}\{x,y\}/\langle fg\rangle\big)\\
&= \dim_{\mathbb{C}}\Big(\bigoplus_{i=1}^{r'+r''}\mathbb{C}\{t_i\}\Big)\Big/(\mathcal{O}_1\oplus\mathcal{O}_2) + \dim_{\mathbb{C}}(\mathcal{O}_1\oplus\mathcal{O}_2)\Big/\big(\mathbb{C}\{x,y\}/\langle fg\rangle\big)\\
&= \dim_{\mathbb{C}}\Big(\bigoplus_{i=1}^{r'}\mathbb{C}\{t_i\}\Big)\Big/\mathcal{O}_1 + \dim_{\mathbb{C}}\Big(\bigoplus_{i=1}^{r''}\mathbb{C}\{t_i\}\Big)\Big/\mathcal{O}_2 + \dim_{\mathbb{C}}\mathbb{C}\{x,y\}\Big/\langle f,g\rangle\\
&= \delta(f)+\delta(g)+i(f,g)\,,
\end{aligned}$$

the latter due to Proposition 3.12. □

We start the computation of $\delta(f)$ by computing the *Hilbert-Samuel function* of the local ring $\mathcal{O}=\mathbb{C}\{x,y\}/\langle f\rangle$,

$$H^1_{\mathcal{O}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}\,, \quad d \mapsto \dim_{\mathbb{C}} \mathcal{O}/\mathfrak{m}^{d+1}\,,$$

where $\mathfrak{m}\subset\mathcal{O}$ denotes the maximal ideal. In the case of hypersurface singularities $\mathcal{O}$ the computation of $H^1_{\mathcal{O}}$ is just an easy exercise:

Lemma 3.33. *Let $f\in\mathbb{C}\{x,y\}$ and $m=\operatorname{mt}(f)$ its multiplicity. Then*

$$H^1_{\mathcal{O}}(d-1) = \dim_{\mathbb{C}}\mathbb{C}\{x,y\}\big/\big\langle f,\mathfrak{m}^d\big\rangle \;=\; md-\frac{m(m-1)}{2}\,,$$

for all $d\geq m$.

Proof. As $f\in\mathfrak{m}^m\setminus\mathfrak{m}^{m+1}$, we have an obvious exact sequence

$$0 \to \mathbb{C}\{x,y\}/\langle x,y\rangle^{d-m} \xrightarrow{\cdot f} \mathbb{C}\{x,y\}/\langle x,y\rangle^{d} \to \mathbb{C}\{x,y\}\big/\big\langle f,\mathfrak{m}^d\big\rangle \to 0\,,$$

and the statement follows since $\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\langle x,y\rangle^k = k(k+1)/2$. □

Proposition 3.34. *Let $f\in\mathfrak{m}\subset\mathbb{C}\{x,y\}$ be reduced. Then the δ-invariant of f can be computed as*

$$\delta(f) = \sum_q \frac{m_q(m_q-1)}{2}\,.$$

Here the sum extends over all points infinitely near to **0** *appearing when resolving the plane curve singularity $\{f=0\}$, and m_q denotes the multiplicity of the strict transform $f_{(q)}$ of f at q.*

Proof. Let $\mathcal{O} = \mathbb{C}\{x,y\}/\langle f\rangle$ and consider the increasing sequence of k-th neighbourhood rings $\mathcal{O} \hookrightarrow \mathcal{O}^{(1)} \hookrightarrow \mathcal{O}^{(2)} \hookrightarrow \ldots \hookrightarrow \mathcal{O}^{(k)} = \overline{\mathcal{O}}$ as introduced above. Then, by the definition of $\mathcal{O}^{(k)}$ (see (3.3.5)) and proceeding by induction, it suffices to show

$$\dim_{\mathbb{C}} \mathcal{O}^{(1)}/\mathcal{O} \;=\; \frac{m(m-1)}{2}\,, \qquad m = \mathrm{mt}(f)\,. \tag{3.4.5}$$

Consider the above morphism (3.3.3),

$$\mathbb{C}\{x,y\}/\langle f\rangle = \mathcal{O} \xrightarrow{\phi} \mathcal{O}^{(1)} = \bigoplus_{i=1}^{s} \mathbb{C}\{u_i,v_i\}\Big/\Big\langle \widetilde{f}(u_i,v_i)\Big\rangle\,,$$

where $\widetilde{f}(u_i,v_i)$ is (a local equation of) the strict transform of f at the point $q_i \in E$. Since $\dim_{\mathbb{C}} \mathcal{O}^{(1)}/\mathcal{O} \le \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O} < \infty$, there is some $d \ge m$ such that $\mathfrak{m}^d\mathcal{O}^{(1)} \subset \mathcal{O}$, $\mathfrak{m} = \langle x,y\rangle$ (we can even choose $d = m$ by Exercise 3.3.4). We obtain an exact sequence of finite dimensional complex vector spaces

$$0 \to \mathbb{C}\{x,y\}/\langle f, \mathfrak{m}^d\rangle \to \bigoplus_{i=1}^{s} \mathbb{C}\{u_i,v_i\}\Big/\Big\langle \widetilde{f}(u_i,v_i),\, u_i^d\Big\rangle \to \mathcal{O}^{(1)}/\mathcal{O} \to 0\,,$$

where the injectivity of the first map $\mathcal{O}/\mathfrak{m}^d \to \mathcal{O}^{(1)}/\mathfrak{m}^d\mathcal{O}^{(1)}$, $x \mapsto (u_1,\ldots,u_s)$, $y \mapsto (u_1v_1,\ldots,u_sv_s)$ is a consequence of Lemma 3.24. This allows to compute (cf. Proposition 3.12 and Lemma 3.33)

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{O}^{(1)}/\mathcal{O} &= d\cdot \sum_{i=1}^{s} i\big(\widetilde{f}(u_i,v_i),u_i\big) - \left(md - \frac{m(m-1)}{2}\right)\\ &= dm - \left(md - \frac{m(m-1)}{2}\right) \;=\; \frac{m(m-1)}{2}\,.\end{aligned}$$

□

It turns out that the δ-invariant is closely related to the Milnor number of a reduced plane curve singularity (as introduced in Section 2). More precisely, if we fix the number of branches of the singularity, the δ-invariant and the Milnor number determine each other. The following formula is due to Milnor [Mil1]. It also holds for arbitrary reduced (not necessarily plane) curve singularities (cf. [BuG]).

Proposition 3.35. *Let $f \in \mathfrak{m} \subset \mathbb{C}\{x,y\}$ be reduced. Then*

$$\mu(f) = 2\delta(f) - r(f) + 1\,, \tag{3.4.6}$$

where $r(f)$ denotes the number of irreducible factors of f.

Before proving this proposition, we introduce polar curves.

Definition 3.36. Let $f \in \mathbb{C}\{x,y\}$ and $(\alpha:\beta) \in \mathbb{P}^1$, then we call

$$P_{(\alpha:\beta)}(f) := \alpha \cdot \frac{\partial f}{\partial x} + \beta \cdot \frac{\partial f}{\partial y} \in \mathbb{C}\{x,y\}$$

the *polar of* f *with respect to* $(\alpha:\beta)$ (or with respect to the line defined by $\ell_{(\alpha:\beta)} := \alpha x + \beta y$). Note that it is defined only up to a non-zero constant.

If $f \in \mathfrak{m}^2$ has an isolated singularity then $P_{(\alpha:\beta)}(f) \in \mathfrak{m} \setminus \{0\}$ defines a plane curve singularity, the *polar curve of* f *w.r.t.* $(\alpha:\beta)$. $P_{(\alpha:\beta)}(f)$ is called a *generic polar* of f if $(\alpha:\beta)$ is generically chosen in $\mathbb{P}^1$.

The polar $P_{(\alpha:\beta)}(f)$ is of interest not only for germs but also for affine curves, that is, for $f \in \mathbb{C}[x,y]$.

If $\varphi(x,y) = (\alpha x + \gamma y, \beta x + \delta y)$, $\alpha\delta - \beta\gamma \neq 0$, is a linear coordinate transformation, then

$$\frac{\partial(f \circ \varphi)}{\partial x} = \alpha \cdot \frac{\partial f}{\partial x} \circ \varphi + \beta \cdot \frac{\partial f}{\partial y} \circ \varphi\,,$$

hence $P_{(1:0)}(f \circ \varphi) = P_{(\alpha:\beta)}(f) \circ \varphi$, more generally

$$P_{(\alpha:\beta)}(f \circ \varphi) = P_{\varphi(\alpha:\beta)}(f) \circ \varphi\,.$$

The following useful lemma relates the Milnor number to the intersection multiplicities of $P_{(\alpha:\beta)}$ and f, respectively the line $\{\ell_{(-\beta:\alpha)} = 0\}$ orthogonal to $\{\ell_{(\alpha:\beta)} = 0\}$.

Lemma 3.37. *Let* $f \in \mathbb{C}\{x,y\}$ *and* $(\alpha:\beta) \in \mathbb{P}^1$, *then*

$$\begin{aligned} i\left(f,\, \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}\right) &= \mu(f) + i\left(-\beta x + \alpha y,\, \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}\right) \\ &= \mu(f) + i\,(-\beta x + \alpha y,\, f) - 1\,. \end{aligned}$$

In particular, the difference $i\big(f, P_{(\alpha:\beta)}(f)\big) - i\big(\ell_{(-\beta:\alpha)}, P_{(\alpha:\beta)}(f)\big)$ *is independent of the chosen point* $(\alpha:\beta) \in \mathbb{P}^1$.

Proof. Let $\varphi : (x,y) \mapsto (\alpha x + \gamma y, \beta x + \delta y)$, $\alpha\delta - \beta\gamma \neq 0$, be a linear coordinate transformation. Then, by the above, $P_{(\alpha:\beta)}(f) \circ \varphi = P_{(1:0)}(f \circ \varphi)$. Moreover, $\ell_{(-\beta:\alpha)} \circ \varphi = (\alpha\delta - \beta\gamma)\ell_{(0:1)}$ and, as for any change of coordinates, $i(f \circ \varphi, g \circ \varphi) = i(f,g)$ and $\mu(f \circ \varphi) = \mu(f)$. Hence, after replacing $f \circ \varphi$ by f, we have to show

$$i\left(f, \frac{\partial f}{\partial x}\right) = \underbrace{i\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}_{=\mu(f)} + i\left(y, \frac{\partial f}{\partial x}\right)\,. \tag{3.4.7}$$

If $\frac{\partial f}{\partial x} = 0$ then both sides are infinite, if $\frac{\partial f}{\partial x} \equiv \text{const} \neq 0$ then both sides are 0. Thus, we may assume that $f \in \mathfrak{m}^2$.

Moreover, since both sides of (3.4.7) are additive with respect to branches of $\frac{\partial f}{\partial x}$, it suffices to show the equality for each branch of $\frac{\partial f}{\partial x}$ in place of $\frac{\partial f}{\partial x}$.

Let $t \mapsto \big(x(t), y(t)\big)$ be a parametrization of any branch of $\frac{\partial f}{\partial x}$. Then we have $\frac{\partial f}{\partial x}\big(x(t), y(t)\big) = 0$, and obtain

$$\begin{aligned} i\left(f, \frac{\partial f}{\partial x}\right) &= \operatorname{ord}\Big(f\big(x(t), y(t)\big)\Big) \ = \ \operatorname{ord}\left(\frac{d}{dt} f\big(x(t), y(t)\big)\right) + 1 \\ &= \operatorname{ord}\left(\frac{\partial f}{\partial y}\big(x(t), y(t)\big) \cdot \frac{d}{dt}\big(y(t)\big)\right) + 1 \ = \ i\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right) + i\left(y, \frac{\partial f}{\partial x}\right), \end{aligned}$$

which also holds if one of the sides is infinite. Furthermore,

$$\begin{aligned} i\left(y, \frac{\partial f}{\partial x}\right) &= \operatorname{ord}\left(\frac{\partial f}{\partial x}(x, 0)\right) = \operatorname{ord}\left(\frac{\partial (f(x,0))}{\partial x}\right) \\ &= \operatorname{ord}\big(f(x, 0)\big) - 1 = i(y, f) - 1\,. \end{aligned}$$

□

The following example shows that the two intersection multiplicities in the preceding lemma vary for different polars, though their difference is constant.

Example 3.37.1. Let $f = y^2 + x^2y + x^5$ then $i\big(f, \frac{\partial f}{\partial x}\big) = 7,\ i\big(y, \frac{\partial f}{\partial x}\big) = 4$, while $i\big(f, \frac{\partial f}{\partial y}\big) = 4,\ i\big(x, \frac{\partial f}{\partial y}\big) = 1$, the difference in both cases being $3 = \mu(f)$. In the following Singular session, these numbers are computed:

```
ring r = 0,(x,y),ds;
poly f = y2+x2y+x5;
poly p1 = diff(f,x);         // first polar df/dx
poly p2 = diff(f,y);         // second polar df/dy
vdim(std(ideal(f,p1)));      // intersection multiplicity of f and p1
//-> 7
vdim(std(ideal(y,p1)));      // intersection multiplicity of y and p1
//-> 4
vdim(std(ideal(f,p2)));      // intersection multiplicity of f and p2
//-> 4
vdim(std(ideal(x,p2)));      // intersection multiplicity of x and p2
//-> 1
LIB "sing.lib";
milnor(f);                   // Milnor number of f
//-> 3
```

Proof of Proposition 3.35. If f defines a smooth germ, that is, $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, then both sides of (3.4.6) vanish, and the statement holds true. Let $f \in \mathfrak{m}^2$.

Step 1. Assume first that f is irreducible. Then y does not divide f, hence it does not divide $\frac{\partial f}{\partial x}$ and, by Lemma 3.37, the claimed equality (3.4.6) is equivalent to

$$i_{\mathbf{0}}\left(f, \frac{\partial f}{\partial x}\right) - i_{\mathbf{0}}\left(y, \frac{\partial f}{\partial x}\right) = 2\delta(f)\,, \tag{3.4.8}$$

where the left-hand side remains unchanged when replacing (x, y) by (y, x). In particular, we may assume in the following that y is not the (unique) tangent of f.

We prove (3.4.8) by induction on the number of blowing ups needed to resolve the singularity of f, the induction base being given by the case of a smooth germ.

Let $m := \mathrm{mt}(f)$. Then $\mathrm{mt}\big(\frac{\partial f}{\partial x}\big) = m - 1$, and we can use the recursive formula in Proposition 3.21 to compute the intersection multiplicity:

$$i_{\mathbf{0}}\left(f, \frac{\partial f}{\partial x}\right) = m(m-1) + i_q\left(\widetilde{f}, \widetilde{\frac{\partial f}{\partial x}}\right),$$

where $\widetilde{f}$, respectively $\widetilde{\frac{\partial f}{\partial x}}$, are local equations of the strict transform of $V(f)$, respectively $V\big(\frac{\partial f}{\partial x}\big)$, at the unique point $q \in E$ corresponding to the unique tangent of f. Considering the blowing up map in the second chart (containing q), that is, in local coordinates $(u, v) \mapsto (uv, v)$, we compute

$$\widetilde{\frac{\partial f}{\partial x}} = \frac{\frac{\partial f}{\partial x}(uv, v)}{v^{m-1}} = \frac{\partial}{\partial u}\left(\frac{f(uv, v)}{v^m}\right) = \frac{\partial \widetilde{f}}{\partial u}.$$

Applying the induction hypothesis to $\widetilde{f}$ and Proposition 3.34, we get

$$i_{\mathbf{0}}\left(f, \frac{\partial f}{\partial x}\right) - i_q\left(v, \frac{\partial \widetilde{f}}{\partial u}\right) = m(m-1) + 2\delta\big(\widetilde{f}\big) = 2\delta(f).$$

Since the tangent cone of f is of the form $(x - \alpha y)^m$, we obtain

$$i_q\left(v, \frac{\partial \widetilde{f}}{\partial u}\right) = m - 1 = i_{\mathbf{0}}\left(y, \frac{\partial f}{\partial x}\right)$$

and, hence, (3.4.8).

Step 2. Now assume that f decomposes as $f = f_1 \cdot \ldots \cdot f_r$ with f_i irreducible. Then we obtain

$$\begin{aligned} i\left(f, \frac{\partial f}{\partial x}\right) &= \sum_{j=1}^{r}\left(i\left(f_j, \frac{\partial f_j}{\partial x}\right) + \sum_{i \neq j} i\big(f_j, f_i\big)\right) \\ &= \sum_{j=1}^{r}\left(2\delta(f_j) + \sum_{i \neq j} i\big(f_j, f_i\big)\right) + \sum_{j=1}^{r} i\left(y, \frac{\partial f_j}{\partial x}\right) \\ &= 2\delta(f) + i\left(y, \frac{\partial f}{\partial x}\right) - r + 1, \end{aligned}$$

due to Step 1, respectively Lemma 3.32. Finally, we conclude (3.4.6) by applying Lemma 3.37. □

κ-Invariant. As before, let $f \in \mathfrak{m} \subset \mathbb{C}\{x,y\}$ be a reduced power series. We define the *κ-invariant* of f as the intersection multiplicity of f with a generic polar, that is,

$$\kappa(f) := i\left(f,\, \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}\right), \quad (\alpha:\beta) \in \mathbb{P}^1 \text{ generic}.$$

The following proposition is a consequence of Lemma 3.37.

Proposition 3.38. *Let $f \in \mathbb{C}\{x,y\}$ be a reduced power series. Then*

$$\kappa(f) = \mu(f) + \mathrm{mt}(f) - 1\,.$$

Example 3.38.1. We check the formula of the preceeding proposition in the case $f = (x^2 - y^3)\cdot(x^3 - y^5)$, by using SINGULAR.

```
LIB "sing.lib";
ring r = 0,(x,y),ds;
poly f = (x2-y3)*(x3-y5);
```

We define a generic polar of f by taking a random linear combination p of the partials:

```
poly p = random(1,100)*diff(f,x) + random(1,100)*diff(f,y); p;
//-> 225x4-21x3y2-135x2y3-35x2y4-90xy5+56y7
```

Note that the coefficients of p vary for every new call of `random`. Finally, we compute the κ-invariant, respectively the right-hand side of the above formula:

```
vdim(std(ideal(f,p)));       // the kappa invariant
//-> 31
milnor(f)+ord(f)-1;          // right-hand side of formula
//-> 31
```

Corollary 3.39. *Let $f \in \mathbb{C}\{x,y\}$ be reduced with irreducible factorization $f = f_1 \cdot \ldots \cdot f_s$. Then*

$$\kappa(f) = \sum_{j=1}^{s} \kappa(f_j) + \sum_{j\neq k} i(f_j, f_k)\,.$$

Proof. By Propositions 3.38, 3.4.6 and Lemma 3.32, we have

$$\begin{aligned}
\kappa(f) &= \mu(f) + \mathrm{mt}(f) - 1 \;=\; 2\delta(f) - r(f) + \mathrm{mt}(f) \\
&= 2\cdot\left(\sum_{j=1}^{s}\delta(f_j) + \sum_{j<k} i(f_j,f_k)\right) - r(f) + \mathrm{mt}(f) \\
&= \sum_{j=1}^{s}\big(2\delta(f_j) - r(f_j) + \mathrm{mt}(f_j)\big) + \sum_{j\neq k} i(f_j,f_k) \\
&= \sum_{j=1}^{s}\kappa(f_j) + \sum_{j\neq k} i(f_j,f_k)\,.
\end{aligned}$$

$\square$

The following lemma generalizes the fact that $\kappa(f) > \tau(f)$ to give a bound for the intersection multiplicity of f with any $g \in \mathbb{C}\{x,y\}$.

Lemma 3.40. *Let $f \in \mathbb{C}\{x,y\}$ be a reduced power series and $\langle f, j(f)\rangle$ the Tjurina ideal of f. Then, for any $g \in \mathfrak{m} \subset \mathbb{C}\{x,y\}$,*

$$i(f,g) > \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f, j(f), g\rangle \,.$$

Proof. Since $i(f,g) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f,g\rangle$, our claim is just that there exist $a,b \in \mathbb{C}\{x,y\}$ such that $a \cdot (\partial f/\partial x) + b \cdot (\partial f/\partial y) \notin \langle f,g\rangle$. Assume the contrary. Then, in particular, there exist $a_1, a_2, b_1, b_2 \in \mathbb{C}\{x,y\}$ such that

$$\frac{\partial f}{\partial x} = a_1 f + b_1 g \,, \qquad \frac{\partial f}{\partial y} = a_2 f + b_2 g \,.$$

Case 1. If $g \in \mathfrak{m} \setminus \mathfrak{m}^2$ then, after an analytic change of coordinates, we can assume $g = x$, and the above equality yields

$$\frac{\partial f}{\partial y}(0,y) = a_2(0,y) f(0,y) \,,$$

which is only possible if both sides are 0, that is, iff $f = x \cdot f'$ for some $f' \in \mathbb{C}\{x,y\}$. Together with the above this implies

$$f' = \frac{\partial f}{\partial x} - x \cdot \frac{\partial f'}{\partial x} = x \cdot \left(a_1 f' + b_1 - \frac{\partial f'}{\partial x} \right) .$$

Hence, $f = x^2 \cdot f''$ for some $f'' \in \mathbb{C}\{x,y\}$ contradicting the assumption that f is reduced.

Case 2. If $g \in \mathfrak{m}^2$ then, after an analytic change of coordinates, we can assume that x is not a factor of f and that g is y-general of order N. Then the Weierstraß preparation theorem and Proposition 3.4 (b) give a decomposition

$$g = \text{unit} \cdot \prod_{i=1}^{N} \Big(y - \varphi_i\big(x^{1/m_i}\big) \Big)$$

for some convergent power series $\varphi_i \in \langle t\rangle \cdot \mathbb{C}\{t\}$. Let m be the least common multiple of the exponents m_i, $i = 1, \ldots, N$. Then $g(x^m, y) \in \mathbb{C}\{x,y\}$ is divisible by some $g' \in \mathfrak{m} \setminus \mathfrak{m}^2$. Defining $F := f(x^m, y) \in \mathbb{C}\{x,y\}$, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial x} &= m x^{m-1} \frac{\partial f}{\partial x} = m x^{m-1} a_1(x^m,y) f(x^m,y) + m x^{m-1} b_1(x^m,y) g(x^m,y) \,, \\
\frac{\partial F}{\partial y} &= \frac{\partial f}{\partial y}(x^m,y) = a_2(x^m,y) f(x^m,y) + b_2(x^m,y) g(x^m,y) \,,
\end{aligned}$$

which give equations $\partial F/\partial x = A_1 F + B_1 g'$, $\partial F/\partial y = A_2 F + B_2 g'$ for some $A_1, A_2, B_1, B_2 \in \mathbb{C}\{x,y\}$. Since $(x,y) \mapsto (x^m, y)$ defines a finite map φ, the composition $F = f \circ \varphi$ is reduced and we get a contradiction as shown in Case 1. $\square$

Semigroup and Conductor. Again, let $f \in \mathbb{C}\{x,y\}$ be a reduced power series, and let $\mathcal{O} := \mathbb{C}\{x,y\}/\langle f\rangle \overset{\varphi}{\hookrightarrow} \bigoplus_{i=1}^r \mathbb{C}\{t_i\} =: \overline{\mathcal{O}}$ denote the normalization, that is, $t_i \mapsto \big(\varphi_i(x)(t_i), \varphi_i(y)(t_i)\big)$ is a parametrization of the i-th branch of the plane curve singularity $\big(V(f), \mathbf{0}\big)$. Then we introduce the product of valuation maps

$$\boldsymbol{v} := (v_1,\dots,v_r) \,:\, \mathcal{O} \longrightarrow \mathbb{Z}^r_{\geq 0}\,, \quad g \longmapsto \big(\mathrm{ord}_{t_i}\, g(\varphi_i(x),\varphi_i(y))\big)_{i=1,\dots,r}\,.$$

Its image $\Gamma(\mathcal{O}) := \boldsymbol{v}(\mathcal{O})$ is a semigroup, called the *semigroup of values* of f. We call the minimal element $\boldsymbol{c} \in \Gamma(\mathcal{O})$ satisfying $\boldsymbol{c} + \mathbb{Z}^r_{\geq 0} \subset \Gamma(\mathcal{O})$ the *conductor* of $\Gamma(\mathcal{O})$, denoted by $\mathrm{cd}(\mathcal{O})$, or $\mathrm{cd}(f)$,

$$\mathrm{cd}(f) = \big(\mathrm{cd}(f)_1,\dots,\mathrm{cd}(f)_r\big)\,.$$

Finally, we define the *conductor ideal* of f (respectively $\mathcal{O}$),

$$I^{cd}(f) := I^{cd}(\mathcal{O}) := \mathrm{Ann}_{\mathcal{O}}\big(\overline{\mathcal{O}}/\mathcal{O}\big) \subset \mathcal{O}\,.$$

Note that $\varphi(I^{cd}(f)) = \big\{g \in \overline{\mathcal{O}} \;\big|\; g\overline{\mathcal{O}} \subset \mathcal{O}\big\}$ is an $\overline{\mathcal{O}}$-ideal. Since $\overline{\mathcal{O}}$ is a principal ideal ring, $\varphi(I^{cd}(f))$ is generated as $\overline{\mathcal{O}}$-ideal by one element. Indeed, it is generated by $\big(t_1^{\mathrm{cd}(f)_1},\dots,t_r^{\mathrm{cd}(f)_r}\big)$.
We recall two facts (see [HeK, Del]).

(1) The semigroup is *symmetric* in the following sense: $\boldsymbol{\alpha} \in \mathbb{Z}^r_{\geq 0}$ is an element of $\Gamma(\mathcal{O})$ iff $\big(\mathrm{cd}(f) - \mathbf{1}\big) - \boldsymbol{\alpha}$ is a *maximal* in $\Gamma(\mathcal{O})$, that is,

$$\big\{\boldsymbol{\beta} \in \Gamma(\mathcal{O}) \;\big|\; \beta_i = \mathrm{cd}(f)_i - 1 - \alpha_i\,,\ \beta_j > \mathrm{cd}(f)_j - 1 - \alpha_j \text{ for } j \neq i\big\} = \emptyset$$

for all $i = 1,\dots,r$. In particular, if $f \in \mathbb{C}\{x,y\}$ is irreducible then

$$\alpha \in \Gamma(\mathcal{O}) \iff \big(\mathrm{cd}(f) - 1\big) - \alpha \notin \Gamma(\mathcal{O})\,. \tag{3.4.9}$$

(2) Let $f = f_1 \cdot \ldots \cdot f_r$ be the irreducible decomposition. Then

$$\mathrm{cd}(f)_k = \mathrm{cd}(f_k) + \sum_{j\neq k} i(f_k, f_j)\,, \quad k = 1,\dots,r\,.$$

It follows that

$$\mathrm{cd}(f) \;=\; \left(2\delta(f_1) + \sum_{j\neq 1} i(f_1,f_j),\dots,2\delta(f_r) + \sum_{j\neq r} i(f_r,f_j)\right)\,. \tag{3.4.10}$$

In particular, for any reduced power series $f \in \mathbb{C}\{x,y\}$,

$$\dim_{\mathbb{C}} \mathcal{O}/I^{cd}(f) = \delta(f)\,, \tag{3.4.11}$$
$$\dim_{\mathbb{C}} \overline{\mathcal{O}}/I^{cd}(f) = \mathrm{cd}(f)_1 + \ldots + \mathrm{cd}(f)_r = 2\delta(f)\,. \tag{3.4.12}$$

Example 3.40.1. (1) $\mathrm{cd}(y^2 - x^{2k+1}) = 2k$,
(2) $\mathrm{cd}(y^2 - x^{2k}) = (k, k)$,
(3) $\mathrm{cd}(y^m - x^m) = (m-1, \ldots, m-1)$.

System of Multiplicity Sequences. Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0}) =: M^{(0)}$ be a singular reduced plane curve germ, and let $\pi_i : M^{(i)} \to M^{(i-1)}$, $i = 1, \ldots, N$, be the blowing up maps introduced in the proof of the desingularization theorem.

(1) Assume that $(C, \mathbf{0})$ is irreducible, and denote by $q^{(i)}$ the unique intersection point of the strict transform $C^{(i)}$ and the exceptional divisor $E^{(i)}$, $i \geq 1$. Then the sequence of positive integers $\big(m_0, m_1, \ldots, m_{n-1}\big)$, $m_0 := \mathrm{mt}(C, \mathbf{0})$, $m_i := \mathrm{mt}(C^{(i)}, q^{(i)})$, $i \geq 1$, $m_n = 1$, is called the *multiplicity sequence* of $(C, \mathbf{0})$.
(2) Assume that $(C, \mathbf{0})$ has the irreducible components $(C_1, \mathbf{0}), \ldots, (C_r, \mathbf{0})$. Then the *system of multiplicity sequences* of $(C, \mathbf{0})$ is given by the following data: For each branch $(C_j, \mathbf{0})$ the extended multiplicity sequence[24]

$$\big(m_{j,0}, m_{j,1}, \ldots, m_{j,n_j-1}, 1, 1, \ldots\big)\,, \quad j = 1, \ldots, r\,,$$

(respectively $(1, 1, \ldots)$ for a smooth branch) together with partitions $\mathcal{P}_i$ of the sets $\big\{(1, i), \ldots, (r, i)\big\}$, $i \geq 0$, defined as follows: (j, i) and (k, i) belong to the same subset iff the strict transforms of $(C_j, \mathbf{0})$ and $(C_k, \mathbf{0})$ intersect in $M^{(i)}$.

The system of multiplicity sequences of a reduced plane curve germ can be illustrated in a diagram as shown in the following

Example 3.40.2. Let $f = (x^2 - y^3)(x^3 - y^5)$ (cf. Figure 3.17 on p. 192). Then the multiplicity sequences of the two branches are $(2, 1, 1)$, respectively $(3, 2, 1, 1)$, and the system of multiplicity sequences can be illustrated in the diagram

$$\boxed{\begin{matrix}2\\3\end{matrix}}\begin{matrix}\text{---}\\\text{---}\end{matrix}\boxed{\begin{matrix}1\\2\end{matrix}}\begin{matrix}\text{---}\\\text{---}\end{matrix}\boxed{\begin{matrix}1\\1\end{matrix}}\begin{matrix}\text{---}\\\text{---}\end{matrix}\begin{matrix}1\\1\end{matrix}\begin{matrix}\text{---}\\\text{---}\end{matrix}\begin{matrix}\cdots\\\cdots\end{matrix}$$

If we consider $g = (y^2 - x^3)(x^3 - y^5)$ instead of f then the multiplicity sequences of the branches are just the same as before, while the partitions change, as illustrated by

$$\boxed{\begin{matrix}2\\3\end{matrix}}\begin{matrix}\text{---}\\\text{---}\end{matrix}\begin{matrix}1\\2\end{matrix}\begin{matrix}\text{---}\\\text{---}\end{matrix}\begin{matrix}1\\1\end{matrix}\begin{matrix}\text{---}\\\text{---}\end{matrix}\begin{matrix}1\\1\end{matrix}\begin{matrix}\text{---}\\\text{---}\end{matrix}\begin{matrix}\cdots\\\cdots\end{matrix}$$

[24] Since additional entries 1 in the extended multiplicity sequence do not give more information, we may consider the extended multiplicity sequence of C_j as infinitely long, or as a sequence of length $\max\{n_j - 1 \mid j = 1, \ldots, r\}$, where n_j is the smallest index with $m_{j,n_j} = 1$.

There are many data equivalent to the system of multiplicity sequences. We just should like to mention

(a) the resolution graph;
(b) the Puiseux pairs of the branches and the pairwise intersection numbers;
(c) the characteristic exponents of the branches and the pairwise intersection numbers;
(d) the iterated torus knots corresponding to the branches and their linking numbers.

In the following, we restrict ourselves to proving the equivalence of the system of multiplicity sequences and the data in (c) (or (b), see Lemma 3.31). For the definition of the resolution graph as well as for more details and proofs, we refer to the textbooks [BrK, DJP, EiN].

Proposition 3.41. *The system of multiplicity sequences of a plane curve germ with isolated singularity determines and is determined by the characteristic exponents of the branches and their pairwise intersection numbers.*

Proof. Since the intersection numbers determine and are determined by the partition set (see Proposition 3.21), we have to study only the case of an irreducible curve germ. We proceed by induction taking the non-singular germ case as base, whereas the induction step actually reduces to the claim that the characteristic exponents of the blown-up germ and its intersection number with the exceptional divisor are determined by the characteristic exponents of the original germ.

Let $\beta_0, \dots, \beta_s$ ($\beta_0 \geq 2$, $s \geq 1$) be the sequence of characteristic exponents of the given plane curve germ, parametrized by

$$x = t^{\beta_0}, \quad y = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \dots, \quad 0 < \beta_0 < \alpha_1 < \alpha_2 < \dots, \quad a_k \neq 0\,.$$

We represent the blow up by a transformation $x := x$, $y := xy$ and obtain a parameterization of the blown-up germ in the form

$$x = t^{\beta_0}, \quad y = a_1 t^{\alpha_1 - \beta_0} + a_2 t^{\alpha_2 - \beta_0} + \dots \tag{3.4.13}$$

If $\beta_1 > 2\beta_0$ then $\alpha_1 \geq 2\beta_0$ and, thus, (3.4.13) represents the Puiseux expansion of the blown up germ. That is, its characteristic exponents are

$$\beta_0,\ \beta_1 - \beta_0,\ \dots,\ \beta_s - \beta_0\,,$$

and the intersection number with the exceptional divisor $E = \{x = 0\}$ is β_0.

Assume now that $\beta_0 < \beta_1 < 2\beta_0$ [25]. Then $\beta_1 = \alpha_1$, the multiplicity of the blown-up germ is $\beta_1 - \beta_0$, and its intersection number with E is again β_0. We reparametrize the blown up germ by setting $y = a_1 \theta^{\beta_1 - \beta_0}$, that is,

[25] Observe that equalities are excluded, since $\frac{\beta_1}{\beta_0} \notin \mathbb{Z}$.

$$t^{\beta_1-\beta_0} + \sum_{k\geq 2} \frac{a_k}{a_1} t^{\alpha_k-\beta_0} = \theta^{\beta_1-\beta_0} \implies \theta = t\Big(1+\sum_{k\geq 2}\frac{a_k}{a_1}t^{\alpha_k-\beta_1}\Big)^{1/(\beta_1-\beta_0)}.$$

Hence,

$$t = \theta \left(1 + \sum_{k\geq 2} \frac{-a_k}{a_1(\beta_1-\beta_0)}\theta^{\alpha_k-\beta_1} + \Phi_1(\theta)\right),$$

where the exponents of θ in Φ_1 are sums of at least two positive exponents of the preceding terms. Subsequently, we have

$$x = \theta^{\beta_0} + \sum_{k\geq 2} \frac{-a_k\beta_0}{a_1(\beta_1-\beta_0)}\theta^{\alpha_k-\beta_1+\beta_0} + \Phi_2(\theta),$$

where the exponents of θ in Φ_2 are of the form $\beta_0 + \sum_{\ell=2}^k j_\ell(\alpha_\ell-\beta_1)$ with $\sum_{\ell=2}^k j_\ell \geq 2$. Since

$$\begin{aligned}&\gcd(\beta_1-\beta_0, \beta_0, \alpha_2-\beta_1+\beta_0, \dots, \alpha_k-\beta_1+\beta_0)\\ &\quad = \gcd\left(\beta_1-\beta_0, \beta_0, \alpha_2-\beta_1+\beta_0, \dots, \alpha_k-\beta_1+\beta_0, \beta_0+\sum_{\ell=2}^k j_\ell(\alpha_\ell-\beta_1)\right),\end{aligned}$$

Φ_2 does not affect the computation of the characteristic exponents. These appear then as

$$\beta_1-\beta_0,\ (\beta_0),\ \beta_2-\beta_1+\beta_0,\ \dots,\ \beta_s-\beta_1+\beta_0,$$

where β_0 is omitted iff $\beta_1-\beta_0$ divides β_0. □

The following, classical result due to K. Brauner [Bra] and O. Zariski [Zar] (see also [BrK, 8.4, Thm. 21]) is fundamental for our treatment of topological singularity types:

Theorem 3.42. *The topological type of a reduced plane curve singularity $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ is completely determined by, and it determines, the system of multiplicity sequences.*

As an immediate corollary, we obtain:

Corollary 3.43. *The multiplicity* mt*, the Milnor number μ, the κ-invariant, the δ-invariant, and the conductor (of the semigroup) are topological invariants of reduced plane curve singularities.*

Proof. δ is a topological invariant due to Theorem 3.42 and Propositions 3.42, 3.34. Since also the number of local branches is a topological invariant, so is the Milnor number (Proposition 3.35). Due to Proposition 3.38, the same holds true for the κ-invariant. Finally, the conductor is a topological invariant, due to formula (3.4.10). □

The *Tjurina number* τ *is not a topological invariant.* This can be seen by considering the following

Example 3.43.1. (a) Let $f = y^3 - x^7$, and let $g \in I = \langle x^\alpha y^\beta \mid 3\alpha + 7\beta > 21\rangle$. Then f and $f+g$ are topologically equivalent, since in both cases the multiplicity sequence reads

$$3 \text{ —— } 3 \text{ —— } 1 \text{ —— } 1 \text{ —— } 1\,.$$

On the other hand, there are exactly two possible analytic types for the plane curve singularity defined by $f+g$: either $f+g$ is analytically equivalent to f (which is the case iff the coefficient for $x^{8/3}$ in the Puiseux expansion of $f+g$ vanishes), or $f+g$ is analytically equivalent to $f+x^5y$ (for a detailed proof, cf. [BrK, pp. 445f]). To show that f and $f+x^5y$ are not analytically equivalent, one can compute the respective Tjurina numbers:

```
ring r=0,(x,y),ds;
LIB "sing.lib";
poly f=y3-x7;
tjurina(f);
//-> 12
tjurina(f+x5y);
//-> 11
```

(b) Let $f = y^4 - x^9$, and let $g \in I = \langle x^\alpha y^\beta \mid 4\alpha + 9\beta > 36\rangle$. Then f and $f+g$ are topologically equivalent, the multiplicity sequences being

$$4 \text{ —— } 4 \text{ —— } 1 \text{ —— } 1 \text{ —— } 1 \text{ —— } 1\,.$$

But there are infinitely many different analytic types possible for $f+g$: for instance, the singularities given by

$$\begin{aligned} f + g_\lambda &= f - 4yx^7 - (2+4\lambda)y^2x^5 + (1-4\lambda+2\lambda^2)x^{10} - 4\lambda^2yx^8 - \lambda^4x^{11} \\ &= \prod_{j=1}^{4}\Big(y - (-1)^{j/2}x^{9/4} - (-1)^j x^{10/4} - \lambda(-1)^{3j/2}x^{11/4}\Big)\,, \quad \lambda \in \mathbb{C}\,, \end{aligned}$$

are pairwise not analytically equivalent (cf. [BrK, pp. 447f]). Anyhow, the latter types, of course, in general cannot be distinguished by the respective Tjurina numbers. The following SINGULAR session computes the Tjurina numbers for f, $f-g_0$ and for $f-g_\lambda$, $\lambda \in \mathbb{C}$ generic.

```
ring r=0,(x,y),ds;
LIB "sing.lib";
poly f=y4-x9;
tjurina(f);
//-> 24
tjurina(f-4yx7-2y2x5+x10);
//-> 21
```

```
ring r1=(0,lam),(x,y),ds;
tjurina(y4-x9-4yx7-(2+4*lam)*y2x5+(1-4*lam+2*lam^2)*x10
          -4*lam^2*yx8-lam^4*x11);
//-> 21
```

Example 3.43.2. We show that the ordinary 4-multiple points defined by $f_t := xy \cdot (x+y) \cdot (x-ty) = 0\}$ $(t \neq 0, -1)$ are not analytically equivalent for different values of t.

Since f_t is homogeneous, the linear part of any analytic isomorphism from f_t to f_s maps f_t to f_s. We have shown in Example 2.55.1 that the cross-ratio is an invariant of a linear isomorphism. Hence, we have to show that the cross-ratio varies with f_t.

```
ring r = (0,t),(x,y),ds;
poly f = xy*(x+y)*(x-t*y);
list L = factorize(f,1);       // L[1][1..4] are the 4 factors of f
int i;
for (i=1; i<=4; i++){
  poly a(i) = subst(subst(L[1][i],x,1),y,0);
  poly b(i) = subst(subst(L[1][i],x,0),y,1);
}
poly r1 = (a(1)*b(3)-a(3)*b(1))*(a(2)*b(4)-a(4)*b(2));
poly r2 = (a(1)*b(4)-a(4)*b(1))*(a(2)*b(3)-a(3)*b(2));
r1/r2;                         // the cross-ratio
//-> 1/(t+1)
```

Hence, f_t is analytically isomorphic to f_s iff $t = s$.

Remark 3.43.3. Let σ be a *topological invariant*, that is, $\sigma(C,z) = \sigma(D,w)$ whenever (C,z) and (D,w) are topologically equivalent. We introduce

$$\sigma(S) := \sigma(C,z)$$

for S the topological type represented by the plane curve germ (C,z). In particular, we introduce in this way

- $\operatorname{mt} S$, the multiplicity of S,
- $\mu(S)$, the Milnor number of S,
- $\delta(S)$, the delta invariant of S,
- $\operatorname{cd}(S)$, the conductor of S.

Recall that the Tjurina number τ is not a topological invariant, but it is an *analytic invariant*, that is, $\tau(C,z) = \tau(D,w)$ whenever (C,z) and (D,w) are analytically equivalent. For an analytic type S represented by the plane curve germ (C,z), we introduce

$$\sigma(S) := \sigma(C,z)$$

if σ is any analytic invariant.

Exercises

Exercise 3.4.1. (1) Find the Puiseux pairs of the germ with local equation $f = y^4 - 2y^2x^3 + x^6 + x^9$.

(2) Compute the Puiseux pairs of the branches of all simple singularities. Moreover, in the case of reducible singularities, compute the intersection number of the branches.

Exercise 3.4.2. Let $f \in \mathbb{C}\{x, y\}$ be reduced and irreducible. Prove that

(1) $\mathit{\Gamma}(\mathcal{O})$ has precisely $\delta(f)$ *gaps*, that is, $\#\bigl(\mathbb{Z}_{\geq 0} \setminus \mathit{\Gamma}(\mathcal{O})\bigr) = \delta(f)$;
(2) $\mathrm{cd}(f) = 2\delta(f)$.

Exercise 3.4.3. (1) Using the computations in the proof of Proposition 3.41, express the multiplicity sequence of an irreducible curve germ via the characteristic exponents $\beta_0, \dots, \beta_s$, and vice versa.

(2) Using Proposition 3.34, prove that, for an irreducible plane curve germ $(C, \mathbf{0})$,

$$\delta(C, \mathbf{0}) = \frac{1}{2} \sum_{k=1}^{s} (\beta_k - 1)(D_{k-1} - D_k)\,,$$

where D_k is defined as in Lemma 3.31 (see [Mil1, page 99].

Exercise 3.4.4. (1) Give an example of an unfolding $F \in \mathbb{C}\{x, y, t\}$ of a reduced power series $f \in \mathbb{C}\{x, y\}$ such that the family of germs defined by $F_t \in \mathbb{C}\{x, y\}$, $t \in (\mathbb{C}, 0)$, is δ-constant but not κ-constant.

(2) Give an example of an unfolding $F \in \mathbb{C}\{x, y, t\}$ of a reduced power series $f \in \mathbb{C}\{x, y\}$ such that F_t, $t \in (\mathbb{C}, 0)$, is defined in a neighbourhood U of $\mathbf{0} \in \mathbb{C}^2$ and satisfies

(a) $\sum_{q \in \{F_t=0\} \cap U} \kappa(F_t, q) = \text{const}$, but $\sum_{q \in \{F_t=0\} \cap U} \delta(F_t, q) \neq \text{const}$,

respectively

(b) $\sum_{q \in \{F_t=0\} \cap U} \kappa(F_t, q) = \text{const}$ and $\sum_{q \in \{F_t=0\} \cap U} \delta(F_t, q) = \text{const}$, but $\sum_{q \in \{F_t=0\} \cap U} \mu(F_t, q) \neq \text{const}$.

Exercise 3.4.5. Show that the minimal embedded resolution of the germ defined by $f \in \mathbb{C}\{x, y\}$ consists of at most $\delta(f) + \mathrm{mt}(f) - 1$ point blowing ups.

II

Local Deformation Theory

Deformation theory is one of the fundamental techniques in algebraic geometry, singularity theory, complex analysis and many other disciplines. We can deform various kinds of objects, for example

- algebraic varieties or complex spaces,
- singularities, i.e., germs of complex spaces,
- morphisms between (germs of) algebraic varieties or complex spaces,
- modules over a ring or sheaves over a complex space, etc.

The basic idea is to perturb a given object slightly so that the deformed object is simpler but still caries enough information about the original object. This latter requirement is algebraically encoded in the concept of flatness. We have already seen in Sections I.1.7 and I.2.1 that flatness implies continuity of certain invariants ("conservation of numbers") and in the present section we shall derive more results showing the usefulness of flatness.

The two main achievements of deformation theory are

- the existence of a *versal deformation* (under certain hypotheses), parameterized by a finite dimensional variety, respectively a complex space (germ), containing essentially all information about all possible small deformations, which depend, a priori, on infinitely many parameters;
- the theory of *infinitesimal deformations* and *obstructions*, a linearization technique, which allows us to reduce many geometric problems to cohomological problems.

In the first section, we treat deformations of complex space germs (X, x). We prove the existence of a versal deformation for isolated singularities of complete intersections and develop the theory of infinitesimal deformations and obstructions for arbitrary isolated singularities.

The second section treats plane curve singularities. There we go much further, considering specific classes of deformations: equimultiple, equinormalizable, and most importantly, *equisingular* ones. Numerically, they can be characterized as preserving certain singularity invariants (the multiplicity, resp.

the δ-invariant, resp. the Milnor number). In turn, geometrically, equisingular deformations are those which preserve the topological type of the singularity. We give a full treatment of equinormalizable (δ-constant) and equisingular (μ-constant) deformations. We use equinormalizable deformations to give a new proof for the smoothness of the μ-constant stratum in a versal deformation.

1 Deformations of Complex Space Germs

In this section, we develop the theory of deformations of complex space germs. Although we use the language of (deformation) functors for precise statements, we always provide explicit descriptions in terms of the defining equations. We elaborate the general theory in the case of a complete intersection where it is particularly transparent because of the non-existence of obstructions.

The key object of the theory is the (vector) *space of the first order deformations* $T^1_{(X,x)}$ which, in the case of hypersurface singularities, is just the Tjurina algebra. For isolated complete intersection singularities, the space $T^1_{(X,x)}$ can be explicitly computed and its basis generates a semiuniversal deformation (a versal deformation of minimal dimension) with linear base space. Geometrically, such a versal deformation of an $(n-k)$-dimensional complete intersection $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ can be viewed as a (germ of a) finite-dimensional complex subspace space of $\mathscr{O}^k_{(\mathbb{C}^n,\mathbf{0})}$ transverse to the orbit of $(X,\mathbf{0})$ under the contact group action.

In general, Grauert's fundamental theorem [Gra1] ensures the existence of a semiuniversal deformation for arbitrary isolated singularities. Yet, the existence of a semiuniversal deformation with a *smooth base space* for non-complete-intersection singularities is conditioned by the vanishing of the *obstruction module* $T^2_{(X,x)}$, which can be viewed as an obstruction space to the lifting of the first order deformations up to second order ones. We give full proofs of its properties and explicit algorithmic descriptions of $T^1_{(X,x)}$ and $T^2_{(X,x)}$ for arbitrary singularities (X,x). The general formal theory for infinitesimal deformations and obstructions together with the cotangent cohomology of a morphism between singularities is presented in Appendix C.

1.1 Deformations of Singularities

We develop now the deformation theory of isolated singularities of complex spaces. The concepts and theorems for this case may serve as a prototype for deformations of other objects, too.

Definition 1.1. Let (X,x) and (S,s) be complex space germs. A *deformation of* (X,x) *over* (S,s) consists of a flat morphism $\phi\colon (\mathscr{X},x) \to (S,s)$ of complex germs together with an isomorphism from (X,x) to the fibre of ϕ, $(X,x) \to (\mathscr{X}_s,x) := (\phi^{-1}(s),x)$.

$(\mathscr{X},x)$ is called the *total space*, (S,s) the *base space*, and $(\mathscr{X}_s,x) \cong (X,x)$ the *special fibre* of the deformation.

We can write a deformation as a Cartesian diagram

$$\begin{tikzcd}
(X,x) \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{X},x) \arrow[d, "\phi \text{ flat}"] \\
\{\mathrm{pt}\} \arrow[r, hook] & (S,s)
\end{tikzcd} \tag{1.1.1}$$

where i is a closed embedding mapping (X,x) isomorphically onto $(\mathscr{X}_s,x)$ and $\{\mathrm{pt}\}$ denotes the reduced point considered as a complex space germ. We denote a deformation by

$$(i,\phi)\colon (X,x) \overset{i}{\hookrightarrow} (\mathscr{X},x) \xrightarrow{\phi} (S,s)\,,$$

or simply by $\phi : (\mathscr{X},x) \to (S,s)$ in order to shorten notation.

Note that we do not only require that there exists an isomorphism mapping the fibre $(\mathscr{X}_s,x)$ to (X,x) but that the isomorphism i is part of the data which we use to identify $(\mathscr{X}_s,x)$ and (X,x). Thus, if $(\mathscr{X}',x) \to (S,s)$ is another deformation of (X,x), then there is a *unique* isomorphism of germs $(\mathscr{X}_s,x) \cong (\mathscr{X}'_s,x)$.

The essential point here is that ϕ is flat, that is, $\mathcal{O}_{\mathscr{X},x}$ is a flat $\mathcal{O}_{S,s}$-module via the induced morphism $\phi^\sharp_x : \mathcal{O}_{S,s} \to \mathcal{O}_{\mathscr{X},x}$. If $\phi\colon \mathscr{X} \to S$ is a small representative of the germ ϕ, then flatness implies that the nearby fibres $\phi^{-1}(t)$ have a close relation to the special fibre $\phi^{-1}(s)$ (see Figure 1.1). By Theorem B.8.13, we have $\dim(\mathscr{X}_s,x) = \dim(\mathscr{X},x) - \dim(S,s)$. Frisch's Theorem I.1.83 says that for a morphism $\phi : \mathscr{X} \to S$ of complex spaces the set of points in $\mathscr{X}$ where ϕ is flat is analytically open. Hence, in our situation, if $\mathscr{X}$ and S are sufficiently small, then $\phi : \mathscr{X} \to S$ is everywhere flat and $\dim(\phi^{-1}(t),y) = \dim(\phi^{-1}(s),x)$ for all $t \in S$ and all $y \in \phi^{-1}(t)$, at least if $\mathscr{X}$ and S are pure dimensional.

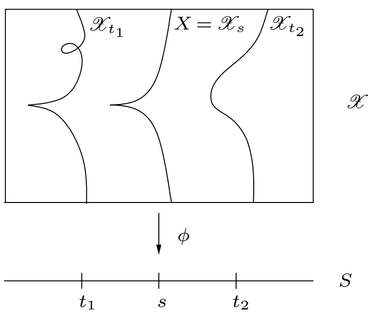


Fig. 1.1. Symbolic picture of a deformation.

Here is an example of a non-flat morphism. The natural map

$$\mathbb{C}\{x\} \longrightarrow \mathbb{C}\{x,y\}/\langle xy\rangle$$

is not flat, since x is a zerodivisor of $\mathbb{C}\{x,y\}/\langle xy\rangle$. Geometrically the dimension of the special fibre of the projection $(\mathbb{C}^2,\mathbf{0}) \supset V(xy) \to (\mathbb{C},0)$ jumps (see Figure 1.2).

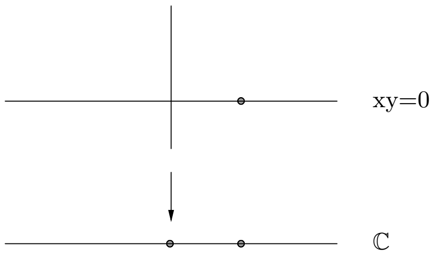


Fig. 1.2. A non-flat morphism.

The algebraic properties of flatness are treated in detail in Appendix B, some consequences for the behaviour of the fibres in Section 1.8. We just recall some **geometric consequences of flatness:**

- $\phi = (\phi_1,\dots,\phi_k) : (\mathscr{X},x) \to (\mathbb{C}^k,\mathbf{0})$ is flat iff $\phi_1,\dots,\phi_k$ is an $\mathcal{O}_{\mathscr{X},x}$-regular sequence.
- If $(\mathscr{X},x)$ is a Cohen-Macaulay singularity, then $\phi_1,\dots,\phi_k \in \mathfrak{m} \subset \mathcal{O}_{\mathscr{X},x}$ is an $\mathcal{O}_{\mathscr{X},x}$-regular sequence iff $\dim_{\mathbb{C}} \mathcal{O}_{\mathscr{X},x}/\langle\phi_1,\dots,\phi_k\rangle = \dim(\mathscr{X},x) - k$.
- In particular, $\phi : (\mathbb{C}^m,\mathbf{0}) \to (\mathbb{C}^k,\mathbf{0})$ is flat iff $\dim\bigl(\phi^{-1}(\mathbf{0}),\mathbf{0}\bigr) = m-k$.

Note that smooth germs, hypersurface and complete intersection singularities, reduced curve singularities and normal surface singularities are Cohen-Macaulay (Exercise I.1.8.5).

Definition 1.2. Given two deformations $(i,\phi)\colon (X,x) \hookrightarrow (\mathscr{X},x) \to (S,s)$ and $(i',\phi')\colon (X,x) \hookrightarrow (\mathscr{X}',x') \to (S',s')$, of (X,x) over (S,s) and (S',s'), respectively. A *morphism of deformations* from (i,ϕ) to (i',ϕ') is a morphism of the diagram (1.1.1) being the identity on $(X,x) \to \{\mathrm{pt}\}$. Hence, it consists of two morphisms (ψ,φ) such that the following diagram commutes

$$\begin{tikzcd}
 & (X,x) \arrow[dl, hook', "i'"'] \arrow[dr, hook, "i"] & \\
(\mathscr{X}',x') \arrow[rr, "\psi"] \arrow[d, "\phi'"'] & & (\mathscr{X},x) \arrow[d, "\phi"] \\
(S',s') \arrow[rr, "\varphi"] & & (S,s)
\end{tikzcd} .$$

Two deformations over the *same* base space (S, s) are *isomorphic* if there exists a morphism (ψ, φ) with ψ an isomorphism and φ the identity map.

It is easy to see that deformations of (X, x) form a category. Usually one considers the (non-full) subcategory of deformations of (X, x) over a fixed base space (S, s) and morphisms (ψ, φ) with $\varphi = \mathrm{id}_{(S,s)}$. Lemma I.1.86 implies that in this category all morphisms are automatically isomorphims.

Before we proceed, let us consider a few examples.

• If $f\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ is a non-constant holomorphic map germ then f is a non-zerodivisor of $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$. In particular, f is flat by Theorem B.8.11 and, therefore, $(i, f) : (X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ is a deformation of the complex space germ $(X, \mathbf{0}) := (f^{-1}(0), \mathbf{0})$ over $(\mathbb{C}, 0)$.

• More generally, let $f := (f_1, \ldots, f_k)\colon (\mathbb{C}^m, \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ be holomorphic and assume that $(X, \mathbf{0}) := (f^{-1}(\mathbf{0}), \mathbf{0})$ is a complete intersection, that is, has dimension $m - k$. Since $\mathcal{O}_{\mathbb{C}^k,\mathbf{0}}$ is a regular local ring, and $\mathcal{O}_{\mathbb{C}^m,\mathbf{0}}$ is Cohen-Macaulay by Corollary B.8.8, we get that f is flat by Theorem B.8.11. This means that $(i, f) : (X, \mathbf{0}) \subset (\mathbb{C}^m, \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ is a deformation of $(X, \mathbf{0})$ over $(\mathbb{C}^k, \mathbf{0})$.

• However, if $\dim(X, \mathbf{0}) > m - k$, then $f = (f_1, \ldots, f_k)\colon (\mathbb{C}^m, \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ is not flat and the defining power series $f_1, \ldots, f_k$ of $(X, \mathbf{0})$ do not induce a deformation of $(X, \mathbf{0})$. For example, let $(X, \mathbf{0}) \subset (\mathbb{C}^3, \mathbf{0})$ be defined by $f_1 = xy$, $f_2 = xz$, $f_3 = yz$, that is, the (germ of the) coordinate axes in $\mathbb{C}^3$, then $\dim(X, \mathbf{0}) = 1$ and hence

$$(X, \mathbf{0}) \subset (\mathbb{C}^3, \mathbf{0}) \xrightarrow{(f_1, f_2, f_3)} (\mathbb{C}^3, \mathbf{0})$$

is not flat and therefore not a deformation of $(X, \mathbf{0})$ (see Figure 1.3).

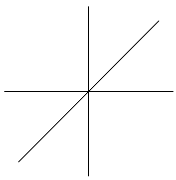


Fig. 1.3. $\{xy = xz = yz = 0\}$.

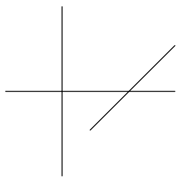


Fig. 1.4. $\{xy - xt = xz = yz = 0\}$

On the other hand, the map

$$(\mathscr{X}, \mathbf{0}) = \big(V(xy - xt, xz, yz), \mathbf{0}\big) \to (\mathbb{C}, 0)\,, \quad (x, y, z, t) \mapsto t$$

is a deformation of $(X, \mathbf{0})$. In fact, we can check that t is a non-zerodivisor of $\mathbb{C}\{x, y, z, t\}/\langle xy - xt, xz, yz\rangle$, either by hand or by the following Singular session:

```
LIB "sing.lib";
ring R = 0,(x,y,z,t),ds;
ideal I=xy-xt,xz,yz;
is_reg(t,I);          // result is 1 iff t is non-zerodivisor mod I
//-> 1
```

We introduce now the concept of induced deformations. They give rise, in a natural way, to morphisms between deformations over different base spaces.

Let $(X,x) \hookrightarrow (\mathscr{X},x) \xrightarrow{\phi} (S,s)$ be a deformation of the complex space germ (X,x) and $\varphi\colon (T,t) \to (S,s)$ a morphism of germs. Then the fibre product (see Definition I.1.46 and p. 58) of ϕ and φ is the following commutative diagram of germs

$$\begin{tikzcd}
 & (X,x) \arrow[dl, hook', "\varphi^* i"'] \arrow[dr, hook, "i"] & \\
(\mathscr{X},x)\times_{(S,s)}(T,t) \arrow[rr, "\widetilde{\varphi}"] \arrow[d, "\varphi^*\phi"'] & & (\mathscr{X},x) \arrow[d, "\phi"] \\
(T,t) \arrow[rr, "\varphi"] & & (S,s)
\end{tikzcd}$$

where $\varphi^*\phi$, resp. $\widetilde{\varphi}$, are induced by the second, resp. first, projection, and

$$\varphi^* i = \left(\widetilde{\varphi}\big|_{(\varphi^*\phi)^{-1}(t)}\right)^{-1} \circ i\,.$$

Definition 1.3. We denote $(\mathscr{X},x)\times_{(S,s)}(T,t)$ by $\varphi^*(\mathscr{X},x)$ and call

$$\varphi^*(i,\phi) := (\varphi^* i, \varphi^*\phi)\colon (X,x) \xhookrightarrow{\varphi^* i} \varphi^*(\mathscr{X},x) \xrightarrow{\varphi^*\phi} (T,t)$$

the *deformation induced by* φ *from* (i,ϕ), or just the *induced deformation* or *pull-back*; φ is called the *base change map.*

By Proposition 1.87, $\varphi^*\phi$ is flat. Hence, $(\varphi^* i, \varphi^*\phi)$ is indeed a deformation of (X,x) over (T,t), and $(\widetilde{\varphi},\varphi)$ is a morphism from (i,ϕ) to $(\varphi^* i,\varphi^*\phi)$.

A typical example of an induced deformation is the restriction to a subspace in the parameter space (S,s) or, as in the following example, the pull-back via a holomorphic map germ onto some subspace of (X,x).

Example 1.3.1. Consider $F(x,y,u,v) = x^2+y^3+uy+v\,,\ \ \mathscr{X} = V(F)\subset\mathbb{C}^4$ and $\phi : (\mathscr{X},\mathbf{0}) \to (S,\mathbf{0}) = (\mathbb{C}^2,\mathbf{0})$, $(x,y,u,v)\mapsto(u,v)$, which defines a deformation of the cusp $V(x^2+y^3)\subset(\mathbb{C}^2,\mathbf{0})$. Let Δ be the *discriminant* of ϕ, that is, the image of the critical points of ϕ,

$$\begin{aligned}\Delta &:= \left\{(u,v)\in\mathbb{C}^2 \ \middle|\ \phi^{-1}(u,v) \text{ is singular}\right\}\\ &= \left\{(u,v)\in\mathbb{C}^2 \ \middle|\ 4u^3+27v^2=0\right\}\,,\end{aligned}$$

This discriminant can be computed via the following SINGULAR session:

```
ring r = 0,(x,y,u,v),ds;
poly F = x2+y3+uy+v;
ideal cF = F,diff(F,x),diff(F,y);   // the critical locus of phi
ideal dF = eliminate(cF,xy);        // the discriminant of phi
dF;
//-> dF[1]=27v2+4u3
```

We can parametrize $\varDelta$ by $\varphi : (\mathbb{C},0) \to (\mathbb{C}^2,\mathbf{0})$, $t \mapsto (-3t^2, 2t^3)$. Then

$$\varphi^*(\mathscr{X},\mathbf{0}) \cong \big\{(x,y,t) \in (\mathbb{C}^3,\mathbf{0}) \mid x^2 + y^3 - 3t^2y + 2t^3 = 0\big\}$$

and $\varphi^*\phi : \varphi^*(\mathscr{X},\mathbf{0}) \to (\mathbb{C},0)$ is the projection $(x,y,t) \mapsto t$. Here the pull-back is a deformation of the cusp with all fibres singular.

Definition 1.4. Let (X,x) be a complex space germ.

(1) $\mathcal{D}ef_{(X,x)}$ denotes the *category of deformations of* (X,x). The objects of $\mathcal{D}ef_{(X,x)}$ are deformations

$$(X,x) \overset{i}{\hookrightarrow} (\mathscr{X},x) \xrightarrow{\phi} (S,s)$$

of (X,x) over some complex germ (S,s) with morphisms (ψ,φ) as defined in Definition 1.2.

(2) $\mathcal{D}ef_{(X,x)}(S,s)$ denotes the *category of deformations of* (X,x) *over* (S,s). It is the subcategory of $\mathcal{D}ef_{(X,x)}$ whose objects are deformations of (X,x) with fixed base space (S,s) and whose morphisms (ψ,φ) satisfy $\varphi = \mathrm{id}_{(S,s)}$. By Lemma I.1.86, any morphism in $\mathcal{D}ef_{(X,x)}(S,s)$ is an isomorphism. A category with this property is called a *groupoid*.

(3) $\underline{\mathcal{D}ef}_{(X,x)}(S,s)$ denotes the set of *isomorphism classes of deformations of* (X,x) *over* (S,s). The elements of $\underline{\mathcal{D}ef}_{(X,x)}(S,s)$ are denoted by

$$\big[(i,\phi)\big] = \big[(X,x) \overset{i}{\hookrightarrow} (\mathscr{X},x) \xrightarrow{\phi} (S,s)\big]\,.$$

For a morphism $\varphi\colon (T,t) \to (S,s)$ of complex germs, the pull-back $\varphi^*(i,\phi)$ is a deformation of (X,x) with base space (T,t) (cf. Definition 1.3 and Proposition 1.87). Since the pull-back of isomorphic deformations are isomorphic, φ^* induces a map

$$\big[\varphi^*\big] : \underline{\mathcal{D}ef}_{(X,x)}(S,s) \longrightarrow \underline{\mathcal{D}ef}_{(X,x)}(T,t)\,.$$

It follows that

$$\underline{\mathcal{D}ef}_{(X,x)} : (\textit{complex germs}) \longrightarrow \mathcal{S}ets\,, \quad (S,s) \mapsto \underline{\mathcal{D}ef}_{(X,x)}(S,s)$$

is a functor, it is called the *deformation functor* of (X,x) or the *functor of isomorphism classes of deformations* of (X,x).

1.2 Embedded Deformations

This section aims at describing the somewhat abstract definitions of the preceding section in more concrete terms, that is, in terms of defining equations and relations. Moreover, we derive a characterization of flatness via lifting of relations.

Let us recall the notion of unfoldings from Section I.2.1 and explain its relation to deformations of a hypersurface germ.

Given $f \in \mathbb{C}\{x_1, \dots, x_n\}$, $f(\mathbf{0}) = 0$, an *unfolding* of f is a power series $F \in \mathbb{C}\{x_1, \dots, x_n, t_1, \dots, t_k\}$ with $F(\boldsymbol{x}, \mathbf{0}) = f(\boldsymbol{x})$, that is,

$$F(\boldsymbol{x}, \boldsymbol{t}) = f(\boldsymbol{x}) + \sum_{|\boldsymbol{\nu}| \geq 1} g_{\boldsymbol{\nu}}(\boldsymbol{x}) \boldsymbol{t}^{\boldsymbol{\nu}} .$$

We identify the power series f and F with holomorphic map germs

$$f \colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0), \quad F \colon (\mathbb{C}^n \times \mathbb{C}^k, \mathbf{0}) \to (\mathbb{C}, 0).$$

Then F induces a deformation of $(X, \mathbf{0}) = (f^{-1}(0), \mathbf{0})$ in the following way

$$\begin{tikzcd}
(X,\mathbf{0}) \arrow[r, hook, "i"] \arrow[d] & (\mathscr{X},\mathbf{0}) \arrow[d, "\phi = pr_2|_{(\mathscr{X},\mathbf{0})}"] & := (F^{-1}(0),\mathbf{0}) \subset (\mathbb{C}^n \times \mathbb{C}^k, \mathbf{0}) \\
\{0\} \arrow[r, hook] & (\mathbb{C}^k, \mathbf{0}) &
\end{tikzcd}$$

where i is the inclusion and ϕ the restriction of the second projection.

Since $(\mathscr{X}, \mathbf{0})$ is a hypersurface, it is Cohen-Macaulay. The fibre dimension satisfies $\dim(\phi^{-1}(\mathbf{0}), \mathbf{0}) = n - 1 = (n + k - 1) - k$, hence ϕ is flat by Theorem B.8.11 and we conclude that (i, ϕ) is a deformation of $(X, \mathbf{0})$. Indeed, each deformation of $(X, \mathbf{0}) = (f^{-1}(0), \mathbf{0})$ over some $(\mathbb{C}^k, \mathbf{0})$ is induced by an unfolding of f. This follows from the next proposition.

We want to show that if $\phi \colon (\mathscr{X}, x) \to (S, s)$ is a deformation of (X, x) and if (X, x) is a subgerm of $(\mathbb{C}^n, \mathbf{0})$, then ϕ factors as

$$(\mathscr{X}, x) \overset{i}{\hookrightarrow} (\mathbb{C}^n, \mathbf{0}) \times (S, s) \xrightarrow{p} (S, s)$$

where i is a closed embedding and p the second projection. That is, the embedding of the fibre $(X, x) \hookrightarrow (\mathbb{C}^n, \mathbf{0})$ can be lifted to an embedding of the deformation ϕ. We show more generally

Proposition 1.5. *Given a Cartesian diagram of complex space germs*

$$\begin{tikzcd}
(X_0, x) \arrow[r, hook] \arrow[d, "f_0"'] \arrow[dr, phantom, "\square"] & (X, x) \arrow[d, "f"] \\
(S_0, s) \arrow[r, hook] & (S, s)
\end{tikzcd}$$

where the horizontal maps are closed embeddings. Assume that f_0 factors as

$$(X_0,x) \overset{i_0}{\hookrightarrow} (\mathbb{C}^n,\mathbf{0}) \times (S_0,s) \xrightarrow{p_0} (S_0,s)$$

with i_0 a closed embedding and p_0 the second projection.[1] *Then there exists a Cartesian diagram*

$$\begin{tikzcd}
(X_0,x) \arrow[r, hook] \arrow[d, hook, "i_0"'] \arrow[dd, bend right=60, "f_0"'] \arrow[dr, phantom, "\square"] & (X,x) \arrow[d, hook, "i"] \arrow[dd, bend left=60, "f"] \\
(\mathbb{C}^n,\mathbf{0}) \times (S_0,s) \arrow[r, hook] \arrow[d, "p_0"'] \arrow[dr, phantom, "\square"] & (\mathbb{C}^n,\mathbf{0}) \times (S,s) \arrow[d, "p"] \\
(S_0,s) \arrow[r, hook] & (S,s)
\end{tikzcd} \tag{1.2.1}$$

with i a closed embedding and p the second projection. That is, the embedding of f_0 over (S_0,s) extends to an embedding of f over (S,s).

Note that we do not require that f_0 or f are flat.

Proof. Let $j\colon (S,s) \hookrightarrow (\mathbb{C}^k,\mathbf{0})$ be an embedding of (S,s) into $(\mathbb{C}^k,\mathbf{0})$. If the embedding of f_0 extends to an embedding of $j \circ f$,

$$(X,x) \overset{i}{\hookrightarrow} (\mathbb{C}^n,\mathbf{0}) \times (\mathbb{C}^k,\mathbf{0}) \to (\mathbb{C}^k,\mathbf{0})\,,$$

then i factors through $(\mathbb{C}^n,\mathbf{0}) \times (S,s)$. Thus, without loss of generality, we may assume $(S,s) = (\mathbb{C}^k,\mathbf{0})$. Let $f = (f_1,\dots,f_k)\colon (X,x) \to (\mathbb{C}^k,\mathbf{0})$ and

$$f_0 = (f_{01},\dots,f_{0k})\colon (X_0,x) \to (S_0,s) \subset (\mathbb{C}^k,\mathbf{0})\,.$$

Then $\widetilde{i}_0$ is of the form $(g_1,\dots,g_n,f_{01},\dots,f_{0k})$ where $\widetilde{i}_0$ is the composition

$$\widetilde{i}_0\colon (X_0,x) \overset{i_0}{\hookrightarrow} (\mathbb{C}^n,\mathbf{0}) \times (S_0,s) \hookrightarrow (\mathbb{C}^n,\mathbf{0}) \times (\mathbb{C}^k,\mathbf{0})\,.$$

Let $\widetilde{g}_j$ be the preimages of g_j under the surjection $\mathcal{O}_{X,x} \to \mathcal{O}_{X_0,x}$. Then

$$i = (\widetilde{g}_1,\dots,\widetilde{g}_n,f_1,\dots,f_k)\colon (X,x) \longrightarrow (\mathbb{C}^n,\mathbf{0}) \times (\mathbb{C}^k,\mathbf{0})$$

extends i_0 such that (1.2.1) commutes. We have to show that i is a closed embedding, that is, the map

$$i^\sharp\colon \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^k,\mathbf{0}} \to \mathcal{O}_{X,x}\,, \quad (x_1,\dots,x_{n+k}) \mapsto (\widetilde{g}_1,\dots,\widetilde{g}_n,f_1,\dots,f_k)$$

is surjective, where $x_1,\dots,x_{n+k}$ generate the maximal ideal of $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^k,\mathbf{0}}$. Let $\mathcal{O}_{S_0,s} = \mathcal{O}_{\mathbb{C}^k,\mathbf{0}}/\langle h_1,\dots,h_r\rangle$, then, since $(X_0,x) \cong (f^{-1}(S_0),x)$, we have

[1] In this situation, we call f_0 an *embedding over* (S_0,s).

$$\mathcal{O}_{X_0,x} = \mathcal{O}_{X,x}/\langle f^\sharp(h_1),\dots,f^\sharp(h_r)\rangle\mathcal{O}_{X,x}\,.$$

Consider the commutative diagram with exact second row

$$\begin{tikzcd} & & & \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^k,\mathbf{0}} \arrow[dl, "i^\sharp"'] \arrow[d, two heads, "i_0^\sharp"] & \\ 0 \arrow[r] & \langle f^\sharp(h_1),\dots,f^\sharp(h_r)\rangle\mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{X_0,x} \arrow[r] & 0\,. \end{tikzcd}$$

For $a\in\mathcal{O}_{X,x}$ there exists $b\in\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^k,\mathbf{0}}$ such that

$$a-i^\sharp(b)\in\langle f^\sharp(h_1),\dots,f^\sharp(h_r)\rangle\mathcal{O}_{X,x}$$

where $f^\sharp(h_i) = i^\sharp(p^\sharp(h_i))$. Since the $p^\sharp(h_i)$ are in the maximal ideal of $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^k,\mathbf{0}}$, it follows that the maximal ideal of $\mathcal{O}_{X,x}$ is generated by the image of the maximal ideal of $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^k,\mathbf{0}}$ under $i^\sharp$. Hence, $i^\sharp$ is surjective and the result follows. $\square$

Applying Proposition 1.5 to a deformation of (X,x) we get

Corollary 1.6. *Let $(X,\mathbf{0})\subset(\mathbb{C}^n,\mathbf{0})$ be a closed subgerm. Then any deformation $(i,\phi)\colon (X,\mathbf{0})\hookrightarrow(\mathscr{X},x)\to(S,s)$ of $(X,\mathbf{0})$ can be embedded, that is, there exists a Cartesian diagram*

$$\begin{tikzcd} (X,\mathbf{0}) \arrow[r, hook, "i"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & (\mathscr{X},x) \arrow[d, hook, "J"] \arrow[dd, bend left=60, "\phi"] \\ (\mathbb{C}^n,\mathbf{0}) \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathbb{C}^n,\mathbf{0})\times(S,s) \arrow[d, "p"] \\ \{s\} \arrow[r, hook] & (S,s) \end{tikzcd}$$

where J is a closed embedding, p is the second projection and j the first inclusion.

In particular, the embedding dimension is semicontinuous under deformations, that is, $\mathrm{edim}\big(\phi^{-1}(\phi(y)),y\big)\le\mathrm{edim}(X,\mathbf{0})$, *for all y in $\mathscr{X}$ sufficiently close to x.*

Summing up, we showed that every deformation $(X,\mathbf{0})\hookrightarrow(\mathscr{X},x)\to(S,s)$ of $(X,\mathbf{0})$ can be assumed to be given as follows: Let $I_{X,\mathbf{0}}=\langle f_1,\dots,f_k\rangle\subset\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ be the ideal of $(X,\mathbf{0})\subset(\mathbb{C}^n,\mathbf{0})$. Then the total space of the deformation of $(X,\mathbf{0})$ is given as

$$(\mathscr{X},x)=V(F_1,\dots,F_k)\subset(\mathbb{C}^n\times S,(\mathbf{0},s))\,,$$

with $\mathcal{O}_{\mathscr{X},x} = \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}/I_{\mathscr{X},x}$, $I_{\mathscr{X},x} = \langle F_1,\dots,F_k\rangle \subset \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}$. The holomorphic map ϕ is given by the projection to the second factor and the image of F_i in $\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}/\mathfrak{m}_{S,s} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$, $i=1,\dots,k$, is equal to f_i.

Let $(S,s)\subset(\mathbb{C}^r,\mathbf{0})$ and denote the coordinates of $\mathbb{C}^n$ by $\boldsymbol{x}=(x_1,\dots,x_n)$ and those of $\mathbb{C}^r$ by $\boldsymbol{t}=(t_1,\dots,t_r)$. Then $f_i = F_i|_{(\mathbb{C}^n,\mathbf{0})}$ and, hence, F_i is of the form[2]

$$F_i(\boldsymbol{x},\boldsymbol{t}) = f_i(\boldsymbol{x}) + \sum_{j=1}^{r} t_j g_{ij}(\boldsymbol{x},\boldsymbol{t}), \quad g_{ij}\in\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^r,\mathbf{0}},$$

that is, F_i is an unfolding of f_i.

In particular, if $(X,\mathbf{0})$ is a hypersurface singularity, that is, if $I_{X,\mathbf{0}} = \langle f\rangle$, then any deformation of $(X,\mathbf{0})$ over a smooth germ $(S,s)=(\mathbb{C}^r,\mathbf{0})$ is induced by an unfolding of f. More generally, the same holds if $(X,\mathbf{0})$ is an $(n-k)$-dimensional complete intersection as we shall see now:

Proposition 1.7. *Let $(X,\mathbf{0})\subset(\mathbb{C}^n,\mathbf{0})$ be a complete intersection germ, and let $f_1,\dots,f_k$ be a minimal set of generators of the ideal of $(X,\mathbf{0})$ in $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$. Then, for any complex germ (S,s) and any lifting $F_i\in\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}$ of f_i, $i=1,\dots,k$, the diagram*

$$(X,\mathbf{0}) \hookrightarrow (\mathscr{X},x) \xrightarrow{p} (S,s)$$

with $(\mathscr{X},x)\subset(\mathbb{C}^n\times S,(\mathbf{0},s))$ the germ defined by $F_1=\ldots=F_k=0$, and p the second projection, is a deformation of $(X,\mathbf{0})$ over (S,s).

Proof. Since $f_1,\dots,f_k$ is a regular sequence, any relation among $f_1,\dots,f_k$ can be generated by the *trivial relations* (also called the *Koszul relations*)

$$(0,\dots,0,-f_j,0,\dots,0,f_i,0\dots,0)$$

with $-f_j$ at place i and f_i at place j. This can be easily shown by induction on k. Another way to see this is to use the Koszul complex of $\boldsymbol{f}=(f_1,\dots,f_k)$: we have

$$H_1(f,\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}) = \{\text{relations between } f_1,\dots,f_k\}/\{\text{trivial relations}\},$$

and, due to Theorem B.6.3, $H_1(f,\mathcal{O}_{\mathbb{C}^n,\mathbf{0}})=0$ if $f_1,\dots,f_k$ is a regular sequence. Since the trivial relations can obviously be lifted, the result follows. □

Example 1.7.1. (1) Let $(X,\mathbf{0})\subset(\mathbb{C}^3,\mathbf{0})$ be the curve germ given by $f_1=xy$, $f_2=xz$, $f_3=yz$. Consider the unfolding of (f_1,f_2,f_3) over $(\mathbb{C},0)$,

$$F_1 = xy - t\,,\ F_2 = xz\,,\ F_3 = yz$$

(see Figure 1.5, p. 233). It is not difficult to check that the sequence

[2] That a system of generators for $I_{\mathscr{X},x}$ can be written in this form follows from the fact that $\mathfrak{m}_{S,s}I_{\mathscr{X},x} = \mathfrak{m}_{S,s}\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}\cap I_{\mathscr{X},x}$, which is a consequence of flatness.

$$0 \longleftarrow \mathcal{O}_{X,\mathbf{0}} \longleftarrow \mathcal{O}_{\mathbb{C}^3,\mathbf{0}} \xleftarrow{(xy,xz,yz)} \mathcal{O}^3_{\mathbb{C}^3,\mathbf{0}} \xleftarrow{\begin{pmatrix} 0 & -z \\ -y & y \\ x & 0 \end{pmatrix}} \mathcal{O}^2_{\mathbb{C}^3,\mathbf{0}} \longleftarrow 0\,,$$

is exact and, hence, a free resolution of $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^3,\mathbf{0}}/\langle f_1, f_2, f_3\rangle$. That is, $(0,-y,x)$ and $(-z,y,0)$ generate the $\mathcal{O}_{\mathbb{C}^3,\mathbf{0}}$-module of relations between xy, xz, yz.

Similarly, we find that $(0,-y,x)$, $(yz,-y^2,t)$, $(xz, t-xy, 0)$ generate the $\mathcal{O}_{\mathbb{C}^3,\mathbf{0}}$-module of relations of F_1, F_2, F_3. The liftable relations for f_1, f_2, f_3 are obtained from these by setting $t = 0$, which shows that the relation $(-z, y, 0)$ cannot be lifted. Hence, $\mathcal{O}_{\mathbb{C}^3\times\mathbb{C},\mathbf{0}}/\langle F_1, F_2, F_3\rangle$ is not $\mathcal{O}_{\mathbb{C},0}$-flat and, therefore, the above unfolding does not define a deformation of $(X, \mathbf{0})$. We check all this in the following Singular session:

```
ring R = 0,(x,y,z,t),ds;
ideal f = xy,xz,yz;
ideal F = xy-t,xz,yz;
module Sf = syz(f);       // the module of relations of f
print(Sf);                // shows the matrix of Sf
//-> 0, -z,
//-> -y,y,
//-> x, 0
syz(Sf);                  // is 0 iff the matrix of Sf injective
//-> _[1]=0
module SF = syz(F);
print(SF);
//-> 0, yz, xz,
//-> -y,-y2,t-xy,
//-> x, t,  0
```

To show that the relation $(-z, y, 0)$ in `Sf` cannot be lifted to `SF`, we substitute `t` by zero in `SF` and show that `Sf` is not contained in the module obtained (`Sf` does not reduce to zero):

```
print(reduce(Sf,std(subst(SF,t,0))));
//-> 0,-z,
//-> 0,y,
//-> 0,0
```

(2) However, if we consider

$$F_1 = xy - tx\,,\ F_2 = xz\,,\ F_3 = yz$$

(see Figure 1.6), we obtain $(-z, -t, x)$, $(-z, y-t, 0)$ as generators of the relations among F_1, F_2, F_3. Since $(0, -y, x) = (-z, 0, x) - (-z, y, 0)$, it follows that any relation among f_1, f_2, f_3 can be lifted. Hence, $\mathcal{O}_{\mathbb{C}^3\times\mathbb{C},\mathbf{0}}/\langle F_1, F_2, F_3\rangle$ is $\mathcal{O}_{\mathbb{C},\mathbf{0}}$-flat and the diagram

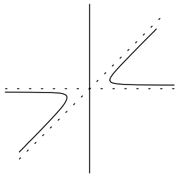


Fig. 1.5. $\{xy - t = xz = yz = 0\}$ (no deformation).

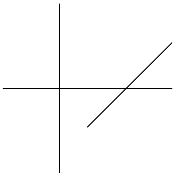


Fig. 1.6. $\{xy - tx = xz = yz = 0\}$ (a deformation).

$$\begin{array}{ccc} (X,\mathbf{0}) & \hookrightarrow & V(F_1,F_2,F_3) \subset (\mathbb{C}^3\times\mathbb{C},(\mathbf{0},0)) \;. \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & (\mathbb{C},0) \end{array}$$

defines a deformation of $(X,0)$.

Note that under the non-flat unfolding (1) the nearby fibre becomes non-connected (see Figure 1.5), while under the flat unfolding (2) the fibre stays connected. Indeed, for each deformation $(X,x) \hookrightarrow (\mathscr{X},x) \to (S,s)$ of a reduced curve singularity, the "nearby fibre" $\mathscr{X}_{\boldsymbol{t}}$, $|\boldsymbol{t}|$ small, is connected (see [BuG]).

Exercises

Exercise 1.2.1. Given $f_1,\dots,f_k \in \mathbb{Q}[\boldsymbol{x}]$, $\widetilde{F}_1,\dots,\widetilde{F}_k \in \mathbb{Q}[\boldsymbol{x},\boldsymbol{t}]$ and an ideal $I \subset \mathbb{Q}[\boldsymbol{t}]$, $\boldsymbol{x} = (x_1,\dots,x_n)$, $\boldsymbol{t} = (t_1,\dots,t_k)$, let F_i denote the image of $\widetilde{F}_i$ in $\mathbb{C}\{\boldsymbol{x},\boldsymbol{t}\}/I\mathbb{C}\{\boldsymbol{x},\boldsymbol{t}\}$. Write a SINGULAR procedure which checks whether the unfolding $F_1,\dots,F_k$ of $f_1,\dots,f_k$ is flat over $\mathbb{C}\{\boldsymbol{t}\}/I\mathbb{C}\{\boldsymbol{t}\}$.

Prove first (using Appendix B) that flatness can be checked by considerung the corresponding morphism of localized polynomial rings instead of the morphism of power series rings.

HINT: Compute the syzygies of $\langle F_1,\dots,F_k\rangle$ and $\langle f_1,\dots,f_k\rangle$ and proceed along the lines of the above example.

Exercise 1.2.2. Let $I_0 \subset \mathbb{C}\{x,y,z,u,v\}$ be the ideal generated by the 2×2-minors of the matrix[3]

$$M_0 = \begin{pmatrix} x & y & z & u \\ y & z & u & v \end{pmatrix}.$$

[3] The variety defined by I_0 in $\mathbb{P}^4$ is called the *rational normal curve* of degree 4 which can be parametrized by $\mathbb{P}^1 \to \mathbb{P}^4$, $(s:t) \mapsto (s^4 : s^3t : s^2t^2 : st^3 : t^4)$. The singularity defined by I_0 in $(\mathbb{C}^5,\mathbf{0})$ is the vertex of the affine *cone over the rational normal curve of degree* 4.

(1) Show that the ideal $I \subset \mathbb{C}\{x,y,z,u,v,a,b,c,d,e,g,h,k\}$ generated by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x+a & y+b & z+c & u+d \\ y+e & z+g & u+h & v+k \end{pmatrix}$$

defines a flat unfolding of I_0.

HINT: This can be shown by using SINGULAR and Exercise 1.2.1.

(2) Consider a 2×4 matrix $\widetilde{M}$ obtained by unfolding (arbitrarily) the entries of M_0. Conclude that the ideal $\widetilde{I}$ which is generated by the 2×2-minors of $\widetilde{M}$ defines a flat unfolding of I_0.

HINT: Use that flatness is preserved under base change, see Proposition I.1.87.

(3) Show that statement (1) does not hold if M_0 is replaced by a 2×4-matrix where the entries are eight independent variables.

1.3 Versal Deformations

A versal deformation of a complex space germ is a deformation which contains basically all information about any possible deformation of this germ. It is one of the fundamental facts of deformation theory that any isolated singularity (X,x) has a versal deformation. We shall prove this theorem for isolated singularities of complete intersections.

In a little less informal way we say that a deformation (i,ϕ) of (X,x) over (S,s) is versal if any other deformation of (X,x) over some base space (T,t) can be induced from (i,ϕ) by some base change $\varphi\colon (T,t) \to (S,s)$. Moreover, if a deformation of (X,x) over some subgerm $(T',t) \subset (T,t)$ is given and induced by some base change $\varphi'\colon (T',t) \to (S,s)$, then φ can be chosen in such a way that it extends φ'. This fact is important, though it might seem a bit technical, as it allows us to construct versal deformations by successively extending over bigger and bigger spaces in a formal manner (see Appendix C for general fundamental facts about formal deformations, in particular, Theorem C.1.6, p. 429, and the sketch of its proof).

Definition 1.8. (1) A deformation $(X,x) \overset{i}{\hookrightarrow} (\mathscr{X},x) \xrightarrow{\phi} (S,s)$ of (X,x) is called *complete* if, for any deformation $(j,\psi)\colon (X,x) \hookrightarrow (\mathscr{Y},y) \to (T,t)$ of (X,x), there exists a morphism $\varphi\colon (T,t) \to (S,s)$ such that (j,ψ) is isomorphic to the induced deformation $(\varphi^* i, \varphi^* \phi)$.

(2) The deformation (i,ϕ) is called *versal* (respectively *formally versal*) if, for a given deformation (j,ψ) as above the following holds: for any closed embedding $k\colon (T',t) \hookrightarrow (T,t)$ of complex germs (respectively of Artinian complex germs) and any morphism $\varphi'\colon (T',t) \to (S,s)$ such that $({\varphi'}^* i, {\varphi'}^* \phi)$ is isomorphic to $(k^* j, k^* \psi)$ there exists a morphism $\varphi\colon (T,t) \to (S,s)$ satisfying

(i) $\varphi \circ k = \varphi'$, and
(ii) $(j, \psi) \cong (\varphi^* i, \varphi^* \phi)$.
That is, there exists a commutative diagram with Cartesian squares

$$\begin{tikzcd}
& (X,x) \arrow[dl, hook', "k^*j"'] \arrow[d, hook, "j"] \arrow[dr, hook, "i"] & \\
k^*(\mathscr{Y},y) \arrow[r, hook] \arrow[d, "k^*\psi"'] \arrow[dr, phantom, "\square"] & (\mathscr{Y},y) \arrow[r, dashed] \arrow[d, "\psi"'] \arrow[dr, phantom, "\square"] & (\mathscr{X},x) \arrow[d, "\phi"] \\
(T',t) \arrow[r, hook, "k"'] \arrow[rr, bend right, "\varphi'"'] & (T,t) \arrow[r, dashed, "\varphi"'] & (S,s)\,.
\end{tikzcd}$$

(3) A (formally) versal deformation is called *semiuniversal* or *miniversal* if, with the notations of (2), the Zariski tangent map $T(\varphi)\colon T_{(T,t)} \to T_{(S,s)}$ is uniquely determined by (i, ϕ) and (j, ψ).

Note that we do not require in (3) that φ itself is uniquely determined (this would be a too restrictive concept for isolated singularities).

A versal deformation is complete (take as (T', t) the reduced point $\{s\}$), but the converse is not true in general. In the literature the distinction between complete and versal deformations is not always sharp, some authors call complete deformations (in our sense) versal. However, the full strength of versal (and, hence, semiuniversal) deformations comes from the property requested in (2).

If we know a versal deformation of (X, x), we know, at least in principle, all other deformations (up to the knowledge of the base change map φ). In particular, we know all nearby fibres and, hence, all nearby singularities which can appear for an arbitrary deformation of (X, x).

An arbitrary complex space germ may not have a versal deformation. It is a fundamental theorem of Grauert [Gra1] that for *isolated singularities* a semiuniversal deformation exists.

Theorem 1.9 (Grauert, 1972). *Any complex space germ (X,x) with isolated singularity*[4] *has a semiuniversal deformation*

$$(X,x) \overset{i}{\hookrightarrow} (\mathscr{X},x) \overset{\phi}{\to} (S,s)\,.$$

We shall prove the formal part of the theorem when (X, x) is an isolated complete intersection. For the general case we refer to [Gra1, Ste, DJP].

Even if we know the existence of a semiuniversal deformation of an isolated singularity, we cannot say anything in advance about its structure for general

[4] More generally, a semiuniversal deformation exists if $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ (see Definition 1.19 and Exercise 1.4.3).

singularities. For instance, we can say nothing about the dimension of the base space of the semiuniversal deformation, which we shortly call the *semiuniversal base space*. It is unknown, but believed, that any complex space germ can occur as a semiuniversal base of an isolated singularity.

Definition 1.10. A singularity (X,x) is called *rigid* iff any deformation of (X,x) over some base space (S,s) is *trivial*, that is, isomorphic to the *product deformation*

$$(X,x) \overset{i}{\hookrightarrow} (X,x) \times (S,s) \xrightarrow{p} (S,s)$$

with i the canonical inclusion and p the second projection.

It follows that (X,x) is rigid iff it has a semiuniversal deformation and the semiuniversal base is a reduced point.

Smooth germs are rigid (Exercise 1.3.1). Further examples of rigid singularities are quotient singularities of dimension ≥ 3 (see [Sch1]) or the singularity at 0 of the union of two planes in $\mathbb{C}^4$ defined by $\langle x,y\rangle \cap \langle z,w\rangle$ (this will follow from the infinitesimal theory in Sections 1.4, 1.5). The existence of rigid singular reduced curve (and normal surface) germs is still an open problem, but one conjectures:

Conjecture 1.11. There exist no rigid singular reduced curve singularities and no rigid singular normal surface singularities.

For results on deformations of reduced curve singularities see [Buc, BuG, Gre2, Ste], for deformations of curve singularities with embedded components we refer to [BrG].

The following properties of versal deformations hold in a much more general deformation theoretic context (see Remark C.1.5.1).

Lemma 1.12. *If a semiuniversal deformation of a complex space germ (X,x) exists, then it is uniquely determined up to (non unique) isomorphism.*

Proof. Let $(X,x) \overset{i}{\hookrightarrow} (\mathscr{X},x) \xrightarrow{\phi} (S,s)$ and $(X,x) \overset{j}{\hookrightarrow} (\mathscr{Y},y) \xrightarrow{\psi} (T,t)$ be semiuniversal deformations of the germ (X,x). By versality, there are morphisms $\varphi : (T,t) \to (S,s)$ and $\varphi' : (S,s) \to (T,t)$ such that $\varphi^*(i,\phi) \cong (j,\psi)$ and $\varphi'^{\,*}(j,\psi) \cong (i,\phi)$ and, hence, $(\varphi \circ \varphi')^*(i,\phi) \cong (i,\phi)$.

Since $\mathrm{id}^*(i,\phi) \cong (i,\phi)$, where id is the identity of (S,s), and since the tangent map $T(\varphi\circ\varphi')$ of $\varphi\circ\varphi'$ is uniquely determined (by semiuniversality), we get

$$T(\varphi\circ\varphi') = T(\varphi)\circ T(\varphi') = T(\mathrm{id})\,,$$

which is the identity. Interchanging the role of φ and φ' we see that $T(\varphi)$ is an isomorphism and hence φ is an isomorphism by the inverse function theorem (Theorem I.1.21). $\square$

We mention the following theorem, which was proved by Flenner [Fle1] in a more general context:

Theorem 1.13. *If a versal deformation of (X,x) exists then there exists also a semiuniversal deformation, and every formally versal deformation of (X,x) is versal.*

For the proof see [Fle1, Satz 5.2]. It is based on the following useful result:

Proposition 1.14. *Every versal deformation of (X,x) differs from the semiuniversal deformation by a smooth factor.*

More precisely, let $\phi : (\mathscr{X},x) \to (S,s)$ be the semiuniversal deformation and $\psi : (\mathscr{Y},y) \to (T,t)$ a versal deformation of (X,x). Then there exists a $p \geq 0$ and an isomorphism

$$\varphi \colon (T,t) \xrightarrow{\cong} (S,s) \times (\mathbb{C}^p, \mathbf{0})$$

such that $\psi \cong (\pi \circ \varphi)^* \phi$ where $\pi \colon (S,s) \times (\mathbb{C}^p, \mathbf{0}) \to (S,s)$ is the projection on the first factor.

Proof. By versality of ψ and semiuniversality of ϕ we get morphisms $(S,s) \to (T,t) \to (S,s)$ such that the tangent map of the composition is the identity. If $\alpha : \mathcal{O}_{S,s} \to \mathcal{O}_{T,t}$ denotes the corresponding ring map, this implies that the induced map $\dot{\alpha} : \mathfrak{m}_{S,s}/\mathfrak{m}_{S,s}^2 \to \mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2$ of cotangent spaces is injective. We may assume that $S \subset \mathbb{C}^n$, $T \subset \mathbb{C}^m$, $s = \mathbf{0}$, $t = \mathbf{0}$, that $\mathcal{O}_{S,s} = \mathbb{C}\{\boldsymbol{s}\}/I = \mathbb{C}\{s_1, \dots, s_n\}/I$ with $I \subset \langle \boldsymbol{s} \rangle^2$, $\mathcal{O}_{T,t} = \mathbb{C}\{\boldsymbol{t}\}/J = \mathbb{C}\{t_1, \dots, t_m\}/J$ with $J \subset \langle \boldsymbol{t} \rangle^2$ (Lemma I.1.24), and that $\dot{\alpha}(s_i) = t_i$, for $i = 1, \dots, n = m - p$, $p := \dim_{\mathbb{C}} \operatorname{Coker}(\dot{\alpha})$.

Let $(s_{n+1}, \dots, s_m)$ be further variables, generating the maximal ideal of $(\mathbb{C}^p, \mathbf{0})$. The map $\gamma : \mathfrak{m}_{T,\mathbf{0}}/\mathfrak{m}_{T,\mathbf{0}}^2 \to \mathfrak{m}_{S \times \mathbb{C}^p, \mathbf{0}}/\mathfrak{m}_{S \times \mathbb{C}^p, \mathbf{0}}^2$, $t_i \mapsto s_i$, $i = 1, \dots, m$, is an isomorphism, inducing an isomorphism

$$\mathcal{O}_{\widetilde{T},\mathbf{0}} := \mathcal{O}_{T,\mathbf{0}}/\mathfrak{m}_{T,\mathbf{0}}^2 \xrightarrow{\cong} \mathcal{O}_{\widetilde{S},\mathbf{0}} := \mathcal{O}_{S \times \mathbb{C}^p, \mathbf{0}}/\mathfrak{m}_{S \times \mathbb{C}^p, \mathbf{0}}^2$$

of analytic algebras. This corresponds to an isomorphism of complex germs $(\widetilde{S}, \mathbf{0}) \xrightarrow{\cong} (\widetilde{T}, \mathbf{0})$, where $(\widetilde{S}, \mathbf{0}) \subset (S \times \mathbb{C}^p, \mathbf{0})$ and $(\widetilde{T}, \mathbf{0}) \subset (T, \mathbf{0})$ are the (fat point) subspaces defined by the squares of the maximal ideals.

Let $\widetilde{\chi}$ be the composition $\widetilde{\chi} : (\widetilde{S}, \mathbf{0}) \xrightarrow{\cong} (\widetilde{T}, \mathbf{0}) \subset (T, \mathbf{0})$. Consider the deformation $\phi \times \mathrm{id} : (\mathscr{X}, x) \times (\mathbb{C}^p, \mathbf{0}) \to (S, \mathbf{0}) \times (\mathbb{C}^p, \mathbf{0})$ of (X,x). By versality of ψ, it can be induced from ψ by a map $\chi : (S, \mathbf{0}) \times (\mathbb{C}^p, \mathbf{0}) \to (T, \mathbf{0})$ such that $\chi|_{(\widetilde{S}, \mathbf{0})} = \widetilde{\chi}$. This implies that the cotangent map of χ, which is γ, is an isomorphism. Hence, by the inverse function theorem, χ is an isomorphism and the result follows. □

Remark 1.14.1. The statements of 1.9 − 1.14 also hold for *multigerms*

$$(X,x) = \coprod_{\ell=1}^{r} (X_\ell, x_\ell)\,,$$

that is, for the disjoint union of finitely many germs (the existence as in Theorem 1.9 is assured if all germs (X_ℓ, x_ℓ) have isolated singular points). Here, a (*versal*, resp. *semiuniversal*) deformation of (X,x) over (S,s) is a multigerm $(\boldsymbol{i},\boldsymbol{\phi}) = \coprod_{\ell=1}^{r}(i_\ell,\phi_\ell)$ such that, for each $\ell = 1,\dots,r$, (i_ℓ,ϕ_ℓ) is a (versal, resp. semiuniversal) deformation of (X_ℓ,x_ℓ) over (S,s).

For the proof of the following theorem, we refer to [Fle1, Fle, Tei] (see also Exercise 1.3.4).

Theorem 1.15 (Openness of versality). *Let $f : X \to S$ be a flat morphism of complex spaces such that $\mathrm{Sing}(f)$ is finite over S. Then the set of points $s \in S$ such that f induces a versal deformation of the multigerm $\big(X, \mathrm{Sing}(f^{-1}(s))\big)$ is analytically open in S.*

It follows from this theorem that if $\phi : (\mathscr{X}, x) \to (S,s)$ is a versal deformation of $\big(\phi^{-1}(s), x\big)$ then, for a sufficiently small representative $\phi : \mathscr{X} \to S$, any multigerm $\phi : \coprod_{x' \in \phi^{-1}(t)}(\mathscr{X}, x') \to (S,t)$, $t \in S$, is a versal deformation of the multigerm $\coprod_{x'\in\phi^{-1}(t)}\big(\phi^{-1}(t), x'\big)$. Note that, due to Theorem I.1.115, $\mathrm{Sing}(f) \cap f^{-1}(s) = \mathrm{Sing}(f^{-1}(s))$ is a finite set.

The analogous statement does not hold for "semiuniversal" in place of "versal".

Although we cannot say anything specific about the semiuniversal deformation of an arbitrary singularity, the situation is different for special classes of singularities. For example, hypersurface singularities or, more generally, complete intersection singularities are never rigid and we can compute explicitly the semiuniversal deformation as we shall show now:

Theorem 1.16. *Let $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ be an isolated complete intersection singularity, and let $f := (f_1,\dots,f_k)$ be a minimal set of generators for the ideal of $(X,\mathbf{0})$. Let $g_1,\dots,g_\tau \in \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$, $g_i = (g_i^1,\dots,g_i^k)$, represent a basis (respectively a system of generators) for the finite dimensional $\mathbb{C}$-vector space*[5]

$$T^1_{(X,\mathbf{0})} := \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} \big/ \big(Df \cdot \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}} + \langle f_1,\dots,f_k\rangle \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\big)\,,$$

and set $F = (F_1,\dots,F_k)$,

$$\begin{aligned} F_1(\boldsymbol{x},\boldsymbol{t}) &= f_1(\boldsymbol{x}) + \sum_{j=1}^{\tau} t_j g_j^1(\boldsymbol{x})\,,\\ &\ \,\vdots \qquad\qquad \vdots \\ F_k(\boldsymbol{x},\boldsymbol{t}) &= f_k(\boldsymbol{x}) + \sum_{j=1}^{\tau} t_j g_j^k(\boldsymbol{x})\,,\end{aligned}$$

$$(\mathscr{X},\mathbf{0}) := V(F_1,\dots,F_k) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0})\,.$$

[5] The vector space $T^1_{(X,x)}$ will be defined for arbitrary complex space germs (X,x) in Definition 1.19. Both definitions coincide by Exercise 1.4.5. For a definition of T^1 in a general deformation theoretic context see Appendix C.

Then $(X,\mathbf{0}) \overset{i}{\hookrightarrow} (\mathscr{X},\mathbf{0}) \xrightarrow{\phi} (\mathbb{C}^\tau,\mathbf{0})$ *with* i,ϕ *being induced by the inclusion* $(\mathbb{C}^n,\mathbf{0}) \subset (\mathbb{C}^n \times \mathbb{C}^\tau,\mathbf{0})$*, respectively the projection* $(\mathbb{C}^n \times \mathbb{C}^\tau,\mathbf{0}) \to (\mathbb{C}^\tau,\mathbf{0})$*, is a semiuniversal (respectively versal) deformation of* $(X,\mathbf{0})$*.*

Here, Df denotes the Jacobian matrix of f,

$$(Df) = \left(\frac{\partial f_i}{\partial x_j}\right) : \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}} \longrightarrow \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\,,$$

that is, $(Df)\cdot\mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}}$ is the submodule of $\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$ spanned by the columns of the Jacobian matrix of f.

Note that $T^1_{(X,\mathbf{0})}$ is an $\mathcal{O}_{X,\mathbf{0}}$-module, called the *Tjurina module* of the complete intersection $(X,\mathbf{0})$. If $(X,\mathbf{0})$ is a hypersurface, then $T^1_{(X,\mathbf{0})}$ is an algebra and called the *Tjurina algebra of* $(X,\mathbf{0})$.

Since the hypersurface case is of special importance we state it explicitly.

Corollary 1.17. *Let* $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ *be an isolated singularity defined by* $f \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ *and* $g_1,\dots,g_\tau \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ *a* $\mathbb{C}$*-basis of the Tjurina algebra*

$$T^1_{(X,\mathbf{0})} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/\big\langle f, \tfrac{\partial f}{\partial x_1},\dots,\tfrac{\partial f}{\partial x_n}\big\rangle\,.$$

If we set

$$F(\boldsymbol{x},\boldsymbol{t}) := f(\boldsymbol{x}) + \sum_{j=1}^{\tau} t_j g_j(\boldsymbol{x})\,, \quad (\mathscr{X},\mathbf{0}) := V(F) \subset (\mathbb{C}^n \times \mathbb{C}^\tau,\mathbf{0})\,,$$

then $(X,\mathbf{0}) \hookrightarrow (\mathscr{X},\mathbf{0}) \xrightarrow{\phi} (\mathbb{C}^\tau,\mathbf{0})$*, with* ϕ *the second projection, is a semiuniversal deformation of* $(X,\mathbf{0})$*.*

Remark 1.17.1. Using the notation of Theorem 1.16, we can choose the basis $g_1,\dots,g_\tau \in \mathcal{O}^k_{\mathbb{C}^n,0}$ of $T^1_{(X,\mathbf{0})}$ such that $g_i = -e_i$, $e_i = (0,\dots,1,\dots,0)$ the i-th canonical generator of $\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$, for $i = 1,\dots,k$ (assuming that $f_i \in \mathfrak{m}^2_{\mathbb{C}^n,\mathbf{0}}$). Then

$$F_i = f_i - t_i + \sum_{j=k+1}^{\tau} t_j g^i_j\,,$$

and we can eliminate $t_1,\dots,t_k$ from $F_1 = \ldots = F_k = 0$. Hence, the semiuniversal deformation of $(X,\mathbf{0})$ is given by

$$\psi : (\mathbb{C}^n \times \mathbb{C}^{\tau-k},\mathbf{0}) \to (\mathbb{C}^k \times \mathbb{C}^{\tau-k},\mathbf{0}) = (\mathbb{C}^\tau,\mathbf{0})$$

with $\psi(\boldsymbol{x},t_1,\dots,t_{\tau-k}) = (G_1(\boldsymbol{x},\boldsymbol{t}),\dots,G_k(\boldsymbol{x},\boldsymbol{t}),t_1,\dots,t_{\tau-k})$,

$$G_i(\boldsymbol{x},\boldsymbol{t}) = f_i(\boldsymbol{x}) + \sum_{j=k+1}^{\tau} t_j g_j(\boldsymbol{x})\,,$$

where $g_j = (g_j^1, \dots, g_j^k)$, $j = k+1, \dots, \tau$, is a basis of the $\mathbb{C}$-vector space

$$\big(\mathfrak{m} \cdot \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\big) \big/ \big((Df) \cdot \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}} + \langle f_1, \dots, f_k \rangle \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\big) ,$$

assuming $f_1, \dots, f_k \in \mathfrak{m}^2_{\mathbb{C}^n,\mathbf{0}}$.

In particular, if $f \in \mathfrak{m}^2_{\mathbb{C}^n,\mathbf{0}}$ and if $1, h_1, \dots, h_{\tau-1}$ is a basis of the Tjurina algebra T_f, then (setting $\boldsymbol{t} := (t_1, \dots t_{\tau-1})$

$$F\colon (\mathbb{C}^n \times \mathbb{C}^{\tau-1}, \mathbf{0}) \longrightarrow (\mathbb{C}^\tau, \mathbf{0}) , \ (\boldsymbol{x}, \boldsymbol{t}) \mapsto \Big(f(\boldsymbol{x}) + \sum_{i=1}^{\tau-1} t_i h_i ,\, \boldsymbol{t} \Big)$$

is a semiuniversal deformation of the hypersurface singularity $(f^{-1}(0), \mathbf{0})$.

Proof of Theorem 1.16. Let $f = (f_1, \dots, f_k)$, and let $g_1, \dots, g_\tau \in \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$ represent a $\mathbb{C}$-basis for the quotient

$$\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} \big/ \big(Df \cdot \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}} + \langle f \rangle \cdot \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\big)$$

(the same arguments work if we start with a system of generators). We want to show the versality of $(X, \mathbf{0}) \overset{i}{\hookrightarrow} (\mathscr{X}, \mathbf{0}) \xrightarrow{\phi} (\mathbb{C}^\tau, \mathbf{0})$, where

$$\mathscr{X} = \left\{ (\boldsymbol{x}, \boldsymbol{s}) \in U \subset \mathbb{C}^n \times \mathbb{C}^\tau \;\middle|\; F(\boldsymbol{x}, \boldsymbol{s}) = f(\boldsymbol{x}) + \sum_{i=1}^{\tau} s_i g_i(\boldsymbol{x}) = 0 \right\} ,$$

$U \subset \mathbb{C}^n \times \mathbb{C}^\tau$ a sufficiently small neighbourhood of $(\mathbf{0}, \mathbf{0})$.

For simplicity, we show only the completeness of (i, ϕ); the proof of the versality is basically the same but with more complicated notation. Moreover, in order to reduce the complexity of notations, we frequently omit the base points of the germs such that $\mathbb{C}^n$ means a sufficiently small neighbourhood of $\mathbf{0} \in \mathbb{C}^n$.

Let $(X, \mathbf{0}) \overset{j}{\hookrightarrow} (\mathscr{Y}, y) \xrightarrow{\psi} (T, t)$ be any deformation of $(X, \mathbf{0})$. We have to show that ψ is induced by a map $\varphi\colon T \to \mathbb{C}^\tau$. By Corollary 1.6, we may assume that ψ is embedded, that is, $\mathscr{Y} \subset \mathbb{C}^n \times T$, $T \subset \mathbb{C}^r$, $t = \mathbf{0}$, and that $\mathscr{Y}$ is defined by k equations $G_j(\boldsymbol{x}, \boldsymbol{t}) = 0$, $j = 1 \dots, k$, with $G_j(\boldsymbol{x}, \mathbf{0}) = f_j(\boldsymbol{x})$. We set $G = (G_1, \dots, G_k)$.

Now, Theorem 1.16 just asserts the existence of a commutative diagram (indices denoting the variables)

$$\begin{tikzcd}
 & & & (X,\mathbf{0}) \arrow[dl, hook'] \arrow[dr, hook] & & \\
\mathbb{C}^n_{\boldsymbol{x}} \times \mathbb{C}^r_{\boldsymbol{t}} \supset & \mathbb{C}^n \times T \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{Y} \arrow[l, hook'] \arrow[rr, "\widetilde{\varphi}"] \arrow[d] \arrow[drr, phantom, "\square"] & & \mathscr{X} \arrow[d] & \subset \mathbb{C}^n_{\boldsymbol{x}} \times \mathbb{C}^\tau_{\boldsymbol{s}} \\
\mathbb{C}^r_{\boldsymbol{t}} \supset & T \arrow[r, equal] & T \arrow[rr, "\varphi"'] & & \mathbb{C}^\tau_{\boldsymbol{s}} \, . &
\end{tikzcd}$$

Note that the map $\mathscr{Y} \to T$ is automatically flat by Proposition 1.7. Hence, it suffices to show the existence of holomorphic map germs

- $\varphi\colon \mathbb{C}^r_{\boldsymbol{t}} \to \mathbb{C}^\tau_{\boldsymbol{s}}$ (the base change map) such that the fibre product

$$\begin{aligned}\mathscr{X} \times_{\mathbb{C}^\tau} T &= \{(\boldsymbol{x},\boldsymbol{s},\boldsymbol{t}) \in \mathbb{C}^n \times \mathbb{C}^\tau \times T \mid F(\boldsymbol{x},\boldsymbol{s}) = 0,\ \boldsymbol{s} = \varphi(\boldsymbol{t})\} \\ &= \{(\boldsymbol{x},\boldsymbol{t}) \in \mathbb{C}^n \times T \mid F(\boldsymbol{x},\varphi(\boldsymbol{t})) = 0\}\end{aligned}$$

  is isomorphic to $\mathscr{Y} = \{(\boldsymbol{x},\boldsymbol{t}) \in \mathbb{C}^n \times T \mid G(\boldsymbol{x},\boldsymbol{t}) = 0\}$ by an isomorphism which is the identity on X and respects the projection to T.

To find the required isomorphism between $\mathscr{X} \times_{\mathbb{C}^\tau} T$ and $\mathscr{Y}$ means to find holomorphic map germs h and H having the following properties:

- $h\colon \mathbb{C}^n_{\boldsymbol{x}} \times \mathbb{C}^r_{\boldsymbol{t}} \to \mathbb{C}^n_{\boldsymbol{x}}$, $h(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{x}$.

Note that this implies that the map $\widetilde{h} : (\boldsymbol{x},\boldsymbol{t}) \mapsto (h(\boldsymbol{x},\boldsymbol{t}),\boldsymbol{t})$ is, for small $\boldsymbol{t}$, a coordinate transformation of $\mathbb{C}^n \times \mathbb{C}^r$, respecting the projection to $\mathbb{C}^r$ and being the identity on X; we require that $\widetilde{h}\big(\mathscr{X} \times_{\mathbb{C}^r} T\big) = \mathscr{Y}$.
Moreover, we ask for a holomorphic map germ

- $H\colon \mathbb{C}^n_{\boldsymbol{x}} \times \mathbb{C}^r_{\boldsymbol{t}} \to \mathrm{Mat}(k \times k, \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^r})$, $H(\boldsymbol{x},\boldsymbol{0}) = 0$ such that $\mathbf{1}_k + H(\boldsymbol{x},\boldsymbol{t})$, which is an invertible matrix for $\boldsymbol{t}$ small, maps the generators $F_j(\boldsymbol{x},\varphi(\boldsymbol{t}))$ of the ideal of $\mathscr{X} \times_{\mathbb{C}^\tau} T$ to the generators $G_j \circ \widetilde{h}$ of the ideal of $\widetilde{h}^{-1}(\mathscr{Y})$.

In other words, we require that φ, h, H satisfy

$$G\big(h(\boldsymbol{x},\boldsymbol{t}),\boldsymbol{t}\big) = \big(\mathbf{1}_k + H(\boldsymbol{x},\boldsymbol{t})\big) \cdot F(\boldsymbol{x},\varphi(\boldsymbol{t}))\,. \tag{1.3.1}$$

We prove the existence of a *formal solution* by a "Potenzreihenansatz". For this purpose, we write $\varphi \in \mathcal{O}^\tau_{\mathbb{C}^r}$, $h \in \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^r}$, $H \in \mathrm{Mat}(k \times k, \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^r})$ as tuples, respectively matrices, of power series

$$\begin{aligned}\varphi(\boldsymbol{t}) &= \varphi_0(\boldsymbol{t}) + \ldots + \varphi_\ell(\boldsymbol{t}) + \ldots \\ h(\boldsymbol{x},\boldsymbol{t}) &= h_0(\boldsymbol{x},\boldsymbol{t}) + \ldots + h_\ell(\boldsymbol{x},\boldsymbol{t}) + \ldots \\ H(\boldsymbol{x},\boldsymbol{t}) &= H_0(\boldsymbol{x},\boldsymbol{t}) + \ldots + H_\ell(\boldsymbol{x},\boldsymbol{t}) + \ldots\end{aligned}$$

where the components of $\varphi_\ell, h_\ell, H_\ell$ are homogeneous polynomials of degree ℓ in $\boldsymbol{t}$ (with coefficients in $\mathbb{C}$ for φ_ℓ, respectively in $\mathcal{O}_{\mathbb{C}^n}$ for h_ℓ and H_ℓ). We set

$$\varphi^\ell(\boldsymbol{t}) = \sum_{i=0}^{\ell} \varphi_i(\boldsymbol{t})\,, \quad h^\ell(\boldsymbol{x},\boldsymbol{t}) = \sum_{i=0}^{\ell} h_\ell(\boldsymbol{x},\boldsymbol{t})\,, \quad H^\ell(\boldsymbol{x},\boldsymbol{t}) = \sum_{i=0}^{\ell} H_\ell(\boldsymbol{x},\boldsymbol{t})\,.$$

Then condition (1.3.1) is equivalent to

$$G\big(h^\ell(\boldsymbol{x},\boldsymbol{t}),\boldsymbol{t}\big) \equiv \big(\mathbf{1} + H^\ell(\boldsymbol{x},\boldsymbol{t})\big) \cdot F\big(\boldsymbol{x},\varphi^\ell(\boldsymbol{t})\big) \mod \langle \boldsymbol{t}\rangle^{\ell+1} \tag{1.3.2}$$

for all ℓ. We construct φ, h, H inductively as power series in $\boldsymbol{t}$ satisfying (1.3.2) for all ℓ. Start with

$$\varphi_0(\boldsymbol{t}) := \boldsymbol{0}\,, \quad h_0(\boldsymbol{x},\boldsymbol{t}) := \boldsymbol{x}\,, \quad H_0(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{0}\,,$$

obviously satisfying (1.3.2) for $\ell = 0$. Assuming (1.3.2) for a given ℓ, we have to construct $\varphi_{\ell+1}, h_{\ell+1}, H_{\ell+1}$ such that (1.3.2) holds for $\ell+1$:

$$G\big(h^{\ell} + h_{\ell+1}, \boldsymbol{t}\big) \equiv \big(\mathbf{1} + H^{\ell} + H_{\ell+1}\big) \cdot F\big(\boldsymbol{x}, \varphi^{\ell} + \varphi_{\ell+1}\big) \mod \langle \boldsymbol{t} \rangle^{\ell+2}.$$

Applying Taylor's formula (up to degree 1, which holds over fields of arbitrary characteristic) to G and F, we obtain (with $h_{\ell+1} = (h_{\ell+1,1}, \dots, h_{\ell+1,n})$ and $\varphi_{\ell+1} = (\varphi_{\ell+1,1}, \dots, \varphi_{\ell+1,\tau})$)

$$\begin{aligned} G\big(h^{\ell} + h_{\ell+1}, \boldsymbol{t}\big) &\equiv G(h^{\ell}, \boldsymbol{t}) + \sum_{i=1}^{n} \frac{\partial G}{\partial x_i}\big(h^{\ell}(\boldsymbol{x}, \boldsymbol{t}), \boldsymbol{t}\big) h_{\ell+1,i} \mod \langle \boldsymbol{t} \rangle^{\ell+2} \\ &\equiv G(h^{\ell}, \boldsymbol{t}) + \sum_{i=1}^{n} \frac{\partial G}{\partial x_i}(\boldsymbol{x}, \mathbf{0}) h_{\ell+1,i} \mod \langle \boldsymbol{t} \rangle^{\ell+2} \\ &\equiv G(h^{\ell}, \boldsymbol{t}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x}) h_{\ell+1,i} \mod \langle \boldsymbol{t} \rangle^{\ell+2}. \end{aligned}$$

The last equality follows since $G(\boldsymbol{x}, \mathbf{0}) = f(\boldsymbol{x})$. Furthermore, we have

$$\begin{aligned} F\big(\boldsymbol{x}, \varphi^{\ell} + \varphi_{\ell+1}\big) &\equiv F(\boldsymbol{x}, \varphi^{\ell}) + \sum_{j=1}^{\tau} \frac{\partial F}{\partial s_j}(\boldsymbol{x}, \varphi^{\ell}) \varphi_{\ell+1,j} \mod \langle \boldsymbol{t} \rangle^{\ell+2} \\ &\equiv F(\boldsymbol{x}, \varphi^{\ell}) + \sum_{j=1}^{\tau} g_s(\boldsymbol{x}) \varphi_{\ell+1,j} \mod \langle \boldsymbol{t} \rangle^{\ell+2}. \end{aligned}$$

Using this and also that $F(\boldsymbol{x}, \varphi^{\ell}) = f(\boldsymbol{x}) \mod \langle \boldsymbol{t} \rangle$, condition (1.3.2) for $\ell+1$ reads

$$\begin{aligned} &G(h^{\ell}, \boldsymbol{t}) - (\mathbf{1} + H^{\ell}) \cdot F(\boldsymbol{x}, \varphi^{\ell}) \\ &\qquad \equiv \sum_{j=1}^{\tau} g_j \varphi_{\ell+1,j} - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x}) h_{\ell+1,i} + \sum_{i=1}^{k} H_{\ell+1,i} f_i \mod \langle \boldsymbol{t} \rangle^{\ell+2}, \end{aligned} \tag{1.3.3}$$

where $H_{\ell+1,i}$ denote the column vectors of $H_{\ell+1}$.

By induction, $G(h^{\ell}, \boldsymbol{t}) - (\mathbf{1} + H^{\ell}) \cdot F(\boldsymbol{x}, \varphi^{\ell}) \in \mathcal{O}^k_{\mathbb{C}^n} \mod \langle \boldsymbol{t}^{\ell+1} \rangle$. By the choice of $g_1, \dots, g_\tau$ as a system of generators of $T^1_{(X,\mathbf{0})}$, we have the equality of $\mathbb{C}$-vector spaces

$$\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} = \sum_{j=1}^{\tau} g_j \mathbb{C} \;\oplus \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} + \sum_{i=1}^{k} f_i \cdot \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} \right). \tag{1.3.4}$$

This implies, by considering the terms of degree $\ell+1$ in $\boldsymbol{t}$, that we can find $\varphi_{\ell+1,j}(\boldsymbol{t})$, $h_{\ell+1,i}(\boldsymbol{x}, \boldsymbol{t})$, $H_{\ell+1,i}(\boldsymbol{x}, \boldsymbol{t})$ satisfying (1.3.2) for $\ell+1$.

That is, we have shown the existence of formal vector-, respectively matrix-valued, power series φ, h, respectively H, satisfying (1.3.1). For this we did only need that G is a formal power series. In other words, we have proved that the deformation

$$(X,\mathbf{0}) \overset{i}{\hookrightarrow} (\mathscr{X},\mathbf{0}) \overset{\phi}{\to} (\mathbb{C}^\tau,\mathbf{0})$$

is "formally complete". However, if G is convergent, then $\varphi_{\ell+1}, h_{\ell+1}, H_{\ell+1}$ can be chosen such that φ, h, H are convergent, too (see [KaS]).

Another way to prove convergence is to use an approximation theorem of Grauert (Theorem 1.18).

To apply Grauert's approximation theorem to formal versal deformations as constructed above, consider the system of equations

$$\begin{aligned}\phi_i(\boldsymbol{x},\boldsymbol{t},h,H,\varphi) &\equiv 0 \mod I_{(T,\mathbf{0})} \subset \mathbb{C}\{\boldsymbol{t}\}\,,\\ \phi(\boldsymbol{x},\boldsymbol{t},h,H,\varphi) &= G(h,\boldsymbol{t}) - (\mathbf{1}+H)\cdot F(\boldsymbol{x},\varphi)\,.\end{aligned}$$

where $\phi = (\phi_1,\dots,\phi_k)$. We have just shown that the assumptions of Theorem 1.18 below hold for this system. Hence, there exists a convergent solution.

To see that (i,ϕ) is semiuniversal if $g_1,\dots,g_\tau$ are a basis of $T^1_{(X,\mathbf{0})}$ we have to show that the tangent map of φ is uniquely determined. That is, we have to show that in (1.3.2), for $\ell=1$, $\varphi_{1,j}$ is uniquely determined mod $\langle\boldsymbol{t}\rangle^2$. Indeed, for fixed $\varphi^\ell, h^\ell, H^\ell$, any $\ell\geq 0$, $\varphi_{\ell+1,j}$ is uniquely determined mod $\langle\boldsymbol{t}\rangle^{\ell+2}$ by (1.3.2) in degree $\ell+1$ iff $g_1,\dots,g_\tau$ are a $\mathbb{C}$-basis of $\mathcal{O}^k_{\mathbb{C}^n\times\mathbb{C}^\tau}/\sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^\tau} + \sum_{i=1}^k f_i\cdot\mathcal{O}^k_{\mathbb{C}^n\times\mathbb{C}^\tau}$, and, therefore, the coefficients $\varphi^\alpha_{\ell+1,j}\in\mathbb{C}$ in $\varphi_{\ell+1,j} = \sum_{|\alpha|=\ell+1}\varphi^\alpha_{\ell+1,j}\boldsymbol{t}^\alpha$ are uniquely determined. $\square$

Note that in the preceding proof, for $\ell>0$, the $h_{\ell,j}$ and $H_{\ell+1,j}$ are not unique and, hence, $\varphi_{\ell+1}$ depends on the previously chosen h_ℓ, H_ℓ. Therefore, the above uniqueness argument in degree 1 cannot be extended to higher degrees and we cannot expect that φ itself is unique.

Theorem 1.18 (Grauert's approximation theorem). *Let $\phi_1,\dots,\phi_k$ be power series in $\mathbb{C}\{\boldsymbol{x},\boldsymbol{t},h,\varphi\}$ and $I\subset\mathbb{C}\{\boldsymbol{t}\}$ an ideal. Suppose that the system of equations*

$$\begin{aligned}\phi_1(\boldsymbol{x},\boldsymbol{t},h,\varphi) &\equiv 0 \ mod\ I\\ &\vdots\\ \phi_k(\boldsymbol{x},\boldsymbol{t},h,\varphi) &\equiv 0 \ mod\ I\end{aligned}\tag{1.3.5}$$

has a solution $(h,\varphi) = \big(h^{\ell_0}(\boldsymbol{x},\boldsymbol{t}),\varphi^{\ell_0}(\boldsymbol{t})\big)$ up to degree ℓ_0 in $\boldsymbol{t}$. Assume further that for $\ell\geq\ell_0$, every solution $\big(h^\ell(\boldsymbol{x},\boldsymbol{t}),\varphi^\ell(\boldsymbol{t})\big)$ up to degree ℓ extends to a solution $\big(h^\ell+h_{\ell+1},\varphi^\ell+\varphi_{\ell+1}\big)$ up to degree $\ell+1$ in $\boldsymbol{t}$, where $h_{\ell+1}\in\mathbb{C}\{\boldsymbol{x}\}[\boldsymbol{t}]$, $\varphi_{\ell+1}\in\mathbb{C}[\boldsymbol{t}]$ are homogeneous polynomials in $\boldsymbol{t}$ of degree $\ell+1$.

Then the system (1.3.5) *has a convergent solution mod I, that is, there exists $h\in\mathbb{C}\{\boldsymbol{x},\boldsymbol{t}\}$, $\varphi\in\mathbb{C}\{\boldsymbol{t}\}$ such that*

$$\phi_1\big(\boldsymbol{x},\boldsymbol{t},h(\boldsymbol{x},\boldsymbol{t}),\varphi(\boldsymbol{t})\big)\equiv\ldots\equiv\phi_k\big(\boldsymbol{x},\boldsymbol{t},h(\boldsymbol{x},\boldsymbol{t}),\varphi(\boldsymbol{t})\big)\equiv 0 \ \ mod\ I\,.$$

Proof. See [Gra1, Gal, GaH, DJP]. □

Supplement to Theorem 1.18. *Given a formal solution* $\bar{h} \in \mathbb{C}\{\boldsymbol{x}\}[[\boldsymbol{t}]]$, $\bar{\varphi} \in \mathbb{C}[[\boldsymbol{t}]]$ *of* (1.3.5) *and a positive integer* $c > 0$. *Then there exists a convergent solution* $h \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$, $\varphi \in \mathbb{C}\{\boldsymbol{t}\}$ *such that*

$$\bar{h} - h \in \langle \boldsymbol{t} \rangle^{c+1}\mathbb{C}\{\boldsymbol{x}\}[[\boldsymbol{t}]]\,, \quad \bar{\varphi} - \varphi \in \langle \boldsymbol{t} \rangle^{c+1}\mathbb{C}[[\boldsymbol{t}]]\,.$$

Proof. Add to the system (1.3.5) the additional equations in $\mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}, h, \varphi\}$

$$\begin{aligned} h - \bar{h}^{(c)}(\boldsymbol{x}, \boldsymbol{t}) &\equiv 0 \mod I \\ \varphi - \bar{\varphi}^{(c)}(\boldsymbol{t}) &\equiv 0 \mod I \end{aligned}$$

where $\bar{h}^{(c)} \in \mathbb{C}\{\boldsymbol{x}\}[\boldsymbol{t}]$, respectively $\bar{\varphi}^{(c)} \in \mathbb{C}[\boldsymbol{t}]$, are the terms of $\bar{h}$, respectively $\bar{\varphi}$, up to degree c in $\boldsymbol{t}$. Now apply Grauert's theorem to this bigger system. □

There are other approximation theorems, the most important one is probably Artin's approximation theorem (see [Art, KPR, DJP]).

Grauert's theorem requires that *every* solution h^ℓ, φ^ℓ up to order ℓ extends to a formal solution. Then it guarantees the existence of convergent solutions $h(\boldsymbol{x}, \boldsymbol{t}), \varphi(\boldsymbol{t})$, where φ is *independent of* $\boldsymbol{x}$. Artin's approximation theorem does only require the existence of one formal solution and then it guarantees the existence of a convergent solution. However, there are examples (see [Gab]) that under the weaker assumption of Artin's theorem we get only $h, \varphi \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$ with φ not independent of $\boldsymbol{x}$.

Artin's theorem has many applications but for the existence of a convergent semiuniversal deformation for arbitrary isolated singularities we need Grauert's theorem (for complete intersections this can be avoided by direct estimates as given in [KaS]).

Example 1.18.1. (1) Let $f(x_1, \ldots, x_n) = x_1^{k+1} + x_2^2 + \ldots + x_n^2$ define an A_k-singularity, then $1, x_1, \ldots, x_1^{k-1}$ is a basis of

$$T^1_{(f^{-1}(0),\boldsymbol{0})} = \mathbb{C}\{\boldsymbol{x}\}/\langle x_1^k, x_2, \ldots, x_n\rangle\,.$$

Therefore, by Remark 1.17.1, $\psi\colon (\mathbb{C}^n \times \mathbb{C}^{k-1}, \boldsymbol{0}) \to (\mathbb{C}^k, \boldsymbol{0})$,

$$(\boldsymbol{x}, \boldsymbol{t}) \longmapsto \Big(f(\boldsymbol{x}) + \sum_{i=1}^{k-1} t_i x_1^i,\, t_1, \ldots, t_{k-1}\Big)$$

is a semiuniversal deformation of $\big(f^{-1}(0), \boldsymbol{0}\big)$.

(2) Let $(X, \boldsymbol{0}) \subset (\mathbb{C}^3, \boldsymbol{0})$ be the isolated complete intersection curve singularity defined by the vanishing of $f_1(\boldsymbol{x}) = x_1^2 + x_2^3$ and of $f_2(\boldsymbol{x}) = x_3^2 + x_2^3$. Then the Tjurina module is $T^1_{(X,\boldsymbol{0})} = \mathbb{C}\{\boldsymbol{x}\}^2/M$, where $M \subset \mathbb{C}\{\boldsymbol{x}\}^2$ is generated by $\binom{x_1}{0}, \binom{x_2^2}{x_2^2}, \binom{0}{x_3}, \binom{f_1}{0}, \binom{0}{f_1}, \binom{f_2}{0}, \binom{0}{f_2}$. We have $\tau = 9$ and a $\mathbb{C}$-basis for

$T^1_{(X,\mathbf{0})}$ is given by $\binom{1}{0}, \binom{0}{1}, \binom{x_2}{0}, \binom{x_3}{0}, \binom{x_2x_3}{0}, \binom{x_2^2}{0}, \binom{0}{x_1}, \binom{0}{x_2}, \binom{0}{x_1x_2}$. Again by Remark 1.17.1, it follows that a semiuniversal deformation of $(X,\mathbf{0})$ is given by $\psi\colon (\mathbb{C}^{10},\mathbf{0}) \to (\mathbb{C}^9,\mathbf{0})$,

$$(\boldsymbol{x},\boldsymbol{t}) \longmapsto \big(f_1(\boldsymbol{x})+t_1x_2+t_2x_3+t_3x_2x_3+t_4x_2^2,\ f_2(\boldsymbol{x})+t_5x_1+t_6x_2+t_7x_1x_2, \boldsymbol{t}\big)\,.$$

We can compute this in a SINGULAR session:

```
ring R = 0,(x(1..3)),ds;
ideal f = x(1)^2+x(2)^3, x(3)^2+x(2)^3;
module M = jacob(f) + f*freemodule(2);
ncols(M);                   // number of generators for M
//-> 7
M = simplify(M,1);          // transform leading coefficients to 1
print(M[5]);                // display the 5th generator of M
//-> [0,x(1)^2+x(2)^3]
print(kbase(std(M)));       // a K-basis for the Tjurina module
//-> x(2)*x(3),x(3),x(2)^2,x(2),1,0,       0,   0,   0,
//-> 0,        0,   0,      0,   0,x(1)*x(2),x(2),x(1),1
```

Exercises

Exercise 1.3.1. Show that smooth complex space germs are rigid.

Exercise 1.3.2. Show that non-smooth hypersurface germs are not rigid.

Exercise 1.3.3. Compute a semiuniversal deformation for

(1) the hypersurface singularity $\{x_1^3 + x_2^5 + x_3^2 + \ldots + x_n^2 = 0\} \subset (\mathbb{C}^n,\mathbf{0})$,
(2) the complete intersection singularity $\{x^2 + y^3 = y^2 + z^2 = 0\} \subset (\mathbb{C}^3,\mathbf{0})$.

In the last exercise we sketch a proof for openness of versality (Theorem 1.15) for an isolated complete intersection singularity $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$:

Exercise 1.3.4. Let $(X,\mathbf{0}) = V(f_1,\ldots,f_k) \subset (\mathbb{C}^n,\mathbf{0})$ be an isolated complete intersection singularity, and let $(i,\phi)\colon (X,\mathbf{0}) \hookrightarrow (\mathscr{X},\mathbf{0}) \to (\mathbb{C}^r,\mathbf{0})$ be a deformation of $(X,\mathbf{0})$ with smooth base, given by an unfolding

$$F(\boldsymbol{x},\boldsymbol{s}) = f(\boldsymbol{x}) + h(\boldsymbol{x},\boldsymbol{s}) : (\mathbb{C}^n,\mathbf{0}) \times (\mathbb{C}^r,\mathbf{0}) \to (\mathbb{C}^k,\mathbf{0})\,, \quad h(\boldsymbol{x},\mathbf{0}) = \mathbf{0}\,,$$

that is, $(\mathscr{X},\mathbf{0}) = V(F_1,\ldots,F_k) \subset (\mathbb{C}^n,\mathbf{0}) \times (\mathbb{C}^r,\mathbf{0})$ and ϕ is the projection on the second factor.

Let $\mathscr{X}$ and S be sufficiently small representatives of $(\mathscr{X},\mathbf{0})$ and $(\mathbb{C}^r,\mathbf{0})$, and let

$$\mathcal{J} := \operatorname{Im}\big(D_{\boldsymbol{x}}(F) : \mathcal{O}^n_{\mathscr{X}} \to \mathcal{O}^k_{\mathscr{X}}\big) \subset \mathcal{O}^k_{\mathscr{X}}\,,$$

where $D_{\boldsymbol{x}}(F)_{ij} = \frac{\partial F_i}{\partial x_j}$, $i = 1,\ldots,k$, $j = 1,\ldots,n$. Define the *relative T^1-sheaf*

$$\mathcal{T}^1_{\mathscr{X}/S} := \mathcal{O}^k_{\mathscr{X}} \Big/ \left(\sum_{i=1}^k F_i\mathcal{O}^k_{\mathscr{X}} + \mathcal{J}\right)$$

and prove the following statements:

(1) $\mathrm{supp}(\mathcal{T}^1_{\mathscr{X}/S}) = \mathrm{Sing}(\phi)$.
(2) $\phi : \mathrm{Sing}(\phi) \to S$ is finite.
(3) $\phi_*\mathcal{T}^1_{\mathscr{X}/S}$ is a coherent sheaf satisfying

$$\left(\phi_*\mathcal{T}^1_{\mathscr{X}/S}\right)_s / \mathfrak{m}_{S,s} \cdot \left(\phi_*\mathcal{T}^1_{\mathscr{X}/S}\right)_s \cong \bigoplus_{p\in\phi^{-1}(s)} T^1_{(X,p)}$$

for each $s \in S$.
(4) For $p \in \mathscr{X}$ the induced map $\phi : (\mathscr{X}, p) \to (S, \phi(p))$ is versal iff the vectors $\frac{\partial h}{\partial s_1}, \dots, \frac{\partial h}{\partial s_r}$, evaluated at $s = \phi(p)$, generate $T^1_{(\phi^{-1}(\phi(p)),p)}$.
(5) The set of points $s \in S$ such that ϕ induces a joint versal deformation of the multigerm $\coprod_{p\in\phi^{-1}(s)} \left(\phi^{-1}(s), p\right)$ is the complement of the support of the sheaf

$$\phi_* \left(\mathcal{O}^k_{\mathscr{X}} \Big/ \sum_{i=1}^r \frac{\partial h}{\partial s_i} \cdot \mathcal{O}_{\mathscr{X}} \right) . \tag{1.3.6}$$

Conclude the openness of versality statement by showing that the support of the sheaf (1.3.6) is a closed analytic set in S.

1.4 Infinitesimal Deformations

In this section we develop infinitesimal deformation theory for *arbitrary singularities*. In particular, we introduce in this generality the vector spaces $T^1_{(X,x)}$ of *first order deformations*, that is, the linearization of the deformations of (X, x) and show how it can be computed. Moreover, we describe the obstructions for lifting an infinitesimal deformation of a given order to higher order. This and the next section can be considered as a concrete special case of the general theory described in Appendix C.1 and C.2.

Infinitesimal deformation theory of first order is the deformation theory over the space T_ε, a “point with one tangent direction”.

Definition 1.19. (1) The complex space germ T_ε consists of one point with local ring $\mathbb{C}[\varepsilon] = \mathbb{C} + \varepsilon \cdot \mathbb{C}$, $\varepsilon^2 = 0$, that is, $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/\langle t^2\rangle$ where t is an indeterminate.
(2) For any complex space germ (X, x) define

$$T^1_{(X,x)} := \underline{\mathcal{D}ef}_{(X,x)}(T_\varepsilon)\,,$$

the set of isomorphism classes of deformations of (X, x) over T_ε. Objects of $\mathcal{D}ef_{(X,x)}(T_\varepsilon)$ are called *infinitesimal deformations of* (X, x) (*of first order*).
(3) We shall see in Proposition 1.25 (see also Lemma C.1.7) that $T^1_{(X,x)}$ carries the structure of a complex vector space, even of an $\mathcal{O}_{X,x}$-module. We call $T^1_{(X,x)}$ the *Tjurina module*, and

$$\tau(X, x) := \dim_{\mathbb{C}} T^1_{(X,x)}$$

the *Tjurina number* of (X, x).

By Theorem 1.9, every isolated singularity (X, x) has a semiuniversal deformation. More generally, any singularity (X, x) with $\tau(X, x) < \infty$ has a semiuniversal deformation (see [Gra1, Ste]); by Exercise 1.4.3, isolated singularities have finite Tjurina number.

The following lemma shows that $T^1_{(X,x)}$ can be identified with the Zariski tangent space to the semiuniversal base of (X, x) (if it exists).

Lemma 1.20. *Let (X, x) be a complex space germ and $\phi : (\mathscr{X}, x) \to (S, s)$ a deformation of (X, x). Then there exists a linear map*[6]

$$T_{S,s} \longrightarrow T^1_{(X,x)} \,,$$

called the Kodaira-Spencer map*, which is surjective if ϕ is versal and bijective if ϕ is semiuniversal.*

Moreover, if (X, x) admits a semiuniversal deformation with smooth base space, then ϕ is semiuniversal iff (S, s) is smooth and the Kodaira-Spencer map is an isomorphism.

Proof. For any complex space germ (S, s) we have $T_{S,s} = \mathrm{Mor}\big(T_\varepsilon, (S, s)\big)$ (see Exercise 1.4.1). Define a map

$$\begin{aligned} \alpha\colon \mathrm{Mor}\big(T_\varepsilon, (S, s)\big) &\longrightarrow T^1_{(X,x)} \,, \\ \varphi &\mapsto \big[\varphi^*\phi\big] \,. \end{aligned}$$

Let us see that α is surjective if ϕ is versal: given a class $[\psi] \in T^1_{(X,x)}$ represented by $\psi : (\mathscr{Y}, x) \to T_\varepsilon$, the versality of ϕ implies the existence of a map $\varphi\colon T_\varepsilon \to (S, s)$ such that $\varphi^*\phi \cong \psi$. Hence, $[\psi] = \alpha(\varphi)$, and α is surjective.

If ϕ is semiuniversal, the tangent map $T\varphi$ of $\varphi\colon T_\varepsilon \to (S, s)$ is uniquely determined by ψ. Since φ is uniquely determined by

$$\varphi^\sharp\colon \mathcal{O}_{S,s} \to \mathcal{O}_{T_\varepsilon} = \mathbb{C}[t]/\langle t^2\rangle$$

and, since $\varphi^\sharp$ is local, we obtain $\varphi^\sharp(\mathfrak{m}^2_{S,s}) = 0$. That is, φ is uniquely determined by

$$\underline{\varphi}^\sharp\colon \mathfrak{m}_{S,s}/\mathfrak{m}^2_{S,s} \longrightarrow \langle t\rangle/\langle t^2\rangle$$

and hence by the dual map $\big(\underline{\varphi}^\sharp\big)^* = T\varphi$. Thus, α is bijective. The linearity of α is shown in Exercise 1.4.1 (2).

If (T, t) is the smooth base space of a semiuniversal deformation of (X, x) then there is a morphism $\varphi : (S, s) \to (T, t)$ inducing the map

$$\alpha : T_{S,s} \to T_{T,t} \cong T^1_{(X,x)}$$

constructed above. Since (S, s) is smooth, φ is an isomorphism iff α is (by the inverse function theorem I.1.21). $\square$

[6] Here, $T_{S,s}$ denotes the Zariski tangent space to (S, s), that is, to S at s.

We are now going to describe $T^1_{(X,x)}$ in terms of the defining ideal of (X,x), without knowing a semiuniversal deformation of (X,x). To do this, we need again embedded deformations, that is, deformations of the inclusion map $(X,x) \hookrightarrow (\mathbb{C}^n, \mathbf{0})$. Slightly more general, we define deformations of a morphism, not necessarily an embedding.

Definition 1.21. Let $f\colon (X,x) \to (S,s)$ be a morphism of complex germs.

(1) A *deformation of* f, or a *deformation of* $(X,x) \to (S,s)$, over a germ (T,t) is a Cartesian diagram

$$\begin{tikzcd}
(X,x) \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & (\mathscr{X},x) \arrow[d, "F"] \arrow[dd, bend left=60, "\phi"] \\
(S,s) \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{S},s) \arrow[d, "p"] \\
\{pt\} \arrow[r, hook] & (T,t)
\end{tikzcd}$$

such that i and j are closed embeddings, and p and ϕ are flat (hence deformations of (X,x), respectively (S,s), over (T,t), but F is not supposed to be flat). We denote such a deformation by (i,j,F,p) or just by (F,p).

A *morphism* between two deformations (i,j,F,p) and (i',j',F',p') of f is a commutative diagram

$$\begin{tikzcd}
& (X,x) \arrow[dl, hook, "i"'] \arrow[dr, hook, "i'"] \arrow[dd] & \\
(\mathscr{X},x) \arrow[rr, "\psi_1"] \arrow[ddd, "F"'] & & (\mathscr{X}',x') \arrow[ddd, "F'"] \\
& (S,s) \arrow[dl, hook, "j"'] \arrow[dr, hook, "j'"] \arrow[dd] & \\
(\mathscr{S},s) \arrow[rr, "\psi_2"] \arrow[ddd, "p"'] & & (\mathscr{S}',s') \arrow[ddd, "p'"] \\
& \{pt\} \arrow[dl, hook] \arrow[dr, hook] & \\
(T,t) \arrow[rr, "\varphi"] & & (T',t')
\end{tikzcd} ,$$

and we denote it by $(\psi_1, \psi_2, \varphi)$. If ψ_1, ψ_2, φ are isomorphisms, then $(\psi_1, \psi_2, \varphi)$ is an isomorphism of deformations of f.

We denote by $\mathcal{D}ef_f = \mathcal{D}ef_{(X,x)\to(S,s)}$ the category of deformations of f, by $\mathcal{D}ef_f(T,t) = \mathcal{D}ef_{(X,x)\to(S,s)}(T,t)$ the (non-full) subcategory of deformations of

f over (T,t) with morphisms as above where $\varphi\colon (T,t)\to(T,t)$ is the identity. Furthermore, we write

$$\underline{\mathcal{D}ef}_f(T,t) = \underline{\mathcal{D}ef}_{(X,x)\to(S,s)}(T,t)$$

for the set of isomorphism classes of such deformations.

(2) A deformation (i,j,F,p) of $(X,x)\to(S,s)$ with $(\mathscr{S},s)=(S,s)\times(T,t)$, $j\colon (S,s)\hookrightarrow(S,s)\times(T,t)$ and $p\colon (S,s)\times(T,t)\to(T,t)$, the canonical embedding and projection, respectively, is called a *deformation of* $(X,x)/(S,s)$ *over* (T,t) and denoted by (i,F) or just by F. A morphism of such deformations is a morphism as in (1) of the form $(\psi,\mathrm{id}_{S,s}\times\varphi,\varphi)$; it is denoted by (ψ,φ).

$\mathcal{D}ef_{(X,x)/(S,s)}$ denotes the category of deformations of $(X,x)/(S,s)$, $\mathcal{D}ef_{(X,x)/(S,s)}(T,t)$ the subcategory of deformations of $(X,x)/(S,s)$ over (T,t) with morphisms being the identity on (T,t), and $\underline{\mathcal{D}ef}_{(X,x)/(S,s)}(T,t)$ the set of isomorphism classes of such deformations.

The difference between (1) and (2) is that in (1) we deform (X,x), (S,s) and f, while in (2) we only deform (X,x) and f but not (S,s) (that is, (S,s) is trivially deformed). Note that

$$\mathcal{D}ef_{(X,x)/\mathrm{pt}} = \mathcal{D}ef_{(X,x)}\ .$$

The following lemma shows that embedded deformations are a special case of Definition 1.21 (2).

Lemma 1.22. *Let $f\colon (X,x)\to(S,s)$ be a closed embedding of complex space germs and let*

$$(\mathscr{X},x)\xrightarrow{F}(\mathscr{S},s)\xrightarrow{p}(T,t)$$

be a deformation of f. Then $F\colon (\mathscr{X},x)\to(\mathscr{S},s)$ is a closed embedding, too.

Proof. Tensorize the exact sequence $\mathcal{O}_{\mathscr{S},s}\xrightarrow{F^\sharp}\mathcal{O}_{\mathscr{X},x}\to\mathrm{Coker}(F^\sharp)\to 0$ with $\otimes_{\mathcal{O}_{T,t}}\mathbb{C}$. Then $\mathrm{Coker}(F^\sharp)/\mathfrak{m}_{T,t}\,\mathrm{Coker}(F^\sharp)=0$, since $f^\sharp\colon \mathcal{O}_{S,s}\to\mathcal{O}_{X,x}$ is surjective. By Nakayama's lemma, the $\mathcal{O}_{\mathscr{X},x}$-module $\mathrm{Coker}(F^\sharp)$ is zero, too. $\square$

Definition 1.23. (1) Let $(X,x)\hookrightarrow(S,s)$ be a closed embedding. The objects of $\mathcal{D}ef_{(X,x)/(S,s)}$ are called *embedded deformations* of (X,x) (in (S,s)).

(2) For an arbitrary morphism $f\colon (X,x)\to(S,s)$ we define

$$T^1_{(X,x)\to(S,s)} := \underline{\mathcal{D}ef}_{(X,x)\to(S,s)}(T_\varepsilon)\,,$$

respectively

$$T^1_{(X,x)/(S,s)} := \underline{\mathcal{D}ef}_{(X,x)/(S,s)}(T_\varepsilon)\,,$$

and call its elements the isomorphism classes of *(first order) infinitesimal deformations of* $(X,x)\to(S,s)$, respectively of $(X,x)/(S,s)$.

We define the vector space structure on $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}$ in Proposition 1.25 (see also Exercise 1.4.4).

Note that for $f\colon (X,x) \hookrightarrow (S,s)$ a closed embedding, two embedded deformations of (X,x) in (S,s) over (T,t) are isomorphic iff they are equal, since $(S,s)\times(T,t) \to (S,s)\times(T,t)$ is the identity. Hence, we can identify in this case $\underline{\mathcal{D}ef}_{(X,x)/(S,s)}(T,t)$ with $\mathcal{D}ef_{(X,x)/(S,s)}(T,t)$.

We are going to describe $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}$ and $T^1_{(X,0)}$ in terms of the equations defining $(X,0) \subset (\mathbb{C}^n,\mathbf{0})$. For $T^1_{(X,\mathbf{0})}$, this generalizes the formulas of Theorem 1.16 and Corollary 1.17 (see Exercise 1.4.5). First, we need some preparations:

Definition 1.24. Let S be a smooth n-dimensional complex manifold and $X \subset S$ a complex subspace given by the coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_S$.

(1) The sheaf $(\mathcal{I}/\mathcal{I}^2)\big|_X$ is called the *conormal sheaf* and its dual

$$\mathcal{N}_{X/S} := \mathscr{H}om_{\mathcal{O}_X}\big((\mathcal{I}/\mathcal{I}^2)\big|_X, \mathcal{O}_X\big)$$

is called the *normal sheaf* of the embedding $X \subset S$.

(2) Let $\Omega^1_X = \big(\Omega^1_S/(\mathcal{I}\Omega^1_S + d\mathcal{I}\cdot\mathcal{O}_S)\big)\big|_X$ be the sheaf of holomorphic 1-forms on X. The dual sheaf $\Theta_X := \mathscr{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is called the sheaf of *holomorphic vector fields* on X.

Recall from Theorem I.1.106 that, for each coherent $\mathcal{O}_X$-sheaf $\mathcal{M}$, there is a canonical isomorphism of $\mathcal{O}_X$-modules

$$\mathscr{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{M}) \xrightarrow{\cong} \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{M}), \quad \varphi \longmapsto \varphi\circ d,$$

where $d\colon \mathcal{O}_X \to \Omega^1_X$ is the exterior derivation and where $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X,\mathcal{M})$ is the *sheaf of* $\mathbb{C}$*-derivations of* $\mathcal{O}_X$ *with values in* $\mathcal{M}$. In particular, we have

$$\Theta_X \cong \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

Moreover, recall (Theorem I.1.106) that the sheaf Ω^1_S is locally free with $\Omega^1_{S,s} = \bigoplus_{i=1}^n \mathcal{O}_{S,s}dx_i$ (where $x_1,\dots,x_n$ are local coordinates of S with center s). As a consequence we have that Θ_S is locally free of rank n and

$$\Theta_{S,s} = \bigoplus_{i=1}^n \mathcal{O}_{S,s}\cdot\frac{\partial}{\partial x_i}$$

where $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ is the dual basis of $dx_1,\dots,dx_n$.

Let $f \in \mathcal{O}_S$ then, in local coordinates, we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i$. In particular, we can define an $\mathcal{O}_S$-linear map $\alpha\colon \mathcal{I} \to \Omega^1_S$, $f \mapsto df$. Due to the Leibniz rule, α induces a map $\alpha : \mathcal{I}/\mathcal{I}^2 \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{O}_X$ yielding the following exact sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\alpha} \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0. \tag{1.4.1}$$

Dualizing (1.4.1), we obtain the exact sequence

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_S \otimes_{\mathcal{O}_S} \mathcal{O}_X \xrightarrow{\beta} \mathcal{N}_{X/S}\,, \tag{1.4.2}$$

where β is the dual of α. In local coordinates, we have for each $x \in X$

$$\Theta_{S,s} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x} = \bigoplus_{i=1}^{n} \mathcal{O}_{X,x} \cdot \tfrac{\partial}{\partial x_i}$$

and the image $\beta\big(\frac{\partial}{\partial x_i}\big) \in \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{I}_x/\mathcal{I}_x^2, \mathcal{O}_{X,x}) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{I}_x, \mathcal{O}_{X,x})$ sends a residue class $[h] \in \mathcal{I}_x/\mathcal{I}_x^2$ to $\big[\frac{\partial h}{\partial x_i}\big] \in \mathcal{O}_{X,x}$. Using these notations we can describe the vector space structure of $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}$ and of $T^1_{(X,\mathbf{0})}$:

Proposition 1.25. *Let $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ be a complex space germ and let $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$. Then*

(1) $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})} \cong \mathcal{N}_{X/\mathbb{C}^n,\mathbf{0}} \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I, \mathcal{O}_{X,\mathbf{0}})\,,$

(2) $T^1_{(X,\mathbf{0})} \cong \mathrm{Coker}(\beta)$, *that is, we have an exact sequence*

$$0 \longrightarrow \Theta_{X,\mathbf{0}} \longrightarrow \Theta_{\mathbb{C}^n,\mathbf{0}} \otimes_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}} \mathcal{O}_{X,\mathbf{0}} \xrightarrow{\beta} \mathcal{N}_{X/\mathbb{C}^n,\mathbf{0}} \longrightarrow T^1_{(X,\mathbf{0})} \longrightarrow 0\,,$$

where $\beta\big(\frac{\partial}{\partial x_i}\big) \in \mathrm{Hom}(I, \mathcal{O}_{X,\mathbf{0}})$ sends $h \in I$ to the class of $\frac{\partial h}{\partial x_i}$ in $\mathcal{O}_{X,\mathbf{0}}$.

Proof. (1) Let $I = \langle f_1, \dots, f_k\rangle \subset \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$, $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$, and $(F_1, \dots, F_k)$ define an embedded deformation of $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ over T_ε. That is, F_i is of the form

$$F_i = f_i + \varepsilon g_i \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} + \varepsilon \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n \times T_\varepsilon,\mathbf{0}}\,, \quad i = 1, \dots, k,$$

and this unfolding defines a deformation of $(X,\mathbf{0})$, which means that it is flat. Another embedded deformation, being defined by $(F'_1, \dots, F'_k)$, $F'_i = f_i + \varepsilon g'_i$, is isomorphic to the embedded deformation defined by $(F_1, \dots, F_k)$ iff the ideals $\langle F_1, \dots, F_k\rangle$ and $\langle F'_1, \dots, F'_k\rangle$ coincide (see the remark after Definition 1.23). But this holds iff there exist two matrices $A, C \in \mathrm{Mat}(k \times k, \mathcal{O}_{\mathbb{C}^n,\mathbf{0}})$ such that

$$(f_1 + \varepsilon g'_1, \dots, f_k + \varepsilon g'_k) = (f_1 + \varepsilon g_1, \dots, f_k + \varepsilon g_k)(\mathbf{1}_k + A + \varepsilon C)\,, \tag{1.4.3}$$

where each column of A is a relation of $f_1, \dots, f_k$. Note that

$$\begin{aligned} \mathcal{N}_{X/\mathbb{C}^n,\mathbf{0}} &= \mathrm{Hom}_{\mathcal{O}_{X,\mathbf{0}}}(I/I^2, \mathcal{O}_{X,\mathbf{0}}) = \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I/I^2, \mathcal{O}_{X,\mathbf{0}}) \\ &\cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I, \mathcal{O}_{X,\mathbf{0}})\,, \end{aligned}$$

where the last isomorphism follows from applying $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(\_, \mathcal{O}_{X,\mathbf{0}})$ to the exact sequence $0 \to I^2 \to I \to I/I^2 \to 0$, and using $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I^2, \mathcal{O}_{X,\mathbf{0}}) = 0$.

Since any infinitesimal deformation of $(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})$ is given by local equations $f_i+\varepsilon g_i$, $i=1,\dots,k$, and, hence, is completely determined by $(g_1,\dots,g_k)$, we can define the following map

$$\begin{aligned}\gamma\colon T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})} &\longrightarrow \mathcal{N}_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}\cong \operatorname{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I,\mathcal{O}_{X,\mathbf{0}})\,,\\ (g_1,\dots,g_k) &\longmapsto \Big(\varphi\colon \sum_{i=1}^k a_if_i\mapsto \sum_{i=1}^k \big[a_ig_i\big]\Big)\,.\end{aligned}$$

First notice that the image $\varphi=\gamma(g_1,\dots,g_k)$ is well-defined: if $\sum_{i=1}^k r_if_i=0$ is any relation between $f_1,\dots,f_k$ then we can lift it by the flatness property (Proposition 1.91) to a relation $\sum_{i=1}^k(r_i+\varepsilon s_i)(f_i+\varepsilon g_i)=0$. Thus, $\varepsilon\cdot\big(\sum_i s_if_i+r_ig_i\big)=0$, which implies $\sum_i r_ig_i\in I$.

Moreover, if $(g_1',\dots,g_k')$ defines an isomorphic embedded first order deformation then we obtain, by comparing the ε-part of (1.4.3),

$$(g_1',\dots,g_k')=(g_1,\dots,g_k)+(h_1,\dots,h_k)+\Big(\sum_{i=1}^k a_{1,i}\cdot g_i\,,\dots,\sum_{i=1}^k a_{k,i}\cdot g_i\Big)$$

with some $h_i\in I$ and some relations $(a_{1,i},\dots,a_{k,i})$ of $(f_1,\dots,f_k)$. As shown above, $\sum_{i=1}^k a_{j,i}g_i\in I$ and, hence, $g_i'-g_i\in I$, which shows that $(g_1,\dots,g_k)$ and $(g_1',\dots,g_k')$ are mapped to the same element in $\operatorname{Hom}_{\mathcal{O}_{X,\mathbf{0}}}(I,\mathcal{O}_{X,\mathbf{0}})$.

Now, if we impose on $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}$ the $\mathbb{C}$-vector space structure from $\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$ then the map γ is, indeed, a linear map. We shall show that it is bijective.

First, we show injectivity: if $\gamma(g_1,\dots,g_k)=0$ then $g_i\in I$, $i=1,\dots,k$, and, therefore, $(f_1+\varepsilon g_1,\dots,f_k+\varepsilon g_k)=(f_1,\dots,f_k)(\mathbf{1}_k+\varepsilon C)$ for some matrix C. In other words, $(f_1+\varepsilon g_1,\dots,f_k+\varepsilon g_k)$ defines a trivial embedded deformation.

To show surjectivity, let $\varphi\in\operatorname{Hom}(I,\mathcal{O}_{X,\mathbf{0}})$. Choose $(g_1,\dots,g_k)\in\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$ representing $\big(\varphi(f_1),\dots,\varphi(f_k)\big)\in\mathcal{O}^k_{X,\mathbf{0}}$ and set $F_i:=f_i+\varepsilon g_i$, $i=1,\dots,k$.

We have to verify the flatness condition for this unfolding. If $\sum_i r_if_i=0$, then $\sum_i r_i\varphi(f_i)=0$, that is, $\sum_i r_ig_i\in I$, and we can write

$$\sum_{i=1}^k r_ig_i=-\sum_{i=1}^k s_if_i\,.$$

Hence, $\sum_i(r_i+\varepsilon s_i)(f_i+\varepsilon g_i)=0$ and $(r_1+\varepsilon s_1,\dots,r_k+\varepsilon s_k)$ is a lifting of the relation $(r_1,\dots,r_k)$. By Proposition 1.91, $F_1,\dots,F_k$ is flat and, therefore, γ is surjective.

(2) Since any abstract, that is, non-embedded, deformation is induced by an embedded deformation (Corollary 1.6), any element of $T^1_{(X,\mathbf{0})}$ is represented by $F_i=f_i+\varepsilon g_i$, $i=1,\dots,k$, as in (1) and, hence, by

$$\gamma(g_1,\dots,g_k)\in\operatorname{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I,\mathcal{O}_{X,\mathbf{0}})\,.$$

We have to show that $(F_1,\dots,F_k)$ defines a trivial abstract deformation iff $\gamma(g_1,\dots,g_k)$ is in the image of β. We know that the deformation defined by

$F_i = f_i + \varepsilon g_i,\ i = 1,\dots,k$, is trivial as abstract deformation iff there is an isomorphism

$$\mathcal{O}_{\mathbb{C}^n\times T_\varepsilon,\mathbf{0}}/\langle F_1,\dots,F_k\rangle \cong \mathcal{O}_{\mathbb{C}^n\times T_\varepsilon,\mathbf{0}}/\langle f_1,\dots,f_k\rangle\,,$$

being the identity modulo ε and being compatible with the inclusion of $\mathcal{O}_{T_\varepsilon,0}$ in $\mathcal{O}_{\mathbb{C}^n\times T_\varepsilon,\mathbf{0}}$. Such an isomorphism is induced by an automorphism φ of $\mathcal{O}_{\mathbb{C}^n\times T_\varepsilon,\mathbf{0}} = \mathbb{C}\{\boldsymbol{x}\}[\varepsilon]$, mapping $x_j \mapsto x_j + \varepsilon\delta_j(\boldsymbol{x})$ and $\varepsilon\mapsto\varepsilon$, such that

$$\big\langle f_1\big(\boldsymbol{x}+\varepsilon\boldsymbol{\delta}(\boldsymbol{x})\big),\dots,f_k\big(\boldsymbol{x}+\varepsilon\boldsymbol{\delta}(\boldsymbol{x})\big)\big\rangle = \big\langle f_1+\varepsilon g_1,\dots,f_k+\varepsilon g_k\big\rangle\,. \tag{1.4.4}$$

By Taylor's formula, $f_i\big(\boldsymbol{x}+\varepsilon\boldsymbol{\delta}(\boldsymbol{x})\big) = f_i(\boldsymbol{x}) + \varepsilon\cdot\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\boldsymbol{x})\delta_j(\boldsymbol{x})$. Setting $\partial := \sum_j \delta_j(\boldsymbol{x})\frac{\partial}{\partial x_j} \in \Theta_{\mathbb{C}^n,\mathbf{0}}$, we have

$$f_i(\boldsymbol{x}+\varepsilon\boldsymbol{\delta}(\boldsymbol{x})) = f_i(\boldsymbol{x}) + \varepsilon\partial(f_i)\,.$$

Then the same argument as in (1) shows that the existence of an automorphism $\varphi\colon x_j\mapsto x_j+\varepsilon\delta_j$ satisfying (1.4.4) is equivalent to the existence of $\partial := \sum_j \delta_j\frac{\partial}{\partial x_j} \in \Theta_{\mathbb{C}^n,\mathbf{0}}$ satisfying

$$\big(\partial(f_1),\dots,\partial(f_k)\big) \equiv (g_1,\dots,g_k) \mod I\,. \tag{1.4.5}$$

If $\partial\in\Theta_{\mathbb{C}^n,\mathbf{0}}$ satisfies (1.4.5), then $\beta(\partial)\in\operatorname{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I,\mathcal{O}_{X,\mathbf{0}})$ maps $\sum_i a_if_i$ to $\sum_i a_i\partial(f_i) \equiv \sum a_ig_i \mod I$, which coincides with the image of $\sum_i a_if_i$ under $\gamma(g_1,\dots,g_k)$. Hence, $\beta(\partial) = \gamma(g_1,\dots,g_k)$.

Conversely, if $\gamma(g_1,\dots,g_k)\in\operatorname{Im}(\beta)$ then there exists a $\partial = \sum_j\delta_j\frac{\partial}{\partial x_j}$ such that $\beta(\partial) = \gamma(g_1,\dots,g_k)$. Hence, $\beta(\partial)(f_i) = \partial(f_i) = g_i$, that is, (1.4.5) holds for ∂ and, therefore, $f_i+\varepsilon g_i$ defines a trivial (abstract) deformation.

Thus, we have shown that $\operatorname{Im}(\beta)$ consists of exactly those embedded deformations which are trivial as abstract deformations. This proves (2). $\square$

Remark 1.25.1. In the proof, we have seen the following:

(1) If $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I,\ I = \langle f_1,\dots,f_k\rangle$, then an embedded deformation of $(X,\mathbf{0})$ over T_ε is given by $F = (F_1,\dots,F_k)$,

$$F_i = f_i + \varepsilon g_i\,,\quad i = 1,\dots,k,$$

$g_i\in\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ representing the image $\varphi(f_i)$ for $\varphi\in\operatorname{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I,\mathcal{O}_{X,\mathbf{0}})$, such that $\sum_i r_ig_i\in I$ for each relation $(r_1,\dots,r_k)$ among $f_1,\dots,f_k$.

F and $F' = (F_1',\dots,F_k')$, $F_i' = f_i+\varepsilon g_i'$, define isomorphic embedded deformations over T_ε iff $g_i - g_i'\in I$. The vector space structure on the space of embedded deformations is given by

$$\begin{aligned} F+F' &= \big(f_1+\varepsilon(g_1+g_1'),\dots,f_k+\varepsilon(g_k+g_k')\big)\,,\\ \lambda F &= (f_1+\varepsilon\lambda g_1,\dots,f_k+\varepsilon\lambda g_k)\,,\quad \lambda\in\mathbb{C}\,. \end{aligned}$$

(2) The embedded deformation defined by F as above is trivial as abstract deformation iff there is a vector field $\partial = \sum_{j=1}^{n} \delta_j \frac{\partial}{\partial x_j} \in \Theta_{\mathbb{C}^n,\mathbf{0}}$ such that

$$g_i = \partial(f_i) \bmod I\,, \quad i = 1, \dots, k\,.$$

In particular, if $I = \langle f \rangle$ defines a hypersurface singularity, then $f + \varepsilon g$ is trivial as abstract deformation iff $g \in \langle f, \frac{\partial f}{\partial x_j} \mid j = 1, \dots, n \rangle$.

The vector space structure on $T^1_{(X,x)}$ is the same as the one above for embedded deformations by taking representatives and it coincides with the one given by Schlessinger's theory (see Exercise 1.4.4).

Now, as we are able to compute $T^1_{(X,x)}$ by using Proposition 1.25, let us mention a few applications.

First of all, $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ is a necessary (Lemma 1.20) and sufficient (Theorem 1.9) condition for the existence of a semiuniversal deformation of (X, x). If (X, x) has an isolated singularity, then $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ (Exercise 1.4.3) but the converse does not hold (see Example 1.26.1, below). Furthermore, we have

Proposition 1.26. *A complex space germ is rigid iff* $T^1_{(X,x)} = 0$.

Proof. (X, x) is rigid iff the semiuniversal deformation exists and consists of a single, reduced point. By Lemma 1.20, together with the existence of a semiuniversal deformation for germs with $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$, this is equivalent to $T^1_{(X,x)} = 0$. □

Example 1.26.1. (1) The simplest known example of an equidimensional (non-smooth) rigid singularity $(X, \mathbf{0})$ is the union of two planes in $(\mathbb{C}^4, \mathbf{0})$, meeting in one point (given by the ideal `I` in the ring `R` below). The product $(X, \mathbf{0}) \times (\mathbb{C}, 0) \subset (\mathbb{C}^5, \mathbf{0})$ (given by the ideal `I` in the ring `R1`) has a non-isolated singularity but is also rigid (hence, has a semiuniversal deformation). We prove these statements using Singular:

```
LIB "deform.lib";
ring R = 0,(x,y,u,v),ds;
ideal I = intersect(ideal(x,y),ideal(u,v));
vdim(T_1(I));          // result is 0 iff V(I) is rigid
//-> 0
ring R1 = 0,(x,y,u,v,w),ds;
ideal I = imap(R,I);
dim_slocus(I);         // dimension of singular locus of V(I)
//-> 1
vdim(T_1(I));
//-> 0
```

(2) An even simpler (but not equidimensional) rigid singularity is the union of the plane $\{x = 0\}$ and the line $\{y = z = 0\}$ in $(\mathbb{C}^3, \mathbf{0})$. This can be checked either by using Singular as above, or, without computer, by showing that the map β in Proposition 1.25 is surjective.

Generalization 1.27. Let $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ be a complex germ and M an $\mathcal{O}_{X,\mathbf{0}}$-module. Define

$$\begin{aligned} T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}(M) &:= \mathrm{Hom}_{\mathcal{O}_{X,\mathbf{0}}}\big(I/I^2, M\big)\,, \\ T^1_{(X,\mathbf{0})}(M) &:= \mathrm{Coker}\Big(\Theta_{\mathbb{C}^n,\mathbf{0}} \otimes_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}} M \xrightarrow{\beta} \mathrm{Hom}_{\mathcal{O}_{X,\mathbf{0}}}\big(I/I^2, M\big)\Big)\,, \end{aligned}$$

with β the $\mathcal{O}_{X,\mathbf{0}}$-linear map defined by $\beta\big(\frac{\partial}{\partial x_i} \otimes m\big) : h \mapsto \frac{\partial h}{\partial x_i}\, m$. Hence,

$$T^1_{(X,\mathbf{0})} = T^1_{X,\mathbf{0}}(\mathcal{O}_{X,\mathbf{0}})\,, \quad T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})} \cong T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}(\mathcal{O}_{X,\mathbf{0}})$$

(see Proposition 1.25).

For $M = V \otimes_{\mathbb{C}} \mathcal{O}_{X,\mathbf{0}}$, V a finite dimensional complex vector space, these modules can be interpreted as modules of infinitesimal deformations. Namely, for any complex germ (T,t) and a finitely generated $\mathcal{O}_{T,t}$-module define the complex germ $(T[M],t)$ by

$$\mathcal{O}_{T[M],t} = \mathcal{O}_{T,t} \oplus \varepsilon M\,, \quad \varepsilon^2 = 0\,,$$

with componentwise addition and obvious multiplication. Then $(T[M],t)$ is an infinitesimal thickening of (T,t) with the same underlying topological space. In particular, for (T,t) the reduced point pt, we get $\mathcal{O}_{\mathrm{pt}[\mathbb{C}]} = \mathbb{C} \oplus \varepsilon\mathbb{C}$, that is, $\mathrm{pt}[\mathbb{C}] = T_\varepsilon$.

For V a finite dimensional complex vector space, $\mathrm{pt}[V]$ is a fat point. In the same way as in Proposition 1.25, we can prove

$$\begin{aligned} T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}\big(\mathcal{O}_{X,\mathbf{0}} \otimes V\big) &\cong \underline{\mathcal{D}ef}_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}\big(\mathrm{pt}[V]\big)\,, \\ T^1_{(X,\mathbf{0})}\big(\mathcal{O}_{X,\mathbf{0}} \otimes V\big) &\cong \underline{\mathcal{D}ef}_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}\big(\mathrm{pt}[V]\big)\,. \end{aligned}$$

How to Compute $T^1_{(X,0)}$.

The proof of Proposition 1.25 provides an algorithm for computing $T^1_{(X,\mathbf{0})}$. An implementation of this algorithm is provided by the SINGULAR library `sing.lib`. The SINGULAR procedure `T_1` computes (and returns) all relevant information about first order deformations which we explain now. We also explain the essential steps of the procedure `T_1`.

Let $(X,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ be given by the ideal $I \subset P := K[[x_1,\dots,x_n]]$, K a field, where we assume (for computational purpose only) that I is given by a set of polynomial generators $f_1,\dots,f_k$. Set $R := P/I$. For the computation of $T^1_{(X,\mathbf{0})}$, consider a presentation of I as P-module,

$$0 \longleftarrow I \longleftarrow P^k \xleftarrow{A} P^p \tag{1.4.6}$$

and note that, for any R-module M,

$$\mathrm{Hom}_R(I/I^2, M) \cong \mathrm{Hom}_P(I/I^2, M) \cong \mathrm{Hom}_P(I, M)\,.$$

Choosing $dx_1, \ldots, dx_n$ as basis of $\Omega^1_{\mathbb{C}^n,\mathbf{0}}$ and the canonical basis of P^k, the right part of the exact sequence in Proposition 1.25 (2) (with $M = R$) is identified with the exact sequence

$$\mathrm{Hom}_P(P^n, M) \xrightarrow{\beta} \mathrm{Hom}_P(I, M) \longrightarrow T^1_{(X,\mathbf{0})}(M) \longrightarrow 0\,, \tag{1.4.7}$$

where β is given by the Jacobian matrix of $(f_1, \ldots, f_n)$,

$$Df = \Big(\frac{\partial f_i}{\partial x_j}\Big) : M^n = \mathrm{Hom}_P(P^n, M) \to \mathrm{Hom}_P(I, M) \subset \mathrm{Hom}_P(P^k, M) = M^k.$$

This sequence can be used to compute $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}(M) = \mathrm{Hom}_P(I, M)$ and $T^1_{(X,\mathbf{0})}(M)$ for any R-module M given by a presentation matrix.

We continue with $M = R$. Applying $\mathrm{Hom}_P(\_\_, R)$ to (1.4.6), we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_P(I, R) = \mathrm{Ker}\, A^t \longrightarrow \mathrm{Hom}_P(P^k, R) \xrightarrow{A^t} \mathrm{Hom}_P(P^p, R)\,,$$

where A^t is the transposed matrix of A. Consider a two-step partial free resolution of the R-module $\mathrm{Im}\, A^t$,

$$\begin{tikzcd}
R^q \arrow[r, "B_2"] & R^r \arrow[r, "B_1"] & \mathrm{Hom}_P(P^k, R) = R^k \arrow[r, "A^t"] & R^p = \mathrm{Hom}_P(P^p, R)\,, \\
 & & \mathrm{Hom}_P(P^n, R) = R^n \arrow[ul, "\ell"] \arrow[u, "Df"'] &
\end{tikzcd}$$

together with the map defined by the Jacobian matrix and with a lifting $\ell : \mathrm{Hom}_P(P^n, R) \to R^r$ thereof. The lifting ℓ exists since the image of the Jacobian map is contained in the normal module $\mathrm{Hom}_P(I, R)$ of I. Finally, we get (keeping notations for B_i and ℓ when lifted to P)

$$\begin{aligned} T^1_{(X,\mathbf{0})} &\cong \mathrm{Im}\, B_1/\,\mathrm{Im}\, Df \cong R^r\big/\big(\mathrm{Im}\,\ell + \mathrm{Im}\, B_2\big) \\ &\cong P^r\big/\big(\mathrm{Im}\,\ell + \mathrm{Im}\, B_2 + I \cdot P^r\big)\,. \end{aligned}$$

Note that if $T^1_{(X,\mathbf{0})}$ is a finite dimensional K-vector space then replacing throughout the above construction $K[[\boldsymbol{x}]]$ by $K[\boldsymbol{x}]_{\langle \boldsymbol{x}\rangle}$ leads to a vector space of the same dimension (and vice versa). If the active basering in a Singular session implements $P = K[\boldsymbol{x}]_{\langle \boldsymbol{x}\rangle}$ then applying `T_1` to an `ideal` implementing I returns a standard basis for the `module t1` $:= \mathrm{Im}\,\ell + \mathrm{Im}\, B_2 + I \cdot P^r$, as elements of the free module P^r. Hence, the columns of `matrix(t1)` generate `t1`. If `T_1` is called with two arguments, e.g. `list L = T_1(I,"");`, then Singular returns a list of three modules:

`L[1]`: (a standard basis for) `t1`,
`L[2]`: a set of generators for $\operatorname{Im} B_1$ (the columns of the matrix B_1 mod I generate the normal module $\operatorname{Hom}_R(I,R) \subset R^k$),
`L[3]`: a set of generators for $\operatorname{Im} A$ (the columns of the matrix A generate the module of relations of I).

In particular, since $A^t \cdot$ `L[2]` $\equiv 0$ mod I, the command

```
reduce(transpose(L[3])*L[2],groebner(I*freemodule(r));
```

returns the zero module. Entering `vdim(T_1(I));` makes SINGULAR display the Tjurina number $\tau = \dim_K T^1_{(X,\mathbf{0})}$, the most important information about $T^1_{(X,\mathbf{0})}$.

The command `kbase(T_1(I));` makes SINGULAR return a basis for the K-vector space $T^1_{(X,\mathbf{0})}$, represented by elements of P^r. Applying `L[2]` to any element of P^r gives an element $(g_1,\ldots,g_k) \in P^k$ such that

$$f_1 + \varepsilon g_1, \ldots, f_k + \varepsilon g_k$$

is an infinitesimal embedded deformation of X, and every embedded deformation of $(X,\mathbf{0})$ over T_ε is obtained in this way.

Moreover, applying `L[2]` to the elements returned by `kbase(L[1]);` we get $(g_1^1,\ldots,g_k^1),\ldots,(g_1^\tau,\ldots,g_k^\tau) \in P^k$ such that

$$f_1 + \varepsilon g_1^i, \ldots, f_k + \varepsilon g_k^i\,, \quad i = 1,\ldots,\tau\,,$$

define embedded deformations of X which represent a basis of abstract deformations of $(X,\mathbf{0})$ over T_ε. This follows from the exact sequence (1.4.7) and Proposition 1.25.

Example 1.27.1. We compute a $\mathbb{C}$-basis of $T^1_{(X,\mathbf{0})}$ for $(X,\mathbf{0}) = V(I)$, the cone over the rational normal curve in $\mathbb{P}^4$ (see Exercise 1.2.2):

```
LIB "deform.lib";
ring R2 = 0,(x,y,z,u,v),ds;
matrix M[2][4] = x,y,z,u,y,z,u,v;
ideal I = minor(M,2); I;
//-> I[1]=-u2+zv
//-> I[2]=-zu+yv
//-> I[3]=-yu+xv
//-> I[4]=z2-yu
//-> I[5]=yz-xu
//-> I[6]=-y2+xz
list L = T_1(I,"");
//-> // dim T_1 = 4
print(L[2]*kbase(L[1]));
//-> 0, u,0, v,
//-> -u,0,0, u,
```

```
//-> 0, 0,u, z,
//-> z, y,0, 0,
//-> 0, x,-z,0,
//-> x, 0,y, 0
```

The four columns of this matrix $(g_i^j)_{i=1..6,j=1..4}$ are a concrete $\mathbb{C}$-basis of $T^1_{(X,\mathbf{0})}$ in the sense that $\mathtt{I[1]}+\varepsilon g_1^j,\dots,\mathtt{I[6]}+\varepsilon g_6^j$, $j=1,\dots,4$, define the corresponding deformations over T_ε.

Remarks and Exercises

Infinitesimal deformations are the first step in formal deformation theory as developed by Schlessinger in a very general context (see Appendix C for a short overview). Schlessinger introduced what is nowadays called the *Schlessinger conditions* $(\mathrm{H}_0)-(\mathrm{H}_4)$ in [Sch]. One can verify that $\underline{\mathcal{D}ef}_{(X,x)}$ satisfies conditions $(\mathrm{H}_0)-(\mathrm{H}_3)$ and, therefore, has a formal versal deformation. Moreover, for every deformation functor satisfying the Schlessinger conditions, the corresponding infinitesimal deformations carry a natural vector space structure. For $T^1_{(X,x)}$ this structure coincides with the one defined above (Exercise 1.4.4). We do not go into the business of formal deformation theory here, but refer to Appendix C. A survey of deformations of complex spaces is given in [Pal2], some aspects of deformations of singularities are covered by [Ste1].

Exercise 1.4.1. (1) Let (S,s) be a complex space germ and let $T_{S,s}$ be the (Zariski) tangent space of (S,s). Show that there is a natural isomorphism of vector spaces $T_{S,s}\cong \mathrm{Mor}(T_\varepsilon,(S,s))$, where Mor denotes the set of morphisms of complex space germs.

(2) Show that the Kodaira-Spencer map defined in Lemma 1.20 is a linear map.

Exercise 1.4.2. Let $X\subset S$ be a complex subspace of a complex manifold with ideal sheaf $\mathcal{I}$. Suppose that X is a *local complete intersection*, that is, $\mathcal{O}_{X,x}=\mathcal{O}_{S,s}/\mathcal{I}_x$ is a complete intersection ring for all $x\in X$.

Show that the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free and that the following sequence

$$0\longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{\alpha} \Omega^1_S\otimes_{\mathcal{O}_S}\mathcal{O}_X\longrightarrow \Omega^1_X\longrightarrow 0$$

is exact. Dualize this to get an exact sequence

$$0\longrightarrow \Theta_X\longrightarrow \Theta_S\otimes_{\mathcal{O}_S}\mathcal{O}_X\xrightarrow{\beta} \mathcal{T}^1_X\longrightarrow 0$$

where $\mathcal{T}^1_X=\mathrm{Coker}(\beta)\cong \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_{X,x},\mathcal{O}_{X,x})$ is concentrated in $\mathrm{Sing}(X)$ and satisfies $\mathcal{T}^1_{X,x}=T^1_{(X,x)}$ for each $x\in X$.

Exercise 1.4.3. Show that $\dim T^1_{(X,x)}<\infty$ if (X,x) has an isolated singularity.

Exercise 1.4.4. Show that the vector space structure on $T^1_{(X,x)}$ defined in this section coincides with the vector space structure given by Lemma C.1.7.
HINT. Use Remark 1.25.1.

Exercise 1.4.5. Show that the formulas for $T^1_{(X,\mathbf{0})}$ from Proposition 1.25 (2) and Theorem 1.16, resp. Corollary 1.17, coincide for a complete intersection singularity, resp. a hypersurface singularity $(X,\mathbf{0})$.

Exercise 1.4.6. Show that for a complete intersection $X \subset U$, $U \subset \mathbb{C}^n$ an open subset, defined by $I_X = \langle f_1, \dots, f_k\rangle$, $k = n - \dim_x X$ for $x \in X$, the singular locus is defined by the ideal

$$\Big\langle f_1, \dots, f_k,\, k-\text{minors of } \big(\tfrac{\partial f_i}{\partial x_j}\big)\Big\rangle \subset \mathcal{O}_U\,.$$

Further, show that the coherent $\mathcal{O}_X$-sheaf

$$\mathcal{T}^1_X := \mathcal{O}_U^k/(Df)\cdot \mathcal{O}_U^n + \langle f_1, \dots, f_k\rangle \mathcal{O}_U^k$$

has support $\operatorname{Sing}(X)$. Finally, show that its stalk $\mathcal{T}^1_{X,x}$ coincides with $T^1_{(X,x)}$ and that $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ if and only if X has an isolated singularity at x.

Exercise 1.4.7. Show that for (X,x) a normal singularity

$$T^1_{(X,x)} \cong \operatorname{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x})\,.$$

1.5 Obstructions

The construction of a semiuniversal deformation for a complex germ (X,x) with $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ can be carried out as follows:

- We start with *first order deformations* and try to *lift these to second order* deformations. In other words, we are looking for possible liftings of a deformation (i,ϕ), $[(i,\phi)] \in \underline{\mathcal{D}ef}_{(X,x)}(T_\varepsilon) = T^1_{(X,x)}$, to a deformation over the fat point point $(T',\mathbf{0})$ containing T_ε, for example to the fat point with local ring $\mathbb{C}[\eta]/\langle \eta^3\rangle$. Or, if we assume the deformations to be embedded (Corollary 1.6), this means that we are looking for a lifting of the first order deformation $f_i + \varepsilon g_i$, $\varepsilon^2 = 0$, to a second order deformation $f_i + \eta g_i + \eta^2 g_i'$, $\eta^3 = 0$, $i = 1, \dots, k$.
- This is exactly what we did when we constructed the semiuniversal deformation of a complete intersection singularity. By induction we showed the existence of a lifting to arbitrarily high order. In general, however, this is not always possible, there are *obstructions* against lifting. Indeed, there is an $\mathcal{O}_{X,x}$-module $T^2_{(X,x)}$ and, for each small extension of T_ε, an *obstruction map*
$$\operatorname{ob}: T^1_{(X,x)} \longrightarrow T^2_{(X,x)}$$
such that the vanishing of $\operatorname{ob}\big([(i,\phi)]\big)$ is equivalent to the existence of a lifting of (i,ϕ) to the small extension, e.g. to second order as above.

- Assuming that the obstruction is zero, we choose a lifting to second order (which is, in general, not unique) and try to lift this to *third order*, that is, to a deformation over the fat point with local ring $\mathbb{C}[t]/\langle t^4\rangle$. Again, there is an obstruction map, and the lifting is possible iff it maps the deformation class to zero.
- Continuing in this manner, in each step, the preimage of 0 under the obstruction map defines homogeneous relations in terms of the elements $t_1, \dots, t_\tau$ of a basis of $(T^1_{(X,x)})^*$, of a given order, which in the limit yield formal power series in $\mathbb{C}[[\boldsymbol{t}]] = \mathbb{C}[[t_1, \dots, t_\tau]]$. If J denotes the ideal in $\mathbb{C}[[\boldsymbol{t}]]$ defined by these power series, the quotient $\mathbb{C}[[\boldsymbol{t}]]/J$ is the local ring of the base space of the (formal) versal deformation. Then $T^1_{(X,x)} = (\langle \boldsymbol{t}\rangle/\langle \boldsymbol{t}\rangle^2)^*$ is the Zariski tangent space to this base space.

This method works for very general deformation functors having an obstruction theory. We collect methods and results from general obstruction theory in Appendix C.2.

We shall now describe the module $T^2_{(X,x)}$ of obstructions to lift a deformation from a fat point $(T, \mathbf{0})$ to an infinitesimally bigger one $(T', \mathbf{0})$.

Let $\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$, with $I = \langle f_1, \dots, f_k\rangle$. Consider a presentation of I,

$$0 \longleftarrow I \xleftarrow{\ \alpha\ } \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} \xleftarrow{\ \beta\ } \mathcal{O}^\ell_{\mathbb{C}^n,\mathbf{0}}\,, \qquad \alpha(e_i) = f_i\,.$$

$\operatorname{Ker}(\alpha) = \operatorname{Im}(\beta)$ is the module of relations for $f_1, \dots, f_k$, which contains the $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$-module of *Koszul relations*

$$\mathrm{Kos} := \langle f_i e_j - f_j e_i \mid 1 \le i < j \le k\rangle\,,$$

$e_1, \dots, e_k$ denoting the standard unit vectors in $\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$. We set $\mathrm{Rel} := \operatorname{Ker}(\alpha)$ and note that Rel/Kos is an $\mathcal{O}_{X,x}$-module: let $\sum_i r_i e_i \in \mathrm{Rel}$, then

$$f_j \cdot \sum_{i=1}^k r_i e_i = f_j \cdot \sum_{i=1}^k r_i e_i - \sum_{i=1}^k r_i f_i e_j = \sum_{i=1}^k r_i \cdot (f_j e_i - f_i e_j) \in \mathrm{Kos}\,.$$

Since $\mathrm{Kos} \subset I\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$, the inclusion $\mathrm{Rel} \subset \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$ induces an $\mathcal{O}_{X,x}$-linear map

$$\mathrm{Rel}/\mathrm{Kos} \longrightarrow \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}/I\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} = \mathcal{O}^k_{X,x}\,.$$

Definition 1.28. We define $T^2_{(X,x)}$ to be the cokernel of Φ, the $\mathcal{O}_{X,x}$-dual of the latter map, that is, we have a defining exact sequence for $T^2_{(X,x)}$:

$$\operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}^k_{X,x}, \mathcal{O}_{X,x}) \xrightarrow{\ \Phi\ } \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathrm{Rel}/\mathrm{Kos}, \mathcal{O}_{X,x}) \to T^2_{(X,x)} \to 0\,. \quad (1.5.1)$$

Proposition 1.29. *Let (X,x) be a complex space germ.*

(1) Let $j:(T,\mathbf{0})\hookrightarrow(T',\mathbf{0})$ be an inclusion of fat points, and let J be the kernel of the corresponding map of local rings $\mathcal{O}_{T',\mathbf{0}}\twoheadrightarrow\mathcal{O}_{T,\mathbf{0}}$. Then there is a map, called the obstruction map,

$$\mathrm{ob}:\underline{\mathcal{D}ef}_{(X,x)}(T,\mathbf{0})\longrightarrow T^2_{(X,x)}\otimes_{\mathbb{C}}J\,,$$

satisfying: a deformation $(i,\phi):(X,x)\hookrightarrow(\mathscr{X},x)\to(T,\mathbf{0})$ admits a lifting $(i',\phi'):(X,x)\hookrightarrow(\mathscr{X}',x)\to(T',\mathbf{0})$ (that is, $j^*(i',\phi')=(i,\phi)$) iff $\mathrm{ob}\big([(i,\phi)]\big)=0$.

(2) If $T^1_{(X,x)}$ is a finite dimensional $\mathbb{C}$-vector space and if $T^2_{(X,x)}=0$, then the semiuniversal deformation of (X,x) exists and has a smooth base space (of dimension $\dim_{\mathbb{C}}T^1_{(X,x)}$*).*

Note that the obstruction map ob is a map between sets (without further structure) as $\underline{\mathcal{D}ef}_{(X,x)}(T,\mathbf{0})$ is just a set.

If (X,x) is a complete intersection then the Koszul relations are the only existing relations. Hence, $\mathrm{Rel}=\mathrm{Kos}$ and $T^2_{(X,x)}=0$. In particular, statement (ii) of Proposition 1.29 confirms the result of Theorem 1.16, which is of course much more specific.

Proof of Proposition 1.29. (1) To simplify notation, we give the proof only for $\mathcal{O}_{T,0}=\mathbb{C}\{t\}/\langle t^p\rangle$ and $\mathcal{O}_{T',0}=\mathbb{C}\{t\}/\langle t^{p+1}\rangle$, with $J=\langle t^p\rangle/\langle t^{p+1}\rangle$, $p\geq 1$.[7]

As before, let $\mathcal{O}_{X,x}=\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$, with $I=\langle f_1,\dots,f_k\rangle$. We can assume a deformation of (X,x) over $(T,0)$ to be embedded, that is, to be given by

$$\boldsymbol{F}(\boldsymbol{x},t)=\boldsymbol{f}(\boldsymbol{x})+t\boldsymbol{g}(\boldsymbol{x},t)\in\big(\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}[t]\big)^k\,,$$

satisfying the flatness condition of Proposition 1.91 mod $\langle t^p\rangle$.

We want to lift such deformation to $(T',0)$. That is, we are looking for $\boldsymbol{g}'\in\mathbb{C}\{\boldsymbol{x}\}^k$ such that

$$\boldsymbol{F}'(\boldsymbol{x},t)=\boldsymbol{F}(\boldsymbol{x},t)+t^p\boldsymbol{g}'(x)\in\big(\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}[t]\big)^k$$

is a deformation (by definition a lifting of $\boldsymbol{F}$) mod $\langle t^{p+1}\rangle$. Due to Proposition 1.91, this means that $\boldsymbol{g}'$ has to satisfy the following condition: for any relation $\boldsymbol{R}=(R_1,\dots,R_k)$ of $\boldsymbol{F}=(F_1,\dots,F_k)$,

$$\boldsymbol{R}(\boldsymbol{x},t)=\boldsymbol{r}(\boldsymbol{x})+t\boldsymbol{h}(\boldsymbol{x},t)\in\big(\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}[t]\big)^k\,,$$

satisfying

[7] This is not a restriction, since any extension $(T,\mathbf{0})\hookrightarrow(T',\mathbf{0})$ of fat points is a composition of finitely many *small extensions*, that is, extensions $(T,\mathbf{0})\hookrightarrow(T',\mathbf{0})$ such that the kernel J of the corresponding map of (Artinian) local rings $\mathcal{O}_{T',\mathbf{0}}\twoheadrightarrow\mathcal{O}_{T,\mathbf{0}}$ is 1-dimensional and satisfies $J^2=0$.

$$\langle \boldsymbol{r}, \boldsymbol{f} \rangle := \sum_{i=1}^{k} r_i f_i = 0\,, \quad \text{and}$$
$$\langle \boldsymbol{R}, \boldsymbol{F} \rangle := \sum_{i=1}^{k} R_i F_i \equiv 0 \mod \langle t^p \rangle\,, \tag{1.5.2}$$

there exists a lifting $\boldsymbol{R}'(\boldsymbol{x}, t) = \boldsymbol{R}(\boldsymbol{x}, t) + t^p \boldsymbol{h}'(\boldsymbol{x}) \in \big(\mathcal{O}_{\mathbb{C}^n,\boldsymbol{0}}[t]\big)^k$, satisfying

$$\langle \boldsymbol{R}', \boldsymbol{F}' \rangle \equiv 0 \mod \langle t^{p+1} \rangle\,.$$

By (1.5.2), $\langle \boldsymbol{R}', \boldsymbol{F} \rangle$ is divisible by t^p, hence, we get

$$\langle \boldsymbol{R}', \boldsymbol{F}' \rangle = t^p \cdot \left(\frac{\langle \boldsymbol{R}', \boldsymbol{F} \rangle}{t^p} + \langle \boldsymbol{h}', t\boldsymbol{g} \rangle + \langle \boldsymbol{r}, \boldsymbol{g}' \rangle \right) \in t^p \mathcal{O}_{X,x}[t]/\langle t^{p+1} \rangle = t^p \mathcal{O}_{X,x}\,.$$

It follows that $\boldsymbol{F}$ admits a lifting $\boldsymbol{F}'$ over $(T', 0)$ iff, for every relation $\boldsymbol{R}$ of $\boldsymbol{F}$, $t^{-p} \cdot \langle \boldsymbol{R}, \boldsymbol{F} \rangle$ is of the form $-\langle \boldsymbol{r}, \boldsymbol{g}' \rangle + t \cdot g''$ mod $\langle t^{p+1} \rangle$, for some $\boldsymbol{g}' \in \mathcal{O}_{X,x}^k$, $g'' \in \mathcal{O}_{X,x}[t]$.

To define the obstruction map ob and to show that the latter holds iff $\mathrm{ob}(F) = 0$, we proceed in two steps:

Step 1. As element of $\mathcal{O}_{X,x}$, $t^{-p} \cdot \langle \boldsymbol{R}, \boldsymbol{F} \rangle$ mod $\langle t \rangle$ depends only on $\boldsymbol{r}$ and $\boldsymbol{F}$, but not on the lifted relation $\boldsymbol{R}$.

Indeed, let $\widetilde{\boldsymbol{R}} = \boldsymbol{r} + t\widetilde{\boldsymbol{h}}$ be another lifting of $\boldsymbol{r}$, satisfying $\langle \widetilde{\boldsymbol{R}}, \boldsymbol{F} \rangle \equiv 0$ mod $\langle t^p \rangle$. Then $t \cdot \langle \boldsymbol{F}, \boldsymbol{h} - \widetilde{\boldsymbol{h}} \rangle \equiv 0$ mod $\langle t^p \rangle$, which implies that $\langle \boldsymbol{F}, \boldsymbol{h} - \widetilde{\boldsymbol{h}} \rangle \equiv 0$ mod $\langle t^{p-1} \rangle$.

Hence, $\boldsymbol{h} - \widetilde{\boldsymbol{h}}$ is a relation of $\boldsymbol{F}$ mod $\langle t^{p-1} \rangle$, which lifts to a relation mod $\langle t^p \rangle$, since $\boldsymbol{F}$ is flat mod $\langle t^p \rangle$. In other words, there exists a $\boldsymbol{h}'' \in \mathcal{O}_{\mathbb{C}^n,\boldsymbol{0}}^k$ such that $\langle \boldsymbol{F}, \boldsymbol{h} - \widetilde{\boldsymbol{h}} + t^{p-1}\boldsymbol{h}'' \rangle \equiv 0$ mod $\langle t^p \rangle$, that is,

$$\langle \boldsymbol{F}, \boldsymbol{R} \rangle - \langle \boldsymbol{F}, \widetilde{\boldsymbol{R}} \rangle = \langle \boldsymbol{F}, t\boldsymbol{h} - t\widetilde{\boldsymbol{h}} \rangle \equiv \langle \boldsymbol{f}, t^p \boldsymbol{h}'' \rangle \mod \langle t^{p+1} \rangle\,.$$

Now, the statement follows, since $\langle \boldsymbol{f}, t^p \boldsymbol{h}'' \rangle$ is 0 as element of $t^p \cdot \mathcal{O}_{X,x}$.

Step 2. We conclude that $\boldsymbol{F}$, representing an element of $\underline{\mathcal{D}ef}_{(X,x)}(T, 0)$, defines a map

$$\mathrm{Rel} \longrightarrow t^p \mathcal{O}_{X,x} = \mathcal{O}_{X,x} \otimes_{\mathbb{C}} J\,, \quad \boldsymbol{r} \longmapsto \langle \boldsymbol{F}, \boldsymbol{R} \rangle\,,$$

where $\boldsymbol{R}$ is any lifting of $\boldsymbol{r}$ to a relation of $\boldsymbol{F}$ mod $\langle t^p \rangle$.

In particular, for $\boldsymbol{r} \in \mathrm{Kos}$, we may choose the Koszul lifting $\boldsymbol{R}$, which satisfies $\langle \boldsymbol{F}, \boldsymbol{R} \rangle = 0 \in \mathcal{O}_{X,x}$ (not only mod $\langle t^p \rangle$). Hence, $\boldsymbol{F}$ defines, via $\boldsymbol{r} \mapsto (\boldsymbol{F}, \boldsymbol{R})$, even an element

$$\langle \boldsymbol{F}, \_ \rangle \in \mathrm{Hom}_{\mathcal{O}_{X,x}}\big(\mathrm{Rel}/\mathrm{Kos}, \mathcal{O}_{X,x} \otimes_{\mathbb{C}} J\big) = \mathrm{Hom}_{\mathcal{O}_{X,x}}\big(\mathrm{Rel}/\mathrm{Kos}, \mathcal{O}_{X,x}\big) \otimes_{\mathbb{C}} J\,.$$

By the above the latter is in the image of $\Phi \otimes \mathrm{id}_J$ (cf. (1.5.1)), iff $\boldsymbol{F}$ admits a lifting $\boldsymbol{F}' = \boldsymbol{F} + t^p \boldsymbol{g}'$ over $(T', 0)$.

We define now ob as

$$\mathrm{ob}(\boldsymbol{F}) := \langle \boldsymbol{F}, \_ \rangle \mod \bigl(\mathrm{Im}(\Phi) \otimes \mathrm{id}_J\bigr) \in T^2_{(X,x)} \otimes_{\mathbb{C}} J\,.$$

Checking that the image only depends on the isomorphism class of $\boldsymbol{F}$ in $\underline{\mathcal{D}ef}_{(X,x)}(T,0)$ (forgetting the embedding), we obtain statement (1).

(2) The existence of a semiuniversal deformation of (X,x) follows since $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ (by Theorem 1.9). To see the smoothness, consider the semiuniversal deformation $(i,\phi) : (X,x) \hookrightarrow (\mathscr{X},x) \to (S,s)$.

Let $(T,\mathbf{0}) \hookrightarrow (T',\mathbf{0})$ be a small extension, and let $\varphi : (T,\mathbf{0}) \to (S,s)$ be any morphism. Since $T^2_{(X,x)} = 0$, if follows from (1) that there exists a lifting (i',ϕ') of $\varphi^*(i,\phi)$ over $(T',\mathbf{0})$. By versality of (i,ϕ), there exists a morphism $\psi : (T',\mathbf{0}) \to (S,s)$, $\psi|_{(T,\mathbf{0})} = \varphi$, such that $\psi^*(i,\phi) \cong (i',\phi')$. This means that the assumptions of the next lemma (formulated for local rings) are satisfied. Hence, (S,s) is smooth. □

Lemma 1.30. *For an analytic K-algebra R the following are equivalent:*

(a) R is regular.
(b) For any surjective morphism $A' \to A$ of Artinian analytic K-algebras and any morphism $\varphi : R \to A$ there exists a morphism $\psi : R \to A'$ such that the following diagram commutes

$$\begin{tikzcd} & R \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] \\ A' \arrow[r, two heads] & A \end{tikzcd}$$

Proof. By Theorem I.1.20 there exists a surjection $\theta : K\langle x_1,\dots,x_n\rangle \twoheadrightarrow R$ with $n = \dim_K \mathfrak{m}/\mathfrak{m}^2$ ($\mathfrak{m} \subset R$ the maximal ideal), which is an isomorphism iff R is regular. If $R \cong K\langle x_1,\dots,x_n\rangle$ then (b) follows easily (e.g., using Lemma I.1.14).

Conversely, if (b) holds then we can lift $\varphi_2 : R \to R/\mathfrak{m}^2 \cong K\langle \boldsymbol{x}\rangle/\langle \boldsymbol{x}\rangle^2$ to $\varphi_3 : R \to R/\mathfrak{m}^3$ and so on, to $\varphi_k : R \to R/\mathfrak{m}^k$, $k \geq 2$.

In the limit we get a morphism $\varphi : \widehat{R} \to K[[\boldsymbol{x}]]$, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R. If $\widehat{\theta}$ denotes the map $K[[\boldsymbol{x}]] \to \widehat{R}$ induced by completing θ, then the composition $\varphi \circ \widehat{\theta} : K[[\boldsymbol{x}]] \to K[[\boldsymbol{x}]]$ is an isomorphism by the inverse function theorem I.1.21. Hence, $\widehat{\theta}$ and therefore θ is an isomorphism, again by the implicit function theorem. □

Statement (2) of Proposition 1.29 can be generalized by applying Laudal's theorem ([Lau, Thm. 4.2]), which relates the base of a formal semiuniversal deformation of (X,x) with the fibre of a formal power series map:

Theorem 1.31 (Laudal). *Let (X,x) be a complex space germ such that $T^1_{(X,x)}$, $T^2_{(X,x)}$ are finite dimensional complex vector spaces. Then there exists a formal power series map*

$$\Psi : T^1_{(X,x)} \longrightarrow T^2_{(X,x)}$$

such that the fibre $\Psi^{-1}(0)$ is the base of a formal semiuniversal deformation of (X,x).

Corollary 1.32. *Let (X,x) be a complex space germ such that $T^1_{(X,x)}$, $T^2_{(X,x)}$ are finite dimensional complex vector spaces, and let (S,s) be the base space of the semiuniversal deformation. Then*

$$\dim_{\mathbb{C}} T^1_{(X,x)} \geq \dim(S,s) \geq \dim_{\mathbb{C}} T^1_{(X,x)} - \dim_{\mathbb{C}} T^2_{(X,x)} ,$$

and $\dim(S,s) = \dim_{\mathbb{C}} T^1_{(X,x)}$ *iff (S,s) is smooth.*

This corollary holds in a general deformation theoretic context (see Appendix C, Proposition C.2.6).

Remark 1.32.1. The $\mathcal{O}_{X,x}$-module $T^2_{(X,x)}$ contains the obstructions against smoothness of the base space of the semiuniversal deformation (if it exists), but it may be strictly bigger. That is, in Corollary 1.32, the dimension of (S,s) may be strictly larger than the difference $\dim_{\mathbb{C}} T^1_{(X,x)} - \dim_{\mathbb{C}} T^2_{(X,x)}$. We illustrate this by a few examples.

(1) The rigid singularity $(X,\mathbf{0})$ of two transversal planes in $(\mathbb{C}^4,\mathbf{0})$ (see Example 1.26.1 (1)) satisfies $\dim_{\mathbb{C}} T^2_{(X,\mathbf{0})} = 4$:

```
LIB "deform.lib";
ring R = 0,(x,y,u,v),ds;
ideal I = intersect(ideal(x,y),ideal(u,v));
vdim(T_2(I));          // vector space dimension of T^2
//-> 4
```

For the rigid singularity $(Y,\mathbf{0}) := (X,\mathbf{0}) \times (\mathbb{C},0) \subset (\mathbb{C}^5,\mathbf{0})$ the module $T^2_{(Y,\mathbf{0})}$ has Krull dimension 1. In particular, it is an infinite dimensional complex vector space:

```
ring R1 = 0,(x,y,u,v,w),ds;
ideal I = imap(R,I);   // (two transversal planes in C^4) x C^1
dim(T_2(I));           // Krull dimension of T^2
//-> 1
```

(2) For the rigid singularity defined by the union of a plane and a transversal line in $(\mathbb{C}^3,\mathbf{0})$ (see Example 1.26.1 (2)) we have $T^2_{(X,\mathbf{0})} = 0$:

```
ring R2 = 0,(x,u,v),ds;
ideal I = intersect(ideal(x),ideal(u,v));
vdim(T_2(I));
//-> 0
```

Example 1.32.2. Let us compute the full semiuniversal deformation of the cone $(X,\mathbf{0}) \subset (\mathbb{C}^5,\mathbf{0})$ over the rational normal curve of degree 4. We get $\dim_{\mathbb{C}} T^1_{(X,\mathbf{0})} = 4$ and $\dim_{\mathbb{C}} T^2_{(X,\mathbf{0})} = 3$. The total space of the semiuniversal deformation has 4 additional variables A, B, C, D (in the ring `Px`), the unfolding of the 6 defining equations of $(X,\mathbf{0})$ is given by the ideal `Fs` and the base space, which is given by the ideal `Js` in $\mathbb{C}\{A,B,C,D\}$, is the union of the 3-plane $\{D=0\}$ and the line $\{B=C=D-A=0\}$ in $(\mathbb{C}^4,\mathbf{0})$:

```
LIB "deform.lib";
ring R = 0,(x,y,z,u,v),ds;
matrix M[2][4] = x,y,z,u,y,z,u,v;
ideal I = minor(M,2);   // rational normal curve in P^4
vdim(T_1(I));
//-> 4
vdim(T_2(I));
//-> 3

list L = versal(I);     // compute semiuniversal deformation
//-> // ready: T_1 and T_2
//-> // start computation in degree 2.
//-> .... (further output skipped) .....

def Px=L[1];
show(Px);
//-> // ring: (0),(A,B,C,D,x,y,z,u,v),(ds(4),ds(5),C);
//-> // minpoly = 0
//-> // objects belonging to this ring:
//-> // Rs                     [0]  matrix 6 x 8
//-> // Fs                     [0]  matrix 1 x 6
//-> // Js                     [0]  matrix 1 x 3

setring Px;
Fs;                     // equations of total space
//-> Fs[1,1]=-u2+zv+Bu+Dv
//-> Fs[1,2]=-zu+yv-Au+Du
//-> Fs[1,3]=-yu+xv+Cu+Dz
//-> Fs[1,4]=z2-yu+Az+By
//-> Fs[1,5]=yz-xu+Bx-Cz
//-> Fs[1,6]=-y2+xz+Ax+Cy
Js;                     // equations of base space
//-> Js[1,1]=BD
//-> Js[1,2]=-AD+D2
//-> Js[1,3]=-CD
```

Hence, the semiuniversal deformation of $(X,\mathbf{0})$ is given by $(\mathscr{X},\mathbf{0}) \to (S,\mathbf{0})$, induced by the projection onto the first factor of $(\mathbb{C}^4,\mathbf{0}) \times (\mathbb{C}^5,\mathbf{0})$,

$$(\mathbb{C}^4,\mathbf{0}) \times (\mathbb{C}^5,\mathbf{0}) \supset V(\mathtt{Fs}) = (\mathscr{X},\mathbf{0}) \to (S,\mathbf{0}) = V(\mathtt{Js}) \subset (\mathbb{C}^4,\mathbf{0})\,.$$

Note that the procedure `versal` proceeds by lifting infinitesimal deformations to higher and higher order (as described in the proof of Proposition 1.29). In general, this process may be infinite (but `versal` stops at a predefined order). However, in many examples, it is finite (as in the example above).

We can further analyse the base space of the semiuniversal deformation by decomposing it into its irreducible components (see Appendix B.1):

```
ring P = 0,(A,B,C,D),dp;
ideal Js = imap(Px,Js);
minAssGTZ(Js);
//-> [1]:
//->    _[1]=D
//-> [2]:
//->    _[1]=C
//->    _[2]=B
//->    _[3]=A-D
```

The output shows that the base space is reduced (the primary and prime components coincide) and that it has two components: a hyperplane and a transversal line.

Exercises

Exercise 1.5.1. Let $(R, \mathfrak{m})$ be a Noetherian local K-algebra such that the canonical map $K \to R/\mathfrak{m}$ is an isomorphism, and let $\widehat{R}$ be an $\mathfrak{m}$-adic completion.

(1) Show that R is regular iff $\widehat{R}$ is regular.

Hint. Show that R and $\widehat{R}$ have the same Hilbert function and, hence, the same dimension and the same embedding dimension.

(2) If R is complete, then $R \cong K[[x_1, \ldots, x_n]]/I$ for some n and some ideal $I \subset K[[x_1, \ldots, x_n]]$ and n can be chosen as $\dim_K \mathfrak{m}/\mathfrak{m}^2$.
(3) Show that Lemma 1.30 generalizes to any complete local ring R as above.

2 Equisingular Deformations of Plane Curve Singularities

In this section, we study deformations of plane curve singularities leaving certain invariants fixed, in particular, the multiplicity, the δ-invariant and the Milnor number. We define these notions also for non-reduced base spaces, especially for fat points, and we develop the theory of the corresponding equimultiple, equinormalizable and equisingular deformations.

We again focus on the issue of versality in our study, and we approach it from two points of view: as deformations of the equation, and as deformations of the parameterization. The second approach culminates in a new proof of the smoothness of the base of a *versal equisingular* deformation. The *equisingularity ideal* plays a central role in the theory. It represents the space of first order equisingular deformations and, geometrically, its quotient by the Tjurina ideal represents the tangent space to the base of the semiuniversal equisingular deformation inside the base of a semiuniversal deformation.

2.1 Equisingular Deformations of the Equation

We study now special deformations of plane curve singularities, requiring that the topological type is preserved. Recall that the topological type of a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ is the equivalence class of $(C, \mathbf{0})$ under local, embedded homeomorphisms (Definition I.3.30), and that the topological type is equivalently characterized by numerical data such as the system of multiplicity sequences (Theorem I.3.42).[8]

To study deformations which do not change the topological type in the nearby germs we must, first of all, specify the point of the nearby fibre where we take the germ. More precisely, we have to introduce the notion of a *deformation with section.*

However, in order to apply the full power of deformation theory, we need deformations over *non-reduced* base spaces. In particular, we have to define first order equisingular deformations, that is, equisingular deformations over the fat point T_ε. Since "constant multiplicity" can be generalized to "equimultiplicity" (along a section) over a non-reduced base, the system of multiplicity sequences is an appropriate invariant for defining equisingular deformations over arbitrary base spaces. This approach was chosen and developed by J. Wahl in his thesis. Based on Zariski's studies in equisingularity [Zar1], he created the infinitesimal theory of equisingular deformations and gave several applications (cf. [Wah, Wah1]).

Throughout the following, let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve singularity, and let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ be a defining power series. We call $f = 0$, or just f the *(local) equation* of $(C, \mathbf{0})$. Deformations of $(C, \mathbf{0})$ (respectively embedded deformations of $(C, \mathbf{0})$) will also be called *deformations of the equation* in contrast to *deformations of the parametrization*, as considered in Section 2.3.

Definition 2.1. A *deformation with section* of $(C, \mathbf{0})$ over a complex germ (T, t_0) consists of a deformation $(i, \phi) : (C, \mathbf{0}) \hookrightarrow (\mathscr{C}, x_0) \to (T, t_0)$ of $(C, \mathbf{0})$ over (T, t_0) and a section of ϕ, that is, a morphism $\sigma : (T, t_0) \to (\mathscr{C}, x_0)$ satisfying $\phi \circ \sigma = \mathrm{id}_{(T,t_0)}$. It is denoted by (i, ϕ, σ) or just by (ϕ, σ).

The category of deformations with section of $(C, \mathbf{0})$ is denoted by $\mathcal{D}ef^{\,sec}_{(C,\mathbf{0})}$, where morphisms are morphisms of deformations which commute with the

[8] It is a general fact from topology (proved by Timourian [Tim] and King [Kin1]) that, if the embedded type of the fibres of a family of hypersurfaces is constant, then the family is even topologically trivial.

sections. Isomorphism classes of deformations with sections over (T, t_0) are denoted by $\underline{\mathcal{D}ef}^{\,sec}_{(C,\mathbf{0})}(T, t_0)$.

It follows from the definition that the section σ is a closed embedding, mapping (T, t_0) isomorphically to $\sigma(T, t_0)$. Moreover, by Corollary 1.6, we may assume the deformation to be embedded, that is, any deformation with section is given by a commutative diagram

$$\begin{tikzcd}
(C,\mathbf{0}) \arrow[r, hook, "i"] \arrow[d] & (\mathscr{C}, x_0) \arrow[r, hook] \arrow[d, shift left, "\phi"] & (\mathbb{C}^2 \times T, (\mathbf{0}, t_0)) \arrow[dl, "\mathrm{pr}"] \\
\{t_0\} \arrow[r, hook] & (T, t_0) \arrow[u, shift left, bend left, "\sigma"] &
\end{tikzcd} \tag{2.1.1}$$

where $(\mathscr{C}, x_0)$ is a hypersurface germ in $(\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$ and pr the natural projection. $(\mathscr{C}, x_0)$ is defined by an unfolding $F \in \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ satisfying $F \circ \sigma = 0$. Hence, F is an element of $\operatorname{Ker}\bigl(\sigma^\sharp\colon \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)} \to \mathcal{O}_{T,t_0}\bigr) =: I_\sigma$, the ideal of $\sigma(T, t_0)$. After fixing local coordinates x, y for $(\mathbb{C}^2, \mathbf{0})$, we get

$$I_\sigma = \langle x - \sigma_1, y - \sigma_2 \rangle, \qquad \sigma_1 := \sigma^\sharp(x),\ \sigma_2 := \sigma^\sharp(y) \in \mathcal{O}_{T,t_0}\,.$$

Hence, I_σ determines the section σ.

The section σ is called the *trivial section* if $\sigma(T, t_0) = (\{\mathbf{0}\} \times T, t_0)$, that is, $I_\sigma = \langle x, y \rangle$. It is called a *singular section* if we have an inclusion of germs $\sigma(T, t_0) \subset (\operatorname{Sing}(\phi), p)$.

Next, we show that the section can be *trivialized*, that is, each embedded deformation with section is isomorphic to an embedded deformation with trivial section, that is, given by a diagram (2.1.1) with σ the trivial section (see Proposition 2.2, below). The proof is based on the relative lifting Lemma I.1.27. In geometric terms, this lemma says that any commutative diagram of morphisms of complex germs (with solid arrows)

$$\begin{tikzcd}
(\mathbb{C}^n \times T, (\mathbf{0}, t_0)) \arrow[rr, dashed] & & (\mathbb{C}^m \times T, (\mathbf{0}, t_0)) \\
(\mathscr{X}, x_0) \arrow[u, hook] \arrow[rr] \arrow[dr] & & (\mathscr{Y}, y_0) \arrow[u, hook] \arrow[dl] \\
& (T, t_0) &
\end{tikzcd}$$

where $(\mathscr{X}, x_0) \to (T, t_0)$ and $(\mathscr{Y}, y_0) \to (T, t_0)$ are induced by the projection, can be completed to a commutative diagram by a dashed arrow. The dashed arrow can be chosen as an isomorphism if $n = m$ and $(\mathscr{X}, x_0) \to (\mathscr{Y}, y_0)$ is an isomorphism (respectively as a closed embedding if $n \le m$ and $(\mathscr{X}, x_0) \to (\mathscr{Y}, y_0)$ is a closed embedding).

Proposition 2.2. *Let* $i : (\mathscr{X}, x_0) \hookrightarrow (\mathbb{C}^n, \mathbf{0}) \times (T, t_0)$ *be a closed embedding, and let* $\mathrm{pr} : (\mathbb{C}^n, \mathbf{0}) \times (T, t_0) \to (T, t_0)$ *be the projection to the second factor. Then each section* $\sigma : (T, t_0) \to (\mathscr{X}, x_0)$ *of* $\mathrm{pr} \circ i$ *can be trivialized. That is, there is an isomorphism*

$$\psi : (\mathbb{C}^n, \mathbf{0}) \times (T, t_0) \xrightarrow{\cong} (\mathbb{C}^n, \mathbf{0}) \times (T, t_0)$$

commuting with pr *such that* $\psi \circ \sigma$ *is the canonical inclusion*

$$\psi \circ \sigma : (T, t_0) \to \{\mathbf{0}\} \times (T, t_0) \subset (\mathbb{C}^n, \mathbf{0}) \times (T, t_0) .$$

Proof. Since $(\sigma(T), x_0) \xrightarrow{\mathrm{pr}} (T, t_0) \hookrightarrow \{\mathbf{0}\} \times (T, t_0)$ is an isomorphism over (T, t_0), the statement follows by applying the relative lifting lemma to the isomorphism of $\mathscr{O}_{T,t_0}$-algebras $\mathscr{O}_{\sigma(T),x_0} \xrightarrow{\cong} \mathscr{O}_{\{\mathbf{0}\}\times(T,t_0)}$. □

Corollary 2.3. *With the above notations, we have*

$$T^{1,sec}_{(C,\mathbf{0})} := \underline{\mathcal{D}ef}^{sec}_{(C,\mathbf{0})}(T_\varepsilon) \cong \mathfrak{m}/\langle f, \mathfrak{m}j(f)\rangle ,$$

where $j(f) \subset \mathbb{C}\{x, y\}$ *denotes the Jacobian ideal and* $\mathfrak{m} \subset \mathbb{C}\{x, y\}$ *the maximal ideal.*

Proof. Since each section can be trivialized, each deformation with section of $(C, \mathbf{0})$ over T_ε is represented by $f + \varepsilon g$ with $g \in \mathfrak{m}$. Such a deformation is trivial iff $g \in \langle f, \mathfrak{m}j(f)\rangle$ as shown in the proof of Proposition 1.25 and Remark 1.25.1. □

Definition 2.4. Let (i, ϕ, σ), $\phi : (\mathscr{C}, x_0) \hookrightarrow (\mathbb{C}^2 \times T, (\mathbf{0}, t_0)) \to (T, t_0)$, be an embedded deformation with section $\sigma : (T, t_0) \to (\mathscr{C}, x_0)$ of $(C, \mathbf{0})$, and let f be an equation for $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ of multiplicity $\mathrm{mt}(f)$. Moreover, let $F \in \mathscr{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ be a defining power series for $(\mathscr{C}, x_0) \subset (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$, and let $I_\sigma \subset \mathscr{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ denote the ideal of $\sigma(T, t_0) \subset (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$. Then (i, ϕ, σ) is called *equimultiple* (or, the deformation (i, ϕ) is called *equimultiple along* σ) iff

$$F \in I_\sigma^{\mathrm{mt}(f)} .$$

Note that this definition is independent of the chosen embedding and local equation.

Definition 2.5. Let T be a complex space, $U \subset \mathbb{C}^2 \times T$ be open and $\sigma : T \to U$, $t \mapsto (\sigma_1(t), \sigma_2(t), t)$, a section of the second projection. We define the *blowing up* of U *along* σ (or the *blowing up of the section* σ) as the complex space

$$\begin{aligned} B\ell_\sigma(U) := B\ell_{\sigma(T)}(U) &:= \left\{(z; L) \in U \times \mathbb{P}^1 \;\middle|\; z - \sigma(t) \in L \times \{t\}\right\} \\ &:= \left\{(x, y, t;\, a_1 : a_2) \in U \times \mathbb{P}^1 \;\middle|\; a_2(x - \sigma_1(t)) = a_1(y - \sigma_2(t))\right\} , \end{aligned}$$

together with the projection $\pi : B\ell_\sigma(U) \to U$. In particular, if σ is the trivial section with $\sigma_1(t) = \sigma_2(t) = 0$ for all $t \in T$, then $B\ell_\sigma(\mathbb{C}^2 \times T) = B\ell_{\mathbf{0}}(\mathbb{C}^2) \times T$.

As previously (when blowing up points), we can cover $U \times \mathbb{P}^1$ by two charts $U \times V_i := \{a_i \neq 0\} \subset U \times \mathbb{P}^1$, $i = 1,2$. For the first chart we obtain (with $v := a_2/a_1$)

$$(U \times V_1) \cap B\ell_\sigma(U) = \big\{(x,y,t,v) \,\big|\, v(x-\sigma_1(t)) = y - \sigma_2(t)\big\}$$

with ideal sheaf $\langle v(x-\sigma_1) - y + \sigma_2\rangle \mathcal{O}_{U\times V_1}$. Setting $u := x - \sigma_1$ and eliminating y, we see that $(U \times V_1) \cap B\ell_\sigma(U)$ is isomorphic to an open subset of $\mathbb{C}^2 \times T$ with coordinates u, v, t. That is, if $U = U_1 \times U_2 \times T$, $U_i \subset \mathbb{C}$ open, then

$$(U \times V_1) \cap B\ell_\sigma(U) = \big\{(u,v,t) \in U_1 \times \mathbb{C} \times T \,\big|\, uv + \sigma_2(t) \in U_2\big\}$$

is an open neighbourhood of $\{0\} \times \mathbb{C} \times T$, and v is an affine coordinate of $\mathbb{C}$, not just a coordinate of the germ $(\mathbb{C}, 0)$. In these coordinates π is given as

$$\pi : (U \times V_1) \cap B\ell_\sigma(U) \to U \subset \mathbb{C}^2 \times T\,,\ (u,v,t) \mapsto \big(u + \sigma_1(t),\, uv + \sigma_2(t),\, t\big)\,.$$

Similarly, we have coordinates $\overline{u}, \overline{v}, t$ in the second chart (with $\overline{u}$ affine) and

$$\pi : (U \times V_2) \cap B\ell_\sigma(U) \to U\,, \quad (\overline{u}, \overline{v}, t) \mapsto \big(\overline{uv} + \sigma_1(t),\, \overline{v} + \sigma_2(t),\, t\big)\,.$$

As $B\ell_\sigma(U)$ can be covered by these two charts, both being isomorphic over T to open subsets in $\mathbb{C}^2 \times T$, we can blow up sections of the composition $B\ell_\sigma(U) \to U \to T$ by choosing coordinates of the charts and proceeding as above. Different coordinates give results which are isomorphic over T.

Furthermore, the construction is local along the sections. Hence, we can blow up finitely many pairwise disjoint sections in an arbitrary order, or simultaneously, and get a blown up complex space, which is unique up to isomorphism over T. By passing to small representatives, we can also blow up sections of morphisms of germs of complex spaces.

For each point $\sigma(t) \in \sigma(T)$ we get $\pi^{-1}(\sigma(t)) = \mathbb{P}^1$ with local equation $u = 0$ in the first chart and with $\overline{v} = 0$ in the second chart. Hence,

$$\mathscr{E} := \pi^{-1}\big(\sigma(T)\big) = \sigma(T) \times \mathbb{P}^1$$

is a divisor in $B\ell_\sigma(U)$, called the *exceptional divisor* of the blowing up (which we describe below in local coordinates).

Now, let (T, t_0) be a germ, let $\sigma : (T, t_0) \to (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$ be a section of the projection to (T, t_0), and let $(\mathscr{C}, x_0)$ be the hypersurface germ of $(\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$ defined by $F \in \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$. Fixing local coordinates, we can write F as

$$F(x,y,\boldsymbol{t}) = \sum_{i,j} a_{ij}(\boldsymbol{t}) \cdot \big(x - \sigma_1(\boldsymbol{t})\big)^i \big(y - \sigma_2(\boldsymbol{t})\big)^j\,, \qquad a_{ij} \in \mathcal{O}_{T,t_0}\,,$$

and $F(x,y,\mathbf{0}) = f(x,y)$. Then F defines an embedded deformation of $(C, \mathbf{0}) = (V(f), \mathbf{0})$ which is equimultiple along σ iff

$$\min\{i+j \mid a_{ij} \neq 0\} = \mathrm{mt}(f)\,.$$

Let $\pi : B\ell_\sigma(\mathbb{C}^2 \times T, (\mathbf{0}, t_0)) \to (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$ be the blowing up along the section σ, which is a germ along the exceptional divisor $\sigma(T) \times \mathbb{P}^1 \subset B\ell_\sigma(U)$ in the blowing up of a small representative $\sigma : T \to U \subset \mathbb{C}^2 \times T$. Assume that F is equimultiple along σ. Then, in the first chart, we have

$$\widehat{F}(u, v, \boldsymbol{t}) := (F \circ \pi)(u, v, \boldsymbol{t}) = \sum_{i,j} a_{ij}(\boldsymbol{t}) u^i (uv)^j = u^{\mathrm{mt}(f)} \cdot \widetilde{F}(u, v, \boldsymbol{t})\,,$$

and, in the second chart,

$$\widehat{F}(\overline{u}, \overline{v}, \boldsymbol{t}) := (F \circ \pi)(\overline{u}, \overline{v}, \boldsymbol{t}) = \overline{v}^{\mathrm{mt}(f)} \cdot \widetilde{F}(\overline{u}, \overline{v}, \boldsymbol{t})\,.$$

The functions $\widetilde{F}(u, v, \boldsymbol{t})$ and $\widetilde{F}(\overline{u}, \overline{v}, \boldsymbol{t})$ (which are defined by these relations) are holomorphic in the respective charts, and they define a unique zero-set in the intersection of these charts.

We define the following (Cartier-)divisors in $B\ell_\sigma(\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$:

- $\widehat{\mathscr{C}}$, the divisor given by $\widehat{F} = 0$, called the *total transform of* $(\mathscr{C}, x_0)$.
- $\widetilde{\mathscr{C}}$, the divisor given by $\widetilde{F} = 0$, called the *strict transform of* $(\mathscr{C}, x_0)$.

As a divisor, we have

$$\widehat{\mathscr{C}} = \widetilde{\mathscr{C}} + \mathrm{mt}(f) \cdot \mathscr{E}\,,$$

and $\widetilde{\mathscr{C}}$ and $\mathscr{E}$ have no common component. The divisor $\widetilde{\mathscr{C}} + \mathscr{E}$ is called the *reduced total transform of* $(\mathscr{C}, x_0)$. In the first chart, it is given by $u \cdot \widetilde{F}(u, v, \boldsymbol{t}) = 0$, in the second by $\overline{v} \cdot \widetilde{F}(\overline{u}, \overline{v}, \boldsymbol{t}) = 0$.

We shall call a family of plane curve singularities equisingular if it is equimultiple and if the reduced total transform in all successive blowing ups (until the special fibre is resolved) are again equimultiple along the singular sections. This is Wahl's [Wah] definition (if the base space is a fat point), and it implies that all fibres are equisingular in the sense of Zariski [Zar1].

Definition 2.6. Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve germ, and let (i, ϕ, σ) be an embedded deformation with section of $(C, \mathbf{0})$ over (T, t_0). If $(C, \mathbf{0})$ is singular, then (i, ϕ, σ) is called an *equisingular deformation of* $(C, \mathbf{0})$ or an *equisingular deformation of the equation of* $(C, \mathbf{0})$ if the following holds: There exist small representatives for (i, ϕ, σ) and a commutative diagram of complex spaces and morphisms

$$\begin{tikzcd}
\mathscr{C}^{(N)} \arrow[r] \arrow[d, hook] & \mathscr{C}^{(N-1)} \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & \mathscr{C}^{(0)} \arrow[d, hook] \arrow[dr, "\phi"] & \\
\mathscr{M}^{(N)} \arrow[r, "\pi_N"] & \mathscr{M}^{(N-1)} \arrow[r, "\pi_{N-1}"] & \cdots \arrow[r, "\pi_1"] & \mathscr{M}^{(0)} \arrow[r] & T \\
M^{(N)} \arrow[r] \arrow[u, hook] \arrow[ur, phantom, "\square"] & M^{(N-1)} \arrow[r] \arrow[u, hook] \arrow[ur, phantom, "\square"] & \cdots \arrow[r] \arrow[ur, phantom, "\square"] & M^{(0)} \arrow[r] \arrow[u, hook] \arrow[ur, phantom, "\square"] & \{t_0\} \arrow[u, hook]
\end{tikzcd} \tag{2.1.2}$$

together with pairwise disjoint sections

$$\sigma_1^{(\ell)}, \dots, \sigma_{k_\ell}^{(\ell)} : T \to \mathscr{C}^{(\ell)} \subset \mathscr{M}^{(\ell)}, \quad \ell = 0, \dots, N-1,$$

of the composition $\mathscr{M}^{(\ell)} \xrightarrow{\pi_\ell} \mathscr{M}^{(\ell-1)} \xrightarrow{\pi_{\ell-1}} \dots \xrightarrow{\pi_1} \mathscr{M}^{(0)} \to T$ with the following properties:

(1) The lower row of (2.1.2) induces a minimal embedded resolution of the plane curve germ $(C, \mathbf{0}) \subset (M^{(0)}, \mathbf{0}) = (\mathbb{C}^2, \mathbf{0})$.
(2) For $\ell = 0$, we have $(\mathscr{M}^{(0)}, x_0) = (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$, $(\mathscr{C}^{(0)}, x_0) = (\mathscr{C}, x_0)$, $k_0 = 1$. Moreover, $\sigma_1^{(0)} : T \to \mathscr{M}^{(0)}$ is the section (induced by) σ, and $(\mathscr{C}^{(0)}, x_0) \hookrightarrow (\mathscr{M}^{(0)}, x_0) \to (T, t_0)$ defines an equimultiple (embedded) deformation of $(C, \mathbf{0})$ along $\sigma_1^{(0)}$.
(3) For $\ell = 1$, we have that $\pi_1 : \mathscr{M}^{(1)} \to \mathscr{M}^{(0)}$ is the blowing up of $\mathscr{M}^{(0)}$ along the section $\sigma_1^{(0)}$, $\mathscr{C}^{(1)}$ is the strict transform of $\mathscr{C}^{(0)} \subset \mathscr{M}^{(0)}$, and $\mathscr{E}^{(1)}$ is the exceptional divisor of π_1.
(4) For $\ell \geq 1$, we require inductively that
   - $\sigma_1^{(\ell)}(t_0), \dots, \sigma_{k_\ell}^{(\ell)}(t_0)$ are precisely the non-nodal singular points of the reduced total transform of $(C, \mathbf{0}) \subset (M^{(0)}, \mathbf{0}) = (\mathbb{C}^2, \mathbf{0})$.
   - $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)} \hookrightarrow \mathscr{M}^{(\ell)} \to T$ induces (embedded) equimultiple deformations along $\sigma_1^{(\ell)}, \dots, \sigma_{k_\ell}^{(\ell)}$, of the respective germs of the reduced total transform $C^{(\ell)} \cup E^{(\ell)}$ of $(C, \mathbf{0})$ in $M^{(\ell)}$.
   - The sections are *compatible*, that is, for each $j = 1, \dots, k_\ell$ there is some $1 \leq i \leq k_{\ell-1}$ such that $\pi_{\ell+1} \circ \sigma_j^{(\ell)} = \sigma_i^{(\ell-1)}$.
   - $\pi_{\ell+1} : \mathscr{M}^{(\ell+1)} \to \mathscr{M}^{(\ell)}$ is the blowing up of $\mathscr{M}^{(\ell)}$ along $\sigma_1^{(\ell)}, \dots, \sigma_{k_\ell}^{(\ell)}$, $\mathscr{C}^{(\ell+1)}$ is the strict transform of $\mathscr{C}^{(\ell)} \subset \mathscr{M}^{(\ell)}$, and $\mathscr{E}^{(\ell+1)}$ is the exceptional divisor of the composition $\pi_1 \circ \dots \circ \pi_{\ell+1}$.

If $(C, \mathbf{0})$ is smooth, each deformation with section is called *equisingular*.

We call a diagram (2.1.2) together with the sections $\sigma_j^{(\ell)}$ such that (1)–(4) hold an *equisingular deformation of the resolution of* $(C, \mathbf{0})$ associated to the embedded deformation with section (i, ϕ, σ).

Remark 2.6.1. (1) The sections $\sigma_i^{(\ell)}$ are also called *equimultiple sections* for the equisingular deformation. By Proposition 2.2, p. 269, all sections can be locally trivialized, that is, for each $p = \sigma_j^{(\ell)}(t_0)$, there are isomorphisms of germs $(\mathscr{M}^{(\ell)}, p) \cong (\mathbb{C}^2, \mathbf{0}) \times (T, t_0)$ over (T, t_0) trivializing the section $\sigma_j^{(\ell)}$.

(2) Considering the restriction of the strict transforms $\mathscr{C}^{(\ell)}$ to the special fibre over t_0, we get a minimal embedded resolution of $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$,

$$\begin{array}{ccccccccc}
M^{(N)} & \xrightarrow{\pi_N} & M^{(N-1)} & \xrightarrow{\pi_{N-1}} & \cdots & \xrightarrow{\pi_2} & M^{(1)} & \xrightarrow{\pi_1} & (\mathbb{C}^2, \mathbf{0}) \\
\cup & & \cup & & & & \cup & & \cup \\
C^{(N)} & \longrightarrow & C^{(N-1)} & \longrightarrow & \cdots & \longrightarrow & C^{(1)} & \longrightarrow & (C, \mathbf{0}) .
\end{array}$$

π_1 is the blowing up of the origin, and $\pi_{\ell+1}$, $\ell = 1,\dots,N-1$, is the simultaneous blowing up of all non-nodal singularities $p_j = \sigma_j^{(\ell)}(t_0)$, $j = 1,\dots,k_\ell$ of the respective reduced total transforms of $(C,\mathbf{0})$. However, it is not important that we blow up the points simultaneously. As the construction is local, we can blow the points up successively in any order, the result is always isomorphic. In the same way, $\pi_{\ell+1} : \mathscr{M}^{(\ell+1)} \to \mathscr{M}^{(\ell)}$ can either blow up $\mathscr{M}^{(\ell)}$ simultaneously along the sections $\sigma_j^{(\ell)}$ or successively in an arbitrary order.

(3) By semicontinuity of the multiplicity[9], equimultiplicity of the reduced total transform $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)}$ along $\sigma_i^{(\ell)}$ is equivalent to equimultiplicity of the strict transform $\mathscr{C}^{(\ell)}$ *and* of the reduced exceptional divisor $\mathscr{E}^{(\ell)}$ along $\sigma_i^{(\ell)}$. Indeed, if we want to preserve the topological type of the singularities along σ in the nearby fibres, it is not sufficient to require only equimultiplicity of the strict transforms as is shown in Example 2.6.2, below.

(4) If the germ $(C,\mathbf{0})$ is smooth, then each (embedded) deformation of $(C,\mathbf{0})$, $(C,\mathbf{0}) \hookrightarrow (\mathscr{C},x_0) \to (T,t_0)$, with section $\sigma : (T,t_0) \to (\mathscr{C},x_0)$ is equimultiple along σ.

If the reduced total transform in the special fibre $C^{(\ell)} \cup E^{(\ell)}$, $\ell \geq 1$, has a node at $q \in C^{(\ell)} \cap E^{(\ell)}$, that is, if $C^{(\ell)}, E^{(\ell)}$ are smooth and intersect transversally at q, then there exists a unique section σ_q such that $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)}$ is equimultiple along σ_q.

This implies that the definition of equisingularity remains unchanged if, in Definition 2.6, we start with any (not necessarily minimal) embedded resolution as special fibre (in the bottom row of diagram (2.1.2)).

(5) It follows also that, for $\ell \geq 1$ and $q \in C^{(\ell)} \cap E^{(\ell)}$,

$$(\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)}, q) \hookrightarrow (\mathscr{M}^{(\ell)}, q) \to (T,t_0)$$

is an equisingular embedded deformation of the germ $(C^{(\ell)} \cup E^{(\ell)}, q)$.

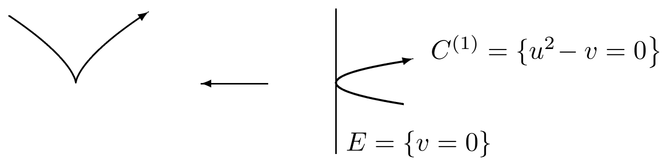
(6) By Proposition 2.8 on page 275, the sections $\sigma_j^{(\ell)}$ are uniquely determined. Since the minimal resolution is unique (Exercise I.3.3.1), it follows that the associated equisingular deformation of the resolution is uniquely determined (up to isomorphism) by (i,ϕ,σ). By (4), the same holds if the lower row of (2.1.2) is any (not necessarily minimal) embedded resolution of $(C,\mathbf{0})$.

Example 2.6.2. Consider the one-parameter deformation of the cusp given by $F := x^2 - y^3 - t^2y^k$, $k \geq 0$. For $k \geq 2$, the deformation given by F is equimultiple along the trivial section $\sigma : t \mapsto (0,0,t)$ (and σ is the unique equimultiple section), while, for $k \leq 1$ there is no equimultiple section.

After blowing up σ, we obtain (in the second chart) the reduced total transform $\{v(u^2 - v - t^2v^{k-2}) = 0\}$. In the special fibre we get the reduced total transform of the cusp, which is the union of the smooth germ

[9] For hypersurfaces, this is easy: if $F_{\boldsymbol{t}}(\boldsymbol{x}) = F(\boldsymbol{x},\boldsymbol{t}) = f(\boldsymbol{x}) + g_{\boldsymbol{t}}(\boldsymbol{x})$, $g_0(\boldsymbol{x}) = 0$, is an unfolding of f then, for $\boldsymbol{t}$ sufficiently close to $\mathbf{0}$, the terms of lowest order of f cannot be cancelled by terms of $g_{\boldsymbol{t}}$. Hence, $\mathrm{mt}(f) \geq \mathrm{mt}\, F_{\boldsymbol{t}}$ for $\boldsymbol{t}$ close to $\mathbf{0}$.

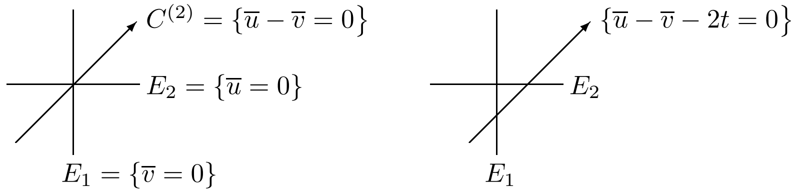
$C^{(1)} = \{u^2 - v = 0\}$ and the exceptional divisor $E = \{v = 0\}$, intersecting with multiplicity 2 at the origin:



For $k \geq 3$, the blown-up deformation has the trivial section $\sigma^{(1)}$ as unique equimultiple section.

For $k = 2$ there are two different equimultiple sections through the origin being compatible with σ. Indeed, a section $\sigma^{(1)}$ is compatible with the trivial section σ iff its image lies in the exceptional divisor $\mathscr{E}^{(1)} = \{v = 0\}$. In other words, a section $\sigma^{(1)}$ through the origin is compatible with σ iff it is given by an ideal $\langle u - t\alpha, v\rangle$, $\alpha \in \mathbb{C}\{t\}$. Since the ideal of the reduced total transform $v(u^2 - v - t^2)$ is contained in $\langle u - t, v\rangle^2$ and in $\langle u + t, v\rangle^2$, we get two equimultiple sections $\sigma^{(1)}_{\pm}$ given by the ideals $\langle u \pm t, v\rangle$. Geometrically, the reduced total transform of the special fibre (which is an A_3-singularity) is deformed into the union of a line and a parabola, meeting transversally in two points, and the equimultiple sections are the singular sections through the nodes. [10] After blowing up $\sigma^{(1)}$ (respectively one of the sections $\sigma^{(1)}_{\pm}$), the reduced total transform in the special fibre is the union of three concurrent lines.

Hence, for $k = 2$, we find no equimultiple section through the origin of the respective reduced total transform $\{\overline{u}\overline{v}(\overline{u} \mp 2t - \overline{v}) = 0\}$ $(\overline{u} = u \pm t)$. Geometrically, this is caused by the fact that the D_4-singularity (of multiplicity 3) in the special fibre is deformed into three nodes (each of multiplicity 2) in the nearby fibres:



If $k \geq 3$, the reduced total transform of F, $\overline{u}\overline{v}(\overline{u} - \overline{v} - t^2\overline{u}^{k-3}\overline{v}^{k-2})$, is contained in $\langle \overline{u}, \overline{v}\rangle^3$. Hence, it defines an equimultiple deformation along the trivial section $\sigma^{(2)}$.

[10] Note that replacing t^2 by t in the definition of F, there is no equimultiple section of the strict transform in case $k = 2$. At first glance, this might seem strange, since fibrewise the A_3-singularity is still deformed into 2 nodes. But there is a monodromy phenomenon which cannot be observed in the real pictures: a loop around the origin in the base of the deformation interchanges the nodes of the nearby fibres. Algebraically, this corresponds to the fact that there is no square root of t in $\mathbb{C}\{t\}$. See also Figure 2.7.

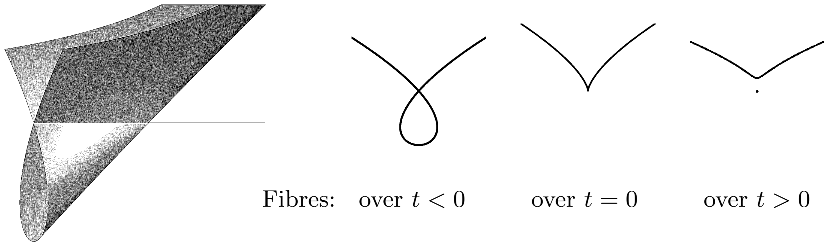


Fig. 2.7. The deformation of the cusp given by $x^2 - y^3 + ty^2$ is equimultiple along the trivial section but not equisingular. Note that the real pictures are misleading: the complex fibres are always connected.

We conclude that F defines an equisingular deformation iff $k \geq 3$: it is even a trivial deformation, since $F = x^2 - y^3(1 - t^2y^{k-3})$.

Finally, the case $k = 2$ shows that it is *not sufficient to require equimultiplicity of the strict transforms* $\mathscr{C}^{(\ell)}$, $\ell \geq 0$. Indeed, the strict transforms $\mathscr{C}^{(2)}_{\pm}$, given by $(\overline{u} \mp 2t - \overline{v})$, are equimultiple along the section $\sigma^{(2)}_{\pm}$ with ideal $\langle \overline{u}, \overline{v} \pm 2t \rangle$, and the latter is compatible with $\sigma^{(1)}_{\pm}$ (since its image lies in the exceptional divisor $\mathscr{E}^{(2)}_{\pm} = \{\overline{u} = 0\}$).

Definition 2.7. A deformation (i, ϕ) of $(C, \mathbf{0})$ over (T, t_0),

$$(C, \mathbf{0}) \overset{i}{\hookrightarrow} (\mathscr{C}, x_0) \xrightarrow{\phi} (T, t_0)\,,$$

is called *equisingular* (or an *ES-deformation*) if there exists an embedded deformation with section (i, ϕ, σ) inducing (i, ϕ) such that (i, ϕ, σ) is equisingular in the sense of Definition 2.6. Two equisingular deformations of $(C, \mathbf{0})$ over (T, t_0) are isomorphic if they are isomorphic as deformations over (T, t_0). The set of isomorphism classes of equisingular deformations of $(C, \mathbf{0})$ over (T, t_0) is denoted by $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}(T, t_0)$, and

$$\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})} \colon (\textit{complex germs}) \longrightarrow \mathcal{S}ets\,, \quad (T, t_0) \longmapsto \underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}(T, t_0)$$

is called the *equisingular deformation functor.*

Proposition 2.8. *Let* (i, ϕ) *be an equisingular deformation of* $(C, \mathbf{0})$ *over* (T, t_0)*. Then the system of equimultiple sections* $\sigma^{(\ell)}_i$*,* $\ell \geq 0$*, for the diagram* (2.1.2) *is uniquely determined.*

Proof.[11] This result is basically due to Wahl, who proved it if (T, t_0) is a fat point, and we refer to his proof [Wah, Thm. 3.2]. In general, let $\sigma^{(\ell)}_i$ and

[11] Another proof of Proposition 2.8 is given in [CGL2], where it is shown that uniqueness of the sections fails in positive characteristic.

$\widetilde{\sigma}_i^{(\ell)}$ be equimultiple sections with $\sigma_i^{(\ell)}(t_0) = \widetilde{\sigma}_i^{(\ell)}(t_0) =: p_i$. Then, by Wahl's result, we may assume that $\sigma_i^{(\ell)\,\sharp}(x_\nu) - \widetilde{\sigma}_i^{(\ell)\,\sharp}(x_\nu) \in \mathcal{O}_{T,t_0}$ vanishes modulo an arbitrary power of $\mathfrak{m}_{T,t_0}$, where x_ν denote generators of the maximal ideal of $\mathcal{O}_{\mathscr{M}^{(\ell)},p_i}$. Hence, $\sigma_i^{(\ell)\,\sharp} = \widetilde{\sigma}_i^{(\ell)\,\sharp}$, by Krull's intersection theorem. $\square$

The approach of Wahl to equisingular deformations is slightly different. He considers diagrams as in Definition 2.6, *together with a system of (equimultiple) sections* satisfying all the required properties. Morphisms in this category (denoted by $\mathcal{D}ef^N_{(C,\mathbf{0})}$) are morphisms of diagrams commuting with the given sections. This approach is necessary to show that the corresponding functor of isomorphism classes $\underline{\mathcal{D}ef}^N_{(C,\mathbf{0})}$ satisfies Schlessinger's conditions and, hence, has a formal semiuniversal deformation. By Proposition 2.8, the natural forgetful functor $\underline{\mathcal{D}ef}^N_{(C,\mathbf{0})} \to \underline{\mathcal{D}ef}_{(C,\mathbf{0})}$ is injective, and we denote the image by $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}$.

Next, we want to show that equisingular deformations of reducible plane curve singularities induce equisingular deformations of the respective branches. For the proof we need the following statement which is interesting in its own:

Proposition 2.9. *Let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve singularity and let $\big(\widetilde{C}_i,\widetilde{0}_i\big)$, $i=1,\dots,r$, be reduced (not necessarily plane) curve singularities. Let $\big(\widetilde{C},\widetilde{0}\big) := \coprod_{i=1}^r \big(\widetilde{C}_i,\widetilde{0}_i\big)$ be the (multigerm of the) disjoint union and let $\pi : \big(\widetilde{C},\widetilde{0}\big) \to (C,\mathbf{0})$ be a finite morphism such that, for sufficiently small representatives, π induces an isomorphism*

$$\pi : \widetilde{C} \setminus \{\widetilde{0}\} \xrightarrow{\cong} C \setminus \{\mathbf{0}\}\,.$$

Moreover, let (T,t_0) be an arbitrary complex germ and consider a Cartesian diagram

$$\begin{tikzcd}
\big(\widetilde{C},\widetilde{0}\big) \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \big(\widetilde{\mathscr{C}},\widetilde{x}_0\big) \arrow[d, "\widetilde{\pi}"'] \arrow[dd, bend left=60, "\phi"] \\
(\mathbb{C}^2,\mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \big(\mathbb{C}^2 \times T,(\mathbf{0},t_0)\big) \arrow[d, "p"'] \\
\{t_0\} \arrow[r, hook] & (T,t_0)
\end{tikzcd}$$

with ϕ a flat morphism. Let $(\mathscr{C},x_0) := \widetilde{\pi}\big(\widetilde{\mathscr{C}},\widetilde{x}_0\big)$ be the image of $\widetilde{\pi}$, endowed with its Fitting structure (see Definition I.1.45). Then the Fitting ideal $\operatorname{Fitt}\big(\widetilde{\pi}_*\big(\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}\big)$ *is a principal ideal in $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$, the induced map $(\mathscr{C},x_0) \to (T,t_0)$ is flat, and $(C,\mathbf{0}) \hookrightarrow (\mathscr{C},x_0) \to (T,t_0)$ is an (embedded) deformation of $(C,\mathbf{0})$.*

Furthermore, the Fitting structure is the unique analytic structure on $\widetilde{\pi}\big(\widetilde{\mathscr{C}},\widetilde{x}_0\big)$ such that the projection to (T,t_0) defines a deformation of $(C,\mathbf{0})$. It coincides with the annihilator structure, that is, the ideal in $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ defining $(\mathscr{C},x_0)$ is the kernel of $\widetilde{\pi}^\sharp : \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)} \to \mathcal{O}_{\widetilde{\mathscr{C}},\widetilde{x}_0}$.

Proof. We work with representatives of the above germs which we always assume to be sufficiently small.

By Proposition I.1.70, π and $\widetilde{\pi}$ are finite morphisms. By the finite coherence theorem I.1.67, we may assume that $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}$ has a free resolution $\mathcal{F}_\bullet$ by $\mathcal{O}_{U\times T}$-modules of finite rank ($U \subset \mathbb{C}^2$ a neighbourhood of $\mathbf{0}$). Moreover, we can assume that the matrices in the free resolution $\mathcal{F}_\bullet$ have only entries in $\mathcal{J}(t_0)$, the ideal sheaf of $\{t_0\}$ in $\mathcal{O}_T$.

Step 1. We show that $\operatorname{Fitt}\big(\widetilde{\pi}_*\big(\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}\big)$ is a principal ideal in $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$:

Since the above diagram is Cartesian, tensoring with $\mathbb{C} = \mathcal{O}_T/\mathcal{J}(t_0)$ gives $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}} \otimes_{\mathcal{O}_T} \mathbb{C} = \pi_*\mathcal{O}_{\widetilde{C}}$, and its stalk at $\mathbf{0}$ is a finitely generated $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module of depth 1 (since $(\widetilde{C},\widetilde{0})$ is a reduced curve germ, hence Cohen-Macaulay). The Auslander-Buchsbaum formula (in the form of Corollary B.9.4) implies that each minimal free resolution of $\big(\pi_*\mathcal{O}_{\widetilde{C}}\big)_{\mathbf{0}}$ has length 1.

Since $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}$ is a flat $\mathcal{O}_T$-module (via p_*), tensoring the exact sequence (of $\mathcal{O}_{U\times T}$-modules)

$$\ldots \longrightarrow \mathcal{F}_2 \xrightarrow{M_2} \mathcal{F}_1 \xrightarrow{M_1} \mathcal{F}_0 \longrightarrow \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}} \longrightarrow 0$$

with $\mathbb{C}$ over $\mathcal{O}_T$ leads to an exact sequence of $\mathcal{O}_U$-modules

$$\ldots \longrightarrow \mathcal{F}_2 \otimes_{\mathcal{O}_T} \mathbb{C} \xrightarrow{\overline{M}_2} \mathcal{F}_1 \otimes_{\mathcal{O}_T} \mathbb{C} \xrightarrow{\overline{M}_1} \mathcal{F}_0 \otimes_{\mathcal{O}_T} \mathbb{C} \longrightarrow \pi_*\mathcal{O}_{\widetilde{C}} \longrightarrow 0\,.$$

By the choice of $\mathcal{F}_\bullet$, all $\mathcal{O}_U$-entries of the matrices $\overline{M}_i$ vanish at $\mathbf{0}$. Hence, $\mathcal{F}_\bullet \otimes_{\mathcal{O}_T} \mathbb{C}$ induces a minimal free resolution of the stalk $\big(\pi_*\mathcal{O}_{\widetilde{C}}\big)_{\mathbf{0}}$, which has length 1 by the above. It follows that the germ at $\mathbf{0}$ of $\overline{M}_1$ is injective, that is, we have a short exact sequence of $\mathcal{O}_U$-modules

$$0 \longrightarrow \mathcal{F}_1 \otimes_{\mathcal{O}_T} \mathbb{C} \xrightarrow{\overline{M}_1} \mathcal{F}_0 \otimes_{\mathcal{O}_T} \mathbb{C} \longrightarrow \pi_*\mathcal{O}_{\widetilde{C}} \longrightarrow 0\,.$$

Since the support of $\pi_*\mathcal{O}_{\widetilde{C}}$ is C (hence, of codimension 1 in U), the free modules $\mathcal{F}_1$ and $\mathcal{F}_0$ must have the same rank. Moreover, Proposition B.5.3 implies that we may also assume M_1 to be injective. In particular, $\mathcal{F}itt_{\mathcal{O}_{U\times T}}\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big)$ is a principal ideal in $\mathcal{O}_{U\times T}$, generated by the determinant of M_1.

Step 2. $(C,\mathbf{0}) \hookrightarrow (\mathscr{C},x_0) \to (T,t_0)$ is an (embedded) deformation of $(C,\mathbf{0})$:

Since $\pi_*\mathcal{O}_{\widetilde{C}}|_{U\cap C\setminus\{\mathbf{0}\}} \cong \mathcal{O}_C|_{U\cap C\setminus\{\mathbf{0}\}}$ by assumption, all germs outside $\mathbf{0}$ of $\det(\overline{M}_1)$ are reduced. Hence, $\det(\overline{M}_1)$ is reduced, and $\det(M_1) \otimes_{\mathcal{O}_T} \mathbb{C} = \det(\overline{M}_1)$ generates the ideal of $C \subset U$. It follows that $(\mathscr{C},x_0)$ with the Fitting structure is flat over (T,t_0) and defines a deformation of $(C,\mathbf{0})$.

Step 3. The Fitting and annihilator structure on $\widetilde{\pi}\big(\widetilde{\mathscr{C}},\widetilde{0}\big)$ coincide:

In general, $\mathcal{F}itt := \mathcal{F}itt\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big) \subset \mathcal{A}nn\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big) =: \mathcal{A}nn$. If we tensor the cokernel by $\mathbb{C}$ over $\mathcal{O}_T$, the result is a $\mathcal{O}_U$-sheaf with support at $\mathbf{0} \in C$, since the sheaves $\pi_*\mathcal{O}_{\widetilde{C}}$ and $\mathcal{O}_C$ are isomorphic outside $\mathbf{0}$.

However, we know already that $\mathcal{F}itt \otimes_{\mathcal{O}_T} \mathbb{C} = \mathcal{F}itt\big(\pi_*\mathcal{O}_{\widetilde{C}}\big)$ is a radical ideal. Since $\mathcal{F}itt \otimes_{\mathcal{O}_T} \mathbb{C} \subset \mathcal{A}nn \otimes_{\mathcal{O}_T} \mathbb{C}$ and both have C as zero-set, Hilbert's Nullstellensatz implies that they must coincide. Hence, we have $\mathcal{A}nn / \mathcal{F}itt \otimes_{\mathcal{O}_T} \mathbb{C} = 0$. On the other hand, Proposition B.5.3 gives that the stalk $(\mathcal{A}nn / \mathcal{F}itt)_{x_0}$ is $\mathcal{O}_{T,t_0}$-flat, hence, faithfully flat as $\mathcal{O}_{T,t_0}$ is local. It follows that $\mathcal{A}nn / \mathcal{F}itt = 0$.

Step 4. To see the uniqueness of the analytic structure of $(\mathscr{C}, x_0)$, let $(\mathscr{C}', x_0)$ denote $\widetilde{\pi}\big(\widetilde{\mathscr{C}}, \widetilde{0}\big)$ with any analytic structure such that

$$(C, \mathbf{0}) \hookrightarrow (\mathscr{C}', x_0) \to (T, t_0)$$

is a deformation of $(C, \mathbf{0})$. Then $\mathcal{O}_{\mathscr{C}', x_0} \to \big(\widetilde{\pi}_* \mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0}, t_0)}$ is injective by Proposition B.5.3, since this is so after tensoring with $\mathbb{C}$ over $\mathcal{O}_{T,t_0}$ and since $\big(\widetilde{\pi}_* \mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0}, t_0)}$ is $\mathcal{O}_{T,t_0}$-flat. It follows that the ideal of $(\mathscr{C}', x_0)$ is the kernel of $\mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, t_0)} \twoheadrightarrow \mathcal{O}_{\mathscr{C}', x_0} \hookrightarrow \big(\widetilde{\pi}_* \mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0}, t_0)}$. Since $\big(\widetilde{\pi}_* \mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0}, t_0)}$ is a ring with 1, the kernel is just the annihilator of $\big(\widetilde{\pi}_* \mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0}, t_0)}$ which coincides with the ideal of $(\mathscr{C}, x_0)$ as shown in Step 3 of the proof. $\square$

Corollary 2.10. *With the assumptions of Proposition 2.9, we have:*

(1) Let F be a generator of $\mathrm{Fitt}(\widetilde{\pi}_* \mathcal{O}_{\widetilde{\mathscr{C}}})_{(\mathbf{0}, t_0)}$. *Then F is a non-zerodivisor of $\mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, t_0)}$.*

(2) If (T, t_0) is reduced (respectively Cohen-Macaulay), then also $(\mathscr{C}, x_0)$ is reduced (respectively Cohen-Macaulay). If (T, t_0) and $(\widetilde{C}, \widetilde{x}_0)$ are normal, then $(\widetilde{\mathscr{C}}, \widetilde{x}_0)$ is also normal, and $(\widetilde{\mathscr{C}}, \widetilde{x}_0) \to (\mathscr{C}, x_0)$ is the normalization of $(\mathscr{C}, x_0)$.

Proof. (1) Tensoring $\mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, t_0)} \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, t_0)} \to \mathcal{O}_{\mathscr{C}, x_0} \to 0$ by $\mathbb{C}$ over $\mathcal{O}_{T,t_0}$, we can argue as in Step 1 of the proof of Proposition 2.9 to see that multiplication by F is injective.

(2) If (T, t_0) is reduced, $\mathcal{O}_{\widetilde{\mathscr{C}}, \widetilde{x}_0}$ and, hence, $\mathcal{O}_{\mathscr{C}, x_0}$ (which is a subring by the second part of Proposition 2.9), has no nilpotent elements. If (T, t_0) is Cohen-Macaulay, also $\mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, t_0)}$ and, since F is a non-zerodivisor, $\mathcal{O}_{\mathscr{C}, x_0}$ are Cohen-Macaulay rings (Corollary B.8.3).

If $\big(\widetilde{C}, \widetilde{0}\big)$ is normal, it is smooth and each deformation of $\big(\widetilde{C}, \widetilde{0}\big)$ is trivial. Hence, $\big(\widetilde{\mathscr{C}}, \widetilde{x}_0\big) \cong \big(\widetilde{C}, \widetilde{0}\big) \times (T, t_0)$ which is normal if (T, t_0) is normal. The singular locus $\mathrm{Sing}(\mathscr{C})$ is everywhere of codimension one (since the fibres of $\mathscr{C} \to T$ have isolated singularities). Thus, Sard's theorem, applied to $\widetilde{\pi} : \widetilde{\mathscr{C}} \setminus \widetilde{\pi}^{-1}(\mathrm{Sing}(\mathscr{C})) \to \mathscr{C} \setminus \mathrm{Sing}(\mathscr{C})$, shows that $\widetilde{\pi}$ is generically an isomorphism. The result follows now from the universal property of normalization (see Theorem I.1.95). $\square$

Proposition 2.11. *Let $f = f_1 f_2$, with $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^2, \mathbf{0}}$ non-units, define a reduced plane curve singularity $(C, \mathbf{0})$, and let $F \in \mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, t_0)}$ define an equisingular deformation of $(C, \mathbf{0})$ over an arbitrary complex space germ (T, t_0).*

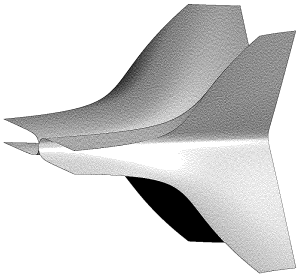


Fig. 2.8. The deformation of an A_3-singularity given by $x^2 - y^4 + tx^2y^2$ is equisingular along the trivial section. It splits into the equisingular deformations of the smooth branches given by $x\sqrt{1+ty^2} - y^2$ and $x\sqrt{1+ty^2} + y^2$ (real picture).

Then F decomposes as $F = F_1F_2$, where $F_1, F_2 \in \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ define equisingular deformations of the plane curve germs at $\mathbf{0}$ defined by f_1 and f_2, respectively. Moreover, F_1 and F_2 are unique up to multiplication by units.

Proof. Since F defines an (embedded) equisingular deformation of $(C, \mathbf{0})$, Definition 2.6 gives rise to a Cartesian diagram

$$\begin{tikzcd}
\big(C^{(N)}, 0^{(N)}\big) \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \big(\mathscr{C}^{(N)}, 0^{(N)}\big) \arrow[d, "\widetilde{\pi}"'] \\
(\mathbb{C}^2, \mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathbb{C}^2 \times T, (\mathbf{0}, t_0)) \arrow[d, "p"'] \\
\{t_0\} \arrow[r, hook] & (T, t_0)\,,
\end{tikzcd} \tag{2.1.3}$$

where $(C^{(N)}, 0^{(N)})$ is the multigerm of the strict transform of $(C, \mathbf{0}) = V(f)$ at the intersection points with the exceptional divisor. Moreover,

$$\widetilde{\pi}\big(\mathscr{C}^{(N)}, 0^{(N)}\big) = V(F) \subset (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))\,,$$

and $\pi : \big(C^{(N)}, 0^{(N)}\big) \to (C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ is a resolution of the singularity of $(C, \mathbf{0})$. In particular, outside the special fibre π is an isomorphism onto $(C, \mathbf{0}) \setminus \{\mathbf{0}\}$, the multigerm $\big(C^{(N)}, 0^{(N)}\big)$ is smooth, and it can be written as the disjoint union of (multi)germs $\big(C^{(N)}, 0^{(N)}\big) = \big(C_1^{(N)}, 0_1^{(N)}\big) \amalg \big(C_2^{(N)}, 0_2^{(N)}\big)$ such that $\pi\big(C_i^{(N)}, 0_i^{(N)}\big) = (C_i, \mathbf{0}) := V(f_i)$ for $i = 1, 2$.

Hence, $\big(\mathscr{C}^{(N)}, 0^{(N)}\big) = \big(\mathscr{C}_1^{(N)}, 0_1^{(N)}\big) \amalg \big(\mathscr{C}_2^{(N)}, 0_2^{(N)}\big)$ and the composition

$$\big(\mathscr{C}_i^{(N)}, 0_i^{(N)}\big) \xrightarrow{\widetilde{\pi}} (\mathbb{C}^2 \times T, (\mathbf{0}, t_0)) \longrightarrow (T, t_0)$$

is flat and specializes to $\big(C_i^{(N)},0_i^{(N)}\big) \xrightarrow{\pi} (\mathbb{C}^2,\mathbf{0}) \to \{t_0\}$. We get diagrams analogous to (2.1.3) for $\big(C_i^{(N)},0_i^{(N)}\big) \hookrightarrow \big(\mathscr{C}_i^{(N)},0_i^{(N)}\big)$, $i=1,2$, and all these diagrams satisfy the assumptions of Proposition 2.9. Applying the latter yields $F_1,F_2 \in \mathcal{O}_{\mathbb{C}^2\times T,0}$ such that $V(F_i) = \widetilde{\pi}\big(\mathscr{C}_i^{(N)},0_i^{(N)}\big) =: (\mathscr{C}_i,x_0)$.

Since, as a set, $(\mathscr{C},x_0)=(\mathscr{C}_1,x_0)\cup(\mathscr{C}_2,x_0)$, and since the structures defined by F, respectively by F_1F_2, define both deformations of $(C,\mathbf{0})$, the uniqueness statement of Proposition 2.9 implies $\langle F\rangle = \langle F_1F_2\rangle$. That is, $F=F_1F_2$ up to multiplication by units.

It is clear that the diagram (2.1.2) for $(C,\mathbf{0}) \hookrightarrow (\mathscr{C},x_0) \to (T,t_0)$ induces diagrams for $(C_i,\mathbf{0}) \hookrightarrow (\mathscr{C}_i,x_0) \to (T,t_0)$, $i=1,2$. Since the strict transforms of $(\mathscr{C},x_0)$ are equimultiple along the sections in the diagram, and since the multiplicities of the strict transforms of $(\mathscr{C}_1,x_0)$ and $(\mathscr{C}_2,x_0)$ add up to the multiplicity of the respective strict transform of $(\mathscr{C},x_0)$, it follows from semicontinuity of multiplicities that the strict transforms of $(\mathscr{C}_1,x_0),(\mathscr{C}_2,x_0)$ are equimultiple along the sections, too. As the reduced exceptional divisors are equimultiple along the sections in the diagram, the reduced total transforms of $(\mathscr{C}_1,x_0),(\mathscr{C}_2,x_0)$ are equimultiple along the sections, that is, the deformations defined by F_1, F_2 are equisingular. □

Remark 2.11.1. Conversely, in general, not every product of equisingular deformations of the branches defines an equisingular deformation of $(C,\mathbf{0})$ (even if the singular sections coincide). However, if $f=f_1\cdot\ldots\cdot f_s$ and if the germs defined by the factors f_i have pairwise no common tangent direction, then every product of equisingular deformations along a (unique) singular section σ defines an equisingular deformation of $(C,\mathbf{0})$.

To show this, we may assume that $s=2$ and that σ is the trivial section. Let $F_1=f_1+h_1$, $F_2=f_2+h_2$ define equisingular deformations of $V(F_1)$, $V(F_2)$ along σ. Then the product F_1F_2 obviously defines an equimultiple deformation along σ. Since no branch of $V(f_1)$ has the same tangent direction as a branch of $V(f_2)$, the equimultiple sections $\sigma_j^{(\ell)}$, $\ell\geq 1$, for the equisingular deformation defined by F_1 are disjoint to the equimultiple sections for the equisingular deformation defined by F_2. As the strict transform of f_2 at $\sigma_j^{(\ell)}(t_0)$ is a unit, the multiplicity of the strict transform of F_1F_2 along such a section $\sigma_j^{(\ell)}$ equals the multiplicity of the strict transform of f_1+h_1 along $\sigma_j^{(\ell)}$. Thus, F_1F_2 defines an equimultiple deformation of the strict transform of $V(f_1f_2)$ along $\sigma_j^{(\ell)}$. As the deformation along $\sigma_j^{(\ell)}$ of the reduced exceptional divisor induced by F_1F_2 coincides with the one induced by F_1, and as the analogous statements hold for F_2, Remark 2.6.1 (3) implies that F_1F_2 defines an equisingular deformation of $V(f_1f_2)$.

Proposition 2.12. *Let $\phi:(\mathscr{C},x_0)\hookrightarrow(\mathscr{M},x_0)\to(T,t_0)$ be an embedded equisingular deformation of $(C,\mathbf{0})$ along the section $\sigma:(T,\mathbf{0})\to(\mathscr{C},\mathbf{0})$ with $(T,\mathbf{0})$ reduced. Assume further that $(C,\mathbf{0})=(C_1,\mathbf{0})\cup(C_2,\mathbf{0})$ where $(C_1,\mathbf{0})$ and $(C_2,\mathbf{0})$ are reduced plane curve singularities without common components, and* *let* $\phi_i : (\mathscr{C}_i, x_0) \hookrightarrow (\mathscr{M}, x_0) \to (T, t_0)$ *be the induced deformations of* $(C_i, \mathbf{0})$ *along* σ*,* $i = 1, 2$ *(Proposition 2.11). Then, for a sufficiently small representative* $\mathscr{C} \to \mathscr{M} \to T$*, the following holds:*

(1) The number of branches of $\mathscr{C}$ *is constant along* σ*, that is,*

$$r(\mathscr{C}_t, \sigma(t)) = r(C, \mathbf{0}), \quad t \in T,$$

where $\mathscr{C}_t = \phi^{-1}(t)$*.*

(2) The intersection multiplicity of $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *is constant along* σ*, that is,*

$$i_{\sigma(t)}(\mathscr{C}_{1,t}, \mathscr{C}_{2,t}) = i_{\mathbf{0}}(C_1, C_2), \quad t \in T,$$

where $\mathscr{C}_{i,t} = \phi_i^{-1}(t)$*.*

We call families $\mathscr{C}_1 \to T$ and $\mathscr{C}_2 \to T$ satisfying property (2) *equiintersectional along* σ.[12]

Proof. (1) We use the notations of Definition 2.6 and consider the induced sequence over $t \in T$, $\mathscr{C}_t^{(N)} \to \mathscr{C}_t^{(N-1)} \to \ldots \to \mathscr{C}_t^{(0)} = \mathscr{C}_t$. Since the space $\mathscr{C}_t^{(N)}$ has $r = r(C, \mathbf{0})$ connected components, $r(\mathscr{C}_t, \sigma(t)) \geq r$. If we would have $r(\mathscr{C}_t, \sigma(t)) > r$, then the map $\mathscr{C}_t^{(N)} \to \mathscr{C}_t$ cannot be surjective on all branches and, hence, there exists some ℓ such that the number of points in $\mathscr{C}_t^{(\ell)} \cap \mathscr{E}_t^{(\ell)}$ exceeds the number of points in $\mathscr{C}_{\mathbf{0}}^{(\ell)} \cap \mathscr{E}_{\mathbf{0}}^{(\ell)}$. Then there is some $1 \leq j \leq k_\ell$ such that $\mathrm{mt}\big(\mathscr{C}_t^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_j^{(\ell)}(t)\big) < \mathrm{mt}\big(\mathscr{C}_{\mathbf{0}}^{(\ell)} \cup \mathscr{E}_{\mathbf{0}}^{(\ell)}, \sigma_j^{(\ell)}(\mathbf{0})\big)$ contradicting equisingularity.

(2) It follows from Proposition I.3.21, p. 190, that

$$i_{\mathbf{0}}(C_1, C_2) = \sum_q \mathrm{mt}(C_1^{(\ell)}, q)\, \mathrm{mt}(C_2^{(\ell)}, q), \tag{2.1.4}$$

where q runs through all infinitely near points belonging to $(C, \mathbf{0})$. Note that $\mathrm{mt}(C_i^{(\ell)}, q) = 0$ if $q \notin C_i^{(\ell)}$.

Since $\mathrm{mt}(C_{i,t}^{(\ell)}, \sigma(t))$ and $r(C_{i,t}^{(\ell)}, \sigma(t))$ (by (1)) are constant, the induced sequence

$$\mathscr{C}_t^{(N)} \to \mathscr{C}_t^{(N-1)} \to \ldots \to \mathscr{C}_t^{(0)} = \mathscr{C}_t$$

is an embedded resolution of $\mathscr{C}_t$. Since, by definition of equisingularity, $\mathrm{mt}\big(\mathscr{C}_{i,t}^{(\ell)}, \sigma_j^{(\ell)}(t)\big)$ is constant in t (for $i = 1, 2$ and all ℓ and j), we get the equality $i_{\sigma(t)}(\mathscr{C}_{1,t}, \mathscr{C}_{2,t}) = i_{\mathbf{0}}(C_1, C_2)$ by applying (2.1.4) to $\mathscr{C}_{1,t}$ and $\mathscr{C}_{2,t}$. □

[12] This notion is generalized to non-reduced base spaces (T, t_0) in Definition 2.65, p. 364.

Proposition 2.13. *Let $(C_i, \mathbf{0})$, $i = 1, \ldots, r$, be the branches of the reduced plane curve singularity $(C, \mathbf{0})$. Let $(\mathscr{C}_i, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be embedded deformations of $(C_i, \mathbf{0})$ given by $F_i \in \mathcal{O}_{\mathscr{M},\mathbf{0}}$, and let $(\mathscr{C}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be the deformation of $(C, \mathbf{0})$ given by $F = F_1 \cdot \ldots \cdot F_r$. Let $(T, \mathbf{0})$ be reduced. Then $(\mathscr{C}, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along a section $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, \mathbf{0})$ iff for a sufficiently small representative T of $(T, \mathbf{0})$ the following holds:*

(1) the number of branches of $\mathscr{C}$ is constant along σ, that is, $r(\mathscr{C}_t, \sigma(t)) = r(C, \mathbf{0})$ for $t \in T$,
(2) the pairwise intersection multiplicity of $\mathscr{C}_i$ and $\mathscr{C}_j$ is constant along σ, that is, $i_{\sigma(t)}(\mathscr{C}_{i,t}, \mathscr{C}_{j,t}) = i_{\mathbf{0}}(C_i, C_j)$ for $i \neq j$ and $t \in T$, and
(3) $(\mathscr{C}_i, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along σ for $i = 1, \ldots, r$.

Proof. If $(\mathscr{C}, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along σ, then (1)–(3) follow from Propositions 2.11 and 2.12. For the converse, we use the notation as in Proposition 2.12. If $r(\mathscr{C}_t, \sigma(t))$ is constant then $\mathscr{C}_t^{(N)} \to \mathscr{C}_t$ is an embedded resolution of $\mathscr{C}_t$, since $\mathrm{mt}(\mathscr{C}_t, \sigma(t))$ is constant by (3). Since $(\mathscr{C}_i, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along σ, the multiplicity $\mathrm{mt}\big(\mathscr{C}_{i,t}^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_k^{(\ell)}(t)\big)$ is constant for all ℓ, k such that $\sigma_k^{(\ell)}(t)$ belongs to $\mathscr{C}_{i,t}^{(\ell)}$. Since the intersection multiplicity $i_{\sigma(t)}(\mathscr{C}_{i,t}, \mathscr{C}_{j,t})$ is constant, we have that $\mathrm{mt}\big(\mathscr{C}_{i,t}^{(\ell)} \cup \mathscr{C}_{j,t}^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_k^{(\ell)}(t)\big)$ is constant if $\sigma_k^{(\ell)}(t)$ belongs to $\mathscr{C}_{i,t}^{(\ell)} \cap \mathscr{C}_{j,t}^{(\ell)}$ by (2.1.4). It follows that $\mathrm{mt}\big(\mathscr{C}_t^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_k^{(\ell)}(t)\big)$ is constant for all $\ell = 0, \ldots, N-1$ and $1 \leq k \leq k_\ell$. $\square$

2.2 The Equisingularity Ideal

In this section, we study first order equisingular deformations, that is, equisingular deformations over the fat point $T_\varepsilon = \big(\{0\}, \mathbb{C}[\varepsilon]/\varepsilon^2\big)$. The main result is the following proposition:

Proposition 2.14. *Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve singularity with local equation $f \in \mathbb{C}\{x, y\}$. Then the following holds:*

(1) The set

$$I^{es}(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} \text{there exists a section } \sigma \text{ such that } f + \varepsilon g \\ \text{defines an equisingular deformation of} \\ (C, \mathbf{0}) \text{ over } T_\varepsilon \text{ along } \sigma \end{array} \right\}$$

is an ideal containing the Tjurina ideal $\langle f, j(f) \rangle$, where $j(f) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.

(2) The subset

$$I^{es}_{\mathit{fix}}(f) := \left\{ g \in I^{es}(f) \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation} \\ \text{of } (C, \mathbf{0}) \text{ along the trivial section over } T_\varepsilon \end{array} \right\}.$$

of $I^{es}(f)$ is an ideal in $\mathbb{C}\{x, y\}$ containing $\langle f, \mathfrak{m} j(f) \rangle$. Moreover, as complex vector subspace of $\mathbb{C}\{x, y\}$, $I^{es}(f)$ is spanned by $I^{es}_{\mathit{fix}}(f)$ and the transversal 2-plane spanned by the partials $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Furthermore, we have

$$\mathrm{mt}(g) \geq \begin{cases} \mathrm{mt}(f) - 1 & \text{if } g \in I^{es}(f)\,, \\ \mathrm{mt}(f) & \text{if } g \in I^{es}_{\mathit{fix}}(f)\,. \end{cases}$$

Definition 2.15. The ideal $I^{es}(f) \subset \mathbb{C}\{x,y\}$ is called the *equisingularity ideal* of $f \in \mathbb{C}\{x,y\}$. $I^{es}_{\mathit{fix}}(f)$ is called the *fixed equisingularity ideal* of f.

We prove Proposition 2.14 by induction on the Milnor number, making blowing-ups as induction steps.

Recall that just requiring the multiplicities of the strict transforms in the blown up family to stay constant is not sufficient to get equisingularity of the original deformation. Indeed, the equisingularity condition in the induction step translates to an equisingularity condition for the strict transform *plus extra conditions on the intersection with fixed smooth germs*, namely the components of the exceptional divisor. This corresponds to the requirement that the multiplicities of the reduced *total* transforms are constant in the definition of equisingularity. Therefore, we have to consider a slightly more general situation in the induction step. This is the reason for introducing the ideals $I^{es}_L(f)$ and $I^{es}_{\mathit{fix},L_1\dots L_k}(f)$ below.

The following example of the cusp $f = x^2 - y^3$ might be helpful for understanding the general situation: A first order (equisingular) deformation of the strict transform $C^{(1)} = \{u^2 - v = 0\}$ corresponds to an equisingular deformation $f + \varepsilon g$ of the cusp along the trivial section exactly if its equation is of the form $u^2 - v + \varepsilon g(uv, v)/v^2$ and if there is a section $\sigma_\alpha : T_\varepsilon \to E$ such that the intersection multiplicity with $E = \{v = 0\}$ along σ_α is constant. In other words, if $I_{\sigma_\alpha} = \langle v, u - \varepsilon\alpha\rangle$ with $\alpha \in \mathbb{C}$, then we require

$$\mathrm{ord}_t\Big((t+\varepsilon\alpha)^2 + \varepsilon g^{(1)}(t+\varepsilon\alpha, 0)\Big) = 2\,, \tag{2.2.5}$$

for $g^{(1)}(u,v) := g(uv,v)/v^2$. Now, we must continue blowing up. Note that

$$(t+\varepsilon\alpha)^2 + \varepsilon g^{(1)}(t+\varepsilon\alpha, 0) = t^2 + 2\varepsilon\alpha t + \varepsilon g^{(1)}(t,0)\,.$$

Hence, replacing $g^{(1)}$ by $g^{(1)} - 2\alpha u = g^{(1)} - \alpha\frac{\partial(u^2-v)}{\partial u}$, we may assume that $\alpha = 0$, that is, σ_α is the trivial section. Then the above condition on the intersection multiplicity is equivalent to $i_{\mathbf{0}}(g^{(1)} - 2\alpha u, E) \geq 2 = i_{\mathbf{0}}(u^2 - v, E)$.

Similarly, a first order (equisingular) deformation of $C^{(2)}$ corresponds to an (equisingular) deformation of $C^{(1)}$ satisfying (2.2.5) for $\alpha = 0$ iff its equation has the form $\overline{u} - \overline{v} + \varepsilon g^{(1)}(\overline{u}, \overline{uv})/\overline{u}$ and the intersection point with the components of the exceptional divisor does not move. The latter means that

$$\begin{aligned} \mathrm{ord}_t\big(t + \varepsilon g^{(2)}(t,0)\big) &\geq 1 = i_{\mathbf{0}}(\overline{u} - \overline{v}, E_1)\,, \\ \mathrm{ord}_t\big(-t + \varepsilon g^{(2)}(0,t)\big) &\geq 1 = i_{\mathbf{0}}(\overline{u} - \overline{v}, E_2)\,, \end{aligned}$$

where $g^{(2)}(\overline{u}, \overline{v}) := g^{(1)}(\overline{u}, \overline{uv})/\overline{u}$.

This example suggests that in the inductive proof we should not only consider $I^{es}(f)$ but also the following auxiliary objects: let $L, L_1, \ldots, L_k \subset (\mathbb{C}^2, \mathbf{0})$ denote smooth germs (respectively their local equations) through the origin with different tangent directions. Consider the sections $\sigma_\alpha : T_\varepsilon \to L$ given by the ideal $I_{\sigma_\alpha} := \langle x - \varepsilon\alpha\ell_1, y - \varepsilon\alpha\ell_2\rangle$, $\alpha \in \mathbb{C}$, where $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \mathbb{C}^2$ is a fixed tangent vector to L, and let $\sigma_0 : T_\varepsilon \to (\mathbb{C}^2, \mathbf{0})$ be the trivial section. Define

$$I_L^{es}(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation} \\ \text{of } (C, \mathbf{0}) \text{ with singular section } \sigma_\alpha \text{ in } L \text{ and} \\ i_{\sigma_\alpha}(f + \varepsilon g, L) = i_{\mathbf{0}}(f, L) \end{array} \right\},$$

$$I_{\mathit{fix},L_1..L_k}^{es}(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation} \\ \text{with trivial singular section } \sigma_0 \text{ and} \\ i_{\sigma_0}(f + \varepsilon g, L_j) = i_{\mathbf{0}}(f, L_j) \text{ for } j = 1, \ldots, k \end{array} \right\}.$$

Here i_{σ_α} denotes the intersection multiplicity along σ_α, that is,

$$i_{\sigma_\alpha}(f + \varepsilon g, L) := \operatorname{ord}_t\bigl(f(t\boldsymbol{\ell} - \varepsilon\alpha\boldsymbol{\ell}) + \varepsilon g(t\boldsymbol{\ell} - \varepsilon\alpha\boldsymbol{\ell})\bigr),$$

and we assume that the intersection multiplicities $i_{\mathbf{0}}(f, L)$ and $i_{\mathbf{0}}(f, L_j)$, $j = 1, \ldots, k$, are finite.

Proof of Proposition 2.14. We show that, for any smooth germs $L, L_1, \ldots, L_k$ ($k \geq 0$) as above, $I_{\mathit{fix},L_1\ldots L_k}^{es}(f)$, $I_L^{es}(f)$ and $I^{es}(f)$ are ideals in $\mathbb{C}\{x,y\}$.

Step 1. We show that it suffices to prove the claim for $I_{\mathit{fix},L_1\ldots L_k}^{es}(f)$, $k \geq 0$.

Step 1a. Assume that $I_{\mathit{fix},L}^{es}(f)$ is an ideal. Then $I_L^{es}(f)$ is an ideal, spanned as a linear space by $I_{\mathit{fix},L}^{es}(f)$ and f'_L, the derivative of f in the direction of L. Furthermore, f'_L does not belong to $I_{\mathit{fix},L}^{es}(f)$.

Indeed, $f + \varepsilon g$ defines an equisingular deformation with singular section σ_α, $\alpha \in \mathbb{C}$, iff the deformation induced by

$$\begin{aligned} &f(x - \varepsilon\alpha\ell_1, y - \varepsilon\alpha\ell_2) + \varepsilon g(x - \varepsilon\alpha\ell_1, y - \varepsilon\alpha\ell_2) \\ &\qquad \equiv f(x,y) - \varepsilon \cdot \Bigl(\alpha \cdot \underbrace{\bigl(\ell_1 \tfrac{\partial f}{\partial x}(x,y) + \ell_2 \tfrac{\partial f}{\partial y}(x,y)\bigr)}_{=:\, f'_L(x,y)} - g(x,y)\Bigr), \end{aligned}$$

is equisingular along the trivial section. We conclude that $I_L^{es}(f)$ is spanned as a linear space by $I_{\mathit{fix},L}^{es}(f)$ and by f'_L.

To show that $I_L^{es}(f)$ is an ideal, we show that $\mathfrak{m} \cdot f'_L \subset I_{\mathit{fix},L}^{es}(f)$. Indeed,

$$\mathfrak{m} \cdot f'_L \subset j_{\mathit{fix},L}(f) := \bigl\{ g \in \mathbb{C}\{x,y\} \bigm| i_{\mathbf{0}}(g, L) \geq i_{\mathbf{0}}(f, L) \bigr\} \cap \mathfrak{m} \cdot j(f),$$

since $i_{\mathbf{0}}(f'_L, L) = i_{\mathbf{0}}(f, L) - 1$. On the other hand, $j_{\mathit{fix},L}(f) \subset I_{\mathit{fix},L}^{es}(f)$, since the ideal $j_{\mathit{fix},L}(f)$ just describes the infinitesimal locally trivial deformations of first order with trivial singular section and fixed intersection multiplicity with L. Since $i_{\sigma_0}(f + \varepsilon f'_L, L) = i_{\mathbf{0}}(f, L) - 1$, we have $f'_L \notin I_{\mathit{fix},L}^{es}(f)$.

Step 1b. Assuming that $I^{es}_{\mathit{fix}}(f)$ is an ideal, we see in the same way that $I^{es}(f)$ is spanned as a linear space by $I^{es}_{\mathit{fix}}(f)$ and the transverse 2-plane spanned by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. We deduce that $I^{es}(f)$ is an ideal, the sum of the ideals $I^{es}_{\mathit{fix}}(f)$ and $j(f)$, since we have the linear decomposition $j(f) = \mathbb{C}\frac{\partial f}{\partial x} + \mathbb{C}\frac{\partial f}{\partial y} + \mathfrak{m}j(f)$. The inclusion $\mathfrak{m}j(f) \subset I^{es}_{\mathit{fix}}(f)$ results from the fact that $\mathfrak{m}j(f)$ describes the infinitesimal locally trivial deformations of first order with trivial singular section.

Observe that $\mathrm{mt}(g) \geq \mathrm{mt}(f)$ for all elements $g \in I^{es}_{\mathit{fix}}(f)$, since all germs equisingular to f have the same multiplicity at the singular point. In view of the preceding result, this yields, in particular, that $\mathrm{mt}(g) \geq \mathrm{mt}(f) - 1$ for all elements $g \in I^{es}(f)$, since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ satisfy the latter inequality.

Step 2. We prove that $I^{es}_{\mathit{fix},L_1\dots L_k}(f)$, $k \geq 0$, is an ideal. To do so, we proceed by induction on the number of blowing ups needed to resolve f.

Step 2a. As base of induction, we consider the case of a non-singular germ $f \in \mathbb{C}\{x, y\}$. Then

$$I^{es}_{\mathit{fix},L_1\dots L_k}(f) = \bigl\{g \in \mathbb{C}\{x, y\} \bigm| i_{\mathbf{0}}(g, L_j) \geq i_{\mathbf{0}}(f, L_j)\,,\ j = 1, \dots, k\bigr\}\,,$$

which obviously defines an ideal in $\mathbb{C}\{x, y\}$.

Step 2b. Assume that f is singular. Let $\pi : M \to (\mathbb{C}^2, \mathbf{0})$ be the blowing up of the origin, and let $E \subset M$ be the exceptional divisor. Denote by $\widetilde{L}_1, \dots, \widetilde{L}_k \subset M$ the strict transforms of $L_1, \dots, L_k$, by $\widetilde{C}$ the strict transform of the germ $(C, \mathbf{0})$, and by $q_1, \dots, q_s$ the intersection points of $\widetilde{C}$ with E. Let $m := \mathrm{mt}(C, \mathbf{0})$, and let $\widetilde{f}_i \in \mathcal{O}_{M,q_i}$ be a local equation for the germ $\bigl(\widetilde{C}, q_i\bigr)$, $i = 1, \dots, s$.

From Definition 2.6, we see that $f + \varepsilon g$ is the defining equation of an equisingular deformation of $(C, \mathbf{0})$ with trivial singular section and fixed intersection multiplicities with $L_1, \dots, L_k$ iff it is mapped under the injective morphism

$$(\pi \times \mathrm{id}_{T_\varepsilon})^\sharp : \mathcal{O}_{\mathbb{C}^2 \times T_\varepsilon, (\mathbf{0},0)} \hookrightarrow \mathcal{O}_{M \times T_\varepsilon, (q_i,0)}$$

to the product of the m-th power of the equation of the exceptional divisor E and the equation of an equisingular deformation of the germ $\bigl(\widetilde{C}, q_i\bigr)$ satisfying the following conditions:

- if one of $\widetilde{L}_1, \dots, \widetilde{L}_k$ passes through q_i, then the equisingular deformation of $\bigl(\widetilde{C}, q_i\bigr)$ has trivial singular section and fixed intersection multiplicities with $\widetilde{L}_1, \dots, \widetilde{L}_k$ and E (cf. Proposition I.3.21),
- if none of $\widetilde{L}_1, \dots, \widetilde{L}_k$ passes through q_i, then the equisingular deformation of $\bigl(\widetilde{C}, q_i\bigr)$ has singular section with values in E and it has fixed intersection multiplicity with E.

Correspondingly, $I^{es}_{\mathit{fix},L_1\dots L_k}(f)$ is the preimage of $\bigoplus_{i=1}^{s} E^m \cdot I_i$ under

$$\pi^\sharp : \mathbb{C}\{x,y\} \to \bigoplus_{i=1}^{s} \mathcal{O}_{M,q_i} ,$$

where

$$I_i := \begin{cases} I^{es}_{\mathit{fix},\widetilde{L}_1\dots\widetilde{L}_k E}\big(\widetilde{f}_i\big), & \text{if } q_i \in \widetilde{L}_1 \cup \ldots \cup \widetilde{L}_k, \\ I^{es}_E\big(\widetilde{f}_i\big), & \text{if } q_i \notin \widetilde{L}_1 \cup \ldots \cup \widetilde{L}_k \ . \end{cases}$$

Finally, since resolving $\widetilde{f}_i$, $i = 1,\dots,s$, needs less blowing ups than resolving f, the induction hypothesis and the result of Step 1a assure that $I_i \subset \mathcal{O}_{M,q_i}$, $i = 1,\dots,s$, are ideals. Hence, $I^{es}_{\mathit{fix},L_1\dots L_k}(f)$ is an ideal, too. $\square$

Example 2.15.1. We reconsider the proof of Proposition 2.14 to compute the equisingularity ideal for A_μ- and D_μ-singularities.

(1) Let $f_\mu := x^2 - y^{\mu+1} \in \mathbb{C}\{x,y\}$, $\mu \geq 1$. Then

$$I^{es}(f_\mu) = \langle f_\mu, j(f_\mu)\rangle = \langle x, y^\mu\rangle = \big\{ g \in \mathbb{C}\{x,y\} \,\big|\, i_{\mathbf{0}}(x,g) \geq \mu \big\} , \qquad (2.2.6)$$

and, for $L := \{y = 0\}$, we get

$$I^{es}_L(f_\mu) = \langle x, y^{\mu+1}\rangle , \quad I^{es}_{\mathit{fix},L}(f_\mu) = I^{es}_{\mathit{fix}}(f_\mu) = \langle f_\mu, \mathfrak{m}\cdot j(f_\mu)\rangle .$$

Indeed, as L is transversal to $\{f_\mu = 0\}$, each equimultiple deformation along the trivial section preserves the intersection multiplicity with L (which is 2), hence, $I^{es}_{\mathit{fix},L}(f_\mu) = I^{es}_{\mathit{fix}}(f_\mu)$. The proof of Proposition 2.14 then shows that, as a linear space, $I^{es}_L(f_\mu)$ is spanned by $I^{es}_{\mathit{fix}}(f_\mu)$ and the derivative $\frac{\partial f}{\partial x} = 2x$. Now, we proceed by induction on μ: For $\mu = 1$, equisingularity is equivalent to equimultiplicity. Hence, $I^{es}_{\mathit{fix}}(f_1) = \mathfrak{m}^2 = \langle f_1, \mathfrak{m}\cdot j(f_1)\rangle$ and $I^{es}(f_1) = \mathfrak{m} = \langle f_1, j(f_1)\rangle$.

For $\mu = 2$, the considerations right before the proof of Proposition 2.14 show that $g \in \mathbb{C}\{x,y\}$ defines an equisingular deformation of the cusp along the trivial section iff $g^{(2)} \in \langle \overline{u}, \overline{v}\rangle$, which is equivalent to $g^{(1)} - 2\alpha u \in \langle u^2, v\rangle$, and thus to $g \in \langle x^2, xy, y^3\rangle = \langle f_\mu, \mathfrak{m}\cdot j(f_2)\rangle$. As $I^{es}(f_2)$ is spanned by $I^{es}_{\mathit{fix}}(f_2)$ and the two partials of f_2, we get $I^{es}(f_2) = \langle f_2, j(f_2)\rangle$.

For $\mu \geq 3$, let $\pi : M \to (\mathbb{C}^2, \mathbf{0})$ be the blowing up of the origin. Then there is a unique intersection point of the strict transform of $\{f_\mu = 0\}$ with the exceptional divisor $E = \pi^{-1}(\mathbf{0})$. Locally at this point, the exceptional divisor is given by $\{v = 0\}$, and the strict transform is given by $\{u^2 - v^{\mu-1} = 0\}$. Together with the induction hypothesis, the proof of Proposition 2.14 shows that $I^{es}_{\mathit{fix}}(f_\mu)$ is the preimage of $v^2 \cdot \langle u, v^{\mu-1}\rangle$ under $\pi^\sharp : (x,y) \mapsto (uv, v)$. Thus, $I^{es}_{\mathit{fix}}(f_\mu) = \langle x^2, xy, y^{\mu+1}\rangle$. Finally, $I^{es}(f_\mu)$ is spanned by $I^{es}_{\mathit{fix}}(f_\mu)$ and the two partials of f_μ. Thus, $I^{es}(f_\mu) = \langle x, y^\mu\rangle$.

(2) Let $g_\mu := y(x^2 - y^{\mu-2}) \in \mathbb{C}\{x,y\}$, $\mu \geq 4$. Then

$$I^{es}(g_\mu) = \langle g_\mu, j(g_\mu)\rangle , \quad I^{es}_{\mathit{fix}}(g_\mu) = \langle g_\mu, \mathfrak{m}\cdot j(g_\mu)\rangle = \langle x^3, x^2y, xy^2, y^{\mu-1}\rangle .$$

For $\mu = 4$, equisingularity is again equivalent to equimultiplicity. Hence, we get $I^{es}_{\mathit{fix}}(g_4) = \mathfrak{m}^3 = \langle g_4, \mathfrak{m} \cdot j(g_4)\rangle$ and $I^{es}(g_4) = \langle g_4, j(g_4)\rangle$.

Now, let $\mu \geq 5$, and let $\pi : M \to (\mathbb{C}^2, \mathbf{0})$ be the blowing up of the origin. Then there is a unique non-nodal singular point of the reduced total transform of $\{g_\mu = 0\}$ on the exceptional divisor $E = \pi^{-1}(\mathbf{0})$. Locally at this point, the exceptional divisor is given by $\{v = 0\}$, and the strict transform is given by $\{u^2 - v^{\mu-4} = 0\}$. For $\mu \geq 6$, the proof of Proposition 2.14, together with Case (1), gives that $I^{es}_{\mathit{fix}}(g_\mu)$ is the preimage of $v^3 \cdot \langle u, v^{\mu-4}\rangle$ under $\pi^\sharp : (x, y) \mapsto (uv, v)$. Thus, $I^{es}_{\mathit{fix}}(g_\mu) = \langle x^3, x^2y, xy^2, y^{\mu-1}\rangle = \langle g_\mu, \mathfrak{m}j(g_\mu)\rangle$, and $I^{es}(g_\mu) = \langle g_\mu, j(g_\mu)\rangle$.

It remains the case $\mu = 5$. Here, $I^{es}_{\mathit{fix}}(g_5)$ is the preimage under $\pi^\sharp$ of the ideal $v^3 \cdot I^{es}_{\mathit{fix},E}(u^2 - v)$ (with $E = \{v = 0\}$). As $I^{es}_{\mathit{fix},E}(u^2 - v) = \langle u^2, v\rangle$, this gives $I^{es}_{\mathit{fix}}(g_5) = \langle x^3, x^2y, xy^2, y^4\rangle$. Hence, $I^{es}(g_5) = \langle g_5, j(g_5)\rangle$.

This example shows that, for f defining an A_k- or a D_k-singularity, the equisingularity and the Tjurina ideal coincide. The same holds for f defining a singularity of type E_6, E_7, E_8, which we leave as an exercise:

Lemma 2.16. *If $f \in \mathbb{C}\{x, y\}$ defines an ADE-singularity, then*

$$I^{es}(f) = \langle f, j(f)\rangle\,, \quad I^{es}_{\mathit{fix}}(f) = \langle f, \mathfrak{m}j(f)\rangle\,.$$

The next proposition gives a general description of the equisingularity ideal in the case of reduced semiquasihomogeneous, respectively Newton nondegenerate (NND, see Definition I.2.15), plane curve singularities. Note that, for the NND polynomial $f = (x^2 - y^3)(y^2 - x^3)$, we get $I^{es}(f) = \langle f, j(f)\rangle$ and $I^{es}_{\mathit{fix}}(f) = \langle f, \mathfrak{m} \cdot j(f)\rangle$, but f does not define an ADE-singularity. Hence, the inverse implication in Lemma 2.16 does not hold.

Proposition 2.17. *Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve singularity with local equation $f \in \mathbb{C}\{x, y\}$.*

(1) If $f = f_0 + f'$ is semiquasihomogeneous with principal part f_0 being quasihomogeneous of type $(w_1, w_2; d)$, then

$$I^{es}(f) = \langle j(f),\, x^\alpha y^\beta \mid w_1\alpha + w_2\beta \geq d\rangle\,.$$

(2) If f is Newton non-degenerate with Newton diagram $\Gamma(f, \mathbf{0})$ at the origin, then the equisingularity ideal is generated by $j(f)$ and the monomials corresponding to points $(\alpha, \beta) \in \mathbb{N}^n$ on or above $\Gamma(f, \mathbf{0})$.[13]

[13] These monomials are just the monomials of Newton order ≥ 1, where we say that a monomial has *Newton order* $\delta \in \mathbb{R}$ (w.r.t. f) iff it corresponds to a point on the hypersurface $\delta \cdot \Gamma(f, \mathbf{0}) \subset \mathbb{R}^2$.

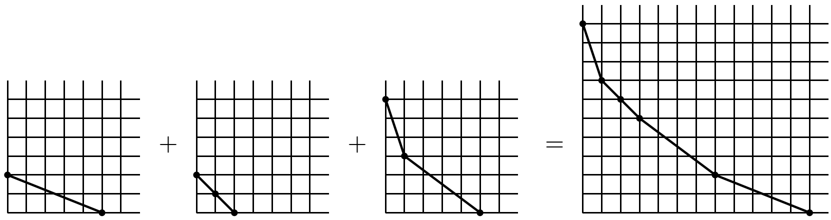


Fig. 2.9. Newton diagram of $f = (x^5 - y^2)(x^2 + xy - y^2)(y^6 - y^3x + x^5)$.

Remark 2.17.1. In fact, for a reduced Newton non-degenerate power series f, we prove the following: let $J, I^s(f) \subset \mathbb{C}\{x,y\}$ be the ideals

$$J = \langle x^\alpha y^\beta \mid x^\alpha y^\beta \text{ has Newton order } \geq 1 \rangle$$

and

$$I^s(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation of} \\ (C,\mathbf{0}) \text{ where the equimultiple sections through} \\ \text{all the infinitely near non-nodes of the reduced} \\ \text{total transform of } (C,\mathbf{0}) \text{ are trivial sections} \end{array} \right\}.$$

Then

$$I^{es}(f) = \langle j(f), I^{es}_{\mathit{fix}}(f) \rangle = \langle j(f), J \rangle = \langle j(f), I^s(f) \rangle .$$

The proof uses the following general facts for the Newton diagram at the origin of a power series $f \in \mathbb{C}\{x,y\}$ (see, e.g., [BrK, §8.4], [DJP, §5.1]):

- if f is irreducible then $\Gamma(f,\mathbf{0})$ has at most one facet;
- if $f = f_1 \cdot \ldots \cdot f_s$ then $\Gamma(f,\mathbf{0})$ is obtained by gluing the facets of $\Gamma(f_i,\mathbf{0})$ (suitably displaced, such that the resulting diagram looks like the graph of a convex piece-wise linear function, see Figure 2.9);
- the blowing up map $\pi^\sharp$ maps monomials of Newton order 1 (resp. ≥ 1, resp. ≤ 1) with respect to f to monomials of Newton order 1 (resp. ≥ 1, resp. ≤ 1) with respect to the total transform of f.

Proof of Proposition 2.17. As in the proof of Proposition 2.14 we proceed by induction, making blowing-ups as induction steps. We simultaneously treat the case of semiquasihomogeneous and Newton non-degenerate singularities.

Actually, what we suppose is that f is reduced and either Newton non-degenerate, or a product of type $f = xf'$ (respectively $f = yf'$), or $f = xyf'$, with f' Newton non-degenerate. In the latter ("non-convenient") cases, we consider a modified Newton diagram $\Gamma(f,\mathbf{0})$ to define the Newton order. For this, we omit vertical and horizontal facets (if they exist) and extend (if necessary) the facets of maximal and minimal slope such that they touch the x- and y-axis, respectively.

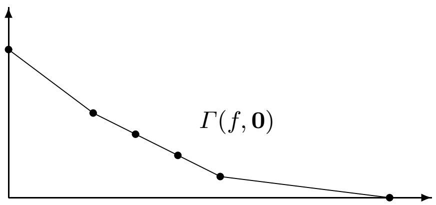


Fig. 2.10. Newton diagram $\varGamma(f,\mathbf{0})$ of a unitangential NND singularity.

Proposition 2.14 shows that we always have $I^{es}(f) = \langle j(f), I^{es}_{fix}(f)\rangle$ and $\langle j(f), I^s(f)\rangle \subset I^{es}(f)$. Thus, it suffices to prove that, under the given assumptions, $\langle j(f), I^{es}_{fix}(f)\rangle$ is contained in the ideal generated by $j(f)$ and the monomials of Newton order ≥ 1 w.r.t. f, and that the latter ideal is contained in $\langle j(f), I^s(f)\rangle$.

Step 1. We show that $I^s(f)$ contains the ideal generated by all monomials of Newton order ≥ 1 w.r.t. f.

Case A. f defines an ordinary singularity (including f smooth).

Then, $I^s(f) = I^{es}_{fix}(f) = \langle x, y\rangle^{\mathrm{mt}(f)}$, hence the statement.

Case B. f is singular and unitangential.

Since f is either SQH or NND, the tangent can only be x or y. Let us assume that it is y, that is, the Newton diagram has no facet of slope ≤ -1 (see also Figure 2.10). In particular, when blowing-up the origin, it suffices to consider the chart $x = u$, $y = uv$.

Now, let $g = x^\alpha y^\beta$ be a monomial of Newton order ≥ 1. Then, in particular, $f + \varepsilon g$ is equimultiple along the trivial section and its reduced total transform is given by

$$u \cdot \left(\frac{f(u,uv)}{u^{\mathrm{mt}(f)}} + \varepsilon \cdot \frac{g(u,uv)}{u^{\mathrm{mt}(f)}} \right) = u\widetilde{f}(u,v) + \varepsilon \cdot \frac{g(u,uv)}{u^{\mathrm{mt}(f)-1}} \,. \tag{2.2.7}$$

Since $g(u,uv)$ is a monomial of Newton order ≥ 1 w.r.t. the total transform $u^{\mathrm{mt}(f)}\widetilde{f}$, the induction hypothesis gives that (2.2.7) defines an equisingular deformation with all equimultiple sections through non-nodes of the reduced total transform of $u\widetilde{f}$ being trivial sections. Hence, $g \in I^s(f)$.

Case C. f has at least two tangential components.

It might happen that the Newton diagram has facets of slope < -1 (all corresponding to branches with tangent x), > -1 (tangent y), and $= -1$ (tangent $\alpha x + \beta y$, $\alpha, \beta \neq 0$). Since f is NND, the last-mentioned branches define an ordinary singularity, hence, impose no equisingularity condition to the respective strict transforms.

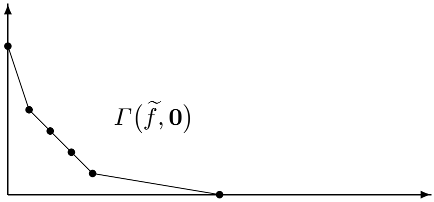


Fig. 2.11. Newton diagram at the origin of the strict transform.

Thus, we can conclude as in Case B, considering both charts of the blowing-up map and noting that the Newton diagram at the origin of the respective strict transform equals the Newton diagram of the strict transform of the respective tangential component.

Step 2. We prove that $I^{es}_{\mathit{fix}}(f)$ coincides with the ideal J generated by xf_y, yf_x and the monomials of Newton order ≥ 1 w.r.t. f.

What we actually claim is that $I^{es}_{\mathit{fix}}(f) \subset J$ (the other inclusion is given by Step 1 and Proposition 2.14). To see this, note that the inclusion $I^{es}_{\mathit{fix}}(f) \subset J$ holds true for ordinary singularities (see Case A, above). It remains to consider Cases B,C.

Case B. f is singular and unitangential.

Let us assume that $f + \varepsilon g$, $g \in \mathbb{C}\{x, y\}$, defines an equisingular deformation with trivial singular section. In particular, this implies that the reduced total transform (2.2.7) is equisingular (with singular section in the exceptional divisor $\{u = 0\}$). Proposition 2.11 implies that

$$u \cdot \left(\widetilde{f}(u,v) + \varepsilon \cdot \frac{g(u,uv)}{u^{\mathrm{mt}(f)}} \right) = \Big(u + \varepsilon g_1(u,v) \Big) \cdot \Big(\widetilde{f}(u,v) + \varepsilon g_2(u,v) \Big) \,,$$

such that both factors define equisingular deformations with singular section in $\{u = 0\}$. Hence, $g_1 \in u \cdot \mathbb{C}\{u, v\}$, and Proposition 2.14 (respectively its proof) and the induction hypothesis give

$$\frac{g(u,uv)}{u^{\mathrm{mt}(f)}} \in \left\langle v\widetilde{f}_u, \widetilde{f}_v, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } \widetilde{f} \right\rangle . \tag{2.2.8}$$

Those monomials in g leading to terms of Newton order ≥ 1 w.r.t. $\widetilde{f}$ have Newton order ≥ 1 w.r.t. f, hence, are contained in $I^{es}_{\mathit{fix}}(f)$ by Step 1. Moreover, we compute that

$$\widetilde{f}_v = \frac{\partial}{\partial v} \left(\frac{f(u,uv)}{u^{\mathrm{mt}(f)}} \right) = \frac{u \cdot f_y(u,uv)}{u^{\mathrm{mt}(f)}}$$

is the image for $g = xf_y$, which is in $I^{es}_{\mathit{fix}}(f)$, too.

The latter proves the claim as long as $\widetilde{f}$ has no component with tangent u (then $v\widetilde{f}_u$ has Newton order ≥ 1). On the other hand, if some components have tangent u, then some of the higher equimultiple sections of (2.2.7) have to be in the strict transform of $\{u = 0\}$.

More precisely, let $-\rho$ be the slope of the steepest facet in $\Gamma\big(\widetilde{f}, \mathbf{0}\big)$ (see also Figure 2.11), then we have the above restriction for $N := \lceil \rho \rceil$ successive equimultiple sections (including the present one).

Of course, the latter imposes $N-1$ independent conditions to $g(u, uv)$. Since $u\widetilde{f}_u$ has Newton order ≥ 1 w.r.t. $\widetilde{f}$, this means that we have to exclude $v\widetilde{f}_u, \ldots, v^{N-1}\widetilde{f}_u$ on the right-hand side of (2.2.8). But, due to the choice of N, $v^N\widetilde{f}_u$ has Newton order ≥ 1. Hence, we conclude that

$$g \in \big\langle x f_y, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f \big\rangle \,.$$

Case C. f has at least two tangential components.

We can suppose that $f = f_1 f_2 f_3$, where f_1 has tangent x, f_2 defines an ordinary singularity, and f_3 has tangent y. Then, again by Proposition 2.11, any equisingular deformation $f + \varepsilon g$ with trivial singular section splits as

$$f + \varepsilon g = (f_1 + \varepsilon g_1) \cdot (f_2 + \varepsilon g_2) \cdot (f_3 + \varepsilon g_3)\,,$$

where (in view of the above)

$$\begin{aligned} g_1 &\in \big\langle y f_{1,x}, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f_1 \big\rangle\,, \\ g_2 &\in \langle x, y \rangle^{\mathrm{mt}(f_2)}\,, \\ g_3 &\in \big\langle x f_{3,y}, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f_3 \big\rangle\,. \end{aligned}$$

In particular, since products of terms of Newton order ≥ 1 w.r.t. f_1, f_2, f_3, respectively, have Newton order ≥ 1 w.r.t. f, it is not difficult to see that the latter implies that

$$\begin{aligned} g \in \big\langle & y^{1+\mathrm{mt}(f_2)+\mathrm{mt}(f_3)} f_{1,x},\; x^{\mathrm{mt}(f_1)+\mathrm{mt}(f_2)+1} f_{3,y}, \\ & \qquad\qquad \text{terms of Newton order} \geq 1 \text{ w.r.t. } f \big\rangle\,. \\ = \big\langle & y f_x, x f_y, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f \big\rangle\,. \end{aligned}$$

Step 3. Combining Steps 1, 2 and Proposition 2.14, we conclude that

$$I^{es}(f) = \langle j(f), I^{es}_{\mathit{fix}}(f) \rangle \subset \langle j(f), J \rangle \subset \langle j(f), I^s(f) \rangle \subset I^{es}(f)\,.$$

Hence all inclusions are equalities, which implies the statement of the proposition. $\square$

The proof of Proposition 2.17 shows that for f Newton non-degenerate, the equisingularity ideal is generated by the Tjurina ideal and the ideal $I^s(f)$.

This is caused by the fact that the equimultiple sections $\sigma_j^{(\ell)}$ through all infinitely near non-nodes can be simultaneously trivialized in this case. That is, each equisingular deformation of a reduced Newton non-degenerate plane curve singularity is isomorphic to an equisingular deformation where all the equimultiple sections $\sigma_j^{(i)}$ through the infinitely near non-nodes of the reduced total transform are globally trivial sections (see Proposition 2.69).

If f is Newton degenerate, however, this is not necessarily the case as the following example shows:

Example 2.17.2. Consider the Newton degenerate polynomial

$$f = (x-2y)^2(x-y)^2x^2y^2 + x^9 + y^9,$$

defining a germ consisting of four transversal cusps. Blowing up the origin, we get four intersection points $q_{1,1},\dots,q_{1,4}$ of the strict transform with the exceptional divisor E_1, the germ of the strict transform being smooth (and tangential to E_1) at each $q_{1,j}$, see Figure 2.12. Thus, each deformation of the (multigerm of the) strict transform is equisingular along some sections $\sigma_1^{(1)},\dots,\sigma_4^{(1)}$. However, since the cross-ratio of the four intersection points $q_{1,1},\dots,q_{1,4}$ is preserved under isomorphisms, usually the sections can not be simultaneously trivialized. Consider, for instance, the 1-parameter deformation given by

$$F(x,y,t) = (x-2y)^2(x-y)^2(x+ty)^2y^2 + x^9 + y^9, \quad t \in \mathbb{C} \text{ close to } 0\,.$$

After blowing up the trivial section, the strict transform of F is a locally trivial deformation of the strict transform of f along sections $\sigma_1^{(1)},\dots,\sigma_4^{(1)}$ in the exceptional divisor, where the cross-ratio of $\sigma_1^{(1)}(t),\dots,\sigma_4^{(1)}(t)$ varies in t. Although the family defined by F is topologically trivial, it cannot be isomorphic (not even C^1-diffeomorphic) to an equisingular deformation with trivial equimultiple sections. The induced deformation over T_ε is defined by $f+\varepsilon g$, with

$$g = 2(x^5y^3 - 6x^4y^4 + 13x^3y^5 - 12x^2y^6 + 4xy^7)\,.$$

We will see below, that, indeed, $g \in I^{es}(f) \setminus \langle f, j(f), I^s(f)\rangle$ (see Example 2.63.1).

We use the previous discussion of the equisingularity ideal to show that a generic element $g \in I^{es}(f)$ intersects f with the same multiplicity $\kappa(f)$ as a generic polar of f, $\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}$, $(\alpha:\beta) \in \mathbb{P}^1$ generic, does. Indeed, this is an immediate consequence of the following lemma (since $j(f) \subset I^{es}(f)$):

Lemma 2.18. *Let $f \in \mathbb{C}\{x,y\}$ be reduced, and let $g \in I^{es}(f)$. Then*

$$i(f,g) \geq \kappa(f)\,.$$

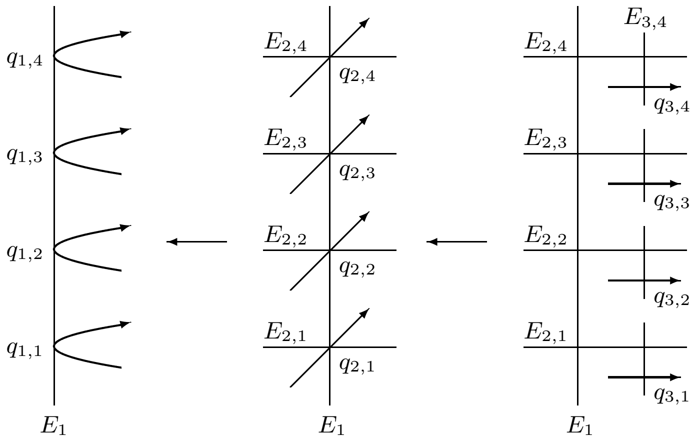


Fig. 2.12. Resolution of $\big\{(x-2y)^2(x-y)^2x^2y^2 + x^9 + y^9 = 0\big\}$.

Moreover, with the notations introduced in the proof of Proposition 2.14, we have

$$i(f,g) \geq \begin{cases} \kappa(f) + \mathrm{mt}(f), & \text{if } g \in I^{es}_{fix}(f)\,, \\ \kappa(f) + i(f,L) - \mathrm{mt}(f)\,, & \text{if } g \in I^{es}_{L}(f)\,, \\ \kappa(f) + \mathrm{mt}(f) + \sum\limits_{j=1}^{k} \big(i(f,L_j) - \mathrm{mt}(f)\big)\,, & \text{if } g \in I^{es}_{fix,L_1\dots L_k}(f)\,. \end{cases}$$

Proof. We proceed by induction on the number of blowing ups needed to resolve the plane curve singularity $\{f=0\}$ *and* to make $\{f=0\}$ transversal to the (strict transforms of) $L, L_1, \dots, L_k$.

Step 1. As base of induction, we have to show for a non-singular f and transverse smooth germs $L, L_1, \dots, L_k$ that $i(f,g) \geq 0$ if $g \in I^{es}(f)$ or if $g \in I^{es}_L(f)$, and that $i(f,g) \geq 1$ if $g \in I^{es}_{fix}(f)$ or if $g \in I^{es}_{fix,L_1\dots L_k}(f)$. But this is obvious.

Step 2. We show that it suffices to prove the statement for $g \in I^{es}_{fix,L_1\dots L_k}(f)$. Thus, let us assume that, for each $g \in I^{es}_{fix,L_1\dots L_k}(f)$,

$$i(f,g) \geq \kappa(f) + \mathrm{mt}(f) + \sum_{j=1}^{k} \big(i(f,L_j) - \mathrm{mt}(f)\big)\,, \tag{2.2.9}$$

The case $k=0$ implies that $i(f,g) \geq \kappa(f) + \mathrm{mt}(f)$ for each $g \in I^{es}_{fix}(f)$. Moreover, due to Proposition 2.14, the equisingularity ideal $I^{es}(f)$ is generated as a linear space by $I^{es}_{fix}(f)$ and the partial derivatives of f. As $i(f,g) \geq \kappa(f) + \mathrm{mt}(f)$ for $g \in I^{es}_{fix}(f)$, and as each element of $j(f)$ intersects f with multiplicity at least $\kappa(f)$, we get $i(f,g) \geq \kappa(f)$ for each $g \in I^{es}(f)$.

Finally, we have seen in Step 1a of the proof of Proposition 2.14 that $\mathfrak{m} I^{es}_L(f) \subset I^{es}_{fix,L}(f)$. Thus, for each $g \in I^{es}_L(f)$, and for each $h \in \mathfrak{m}$, we get by (2.2.9)

$$\begin{aligned} i(f,g)+i(f,h) = i(f,hg) &\geq \kappa(f)+\mathrm{mt}(f)+i(f,L)-\mathrm{mt}(f) \\ &= \kappa(f)+i(f,L)\,. \end{aligned}$$

If we choose $h \in \mathfrak{m}$ generically, then $i(f,h)=\mathrm{mt}(f)$ (Exercise I.3.2.1), and we obtain the wanted inequality $i(f,g) \geq \kappa(f)+i(f,L)-\mathrm{mt}(f)$ for $g \in I_L^{es}(f)$.

Step 3. Let $f \in \mathbb{C}\{x,y\}$ be an arbitrary reduced element defining a curve germ $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$, and let $g \in I^{es}_{\mathit{fix},L_1\ldots L_k}(f)$. Further, let $\pi : M \to (\mathbb{C}^2,\mathbf{0})$ be the blowing up of the origin, and let $E \subset M$ be the exceptional divisor. Denote by $\widetilde{L}_1,\ldots,\widetilde{L}_k \subset M$ the strict transforms of $L_1,\ldots,L_k$, by $\widetilde{C}$ the strict transform of the germ $(C,\mathbf{0})$, and by $q_1,\ldots,q_s$ the intersection points of $\widetilde{C}$ with E. Since $L_1,\ldots,L_k$ are supposed to be *transversal* smooth germs, each q_i is contained in at most one of the $\widetilde{L}_j$, and each $\widetilde{L}_j$ contains at most one of the q_i. Thus, we may assume that for some $0 \leq \ell \leq \min\{k,s\}$, we have

$$q_i \in \widetilde{L}_j \iff i=j\leq \ell\,. \tag{2.2.10}$$

Now, let $\widetilde{f}_i \in \mathcal{O}_{M,q_i}$, respectively $e_i \in \mathcal{O}_{M,q_i}$, be local equations for the germ of $\widetilde{C}$, respectively of E, at q_i, and denote by $\widehat{g}_i$ the total transform of g at q_i $i=1,\ldots,s$. Then, due to Proposition 2.14, $\mathrm{mt}(g)\geq \mathrm{mt}(f)$, and

$$\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}} \in I_i := \begin{cases} I^{es}_{\mathit{fix},\widetilde{L}_1\ldots\widetilde{L}_k E}\big(\widetilde{f}_i\big)\,, & \text{if } i\leq \ell\,,\\ I^{es}_E\big(\widetilde{f}_i\big)\,, & \text{if } i>\ell\,. \end{cases} \tag{2.2.11}$$

Moreover, due to Proposition I.3.21, and since $\sum_{i=1}^s i\big(\widetilde{f}_i,e_i\big)=\mathrm{mt}(f)$, we have

$$\begin{aligned} i(f,g) &= \mathrm{mt}(f)\cdot\mathrm{mt}(g)+\sum_{i=1}^s i\big(\widetilde{f}_i,\widetilde{g}_i\big)\\ &= \mathrm{mt}(f)^2+\sum_{i=1}^s\left(i\Big(\widetilde{f}_i,\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}}\Big) + \big(\mathrm{mt}(g)-\mathrm{mt}(f)\big)\cdot i\big(\widetilde{f}_i,e_i\big)\right)\\ &\geq \mathrm{mt}(f)^2+\sum_{i=1}^s i\Big(\widetilde{f}_i,\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}}\Big)\,. \end{aligned}$$

Here, by (2.2.11) and the induction hypothesis,

$$i\Big(\widetilde{f}_i,\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}}\Big) \geq \begin{cases} \kappa\big(\widetilde{f}_i\big)+i\big(\widetilde{f}_i,e_i\big)+i\big(\widetilde{f}_i,\widetilde{L}_i\big)-\mathrm{mt}\big(\widetilde{f}_i\big), & \text{if } i\leq\ell\,,\\ \kappa\big(\widetilde{f}_i\big)+i\big(\widetilde{f}_i,e_i\big)-\mathrm{mt}\big(\widetilde{f}_i\big)\,, & \text{if } i>\ell\,. \end{cases}$$

Thus,

$$i(f,g)\geq \mathrm{mt}(f)^2+\sum_{i=1}^s\Big(\kappa\big(\widetilde{f}_i\big)-\mathrm{mt}\big(\widetilde{f}_i\big)\Big)+\sum_{i=1}^s i\big(\widetilde{f}_i,e_i\big)+\sum_{i=1}^{\ell} i\big(\widetilde{f}_i,\widetilde{L}_i\big)\,.$$

The statement follows, since $\sum_{i=1}^{s} i\big(\widetilde{f}_i, e_i\big) = \mathrm{mt}(f)$, since

$$\begin{aligned}\sum_{i=1}^{s}\Big(\kappa\big(\widetilde{f}_i\big) - \mathrm{mt}\big(\widetilde{f}_i\big)\Big) &= \sum_{i=1}^{s}\Big(2\delta\big(\widetilde{f}_i\big) - r\big(\widetilde{f}_i\big)\Big)\\ &= 2\delta(f) - \mathrm{mt}(f)\big(\mathrm{mt}(f)-1\big) - r(f)\\ &= \kappa(f) - \mathrm{mt}(f)^2\end{aligned}$$

(due to Propositions I.3.38, I.3.35 and I.3.34), and since

$$\sum_{i=1}^{\ell} i\big(\widetilde{f}_i, \widetilde{L}_i\big) = \sum_{j=1}^{k}\sum_{i=1}^{s} i\big(\widetilde{f}_i, \widetilde{L}_j\big) = \sum_{j=1}^{k}\Big(i(f, L_j) - \mathrm{mt}(f)\Big),$$

due to the assumption (2.2.10). □

Proposition 2.19. *Let (i, ϕ, σ) be an equisingular deformation over $(\mathbb{C}, 0)$ of the reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$. Moreover, let*

$$\begin{tikzcd} C \arrow[r, hook, "i"] \arrow[d] & \mathscr{C} \arrow[r, hook] \arrow[d, shift left, "\phi"] & B \times T \arrow[dl, "\mathrm{pr}_T"] \\ \{0\} \arrow[r, hook] & T \arrow[u, bend left, "\sigma"] & \end{tikzcd}$$

be a representative for (i, ϕ, σ) with $T \subset \mathbb{C}$, $B \subset \mathbb{C}^2$ neighbourhoods of the origin. Then, for all sufficiently small open neighbourhoods $U \subset B$ of the origin, we can choose an open neighbourhood $W = W(0) \subset T$ such that, for all $t \in W$, we have:

(i)

$$i_U(C, C_t) := \sum_{z \in U} i_z(C, C_t) \geq \kappa(C, \mathbf{0}),$$

where $C_t \times \{t\} = \mathscr{C}_t \subset U \times \{t\}$ is the fibre of ϕ over t.

(ii) If σ is the trivial section, we have even

$$i_U(C, C_t) \geq \kappa(C, \mathbf{0}) + \mathrm{mt}(C, \mathbf{0}).$$

Proof. The hypersurface germ $(\mathscr{C}, \mathbf{0}) \subset (\mathbb{C}^2 \times \mathbb{C}, \mathbf{0})$ is defined by an unfolding $F \in \mathbb{C}\{x, y, t\}$ with $f := F_0 \in \mathbb{C}\{x, y\}$ being a local equation for $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$. We may assume that $F \neq f$, otherwise the left-hand side in (i) and (ii) are infinite. We write

$$F(x, y, t) = f(x, y) + t^m f_m(x, y) + t^{m+1} g(x, y, t), \quad m \geq 1.$$

Then $\overline{F} := f + t^m f_m$ defines an equisingular deformation of $(C, \mathbf{0})$ over the fat point $\big(\{0\}, \mathbb{C}\{t\}/\langle t^{m+1}\rangle\big)$, with the (uniquely determined) singular section

$\overline{\sigma} = \sigma \bmod \langle t^{m+1}\rangle$ given by the ideal $I_{\overline{\sigma}} = \langle x - t^m\alpha, y - t^m\beta\rangle$, $\alpha, \beta \in \mathbb{C}$. Substituting t^m by ε, we get that $f + \varepsilon f_m$ defines an equisingular deformation over T_ε with the singular section being defined by the ideal $\langle x - \varepsilon\alpha, y - \varepsilon\beta\rangle$. If σ is the trivial section, then $\alpha = \beta = 0$. We obtain

$$f_m \in \begin{cases} I^{es}_{fix}(f) & \text{if } \sigma \text{ is the trivial section}, \\ I^{es}(f) & \text{otherwise}, \end{cases}$$

and Lemma 2.18 gives

$$i(f, f_m) \geq \begin{cases} \kappa(f) + \mathrm{mt}(f) & \text{if } \sigma \text{ is the trivial section}, \\ \kappa(f) & \text{otherwise}. \end{cases}$$

It remains to show that for a sufficiently small neighbourhood $U \subset \mathbb{C}^2$ of the origin, we find some $\rho > 0$ such that

$$i(f, f_m) = i_U(C, C_t) = \sum_{\boldsymbol{z} \in U} i_{\boldsymbol{z}}(C, C_t) \quad \text{for } 0 < |t| < \rho .$$

We consider the unfolding of f_m given by $G := f_m + tg$. By Proposition I.3.14, for $U \subset \mathbb{C}^2$ a sufficiently small neighbourhood of the origin, we find some open neighbourhood $W \subset \mathbb{C}$ of 0 such that G converges on $U \times W$ and such that, for each $t \in W$,

$$i(f, f_m) = i_U(C, D_t) = \sum_{\boldsymbol{z} \in U} i_{\boldsymbol{z}}(C, D_t), \qquad D_t := V(G_t).$$

Now, the statement follows, since from the definition of the intersection multiplicity we get

$$\begin{aligned} i_{\boldsymbol{z}}(C, C_t) &= i(f \circ \phi_{\boldsymbol{z}},\, f \circ \phi_{\boldsymbol{z}} + t^m f_m \circ \phi_{\boldsymbol{z}} + t^{m+1} g \circ \phi_{\boldsymbol{z}}) \\ &= i(f \circ \phi_{\boldsymbol{z}},\, f_m \circ \phi_{\boldsymbol{z}} + tg \circ \phi_{\boldsymbol{z}}) \;=\; i_{\boldsymbol{z}}(C, D_t) \end{aligned}$$

with $\phi_{\boldsymbol{z}}$ the linear coordinate change $x \mapsto x + z_1$, $y \mapsto y + z_2$, $\boldsymbol{z} = (z_1, z_2)$. $\square$

2.3 Deformations of the Parametrization

We describe now a different approach to equisingular deformations of a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ by considering deformations of the parametrization.

To define deformations of the parametrization (with section) we need deformations of a sequence of morphisms.

Definition 2.20. Let $(X_n, x_n) \xrightarrow{f_n} (X_{n-1}, x_{n-1}) \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} (X_0, x_0)$ be a sequence of morphisms of complex (multi-) germs.

(1) A *deformation of the sequence of morphisms* over a complex germ (T, t_0) is a Cartesian diagram

$$
\begin{tikzcd}
(X_n, x_n) \arrow[r, hook] \arrow[d, "f_n"'] \arrow[dr, phantom, "\square"] & (\mathscr{X}_n, x_n) \arrow[d, "F_n"] \\
(X_{n-1}, x_{n-1}) \arrow[r, hook] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\square"] & (\mathscr{X}_{n-1}, x_{n-1}) \arrow[d, "F_{n-1}"] \\
\vdots \arrow[d, "f_1"'] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, "F_1"] \\
(X_0, x_0) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{X}_0, x_0) \arrow[d, "F_0"] \\
\{t_0\} \arrow[r, hook] & (T, t_0)
\end{tikzcd}
$$

such that the composition $F_0 \circ \ldots \circ F_i : (\mathscr{X}_i, x_i) \to (T, t_0)$, $i = 0, \ldots, n$, is flat (hence a deformation of (X_i, x_i)).

(2) If $(\mathscr{X}'_n, x'_n) \to \ldots \to (\mathscr{X}'_0, x'_0) \to (T', t'_0)$ is another deformation of the sequence over a complex germ (T', t'_0), then a *morphism* from the second deformation to the first one is given by a morphism $\varphi : (T', t'_0) \to (T, t_0)$ and liftings $(\mathscr{X}'_i, x'_i) \to (\mathscr{X}_i, x_i)$, $i = 0, \ldots, n$, such that the obvious diagram commutes.

(3) The category of deformations of the sequence $(X_n, x_n) \to \ldots \to (X_0, x_0)$ is denoted by $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}$.

If we consider only deformations over a fixed germ (T, t_0), then we get the (non-full) subcategory $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T, t_0)$ with morphisms being the identity on (T, t_0). $\underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T, t_0)$ denotes the set of isomorphism classes of deformations in $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T, t_0)$.

(4) We call $T^1_{(X_n,x_n)\to\ldots\to(X_0,x_0)} := \underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T_\varepsilon)$ the set of isomorphism classes of *(first order) infinitesimal deformations* of the sequence $(X_n, x_n) \to \ldots \to (X_0, x_0)$.

(5) Deformations of the sequence $(X_n, x_n) \to \ldots \to (X_0, x_0)$ over (T, t_0) satisfying $(\mathscr{X}_0, x_0) = (X_0, x_0) \times (T, t_0)$, together with morphisms of deformations satisfying that the first lifting

$$(\mathscr{X}'_0, x'_0) = (X_0, x_0) \times (T', t'_0) \to (X_0, x_0) \times (T, t_0) = (\mathscr{X}_0, x_0)$$

is of type $\mathrm{id}_{(X_0,x_0)} \times \varphi$, form a subcategory of $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}$ denoted by $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}$. The set of isomorphism classes of first order deformations of $(X_n, x_n) \to \ldots \to (X_1, x_1)/(X_0, x_0)$ is

$$T^1_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)} := \underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}(T_\varepsilon)\,.$$

(6) The functor

$$\begin{aligned} \underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)} : (\textit{complex germs}) &\longrightarrow \mathcal{S}ets\,, \\ (T,t_0) &\longmapsto \underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T,t_0) \end{aligned}$$

is called the *deformation functor of the sequence* $(X_n,x_n)\to\ldots\to(X_0,x_0)$. In the same way, we define the functor $\underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}$.

Since this functor satisfies Schlessinger's conditions (H_0), (H_1) and (H_2), it follows that $T^1_{(X_n,x_n)\to\ldots\to(X_0,x_0)}$ and $T^1_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}$ are complex vector spaces (see Appendix C).

Remark 2.20.1. Let $\{\mathrm{pt}\}$ denote the reduced complex germ consisting of one point. Then deformations of (X,x) can be identified with deformations of the morphism $(X,x)\to\{\mathrm{pt}\}$, and deformations of (X,x) with section can be identified with deformations of the sequence $\{x\}\to(X,x)\to\{\mathrm{pt}\}$. In other words, the category $\mathcal{D}ef_{(X,x)}$ (respectively $\mathcal{D}ef^{\,sec}_{(X,x)}$) is naturally equivalent to $\mathcal{D}ef_{(X,x)\to\{\mathrm{pt}\}}$ (respectively $\mathcal{D}ef_{\{x\}\to(X,x)\to\{\mathrm{pt}\}}$).

These definitions can obviously be generalized to deformations of diagrams instead of deformations of sequences of morphisms and to multigerms instead of germs and the corresponding deformation functors again satisfy Schlessinger's conditions (H_0), (H_1) and (H_2). We formulate this only for a special case, which is needed below.

Definition 2.21. Let $\big(\overline{X},\overline{x}\big)=\coprod_{j=1}^r\big(\overline{X}_j,\overline{x}_j\big)$ and $(X,x)=\coprod_{i=1}^s(X_i,x_i)$ be multigerms, and let $f:\big(\overline{X},\overline{x}\big)\to(X,x)$ be a morphism mapping the set $\big\{\overline{x}\big\}=\big\{\overline{x}_1,\ldots,\overline{x}_r\big\}$ onto $\{x\}=\{x_1,\ldots,x_s\}$. Then a *deformation of the diagram*

$$\begin{tikzcd} \{\overline{x}\} \arrow[r, hook] \arrow[d] & (\overline{X},\overline{x}) \arrow[d, "f"] \\ \{x\} \arrow[r, hook] & (X,x) \end{tikzcd}$$

over (T,t_0) consists of deformations over (T,t_0) of $\big(\overline{X},\overline{x}\big)$ and of $\{\overline{x}\}\to\{x\}$, which fit into an obvious commutative diagram. As a deformation of a finite set of reduced points is trivially isomorphic to the disjoint union of the same number of copies of (T,t_0), such a deformation is equivalently given by a commutative diagram

$$\begin{tikzcd} (\overline{X},\overline{x}) \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\Box"] & (\overline{\mathscr{X}},\overline{x}) \arrow[d, "F"] \\ (X,x) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\Box"] & (\mathscr{X},x) \arrow[d] \\ \{t_0\} \arrow[r, hook] & (T,t_0) \arrow[u, bend right, "\sigma"'] \arrow[uu, bend right=60, "\overline{\sigma}"'] \end{tikzcd}$$

where $\sigma = \{\sigma_1, \ldots, \sigma_s\}$ and $\overline{\sigma} = \{\overline{\sigma}_1, \ldots, \overline{\sigma}_r\}$ are (multi-)sections satisfying $\sigma_i(t_0) = x_i$, $\overline{\sigma}_j(t_0) = \overline{x}_j$ for all i, j. Moreover, for each $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$ such that $x_i = f(\overline{x}_j)$ we have $\sigma_i = F \circ \overline{\sigma}_j$. In this situation, we say that the (multi-)sections σ and $\overline{\sigma}$ are *compatible*.

We call such deformations *deformations of* $(\overline{X}, \overline{x}) \to (X, x)$ *with compatible sections* (or just *deformations with section*), and denote the corresponding category by $\mathcal{D}ef^{sec}_{(\overline{X},\overline{x})\to(X,x)}$. Recall that the multisections σ and $\overline{\sigma}$ can be trivialized (Proposition 2.2).

We wish to apply all this to deformations of the parametrization of a plane curve singularity. To keep the notations shorter and to avoid overlaps in the notations, from now on we denote the base points of the complex germs appearing by $\mathbf{0}$, $\overline{0}$ or $\overline{0}_i$ (without necessarily referring to an embedding in some $(\mathbb{C}^n, \mathbf{0})$).

Consider the commutative diagram of complex (multi-) germs

$$\begin{tikzcd}
(\overline{C}, \overline{0}) \arrow[d, two heads, "n"'] \arrow[dr, "\varphi"] & \\
(C, \mathbf{0}) \arrow[r, hook, "j"'] & (\mathbb{C}^2, \mathbf{0})
\end{tikzcd}$$

where $(C, \mathbf{0})$ is a reduced plane curve singularity, j the given embedding, n the *normalization*, and $\varphi = j \circ n$ the *parametrization* of $(C, \mathbf{0})$.

If $(C, \mathbf{0}) = (C_1, \mathbf{0}) \cup \ldots \cup (C_r, \mathbf{0})$ is the decomposition of $(C, \mathbf{0})$ into irreducible components, then $(\overline{C}, \overline{0}) = (\overline{C}_1, \overline{0}_1) \amalg \ldots \amalg (\overline{C}_r, \overline{0}_r)$ is a multigerm with $(\overline{C}_i, \overline{0}_i) \cong (\mathbb{C}, 0)$ mapped onto $(C_i, \mathbf{0})$, inducing the normalization of the component $(C_i, \mathbf{0})$. On the level of (semi-) local rings we have

$$\begin{tikzcd}
\mathcal{O}_{\overline{C},\overline{0}} = \bigoplus_{i=1}^{r} \mathcal{O}_{\overline{C}_i,\overline{0}_i} \cong \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\} & \\
\mathcal{O}_{C,\mathbf{0}} \arrow[u, hook, "n^\sharp"] & \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \cong \mathbb{C}\{x, y\}\,. \arrow[l, two heads] \arrow[ul, two heads, "\varphi^\sharp = (\varphi_1^\sharp, \ldots, \varphi_r^\sharp)"']
\end{tikzcd}$$

We fix coordinates x, y for $(\mathbb{C}^2, \mathbf{0})$ and, for each $i = 1, \ldots, r$, a local coordinate t_i of $(\overline{C}_i, \overline{0}_i)$, identifying this germ with $(\mathbb{C}, 0)$. Then the parametrization $\varphi = \{\varphi_i \mid i = 1, \ldots, r\}$ is given by r holomorphic map germs

$$\varphi_i = \varphi|_{(\overline{C}_i,\overline{0}_i)} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, \mathbf{0})\,, \quad t_i \longmapsto \big(x_i(t_i), y_i(t_i)\big)\,.$$

If $f \in \mathbb{C}\{x, y\}$ defines $(C, \mathbf{0})$, f decomposes in r irreducible factors $f_1, \ldots, f_r$ with $(C_i, \mathbf{0}) = \big(V(f_i), \mathbf{0}\big)$. With the identification $\mathcal{O}_{\overline{C},\overline{0}} = \bigoplus_{i=1}^r \mathbb{C}\{t_i\}$, we have

$$\varphi^\sharp = (\varphi_i^\sharp)_{i=1}^r : \mathbb{C}\{x,y\} \to \bigoplus_{i=1}^r \mathbb{C}\{t_i\},$$

with $\varphi_i^\sharp(x) = x_i(t_i)$, $\varphi_i^\sharp(y) = y_i(t_i)$, and $\operatorname{Ker}(\varphi_i^\sharp) = \langle f_i \rangle$, $\operatorname{Ker}(\varphi^\sharp) = \langle f \rangle$.

Remark 2.21.1. Since $(\overline{C}, \overline{0})$ and $(\mathbb{C}^2, \mathbf{0})$ are smooth (multi-)germs, any deformation of these germs is trivial (Exercise 1.3.1). Hence, any deformation of the parametrization $\varphi : (\overline{C}, \overline{0}) \to (\mathbb{C}^2, \mathbf{0})$ over a germ $(T, \mathbf{0})$ is given by a Cartesian diagram and isomorphisms

$$\begin{tikzcd}
(\overline{C},\overline{0}) \arrow[r, hook, "i"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & \big(\overline{\mathscr{C}},\overline{0}\big) \arrow[r, "\cong"] \arrow[d, "\phi"] & \big(\overline{C}\times T,\overline{0}\big) \arrow[d, "\phi"] \\
(\mathbb{C}^2,\mathbf{0}) \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & \big(\mathscr{M},\mathbf{0}\big) \arrow[r, "\cong"] \arrow[d, "\phi_0"] & \big(\mathbb{C}^2\times T,\mathbf{0}\big) \arrow[d, "\mathrm{pr}"] \\
\{\mathbf{0}\} \arrow[r, hook] & (T,\mathbf{0}) \arrow[r, equal] & (T,\mathbf{0}) \arrow[u, bend right, "\sigma"'] \arrow[uu, bend right=60, "\overline{\sigma}"']
\end{tikzcd}$$

with pr the projection, $\big(\overline{\mathscr{C}}, \overline{0}\big) = \coprod_{i=1}^r \big(\overline{\mathscr{C}}_i, \overline{0}_i\big)$, and $\big(\overline{\mathscr{C}}_i, \overline{0}_i\big) \cong \big(\overline{C}_i \times T, \overline{0}_i\big)$. Compatible sections $\overline{\sigma}$ and σ consist of disjoint sections $\overline{\sigma}_i : (T, \mathbf{0}) \to (\overline{\mathscr{C}}_i, \overline{0}_i)$ of $\mathrm{pr} \circ \phi_i$, where $\phi_i : \big(\overline{\mathscr{C}}_i, \overline{0}_i\big) \to (\mathscr{M}, \mathbf{0})$ denotes the restriction of ϕ, and a section σ of pr such that $\phi \circ \overline{\sigma}_i = \sigma$, $i = 1, \ldots, r$. Note that pr and $\mathrm{pr} \circ \phi$ are automatically flat by Corollary I.1.88 and there is no further requirement on ϕ.

Let $(\mathscr{C}, \mathbf{0}) := \phi\big(\overline{\mathscr{C}}, \overline{0}\big)$ with Fitting structure. Then, by Proposition 2.9, the restriction $\phi_0 : (\mathscr{C}, \mathbf{0}) \to (T, \mathbf{0})$ is a deformation of $(C, \mathbf{0})$. Having fixed local coordinates x, y for $(\mathbb{C}^2, \mathbf{0})$ and t_i for $(\overline{C}_i, \overline{0}_i)$, the morphism

$$\phi = \{\phi_i \mid i = 1, \ldots, r\} : (\overline{C} \times T, \overline{0}) \to (\mathbb{C}^2 \times T, \mathbf{0})$$

is given by r holomorphic map germs

$$\phi_i : (\mathbb{C} \times T, \mathbf{0}) \to (\mathbb{C}^2 \times T, \mathbf{0}), \quad (t_i, s) \mapsto \big(\phi_{i,1}(t_i, s), s\big),$$

with $\phi_{i,1}(t_i, s) = \big(X_i(t_i, s), Y_i(t_i, s)\big)$, $X_i(t_i, \mathbf{0}) = x_i(t_i)$, $Y_i(t_i, \mathbf{0}) = y_i(t_i)$.

A section $\overline{\sigma} : (T, \mathbf{0}) \to (\overline{C} \times T, \overline{0})$, $s \mapsto \coprod_{i=1}^r \overline{\sigma}_i(s)$, compatible with the trivial section σ, $\sigma(s) = (\mathbf{0}, s)$, is then given by r holomorphic germs

$$\overline{\sigma}_i : (T, \mathbf{0}) \to (\overline{C}_i \times T, \overline{0}_i), \quad \overline{\sigma}_i(s) = \big(\overline{\sigma}_{i,1}(s), s\big)$$

such that $\big(X_i(\overline{\sigma}_i(s)), Y_i(\overline{\sigma}_i(s))\big) = (0, 0) \in \mathbb{C}^2$.

Definition 2.22. Let $n : (\overline{C}, \overline{0}) \to (C, \mathbf{0})$ be the normalization of the reduced plane curve germ $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$, let $\varphi : (\overline{C}, \overline{0}) \to (\mathbb{C}^2, \mathbf{0})$ be its parametrization, and let $(T, \mathbf{0})$ be a complex space germ.

(1) Objects in the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}(T,\mathbf{0})$, respectively in the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$, are called *deformations of the normalization*, respectively *deformations of the parametrization* of $(C,\mathbf{0})$ over $(T,\mathbf{0})$. They are denoted by (i,j,ϕ,ϕ_0) or just by ϕ.

(2) The corresponding deformations of the normalization $\big(\overline{C},\overline{0}\big)\to(C,\mathbf{0})$, resp. of the parametrization $\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$, with compatible sections are objects in the category $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}(T,\mathbf{0})$, resp. in $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$. Objects in these categories are called *deformations with section of the normalization*, resp. *of the parametrization*, of $(C,\mathbf{0})$. They are denoted by $(\phi,\overline{\sigma},\sigma)$.

(3) $T^{1,sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$, resp. $T^{1,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$, denotes the corresponding vector space of *(first order) infinitesimal deformations* of the normalization, resp. parametrization, with section.

We show now that isomorphism classes of deformations of the normalization and of the parametrization are essentially the same thing.

Proposition 2.23. *If* $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ *is a reduced plane curve singularity, then there is a surjective functor from* $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ *to* $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$, *inducing an isomorphism between the deformation functors* $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ *and* $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$. *The same holds for* $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$, *resp.* $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ *and the corresponding deformation functors.*

Proof. We consider the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}$ and show that the natural forgetful functors from this category to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ and to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ induce isomorphisms for the corresponding deformation functors.

By Proposition 2.9, we have a functor from the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}$ and, by forgetting $(\mathbb{C}^2,\mathbf{0})$, to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$. The relative lifting lemma 1.27 says that, for a given germ $(T,\mathbf{0})$, the functor $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})\to\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}(T,\mathbf{0})$ is surjective (full) and injective on the set of isomorphism classes. Hence, the deformation functors are isomorphic.

To see that the two functors $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}$ and $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ are isomorphic, note that Proposition 2.9 easily implies that the forgetful map $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})/(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})\to\mathcal{D}ef_{(\overline{C},\overline{0})/(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ is an isomorphism of categories. Since $(\mathbb{C}^2,\mathbf{0})$ is a smooth germ, each deformation of $(\mathbb{C}^2,\mathbf{0})$ is trivial (Exercise 1.3.1), hence, $\mathcal{D}ef_{(\overline{C},\overline{0})/(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})\to\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ induces a bijection on the set of isomorphism classes, and similar for $(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})$. Together this implies the required isomorphism.

As sections from the base space to the total space of deformations are not affected by the previous arguments, it follows that $\underline{\mathcal{D}ef}^{sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ and $\underline{\mathcal{D}ef}^{sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ are isomorphic, too. $\square$

As an immediate consequence of Proposition 2.23, we obtain the following corollary:

Corollary 2.24. *Each deformation of the parametrization (with compatible sections) induces an embedded deformation (with section) of the curve germ* $(C,\mathbf{0})$.

Considering deformations over T_ε, this yields vector space homomorphisms

$$T^1_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} \xrightarrow{\alpha'} T^1_{(C,\mathbf{0})}\,, \qquad T^{1,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} \xrightarrow{\beta'} T^{1,sec}_{(C,\mathbf{0})}\,.$$

In Section 2.4 below, we describe these maps α' and β' in explicit terms.

Example 2.24.1. Consider the cusp, parametrized by $\varphi: t \mapsto (t^3, t^2)$, and the deformation of the parametrization $\phi: (t,s) \mapsto (t^3 - s^2 t, t^2 - s^2)$ over $(\mathbb{C},0)$. According to Proposition 2.9, the induced embedded deformation of $(C,\mathbf{0})$ is given by $\operatorname{Ker}(\phi^\sharp: \mathbb{C}\{x,y\} \to \mathbb{C}\{t,s\})$. Hence, the deformation of the equation is given by $\big(V(x^2 - y^3 - s^2y^2), \mathbf{0}\big) \to (\mathbb{C},0)$, $(x,y,s) \mapsto s$, which is the deformation of the cusp into an ordinary double point along the trivial (singular) section $s \mapsto (\mathbf{0}, s)$ with image $\{\mathbf{0}\} \times (\mathbb{C},0)$. The preimage in $(\overline{\mathscr{C}}, \overline{0}) = (\mathbb{C}\times\mathbb{C},\mathbf{0})$ of this image is $\{(s,t) \mid t^2 - s^2 = 0\}$.

It follows that the deformation $(\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{C},\mathbf{0}) \to (\mathbb{C},0)$ admits two sections $s \mapsto \{(s,t) \mid t = \pm s\}$ which both map to the unique singular section of $(\mathscr{C},\mathbf{0}) \to (\mathbb{C},0)$.

Equimultiple Deformations

We are now going to define equimultiple deformations of the parametrization.

The multiplicity of $(C,\mathbf{0})$ satisfies $\operatorname{mt}(C,\mathbf{0}) = \sum_{i=1}^r \operatorname{mt}(C_i,\mathbf{0})$, and the multiplicity of the i-th branch satisfies

$$\operatorname{mt}(C_i,\mathbf{0}) = \min\big\{\operatorname{ord}_{t_i} x_i(t_i), \operatorname{ord}_{t_i} y_i(t_i)\big\} =: \operatorname{ord}(\varphi_i, \overline{0}_i) =: \operatorname{ord}\varphi_i\,,$$

$\operatorname{ord}\varphi_i$ being the *order of the parametrization* of the i-th branch. This follows from Proposition I.3.12 (see also Exercise I.3.2.1 and Proposition I.3.21). We call

$$\mathbf{mt}(C,\mathbf{0}) := \big(\operatorname{mt}(C_1,\mathbf{0}), \dots, \operatorname{mt}(C_r,\mathbf{0})\big)$$

the *multiplicity vector* of $(C,\mathbf{0})$ which, therefore, equals

$$\mathbf{ord}\,\varphi := \mathbf{ord}(\varphi,\overline{0}) := \big(\operatorname{ord}\varphi_1, \dots, \operatorname{ord}\varphi_r\big)\,,$$

the *order of the parametrization* of $(C,\mathbf{0})$.

Note that $\operatorname{ord}\varphi_i = \max\big\{m \mid \varphi_i^\sharp(\mathfrak{m}_{(\overline{C}_i,\overline{0}_i)}) \subset \mathfrak{m}^m_{\mathbb{C}^2,\mathbf{0}}\big\}$, where the right-hand side does not involve any choice of coordinates.

Let $(\phi, \overline{\sigma}, \sigma)$ be a deformation with section of the parametrization of $(C,\mathbf{0})$ over $(T,\mathbf{0})$. We set

$$I_{\overline{\sigma}_i} := \mathrm{Ker}\big(\overline{\sigma}_i^\sharp : \mathcal{O}_{\overline{\mathscr{C}}_i,\overline{0}_i} \to \mathcal{O}_{T,\mathbf{0}}\big)\,, \quad i = 1,\dots,r\,,$$
$$I_\sigma := \mathrm{Ker}\big(\sigma^\sharp : \mathcal{O}_{\mathscr{C},\mathbf{0}} \to \mathcal{O}_{T,\mathbf{0}}\big)\,,$$

which are the ideals of the respective sections. We have $\phi_i^\sharp(I_\sigma) \subset I_{\overline{\sigma}_i}$ and define the order of the deformation of the parametrization of the i-th branch (along $\overline{\sigma}_i$) as

$$\mathrm{ord}(\phi_i,\overline{\sigma}_i,\sigma) := \max\big\{m \;\big|\; \phi_i^\sharp(I_\sigma) \subset I_{\overline{\sigma}_i}^m\big\}\,.$$

The r-tuple

$$\mathbf{ord}\,\phi := \mathbf{ord}(\phi,\overline{\sigma},\sigma) := \big(\mathrm{ord}(\phi_1,\overline{\sigma}_1),\dots,\mathrm{ord}(\phi_r,\overline{\sigma}_r)\big)$$

is called the *order (vector) of the deformation of the parametrization* of $(C,\mathbf{0})$ (along $\overline{\sigma},\sigma$).

Definition 2.25. (1) A deformation of the parametrization with section $(\phi,\overline{\sigma},\sigma) \in \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ is called *equimultiple* if $\mathbf{ord}\,\phi = \mathbf{ord}\,\varphi$.

We denote by $\mathcal{D}ef^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0}) \subset \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ the full subcategory of *equimultiple deformations of the parametrization.* Moreover, the corresponding set of isomorphism classes is denoted by $\underline{\mathcal{D}ef}^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$, and we set

$$T^{1,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} := \underline{\mathcal{D}ef}^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T_\varepsilon)\,.$$

(2) More generally, let $\boldsymbol{m} = (m_1,\dots,m_r)$, $1 \le m_i \le \mathrm{ord}\,\varphi_i$, be an integer vector. Then we say that $(\phi,\overline{\sigma})$ is $\boldsymbol{m}$*-multiple* if $\phi_i^*(I_\sigma) \subset I_{\overline{\sigma}_i}^{m_i}$ for $i = 1,\dots,r$.

$\mathcal{D}ef^{\,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$, $\underline{\mathcal{D}ef}^{\,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$, and $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ have the obvious meaning.

Note that $\mathcal{D}ef^{\,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ coincides with $\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ for $\boldsymbol{m} = (1,\dots,1)$, and with $\mathcal{D}ef^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ for $\boldsymbol{m} = (\mathrm{ord}\,\varphi_1,\dots,\mathrm{ord}\,\varphi_r)$.

If σ and all the $\overline{\sigma}_i$ are trivial sections (which we always may assume by Proposition 2.2), then $\mathrm{ord}(\phi_i,\overline{\sigma}_i)$ is the minimum of the t_i-orders of $X_i(t_i,s)$ and $Y_i(t_i,s)$. If this minimum is attained by, say, X_i, then equimultiple implies that the leading term of (the power series expansion in t_i of) X_i is a unit in $\mathcal{O}_{T,\mathbf{0}}$. Moreover, the deformation is $\boldsymbol{m}$-multiple iff $\mathrm{ord}_{t_i} X_i \ge m_i$ and $\mathrm{ord}_{t_i} Y_i \ge m_i$ for all i. Furthermore, an equimultiple deformation of the parametrization of $(C,\mathbf{0})$ induces an equimultiple deformation (of the equation) of $(C,\mathbf{0})$:

Lemma 2.26. *Let $(\phi,\overline{\sigma},\sigma)$ be an equimultiple deformation of the parametrization of $(C,\mathbf{0})$. Then the induced embedded deformation of each branch of $(C,\mathbf{0})$ and, hence, of $(C,\mathbf{0})$ itself, is equimultiple along σ, too.*

Proof. First, assume that the base $(T,\mathbf{0})$ of the deformation is reduced. For each $t\in T$ near $\mathbf{0}$, ϕ induces a parametrization $\phi_t:\big(\overline{\mathscr{C}}_t,\overline{\sigma}(t)\big)\to\big(\mathbb{C}^2,\sigma(t)\big)$ of the fibre $\big(\mathscr{C}_t,\sigma(t)\big)$ of $(\mathscr{C},\mathbf{0})\to(T,\mathbf{0})$ over t. Since ϕ is equimultiple,

$$\mathbf{mt}(C,\mathbf{0})=\mathbf{ord}(\varphi,\overline{0})=\mathbf{ord}\big(\phi_t,\overline{\sigma}(t)\big)=\mathbf{mt}(\phi_t,\sigma(t))$$

by Exercise I.3.2.1.

For an arbitrary base $(T,\mathbf{0})$, we may assume that $(T,\mathbf{0})\subset(\mathbb{C}^n,\mathbf{0})$, that $\phi:\big(\overline{C}\times\overline{T},\overline{0}\big)\to\big(\mathbb{C}^2\times T,\mathbf{0}\big)$, and that the sections are trivial. Then it is clear that there is an extension $\widetilde{\phi}:\big(\overline{C}\times\mathbb{C}^n,\overline{0}\big)\to\big(\mathbb{C}^2\times\mathbb{C}^n,\mathbf{0}\big)$ of ϕ which is equimultiple along trivial sections and the result follows as before. $\square$

However, the converse of Lemma 2.26 is not true as the following example shows.

Example 2.26.1. (Continuation of Example 2.24.1) The deformation

$$(\mathscr{C},\mathbf{0})=\big(V(x^2-y^3-s^2y^2),\mathbf{0}\big)\longrightarrow(\mathbb{C},0)\,,\quad(x,y,s)\longmapsto s\,,$$

of the cusp to a node is equimultiple along the trivial section σ. It is induced by the deformation of the parametrization

$$\phi:(\mathbb{C}\times\mathbb{C},\mathbf{0})\longrightarrow(\mathbb{C}^2\times\mathbb{C},\mathbf{0})\,,\quad(t,s)\longmapsto(t^3-s^2t,t^2-s^2,s)$$

either along the section $\overline{\sigma}:s\mapsto(s,s)$, or along the section $\overline{\sigma}:s\mapsto(-s,s)$. However, $(\phi,\overline{\sigma},\sigma)$ is not equimultiple: $I_\sigma=\langle x,y\rangle$, $I_{\overline{\sigma}}=\langle t-s\rangle$ (or $\langle t+s\rangle$), $\mathbf{ord}\,\varphi=\mathrm{mt}(C,\mathbf{0})=2$, while $\phi^\sharp(I_\sigma)=\langle t^3-s^2t,(t-s)(t+s)\rangle$ is contained in $I_{\overline{\sigma}}$, but not in $I_{\overline{\sigma}}^2$.

Next, we give an explicit description for the vector space $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ of first order $\boldsymbol{m}$-multiple deformations of the parametrization.

Let $\varphi:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$ be the parametrization of $\phi(\overline{C},\overline{0})=(C,\mathbf{0})=\bigcup_{i=1}^r(C_i,\mathbf{0})$, given by the system of parametrizations for the branches $t_i\mapsto\varphi_i(t_i)=\big(x_i(t_i),y_i(t_i)\big)$, $i=1,\dots,r$. In the following, we identify $\mathcal{O}_{C,\mathbf{0}}$ with $n^\sharp\mathcal{O}_{C,\mathbf{0}}=\varphi^\sharp\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\subset\mathcal{O}_{\overline{C},\overline{0}}$, and also any ideal of $\mathcal{O}_{C,\mathbf{0}}$ with its image in $\mathcal{O}_{\overline{C},\overline{0}}$. Then the subalgebra

$$\mathcal{O}_{C,\mathbf{0}}=\mathbb{C}\left\{\begin{pmatrix}x_1\\ \vdots\\ x_r\end{pmatrix},\begin{pmatrix}y_1\\ \vdots\\ y_r\end{pmatrix}\right\}\subset\bigoplus_{i=1}^r\mathbb{C}\{t_i\}=\mathcal{O}_{\overline{C},\overline{0}}$$

has $\mathbb{C}$-codimension $\delta=\delta(C,\mathbf{0})$. We set

$$\dot{\varphi}:=\dot{\boldsymbol{x}}\cdot\frac{\partial}{\partial x}+\dot{\boldsymbol{y}}\cdot\frac{\partial}{\partial y}\in\mathcal{O}_{\overline{C},\overline{0}}\cdot\frac{\partial}{\partial x}\oplus\mathcal{O}_{\overline{C},\overline{0}}\cdot\frac{\partial}{\partial y}\,,$$

with

$$\dot{\boldsymbol{x}} := \dot{\varphi^\sharp(x)} := \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_r \end{pmatrix}, \quad \dot{\boldsymbol{y}} := \dot{\varphi^\sharp(y)} := \begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_r \end{pmatrix},$$

and with $\dot{x}_i, \dot{y}_i$ denoting the derivatives of x_i, y_i with respect to t_i. Let

$$\mathfrak{m}_{\overline{C},\overline{0}} := \bigoplus_{i=1}^{r} \mathfrak{m}_{\overline{C}_i,\overline{0}_i} = \bigoplus_{i=1}^{r} t_i\mathbb{C}\{t_i\}.$$

be the Jacobson radical of $\mathcal{O}_{\overline{C},\overline{0}}$, and set, for any r-tuple $\boldsymbol{m} = (m_1, \dots, m_r)$ of integers,

$$\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}} := \bigoplus_{i=1}^{r} \mathfrak{m}^{m_i}_{\overline{C}_i,\overline{0}_i} = \bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}.$$

If $1 \leq m_i \leq \operatorname{ord}\varphi_i$ for all $i = 1, \dots, r$, we introduce the complex vector space

$$\begin{aligned} M^{\boldsymbol{m}}_{\varphi} &:= \left(\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}\right) \Big/ \left(\dot{\varphi}\cdot\mathfrak{m}_{\overline{C},\overline{0}} + \mathfrak{m}_{C,\boldsymbol{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{C,\boldsymbol{0}}\frac{\partial}{\partial y}\right) \\ &= \left((\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}/\mathfrak{m}_{C,\boldsymbol{0}})\frac{\partial}{\partial x} \oplus (\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}/\mathfrak{m}_{C,\boldsymbol{0}})\frac{\partial}{\partial y}\right) \Big/ \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}/\mathfrak{m}_{C,\boldsymbol{0}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right). \end{aligned}$$

For $\boldsymbol{0} = (0, \dots, 0)$, we set

$$\begin{aligned} M^{\boldsymbol{0}}_{\varphi} &:= \left(\mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}\right) \Big/ \left(\dot{\varphi}\cdot\mathcal{O}_{\overline{C},\overline{0}} + \mathcal{O}_{C,\boldsymbol{0}}\frac{\partial}{\partial x} \oplus \mathcal{O}_{C,\boldsymbol{0}}\frac{\partial}{\partial y}\right). \\ &= \left((\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\boldsymbol{0}})\frac{\partial}{\partial x} \oplus (\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\boldsymbol{0}})\frac{\partial}{\partial y}\right) \Big/ \mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\boldsymbol{0}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right). \end{aligned}$$

Proposition 2.27. *Using the above notations, the following holds:*

(1) $T^1_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{\boldsymbol{0}}_{\varphi}$ *and* $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{\boldsymbol{m}}_{\varphi}$ *if* $1 \leq m_i \leq \operatorname{ord}\varphi_i$ *for all* $i = 1, \dots, r$*. In particular,*

$$T^{1,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{(1,\dots,1)}_{\varphi}, \quad T^{1,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{(\operatorname{ord}\varphi_1,\dots,\operatorname{ord}\varphi_r)}_{\varphi}.$$

(2) Let $(T, \boldsymbol{0}) = (\mathbb{C}^k, \boldsymbol{0})$ *with local coordinates* $\boldsymbol{s} = (s_1, \dots, s_k)$*. Moreover, let* $\phi : (\overline{C} \times \mathbb{C}^k, \overline{0}) \to (\mathbb{C}^2 \times \mathbb{C}^k, \boldsymbol{0})$ *define an* $\boldsymbol{m}$*-multiple deformation of the parametrization along the trivial sections* $\overline{\sigma}$ *and* σ*, given by* r *holomorphic germs*

$$\phi_i : (\overline{C}_i \times \mathbb{C}^k, \overline{0}_i) \to (\mathbb{C}^2 \times \mathbb{C}^k, \boldsymbol{0}), \quad (t_i, \boldsymbol{s}) \mapsto \big(X_i(t_i, \boldsymbol{s}), Y_i(t_i, \boldsymbol{s}), \boldsymbol{s}\big).$$

Then $(\phi, \overline{\sigma}, \sigma)$ *is a versal (respectively semiuniversal)* $\boldsymbol{m}$*-multiple deformation iff the column vectors*

$$\left(\frac{\partial X_i}{\partial s_j}(t_i, \boldsymbol{0})\frac{\partial}{\partial x} + \frac{\partial Y_i}{\partial s_j}(t_i, \boldsymbol{0})\frac{\partial}{\partial y}\right)_{i=1}^{r} \in \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial y},$$

$j = 1, \dots, k$*, represent a system of generators (respectively a basis) for the vector space* $M^{\boldsymbol{m}}_{\varphi}$*.*

(3) Let $\boldsymbol{a}^j, \boldsymbol{b}^j \in \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}} = \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$ *be such that*

$$\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial x} = \begin{pmatrix} a_1^j \\ \vdots \\ a_r^j \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} b_1^j \\ \vdots \\ b_r^j \end{pmatrix} \frac{\partial}{\partial y}, \quad j = 1, \ldots, k,$$

represent a basis for $M^{\boldsymbol{m}}_\varphi$*. Then the deformation of the parametrization* $\phi : (\overline{C}, \overline{0}) \times (\mathbb{C}^k, \boldsymbol{0}) \to (\mathbb{C}^2, \boldsymbol{0}) \times (\mathbb{C}^k, \boldsymbol{0})$ *given by* $\phi_i = (X_i, Y_i, \boldsymbol{s})$ *with*

$$X_i(t_i, \boldsymbol{s}) = x_i(t_i) + \sum_{j=1}^k a_i^j(t_i) s_j\,,$$
$$Y_i(t_i, \boldsymbol{s}) = y_i(t_i) + \sum_{j=1}^k b_i^j(t_i) s_j\,,$$

$i = 1, \ldots, r$*, is a semiuniversal* $\boldsymbol{m}$*-multiple deformation of the parametrization* φ *over* $(\mathbb{C}^k, \boldsymbol{0})$*.*

In particular, $\boldsymbol{m}$*-multiple deformations of the parametrization are unobstructed and have a smooth semiuniversal base space of dimension* $\dim_{\mathbb{C}}(M^{\boldsymbol{m}}_\varphi)$*.*

Proof. Let $\phi \in \mathcal{D}ef_{(\overline{C},\overline{0}) \to (\mathbb{C}^2,\boldsymbol{0})}(T_\varepsilon)$ be as in Remark 2.21.1, that is, ϕ is given by

$$X_i(t_i, \varepsilon) = x_i(t_i) + \varepsilon a_i(t_i)\,, \quad Y_i(t_i, \varepsilon) = y_i(t_i) + \varepsilon b_i(t_i)\,,$$

with $a_i, b_i \in \mathbb{C}\{t_i\}$, $i = 1, \ldots, r$, $\varepsilon^2 = 0$.

ϕ is trivial iff there exist isomorphisms $(\overline{C} \times T_\varepsilon, \overline{0}) \xrightarrow{\cong} (\overline{C} \times T_\varepsilon, \overline{0})$ and $(\mathbb{C}^2 \times T_\varepsilon, \boldsymbol{0}) \xrightarrow{\cong} (\mathbb{C}^2 \times T_\varepsilon, \boldsymbol{0})$ over T_ε, being the identity modulo ε, such that via these isomorphisms ϕ is mapped to the product deformation (that is, the deformation as above with $a_i, b_i = 0$). On the ring level, these isomorphisms are given as

$$x \longmapsto x + \varepsilon\psi_1(x, y)\,, \quad y \longmapsto y + \varepsilon\psi_2(x, y)\,,$$

$\psi_1, \psi_2 \in \mathbb{C}\{x, y\}$ arbitrary, and as

$$t_i \longmapsto \widetilde{t}_i := t_i + \varepsilon h_i(t_i)\,, \quad i = 1, \ldots, r\,,$$

$h_i \in \mathbb{C}\{t_i\}$ arbitrary, such that

$$x_i(t_i) + \varepsilon a_i(t_i) = x_i(\widetilde{t}_i) + \varepsilon\psi_1\big(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i)\big)\,,$$
$$y_i(t_i) + \varepsilon a_i(t_i) = y_i(\widetilde{t}_i) + \varepsilon\psi_2\big(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i)\big)\,.$$

Using Taylor's formula and $\varepsilon^2 = 0$, we get $x_i(\widetilde{t}_i) = x_i(t_i) + \varepsilon\dot{x}_i(t_i)h_i(t_i)$ and $\varepsilon\psi_1\big(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i)\big) = \varepsilon\psi_1\big(x_i(t_i), y_i(t_i)\big)$, and the analogous equations for $y_i(\widetilde{t}_i)$ and $\varepsilon\psi_2$.

Hence, the necessary and sufficient condition for ϕ to be trivial reads

$$a_i = \dot{x}_i h_i + \psi_1(x_i, y_i)\,, \quad b_i = \dot{y}_i h_i + \psi_2(x_i, y_i)\,,$$

that is,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} \frac{\partial}{\partial y} \in \dot{\varphi} \cdot \mathcal{O}_{\overline{C},\overline{0}} + \mathcal{O}_{C,\mathbf{0}} \cdot \frac{\partial}{\partial x} \oplus \mathcal{O}_{C,\mathbf{0}} \cdot \frac{\partial}{\partial y} \,.$$

Moreover, ϕ is $\boldsymbol{m}$-multiple along the trivial sections iff $a_i, b_i \in t^{m_i}\mathbb{C}\{t_i\}$. ϕ is trivial along the trivial sections iff the above isomorphisms respect the trivial sections, that is, $\psi_1, \psi_2 \in \mathfrak{m}_{\mathbb{C}^2,\mathbf{0}}$ and $h_i \in t_i\mathbb{C}\{t_i\}$. This proves statement (1).

As the proofs of (2) and (3) are similar to (but simpler than) the proofs of the respective statements for equisingular deformations, we omit them here. □

Example 2.27.1. (1) Consider the irreducible plane curve singularity $(C, \mathbf{0})$ parametrized by $\varphi : t \mapsto (t^2, t^7)$. Then

$$M_\varphi^m \cong (t^m\mathbb{C}\{t\})^2 \Big/ (2t, 7t^6) \cdot t^\delta \cdot \mathbb{C}\{t\} + \langle t^2, t^7\rangle^\delta \mathbb{C}\{t^2, t^7\}^2 \,,$$

with $\delta = 0$ if $m = 0$, and $\delta = 1$ if $m > 0$. As a $\mathbb{C}$-vector space, M_φ^0 has the basis $\{(0,t), (0,t^3), (0,t^5)\}$. Hence,

$$X(t,\boldsymbol{s}) = t^2\,, \quad Y(t,\boldsymbol{s}) = t^7 + s_1t + s_2t^3 + s_3t^5\,,$$

defines a semiuniversal deformation of the parametrization of $(C, \mathbf{0})$. Similarly, for $m = 1$,

$$X(t,\boldsymbol{s}) = t^2 + s_1t\,, \quad Y(t,\boldsymbol{s}) = t^7 + s_2t + s_3t^3 + s_4t^5$$

defines a semiuniversal deformation of the parametrization with section, and

$$X(t,\boldsymbol{s}) = t^2\,, \quad Y(t,\boldsymbol{s}) = t^7 + s_1t^3 + s_2t^5$$

a semiuniversal equimultiple deformation of the parametrization.

(2) Consider the reducible plane curve singularity $(C, \mathbf{0})$ given by the local equation $x(x^3 - y^5)$, and let $(x_i(t), y_i(t))$, $i = 1, 2$, be parametrizations for the branches of $(C, \mathbf{0})$. To save indices, we write $\mathcal{O}_{\overline{C},\overline{0}} = \mathbb{C}\{t\} \oplus \mathbb{C}\{t\}$ instead of $\mathbb{C}\{t_1\} \oplus \mathbb{C}\{t_2\}$ and

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 \end{pmatrix}, \quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}$$

as column vectors in $\mathbb{C}\{t\} \oplus \mathbb{C}\{t\}$. Then $\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\mathbf{0}}$ has dimension $\delta = 9$ and has the $\mathbb{C}$-basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \end{pmatrix}, \begin{pmatrix} 0 \\ t^2 \end{pmatrix}, \begin{pmatrix} 0 \\ t^3 \end{pmatrix}, \begin{pmatrix} 0 \\ t^4 \end{pmatrix}, \begin{pmatrix} 0 \\ t^6 \end{pmatrix}, \begin{pmatrix} 0 \\ t^7 \end{pmatrix}, \begin{pmatrix} 0 \\ t^9 \end{pmatrix}, \begin{pmatrix} 0 \\ t^{12} \end{pmatrix} \right\}.$$

Now, $M_\varphi^{(0,0)}$ is $\big(\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\mathbf{0}}\big)^2$ modulo

$$\left(\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}, \begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}\right)\cdot \mathcal{O}_{\overline{C},\overline{0}} = \left(\begin{pmatrix}0\\ 5t^4\end{pmatrix}, \begin{pmatrix}1\\ 3t^2\end{pmatrix}\right)\cdot \big(\mathbb{C}\{t\}\oplus\mathbb{C}\{t\}\big)\,.$$

We compute a $\mathbb{C}$-basis of $M_\varphi^{(0,0)}$ as

$$\left\{\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^2\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^3\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\right.$$
$$\left.\left(\begin{pmatrix}0\\t^4\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^6\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^9\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\t\end{pmatrix}\right)\right\}.$$

Hence, a semiuniversal deformation of the parametrization of $(C,\mathbf{0})$ is given by:

$$\begin{aligned}\begin{pmatrix}X_1(t,\boldsymbol{s})\\ X_2(t,\boldsymbol{s})\end{pmatrix} &= \begin{pmatrix}s_1\\ t^5+s_2t+s_3t^2+s_4t^3+s_5t^4+s_6t^6+s_7t^9\end{pmatrix},\\ \begin{pmatrix}Y_1(t,\boldsymbol{s})\\ Y_2(t,\boldsymbol{s})\end{pmatrix} &= \begin{pmatrix}t\\ t^3+s_8t\end{pmatrix}.\end{aligned}$$

2.4 Computation of T^1 and T^2

In the previous section, we gave an explicit description of the semiuniversal deformation of the parametrization of a reduced plane curve singularity $j:(C,\mathbf{0})\hookrightarrow(\mathbb{C}^2,\mathbf{0})$. In this section, we consider infinitesimal deformations and obstructions for deformations of the parametrization and for related deformations. We are interested in explicit formulas for T^1 and T^2 in terms of basic invariants of $(C,\mathbf{0})$, because these modules contain important information on the deformation functors. For example, if $T^2=0$, then the semiuniversal deformation has a smooth base space of dimensiom $\dim_{\mathbb{C}} T^1$.

The main tool is the cotangent braid of the normalization of $(C,\mathbf{0})$, $n:\big(\overline{C},\overline{0}\big)\to(C,\mathbf{0})$, and of the parametrization $\varphi:=j\circ n:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$. This can be found in Appendix C.5, as well as the notations to be used and the formula

$$T^i_{X\backslash X\to Y/Y}\cong T^{i-1}_Y(F_*\mathcal{O}_X)\,,\quad i\geq 0\,, \tag{2.4.12}$$

where $F:X\to Y$ is any morphism of complex spaces, respectively of germs of complex spaces.

To simplify notations, throughout this section we usually omit the base points. That is, we write $\overline{C}$ instead of $\big(\overline{C},\overline{0}\big)$, C instead of $(C,\mathbf{0})$, and $\mathbb{C}^2$ instead of $(\mathbb{C}^2,\mathbf{0})$. Furthermore, we set

$$\mathcal{O}=\mathcal{O}_{C,\mathbf{0}}=\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}/\langle f\rangle\,,\quad \overline{\mathcal{O}}=\mathcal{O}_{\overline{C},\overline{0}}=\bigoplus_{i=1}^{r}\mathbb{C}\{t_i\}\,.$$

The maps $n^\sharp:\mathcal{O}\to\overline{\mathcal{O}}$, resp. $\varphi^\sharp:\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\to\overline{\mathcal{O}}$, are the $\mathbb{C}$-algebra maps of n and φ, sending x to $(x_1,\dots,x_r)$ and y to $(y_1,\dots,y_r)$ in $\overline{\mathcal{O}}$. We set

$$\dot{x}_i := \frac{\partial x_i}{\partial t_i}\,, \quad \dot{y}_i := \frac{\partial y_i}{\partial t_i}\,.$$

For computing T^1 and T^2, we also need T^0, which we describe first:

Lemma 2.28. *With the above notations, we have*

(1) $T^0_{\overline{C}\to\mathbb{C}^2} = \big\{(\xi,\eta) \in \mathrm{Der}_{\mathbb{C}}\big(\overline{\mathcal{O}},\overline{\mathcal{O}}\big) \times \mathrm{Der}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big) \;\big|\; \xi \circ \varphi^\sharp = \varphi^\sharp \circ \eta\big\}\,,$
$T^0_{\overline{C}\to C} \xrightarrow{\cong} T^0_C\,.$

(2) $T^0_{\overline{C}/C} = T^0_{\overline{C}/\mathbb{C}^2} = 0\,.$

(3) $T^0_{\overline{C}\backslash\overline{C}\to C/C} = T^0_{\overline{C}\backslash\overline{C}\to\mathbb{C}^2/\mathbb{C}^2} = 0\,.$

(4) $T^0_{\overline{C}\backslash\mathbb{C}^2} = \big\{\eta \in \mathrm{Der}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big) \;\big|\; \varphi^\sharp \circ \eta = 0\big\} = \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \cdot \Big(f\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\Big)\,,$
$T^0_{\overline{C}\backslash C} = 0\,.$

(5) $T^0_C = \mathrm{Der}_{\mathbb{C}}(\mathcal{O},\mathcal{O}) = \mathrm{Hom}_{\mathbb{C}}(\Omega^1_{C,\mathbf{0}},\mathcal{O})\,.$

(6) For each $\overline{\mathcal{O}}$-module N, respectively each $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module M, we have

$$T^0_{\overline{C}}(N) = \bigoplus_{i=1}^r N\frac{\partial}{\partial t_i}\,, \quad T^0_{\mathbb{C}^2}(M) = M\frac{\partial}{\partial x} \oplus M\frac{\partial}{\partial y}\,.$$

Moreover, $T^0_{\overline{C}} = T^0_{\overline{C}}\big(\overline{\mathcal{O}}\big)\,, \quad T^0_{\mathbb{C}^2} = T^0_{\mathbb{C}^2}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big)\,.$

Proof. (1) The first statement is just the definition of $T^0_{\overline{C}\to\mathbb{C}^2}$. The definition of $T^0_{\overline{C}\to C}$ is analogous. From $T^0_{\overline{C}/C} = 0$ (shown in (2)) and from the exact sequence $\dashrightarrow$ in the braid of $\overline{C} \to C$ (see Figure 2.14), it follows that the map $T^0_{\overline{C}\to C} \to T^0_C$ is injective. However, in characteristic 0, every derivation of $\mathcal{O}$ lifts to $\overline{\mathcal{O}}$ (cf. [Del1]), hence, we have an isomorphism.

(2) By definition, we have $T^0_{\overline{C}/C} = \big\{\xi \in \mathrm{Der}_{\mathbb{C}}\big(\overline{\mathcal{O}},\overline{\mathcal{O}}\big) \;\big|\; \xi \circ n^\sharp = 0\big\}$. Each derivation $\xi \in \mathrm{Der}_{\mathbb{C}}\big(\overline{\mathcal{O}},\overline{\mathcal{O}}\big)$ is of the form $\xi = \sum_{i=1}^r h_i \frac{\partial}{\partial t_i}$ for some $h_i \in \mathbb{C}\{t_i\}$. Now, the equality $\xi \circ n^\sharp = 0$ implies that

$$0 = \xi \circ n^\sharp(x) = \xi\big(x_1(t_1),\dots,x_r(t_r)\big) = \big(h_1\dot{x}_1,\dots,h_r\dot{x}_r\big)\,,$$

and, in an analogous manner, $\big(h_1\dot{y}_1,\dots,h_r\dot{y}_r\big) = 0$. Hence, for all $i = 1,\dots,r$, $h_i(\dot{x}_i,\dot{y}_i) = 0$, which implies $h_i = 0$ as $(\dot{x}_i,\dot{y}_i) \neq (0,0)$. The same argument applies to $T^0_{\overline{C}/\mathbb{C}^2}$.

(3) follows from the definition, respectively from the isomorphism (2.4.12).

(4) $T^0_{\overline{C}\backslash C} = \{\xi \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O},\mathcal{O}) \mid n^\sharp \circ \xi = 0\} = 0$, since $n^\sharp$ is injective. The result for $T^0_{\overline{C}\backslash\mathbb{C}^2}$ follows in the same way, since $\mathrm{Ker}\,\varphi^\sharp = \mathcal{O}f$.

(5),(6) are just the definitions. $\square$

In the following, we use that, for (X,x) a smooth germ (respectively a complete intersection germ), we have $T^i_{X,x}(M)=0$ for each finitely generated $\mathcal{O}_{X,x}$-module and $i\geq 1$ (respectively $i\geq 2$). In particular, as plane curve singularities are complete intersections, $T^i_C(M)=0$ for all $i\geq 2$.

The non-zero terms of the braid for the parametrization are shown in Figure 2.13, with $T^1_{\overline{C}\backslash\overline{C}\to\mathbb{C}^2/\mathbb{C}^2}$ being replaced by $T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)$ according to (2.4.12).

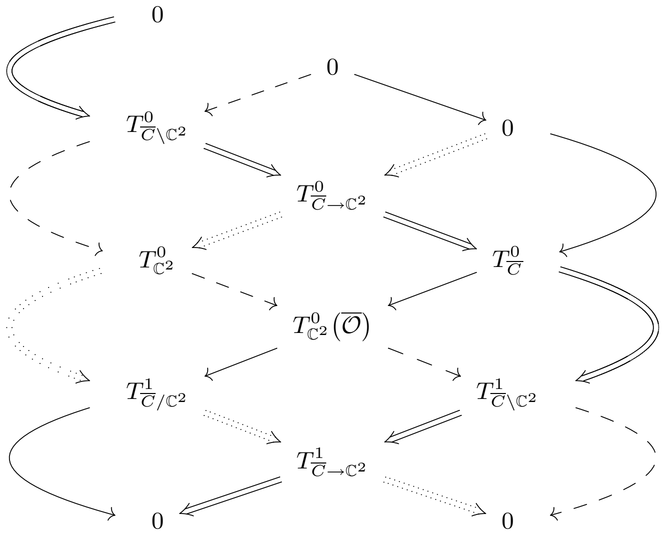


Fig. 2.13. The cotangent braid for the parametrization $\varphi:\overline{C}\to\mathbb{C}^2$.

The maps $\varphi^*: T^0_{\mathbb{C}^2} \dashrightarrow T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)$ and $\varphi': T^0_{\overline{C}} \longrightarrow T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)$ in the braid can be made explicit by using the isomorphisms in Lemma 2.28. Namely,

$$\varphi^*: \mathbb{C}\{x,y\}\frac{\partial}{\partial x}\oplus\mathbb{C}\{x,y\}\frac{\partial}{\partial y}\longrightarrow\bigoplus_{i=1}^r\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\oplus\bigoplus_{i=1}^r\mathbb{C}\{t_i\}\frac{\partial}{\partial y}$$

is componentwise the structure map

$$x\mapsto\big(x_1(t_1),\dots,x_r(t_r)\big)\,,\quad y\mapsto\big(y_1(t_1),\dots,y_r(t_r)\big)\,,$$

while $\varphi'=(\varphi'_1,\dots,\varphi'_r)$ is the tangent map

$$\varphi'_i:\mathbb{C}\{t_i\}\frac{\partial}{\partial t_i}\to\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\oplus\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\,,\qquad\frac{\partial}{\partial t_i}\mapsto\dot{x}_i(t_i)\frac{\partial}{\partial x}+\dot{y}_i(t_i)\frac{\partial}{\partial y}\,.$$

In particular, we have

$$\begin{aligned} \varphi^*(T^0_{\mathbb{C}^2}) &= \mathcal{O}\cdot\frac{\partial}{\partial x}\oplus\mathcal{O}\cdot\frac{\partial}{\partial y} \\ \varphi'(T^0_{\overline{C}}) &= \overline{\mathcal{O}}\cdot\dot{\varphi} = \overline{\mathcal{O}}\cdot\dot{\boldsymbol{x}}\,\frac{\partial}{\partial x}\oplus\overline{\mathcal{O}}\cdot\dot{\boldsymbol{y}}\,\frac{\partial}{\partial y} \end{aligned} \tag{2.4.13}$$

with

$$\dot{\varphi} = \varphi'\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\right) = \dot{\boldsymbol{x}}\cdot\frac{\partial}{\partial x}+\dot{\boldsymbol{y}}\cdot\frac{\partial}{\partial y} = \begin{pmatrix}\dot{x}_1\\ \vdots\\ \dot{x}_r\end{pmatrix}\cdot\frac{\partial}{\partial x}+\begin{pmatrix}\dot{y}_1\\ \vdots\\ \dot{y}_r\end{pmatrix}\cdot\frac{\partial}{\partial y}\,.$$

Using the results of Lemma 2.28 and the isomorphism (2.4.12), the braid for the normalization looks as displayed in Figure 2.14.

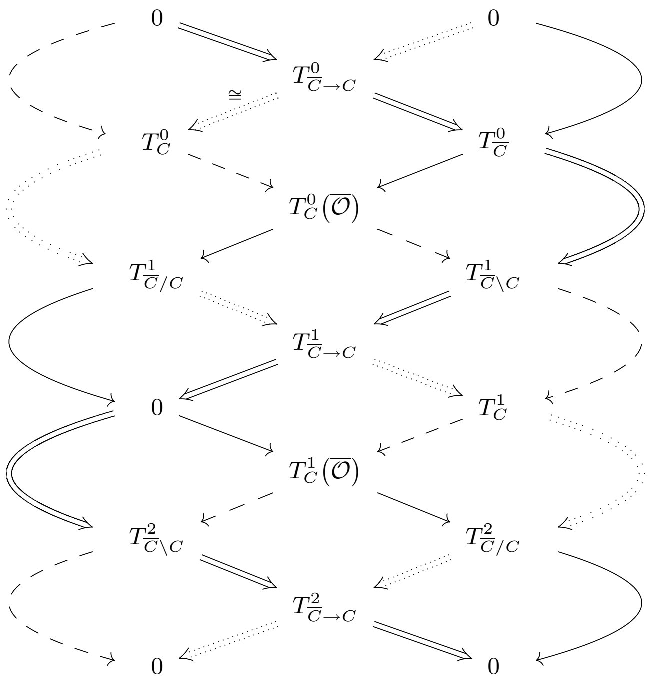


Fig. 2.14. The cotangent braid for the normalization $n:\overline{C}\to C$.

Since we have $T^0_C(M)\subset T^0_{\mathbb{C}^2}(M)$ for each $\mathcal{O}$-module M, we can give the following description of $n^*: T^0_C \dashrightarrow T^0_C(\overline{\mathcal{O}})$ and $n': T^0_{\overline{C}}\longrightarrow T^0_C(\overline{\mathcal{O}})$:

$$n^*:\mathcal{O}\frac{\partial}{\partial x}\oplus\mathcal{O}\frac{\partial}{\partial y}\supset T^0_C\longrightarrow T^1_{\overline{C}\backslash\overline{C}\to C/C}\cong T^0(\overline{\mathcal{O}})\subset\overline{\mathcal{O}}\frac{\partial}{\partial x}\oplus\overline{\mathcal{O}}\frac{\partial}{\partial y}$$

is given by $x \mapsto \big(x_1(t_1),\dots,x_r(t_r)\big)$, $y \mapsto \big(y_1(t_1),\dots,y_r(t_r)\big)$, and

$$n' : \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\}\frac{\partial}{\partial t_i} \cong T^0_{\overline{C}} \longrightarrow T^1_{\overline{C}\backslash \overline{C}\to C/C} \subset \bigoplus_{i=1}^{r}\mathbb{C}\{t_i\}\frac{\partial}{\partial x} \oplus \bigoplus_{i=1}^{r}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\,,$$

is given by $\frac{\partial}{\partial t_i} \mapsto \dot{x}_i(t_i)\frac{\partial}{\partial x} + \dot{y}_i(t_i)\frac{\partial}{\partial y}$.

Lemma 2.29. *With the notations introduced above, we have*

$$T^i_{\overline{C}\backslash C} \cong T^{i-1}_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,, \quad i \geq 0\,.$$

Proof. $T^i_{\overline{C}\backslash C}$ appears in the exact sequence of complex vector spaces

$$0 \dashrightarrow T^0_C \dashrightarrow T^0_C\big(\overline{\mathcal{O}}\big) \dashrightarrow T^1_{\overline{C}\backslash C} \dashrightarrow T^1_C \dashrightarrow T^1_C\big(\overline{\mathcal{O}}\big) \dashrightarrow T^2_{\overline{C}\backslash C} \dashrightarrow \cdots$$

of the cotangent braid for the normalization (see Figure 2.14). Moreover, by Appendix C.4, we have the long T^i_C-sequence induced by the exact sequence $0 \to \mathcal{O} \to \overline{\mathcal{O}} \to \overline{\mathcal{O}}/\mathcal{O} \to 0$ of $\mathcal{O}$-modules. Since $T^i_C(\mathcal{O}) = T^i_C$, we can replace $T^i_{\overline{C}\backslash C}$ by $T^{i-1}_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)$ in the above exact sequence, whence the result. □

The following proposition is the main result of this section. As usually, τ denotes the Tjurina number, δ the δ-invariant, mt the multiplicity, and r the number of branches of $(C,\mathbf{0})$.

Proposition 2.30. *Let $(C,\mathbf{0}) \overset{j}{\hookrightarrow} (\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve singularity, defined by $f \in \mathbb{C}\{x,y\}$. Let $n : \big(\overline{C},\overline{0}\big) \to (C,\mathbf{0})$ be the normalization, and let $\varphi := j \circ n$ be the parametrization of $(C,\mathbf{0})$. Then the following holds:*

(1) (i) $T^1_{\overline{C}\backslash\mathbb{C}^2} \cong \big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial x} \oplus \big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial y}$ is a complex vector space of dimension 2δ.
(ii) $T^2_{\overline{C}\backslash\mathbb{C}^2} = 0$.

(2) (i) $T^1_{\overline{C}/\mathbb{C}^2} \cong \Big(\overline{\mathcal{O}}\frac{\partial}{\partial x} \oplus \overline{\mathcal{O}}\frac{\partial}{\partial y}\Big) \Big/ \overline{\mathcal{O}}\Big(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\Big)$ is an $\overline{\mathcal{O}}$-module of rank one.
(ii) $T^2_{\overline{C}/\mathbb{C}^2} = 0$.

(3) (i) $T^1_{\overline{C}\to\mathbb{C}^2} \cong \Big(\big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial x} \oplus \big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial y}\Big) \Big/ \big(\overline{\mathcal{O}}/\mathcal{O}\big)\Big(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\Big)$ is a $\mathbb{C}$-vector space of dimension $2\delta - \dim_{\mathbb{C}}(T^0_{\overline{C}}/T^0_C) = \tau - \delta$.
(ii) $T^2_{\overline{C}\to\mathbb{C}^2} = 0$.

(4) (i) $T^1_C \cong \mathcal{O} \Big/ \Big(\mathcal{O}\frac{\partial f}{\partial x} + \mathcal{O}\frac{\partial f}{\partial y}\Big)$ is a $\mathbb{C}$-vector space of dimension τ.
(ii) $T^2_C = 0$.

(5) (i) $T^1_{\overline{C}\to C} \cong T^1_{\overline{C}\to\mathbb{C}^2}$ has $\mathbb{C}$-dimension $\tau - \delta$.
(ii) $T^2_{\overline{C}\to C} \cong \overline{\mathcal{O}}/\mathcal{O}$ has $\mathbb{C}$-dimension δ.

(6) (i) $T^1_{\overline{C}\setminus C} \cong T^1_{\overline{C}\setminus \mathbb{C}^2}$ *has* $\mathbb{C}$*-dimension* 2δ*.*

(ii) $T^2_{\overline{C}\setminus C} \cong \overline{\mathcal{O}}/\mathcal{O}$ *has* $\mathbb{C}$*-dimension* δ*.*

(7) (i) $T^0_C\big(\overline{\mathcal{O}}\big) \cong \overline{\mathcal{O}}\left(\dot{\overline{\boldsymbol{x}}}\frac{\partial}{\partial x} + \dot{\overline{\boldsymbol{y}}}\frac{\partial}{\partial y}\right)$ *is a free* $\overline{\mathcal{O}}$*-module of rank* 1*. Here,* $\dot{\overline{\boldsymbol{x}}} = \big(\dot{\overline{x}}_1, \ldots, \dot{\overline{x}}_r\big)$, $\dot{\overline{\boldsymbol{y}}} = \big(\dot{\overline{y}}_1, \ldots, \dot{\overline{y}}_r\big)$, *where*

$$\dot{\overline{x}}_i := \frac{\dot{x}_i}{\gcd(\dot{x}_i, \dot{y}_i)} = t_i^{-m_i+1}\dot{x}_i(t_i),\ \ \dot{\overline{y}}_i := \frac{\dot{y}_i}{\gcd(\dot{x}_i, \dot{y}_i)} = t_i^{-m_i+1}\dot{y}_i(t_i),$$

where $m_i = \min\{\operatorname{ord}_{t_i} x_i(t_i), \operatorname{ord}_{t_i} y_i(t_i)\}$.

(ii) $T^1_C\big(\overline{\mathcal{O}}\big) \cong T^2_{\overline{C}\setminus C}$ *is of* $\mathbb{C}$*-dimension* δ*.*

(8) (i) $T^1_{\overline{C}/C} \cong \overline{\mathcal{O}}\left(\dot{\overline{\boldsymbol{x}}}\frac{\partial}{\partial x} + \dot{\overline{\boldsymbol{y}}}\frac{\partial}{\partial y}\right)\Big/\overline{\mathcal{O}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right)$ *is a* $\mathbb{C}$*-vector space of dimension* $\mathrm{mt} - r$ *.*

(ii) $T^2_{\overline{C}/C}$ *has* $\mathbb{C}$*-dimension* $2\delta + \mathrm{mt} - r$*.*

Proof. (1) (i) From the exact sequence $\dashrightarrow$ in the cotangent braid for the parametrization, we get $T^1_{\overline{C}\setminus\mathbb{C}^2} = \operatorname{Coker}\big(\varphi^* : T^0_{\mathbb{C}^2} \to T^0_{\mathbb{C}^2}(\overline{\mathcal{O}})\big)$, and then the formula follows from the explicit description of φ^*. (ii) is also a consequence of the same exact sequence, noting that $T^1_{\mathbb{C}^2}(\varphi_*\mathcal{O}_{\overline{C}}) = 0 = T^2_{\mathbb{C}^2}$, since $(\mathbb{C}^2, \mathbf{0})$ is smooth.

(2) (i) and (ii) follow in the same way from the exact sequence $\longrightarrow$ in the cotangent braid for the parametrization and the explicit description of φ'.

For the next statements, consider the exact sequences $\Longrightarrow$ in the braids for the normalization and for the parametrization. From these we obtain the rows in the following commutative diagram with exact rows and columns (with $I = \mathcal{O}f$)

$$\begin{tikzcd}
0 \arrow[r] & T^0_{\overline{C}\to C} \arrow[r] & T^0_{\overline{C}} \arrow[r] & T^1_{\overline{C}\setminus C} \arrow[r] \arrow[d, hook] & T^1_{\overline{C}\to C} \arrow[r] \arrow[d, hook] & 0 \\
 & T^0_{\overline{C}\to\mathbb{C}^2} \arrow[r] \arrow[u, two heads, "\alpha"'] & T^0_{\overline{C}} \arrow[r] \arrow[u, equal] & T^1_{\overline{C}\setminus\mathbb{C}^2} \arrow[r] \arrow[d, "d^*"] & T^1_{\overline{C}\to\mathbb{C}^2} \arrow[r] & 0 \\
 & & & \operatorname{Hom}_{\mathcal{O}}\big(I/I^2, \overline{\mathcal{O}}/\mathcal{O}\big) \arrow[d, two heads] & & \\
 & & & T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,. & &
\end{tikzcd} \tag{2.4.14}$$

To define the map α, note that $(\xi, \rho) \in T^0_{\overline{C}\to\mathbb{C}^2}$ satisfies $\xi \circ \varphi^* = \varphi^* \circ \rho$. Since $\ker \varphi^\sharp = I$, we get $\rho(I) \subset I$. Hence, ρ induces a derivation η of $\mathcal{O}_C$, and we define $\alpha(\xi, \rho) = (\xi, \eta)$. As $\varphi^* = n^* \circ j^\sharp$, we have $\xi \circ n^\sharp = n^\sharp \circ \eta$, and α is well-defined.

To see that the map α is surjective, apply the functor $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\_\_,\mathcal{O})$ to the surjection $\Omega^1_{\mathbb{C}^2,\mathbf{0}} \twoheadrightarrow \Omega^1_{C,\mathbf{0}}$, and deduce that $\mathrm{Hom}_{\mathcal{O}}(\Omega^1_{C,\mathbf{0}},\mathcal{O})$ injects into $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\mathcal{O})$. On the other hand, applying $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\_\_)$ to the exact sequence $0 \to I \to \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \to \mathcal{O} \to 0$ gives rise to the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},I) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}) \to 0$$

since $\Omega^1_{\mathbb{C}^2,\mathbf{0}}$ is free. Thus, each $\eta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O},\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\Omega^1_{C,\mathbf{0}},\mathcal{O})$ lifts to an element $\rho \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}})$, which shows that α is surjective.

The third column results from applying $\mathrm{Hom}_{\mathcal{O}}(\_\_,\overline{\mathcal{O}}/\mathcal{O})$ to the defining exact sequence of $\Omega^1_{C,\mathbf{0}}$,

$$0 \longrightarrow I/I^2 \xrightarrow{\ d\ } \Omega^1_{\mathbb{C}^2,\mathbf{0}} \otimes \mathcal{O} \longrightarrow \Omega^1_{C,\mathbf{0}} \longrightarrow 0\,, \tag{2.4.15}$$

with d induced by the exterior derivation. Note that (see Lemma 2.29)

$$\begin{aligned} T^1_{\overline{C}\setminus C} &\cong T^0_C\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{C,\mathbf{0}},\overline{\mathcal{O}}/\mathcal{O}\big)\,,\\ T^1_{\overline{C}\setminus \mathbb{C}^2} &\cong T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2,\mathbf{0}}\otimes\mathcal{O},\overline{\mathcal{O}}/\mathcal{O}\big)\,, \end{aligned}$$

and (see Proposition 1.25 and Generalization 1.27)

$$\begin{aligned} T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big) &\cong \mathrm{Coker}\,\big(d^*\colon \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2,\mathbf{0}}\otimes\mathcal{O},\overline{\mathcal{O}}/\mathcal{O}\big) \to \mathrm{Hom}_{\mathcal{O}}\big(I/I^2,\overline{\mathcal{O}}/\mathcal{O}\big)\big)\,,\\ T^1_{C/\mathbb{C}^2}\big(\overline{\mathcal{O}}/\mathcal{O}\big) &\cong \mathrm{Hom}_{\mathcal{O}}\big(I/I^2,\overline{\mathcal{O}}/\mathcal{O}\big)\,. \end{aligned}$$

The last column in (2.4.14) is induced by the previous one. The commutativity is obvious.

(3) Consider the cotangent braid for the parametrization to conclude that

$$T^1_{\overline{C}\to\mathbb{C}^2} = \mathrm{Coker}\big(T^0_{\overline{C}} \to T^1_{\overline{C}\setminus\mathbb{C}^2}\big) = \mathrm{Coker}\big(T^0_{\overline{C}} \xrightarrow{\varphi'} T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)/\varphi^*(T^0_{\mathbb{C}^2})\big)\,,$$

and then use (2.4.13) to get the first formula for $T^1_{\overline{C}\to\mathbb{C}^2}$.

To compute its dimension, we use the diagram (2.4.14), statement (1) (i), and that $T^0_{\overline{C}\to\mathbb{C}^2} \cong T^0_{\overline{C}}$ by Lemma 2.28:

$$\begin{aligned} \dim_{\mathbb{C}} T^1_{\overline{C}\to\mathbb{C}^2} &= \dim_{\mathbb{C}} T^1_{\overline{C}\setminus\mathbb{C}^2} - \dim_{\mathbb{C}} \mathrm{Im}(T^0_{\overline{C}} \to T^1_{\overline{C}\setminus\mathbb{C}^2})\\ &= 2\delta - \dim_{\mathbb{C}} \mathrm{Im}(T^0_{\overline{C}} \to T^1_{\overline{C}\setminus\mathbb{C}^2}) \;=\; 2\delta - \dim_{\mathbb{C}}(T^0_{\overline{C}}/T^0_C)\,. \end{aligned}$$

A formula of Deligne (for the dimension of smoothing components for not necessarily plane curve singularities, see [Del1, GrL]) gives, in our situation,

$$\dim_{\mathbb{C}}(T^0_{\overline{C}}/T^0_C) = 3\delta - \tau\,.$$

(For an independent proof, see Lemma 2.32.) This proves (3) (i). The vanishing of $T^2_{\overline{C}\to\mathbb{C}^2}$ follows from the cotangent braid for the parametrization.

(4) follows from Propositions 1.25 and 1.29 (see also Corollary 1.17).

(5) By Proposition 2.23, we know that $\underline{\mathcal{D}ef}_{\overline{C}\to C}(T) \cong \underline{\mathcal{D}ef}_{\overline{C}\to\mathbb{C}^2}(T)$ for each complex germ T, in particular, for $T = T_\varepsilon$. Hence, $T^1_{\overline{C}\to C} \cong T^1_{\overline{C}\to\mathbb{C}^2}$, which proves (i).

To show (ii), we notice that the same argument proves $T^1_{\overline{C}\backslash C} \cong T^1_{\overline{C}\backslash\mathbb{C}^2}$. From the commutative diagram (2.4.14), it follows that d^* is the zero map and

$$\mathrm{Hom}_{\mathcal{O}}\big(I/I^2, \overline{\mathcal{O}}/\mathcal{O}\big) \xrightarrow{\cong} T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,,$$

and, since $I/I^2 \cong \mathcal{O}f$, we get

$$T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \overline{\mathcal{O}}/\mathcal{O}\,,$$

which has $\mathbb{C}$-dimension δ. Furthermore, by Lemma 2.29, and by the braid for the normalization, we get

$$T^2_{\overline{C}\to C} \cong T^2_{\overline{C}\backslash C} \cong T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,,$$

whence (ii). [14]

(6) We proved in (5) that $T^1_{\overline{C}\backslash C} \cong T^1_{\overline{C}\backslash\mathbb{C}^2}$, the latter being isomorphic to $\mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2,\mathbf{0}} \otimes \mathcal{O}, \overline{\mathcal{O}}/\mathcal{O}\big) \cong \overline{\mathcal{O}}/\mathcal{O} \oplus \overline{\mathcal{O}}/\mathcal{O}$, which shows (i). (ii) was already proved in (5).

(7) Applying $\mathrm{Hom}_{\mathcal{O}}\big(\_\_, \overline{\mathcal{O}}\big)$ to the sequence (2.4.15), we deduce that $T^0_C\big(\overline{\mathcal{O}}\big)$ is a torsion free, hence free, $\overline{\mathcal{O}}$-module of rank 1, which equals the kernel of the map

$$\overline{\mathcal{O}}\frac{\partial}{\partial x} \oplus \overline{\mathcal{O}}\frac{\partial}{\partial y} \cong \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}, \overline{\mathcal{O}}\big) \xrightarrow{d^*} \mathrm{Hom}_{\mathcal{O}}\big(I/I^2, \overline{\mathcal{O}}\big) \cong \overline{\mathcal{O}}$$

given by the Jacobian matrix $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. By the chain rule,

$$\frac{\partial}{\partial x}(x_i, y_i)\dot{x}_i + \frac{\partial}{\partial y}(x_i, y_i)\dot{y}_i \;=\; 0\,.$$

Hence, $\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}$ is contained in $T^0_C\big(\overline{\mathcal{O}}\big)$, and it is a non-zerodivisor (in characteristic 0). Therefore, $T^0_C\big(\overline{\mathcal{O}}\big)$ is generated by $\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}$, which proves (i). (ii) follows from the braid for the normalization and from (8) (ii).

(8) (ii) follows from taking the alternating sum of dimensions in the exact sequence $\Longrightarrow$ of the cotangent braid for the normalization.

[14] The fact that the homomorphism $d^* : T^1_{\overline{C}\backslash\mathbb{C}^2} \to \mathrm{Hom}_{\mathcal{O}}(I/I^2, \overline{\mathcal{O}}/\mathcal{O})$ in the diagram (2.4.14) is the zero map is equivalent to the fact that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ annihilate $\overline{\mathcal{O}}/\mathcal{O}$ which is proved here by using deformation theory. This fact can, of course, be proved directly and gives then another proof of (5).

(i) From the cotangent braid for the normalization and from (7) (ii), we have

$$T^1_{\overline{C}/C} \cong \operatorname{Coker}\left(\overline{\mathcal{O}} \cong T^0_{\overline{C}} \xrightarrow{n'} T^0_C(\overline{\mathcal{O}}) \cong \overline{\mathcal{O}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\right).$$

The statement follows from the description of n', noting that, in characteristic 0, $\operatorname{ord}_{t_i}\big(\gcd(\dot{x}_i, \dot{y}_i)\big) = m_i - 1$, where m_i is the multiplicity of the i-th branch. Hence,

$$T^1_{\overline{C}/C} \cong \bigoplus_{i=1}^r \mathbb{C}\{t_i\}/\big\langle \gcd(\dot{x}_i, \dot{y}_i)\big\rangle,$$

which is of $\mathbb{C}$-dimension $\operatorname{mt} - r$. □

The proof of Proposition 2.30 (5) and the footnote on page 315 yield the following lemma which is of independent interest:

Lemma 2.31. *For a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ defined by $f \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$, the Jacobian ideal $j(f) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle \subset \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$ satisfies*

$$j(f) \cdot \mathcal{O}_{\overline{C},\overline{0}} \subset \mathcal{O}_{C,\mathbf{0}}\,, \text{ that is, } j(f) \cdot \mathcal{O}_{C,\mathbf{0}} \subset I^{cd}(C, \mathbf{0}),$$

where $I^{cd}(C, \mathbf{0}) = \operatorname{Ann}_{\mathcal{O}_{C,\mathbf{0}}}\big(\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\mathbf{0}}\big)$ is the conductor ideal.

Next, we give an independent proof of Deligne's formula, used in the proof of Proposition 2.30, for plane curve singularities:

Lemma 2.32. *For a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$, we have*

$$\dim_{\mathbb{C}}\big(T^0_{\overline{C}}/T^0_C\big) = 3\delta(C, \mathbf{0}) - \tau(C, \mathbf{0}).$$

Proof. We use the notations of Proposition 2.30. As each derivation of $\mathcal{O}$ lifts uniquely to $\overline{\mathcal{O}}$, the modules T^0_C and $T^0_{\overline{C}\to C}$ have the same image in $T^0_{\overline{C}}$. The latter image consists of derivations $\xi = \sum_{i=1}^r h_i \frac{\partial}{\partial t_i} \in \operatorname{Der}_{\mathbb{C}}(\overline{\mathcal{O}}, \overline{\mathcal{O}})$ such that there exists an $\eta \in \operatorname{Der}_{\mathbb{C}}(\mathcal{O}, \mathcal{O})$ satisfying $\xi \circ n^* = n^* \circ \eta$.

η is of the form $g_1 \frac{\partial}{\partial x} + g_2 \frac{\partial}{\partial y}$, $g_1, g_2 \in \mathcal{O}$, such that $g_1 \frac{\partial f}{\partial x} + g_2 \frac{\partial f}{\partial y} = 0$. Evaluating $\xi \circ n^*$ and $n^* \circ \eta$ at x and at y, we obtain

$$h_i \cdot \dot{x}_i = g_1(x_i, y_i)\,, \quad h_i \cdot \dot{y}_i = g_2(x_i, y_i)\,, \quad i = 1, \ldots, r\,.$$

The condition $h_i \left(\dot{x}_i \frac{\partial f}{\partial x} + \dot{y}_i \frac{\partial f}{\partial y}\right) = 0$ is fulfilled as $\dot{x}_i \frac{\partial f}{\partial x} + \dot{y}_i \frac{\partial f}{\partial y} = 0$ by the chain rule. Hence, identifying $T^0_{\overline{C}}$ with $\overline{\mathcal{O}}$, we get

$$\operatorname{Im}\big(T^0_{\overline{C}\to C} \to T^0_{\overline{C}}\big) \cong \big\{h \in \overline{\mathcal{O}} \,\big|\, h \cdot \dot{\boldsymbol{x}} \in \mathcal{O},\, h \cdot \dot{\boldsymbol{y}} \in \mathcal{O}\big\}.$$

Now, we have to use local duality. Let ω denote the *dualizing module* (or *canonical module* of $\mathcal{O}$), see [HeK1]. The dualizing module may be realized as a *fractional ideal*, that is, an $\mathcal{O}$-ideal in $\operatorname{Quot}\big(\overline{\mathcal{O}}\big)$, such that

$$I \mapsto \mathrm{Hom}_{\mathcal{O}}(I,\omega) = \omega : I = \big\{h \in \mathrm{Quot}\,\big(\overline{\mathcal{O}}\big) \;\big|\; hI \subset \omega\big\}$$

defines an inclusion preserving functor on the set of fractional ideals satisfying, in particular,

$$\omega : \omega = \mathcal{O}\,, \quad \omega : (\omega : I) = I\,, \quad \dim_{\mathbb{C}} I/J = \dim_{\mathbb{C}}(\omega : J)/(\omega : I)$$

for each fractional ideals I, J. Since $(C, \mathbf{0})$ is a plane curve singularity, hence Gorenstein, we have $\omega \cong \mathcal{O}$.

An explicit description of the dualizing module ω can be given by means of meromorphic differential forms. Let $\Omega^1_{\overline{C},\overline{0}}(\overline{0})$ denote the germs of meromorphic 1-forms on $\big(\overline{C}, \overline{0}\big)$ with poles only at $\overline{0}$. Set

$$\omega^R_{C,\mathbf{0}} := n_* \left\{ \alpha \in \Omega^1_{\overline{C},\overline{0}}(\overline{0}) \;\middle|\; \sum_{i=1}^{r} \mathrm{res}_{\overline{0}_i}(f\alpha) = 0 \text{ for all } f \in \mathcal{O} \right\}\,,$$

which are Rosenlicht's regular differential forms (see [Ser3, IV.9]). We have canonical mappings

$$\mathcal{O} \xrightarrow{\;d\;} \Omega^1_{C,\mathbf{0}} \longrightarrow n_*\Omega^1_{\overline{C},\overline{0}} \hookrightarrow \omega^R_{C,\mathbf{0}}$$

with d the exterior derivation. Exterior multiplikation with df provides (for plane curve singularities) an isomorphism

$$\wedge\, df : \omega^R_{C,\mathbf{0}} \xrightarrow{\;\cong\;} \mathcal{O}dx \wedge dy$$

(see [Ser3, Ch. II]). Let $\Omega_{C,\mathbf{0}}$ ($\cong \Omega^1_{C,\mathbf{0}}/\text{torsion}$) denote the image of $\Omega^1_{C,\mathbf{0}}$ in $\omega^R_{C,\mathbf{0}}$. Then

$$\wedge\, df : \Omega_{C,\mathbf{0}} \xrightarrow{\;\cong\;} \left\langle \frac{\partial f}{\partial x} dx \wedge dy, \frac{\partial f}{\partial y} dx \wedge dy \right\rangle \subset \mathcal{O}dx \wedge dy\,,$$

and, hence,

$$\dim_{\mathbb{C}} \omega^R_{C,\mathbf{0}}/\Omega_{C,\mathbf{0}} = \dim_{\mathbb{C}} \mathcal{O} \Big/ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \tau\,.$$

All this can be understood in terms of fractional ideals. We can identify the meromorphic differential forms on $\big(\overline{C}, \overline{0}\big)$ with $\mathrm{Quot}\big(\overline{\mathcal{O}}\big)$ by mapping $g(t_i)dt_i \mapsto g(t_i)t_i$. Under this identification, we get ideals in $\mathrm{Quot}\big(\overline{\mathcal{O}}\big)$ corresponding to $\omega^R_{C,\mathbf{0}}$, to $\Omega^1_{\overline{C},\overline{0}}$, respectively to $\Omega_{C,\mathbf{0}}$. We denote these fractional ideals by ω, $\overline{\Omega}$, respectively Ω. Note that, as $\Omega_{C,\mathbf{0}}$ is generated by dx and dy, we obtain $\Omega = \langle \dot{\boldsymbol{x}}, \dot{\boldsymbol{y}} \rangle \mathcal{O} \subset \overline{\mathcal{O}}$, and, hence,

$$\mathrm{Im}\,\big(T^0_{\overline{C}\to C} \to T^0_{\overline{C}}\big) = \big\{h \in \mathrm{Quot}\big(\overline{\mathcal{O}}\big) \;\big|\; h\Omega \subset \mathcal{O}\big\} = \mathcal{O} : \Omega\,.$$

To compute the dimension of $T^0_{\overline{C}}/T^0_C = \overline{\mathcal{O}}/(\mathcal{O} : \Omega)$, we use that

$$\dim_{\mathbb{C}}\bigl(\overline{\mathcal{O}}/(\mathcal{O}:\Omega)\bigr) = \dim_{\mathbb{C}}\bigl(\mathcal{O}:(\mathcal{O}:\Omega)/(\mathcal{O}:\overline{\mathcal{O}})\bigr) = \dim_{\mathbb{C}}(\Omega/I^{cd})\,,$$

where $I^{cd} = \mathcal{O}:\overline{\mathcal{O}}$ is the conductor ideal. Furthermore, we use that the residue map $\mathrm{res}: \overline{\mathcal{O}} \times \omega \to \mathbb{C}$, $(h,\alpha) \mapsto \sum_{i=1}^{r} \mathrm{res}_{\overline{\mathcal{O}}_i}(h\alpha)$ induces a non-degenerate pairing between $\overline{\mathcal{O}}/\mathcal{O}$ and $\omega/\overline{\mathcal{O}}$. In particular,

$$\dim_{\mathbb{C}}\bigl(\omega/\overline{\mathcal{O}}\bigr) = \dim_{\mathbb{C}}\bigl(\overline{\mathcal{O}}/\mathcal{O}\bigr) = \delta\,.$$

We have the inclusions $I^{cd} \subset \Omega \subset \overline{\mathcal{O}} \subset \omega$. As $(C,\mathbf{0})$ is a plane curve singularity, $\dim_{\mathbb{C}}\bigl(\overline{\mathcal{O}}/I^{cd}\bigr) = 2\delta$ (see I.(3.4.12)), and we get

$$\begin{aligned}\dim_{\mathbb{C}}\bigl(\Omega/I^{cd}\bigr) &= \dim_{\mathbb{C}}\bigl(\overline{\mathcal{O}}/I^{cd}\bigr) + \dim_{\mathbb{C}}\bigl(\omega/\overline{\mathcal{O}}\bigr) - \dim_{\mathbb{C}}\bigl(\omega/\Omega\bigr)\\ &= 2\delta + \delta - \tau \;=\; 3\delta - \tau\,,\end{aligned}$$

proving the statement of the lemma. □

We continue by describing the vector space homomorphisms

$$T^1_{\overline{C}\to\mathbb{C}^2} \xrightarrow{\alpha'} T^1_C\,, \qquad T^{1,sec}_{\overline{C}\to\mathbb{C}^2} \xrightarrow{\beta'} T^{1,sec}_C$$

(see page 302) in explicit terms, see (2.4.16) on page 319.

Let $(C,\mathbf{0})$ be given by the local equation $f \in \mathbb{C}\{x,y\}$ with irreducible decomposition $f = f_1 \cdot \ldots \cdot f_r$. Let $(C_i,\mathbf{0})$ be the branch of $(C,\mathbf{0})$ defined by f_i, $i = 1,\ldots,r$. Further, let $x_i(t_i), y_i(t_i) \in \mathbb{C}\{t_i\}$ define a parametrization $\varphi_i : (\mathbb{C},\mathbf{0}) \to (C_i,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ of $(C_i,\mathbf{0})$. In addition to the notations introduced before in this section, we set

$$\mathfrak{m} := \mathfrak{m}_{C,\mathbf{0}}\,, \quad \overline{\mathfrak{m}} = \mathfrak{m}_{\overline{C},\bar{0}} = \bigoplus_{i=1}^{r} t_i\mathbb{C}\{t_i\}\,.$$

Every deformation of the parametrization $\varphi = (\varphi_1,\ldots,\varphi_r)$ of $(C,\mathbf{0})$ is given by a deformation of the φ_i. Over T_ε, it is defined by

$$\begin{aligned}X_i(t_i,\varepsilon) &= x_i(t_i) + \varepsilon a_i(t_i)\,,\\ Y_i(t_i,\varepsilon) &= y_i(t_i) + \varepsilon b_i(t_i)\,,\end{aligned}$$

with

$$\boldsymbol{a} = \begin{pmatrix} a_1\\ \vdots\\ a_r\end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} b_1\\ \vdots\\ b_r\end{pmatrix} \in \overline{\mathcal{O}} = \mathcal{O}_{\overline{C},\bar{0}} = \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\}\,.$$

If we consider deformations with (trivial) sections, we assume that $\boldsymbol{a},\boldsymbol{b} \in \overline{\mathfrak{m}}$. As each section can be trivialized (Proposition 2.2), this no loss of generality.

Lemma 2.33. *Let $x_i(t_i) + \varepsilon a_i(t_i)$, $y_i(t_i) + \varepsilon b_i(t_i)$, $i = 1,\ldots,r$, define a deformation of the parametrization of $(C,\mathbf{0})$ over T_ε. Then the induced deformation of the equation is given by*

$$f - \varepsilon(g + hf)$$

for some $h \in \mathbb{C}\{x,y\}$ *and for* $g \in \mathbb{C}\{x,y\}$ *a representative of*

$$\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y} \in \mathcal{O} = \mathbb{C}\{x,y\}/\langle f\rangle\,.$$

Moreover, $g \in \langle x,y\rangle\mathbb{C}\{x,y\}$ *if* $\boldsymbol{a},\boldsymbol{b} \in \overline{\mathfrak{m}}$.

Here, $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$ has to be interpreted as an element of $\overline{\mathcal{O}}$ via

$$\boldsymbol{a}\frac{\partial f}{\partial x} = \begin{pmatrix} a_1(t_1)\frac{\partial f}{\partial x}\big(x_1(t_1), y_1(t_1)\big) \\ \vdots \\ a_r(t_r)\frac{\partial f}{\partial x}\big(x_r(t_r), y_r(t_r)\big) \end{pmatrix}$$

and similarly for $\boldsymbol{b}\frac{\partial f}{\partial y}$. By Lemma 2.31, we know that $\frac{\partial f}{\partial x}\cdot\overline{\mathcal{O}} \subset \mathcal{O}$ and $\frac{\partial f}{\partial x}\cdot\overline{\mathfrak{m}} \subset \mathfrak{m}$ and that the analogous statements hold for $\frac{\partial f}{\partial y}$. Hence, a representative g can be chosen as in Lemma 2.33.

If we write $f = f_i \cdot \widehat{f_i}$ then

$$a_i(t_i)\frac{\partial f}{\partial x}\big(x_i(t_i), y_i(t_i)\big) = a_i(t_i)\frac{\partial f_i}{\partial x}\big(x_i(t_i), y_i(t_i)\big)\cdot \widehat{f_i}\big(x_i(t_i), y_i(t_i)\big)$$

(since $f_i\big(x_i(t_i), y_i(t_i)\big) = 0$) and similarly for $\frac{\partial f_i}{\partial y}$.

In Proposition 2.30, we computed $T^1_{\overline{C}\to\mathbb{C}^2}$ and $T^{1,sec}_{\overline{C}\to\mathbb{C}^2}$, and in Proposition 1.25 we showed that $T^1_{C/\mathbb{C}^2} \cong \mathrm{Hom}_{\mathbb{C}\{x,y\}}(\langle f\rangle, \mathcal{O}) \cong \mathcal{O}$. The same argument yields $T^{1,sec}_{C/\mathbb{C}^2} \cong \mathfrak{m}$.

It follows that the homomorphism α', resp. β', is given by the class mod $\mathcal{O}\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$, resp. mod $\mathfrak{m}\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$, of

$$\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \longmapsto \boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}\,. \tag{2.4.16}$$

Proof of Lemma 2.33. Let $F_i = f_i + \varepsilon g_i$ define the deformation of $(C_i, \mathbf{0})$ induced by $x_i(t_i) + \varepsilon a_i(t_i)$, $y_i(t_i) + \varepsilon b_i(t_i)$. Then

$$\begin{aligned} 0 &= F_i(x_i + \varepsilon a_i,\, y_i + \varepsilon b_i) \\ &= F_i(x_i, y_i) + \varepsilon\left(a_i\frac{\partial F_i}{\partial x}(x_i, y_i) + b_i\frac{\partial F_i}{\partial y}(x_i, y_i)\right) \\ &= \varepsilon g_i(x_i, y_i) + \varepsilon\cdot\left(a_i\frac{\partial f_i}{\partial x}(x_i, y_i) + b_i\frac{\partial f_i}{\partial y}(x_i, y_i)\right)\,. \end{aligned}$$

It follows that the right-hand side vanishes on the branch $(C_i, \mathbf{0})$. Hence, we get, for some $h_i \in \mathbb{C}\{x,y\}$,

$$-g_i = k_i + h_i f_i\,,$$

where $k_i \in \mathbb{C}\{x,y\}$ is a representative of

$$a_i \frac{\partial f_i}{\partial x} + b_i \frac{\partial f_i}{\partial y} \in \mathcal{O}_{C_i,\mathbf{0}} = \mathbb{C}\{x,y\}/\langle f_i \rangle\,.$$

This shows already the claim in the unibranch case. For the case of several branches, the deformation of $(C,\mathbf{0})$ is given by

$$F = F_1 \cdot \ldots \cdot F_r = f_1 \cdot \ldots \cdot f_r + \varepsilon \cdot \sum_{i=1}^{r} g_i \widehat{f}_i\,.$$

Consider the image of $g_i\widehat{f}_i$ in $\bigoplus_{j=1}^r \mathbb{C}\{t_j\}$. Since $\widehat{f}_i\big(x_j(t_j),y_j(t_j)\big) = 0$ for $j \neq i$, only the i-th component is non-zero and we get

$$\begin{aligned} g_i\widehat{f}_i\big(x_i(t_i),y_i(t_i)\big) = -a_i(t_i)&\frac{\partial f_i}{\partial x}\big(x_i(t_i),y_i(t_i)\big) \cdot \widehat{f}_i\big(x_i(t_i),y_i(t_i)\big) \\ &+b_i(t_i)\frac{\partial f_i}{\partial y}\big(x_i(t_i),y_i(t_i)\big) \cdot \widehat{f}_i\big(x_i(t_i),y_i(t_i)\big) \end{aligned}$$

which is the i-th component of $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$. $\square$

We close this section by computing T^1 for deformations with section. In addition to the above short hand notations, we introduce

$$J := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \mathcal{O}\,,$$

where $\mathcal{O} = \mathbb{C}\{x,y\}/\langle f \rangle$.

Proposition 2.34. *(1) We have the following isomorphisms of $\mathcal{O}$-modules:*

(i) $T^{1,sec}_{\overline{C}\to C} \cong \left((\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial x} \oplus (\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial y}\right) \Big/ (\overline{\mathfrak{m}}/\mathfrak{m}) \left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right)$,

(ii) $T^{1,sec}_{C} \cong \mathfrak{m}/\mathfrak{m}J$,

(iii) $T^{1,sec}_{\overline{C}/C} \cong \overline{\mathfrak{m}}\left(\dot{\overline{\boldsymbol{x}}}\frac{\partial}{\partial x} + \dot{\overline{\boldsymbol{y}}}\frac{\partial}{\partial y}\right) \Big/ \overline{\mathfrak{m}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right)$, *where*

$$(\dot{\overline{\boldsymbol{x}}}, \dot{\overline{\boldsymbol{y}}}) = t^{-\boldsymbol{m}+\boldsymbol{1}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{y}}))\,, \quad t^{-\boldsymbol{m}+\boldsymbol{1}} = (t_1^{-m_1+1}, \ldots, t_r^{-m_r+1})\,,$$

with $m_i = \min\{\mathrm{ord}_{t_i} x_i(t_i), \mathrm{ord}_{t_i} y_i(t_i)\}$.

(2) There are exact sequences of $\mathcal{O}$-modules

$$0 \to T^{1,sec}_{\overline{C}/C} \to T^{1,sec}_{\overline{C}\to C} \to T^{1,sec}_{C} \to \mathfrak{m}/\overline{\mathfrak{m}}J \to 0\,,$$
$$0 \to T^{1}_{\overline{C}/C} \to T^{1}_{\overline{C}\to C} \to T^{1}_{C} \to \mathcal{O}/\overline{\mathcal{O}}J \to 0\,.$$

With respect to the isomorphisms in (1), the map $T^{1,sec}_{\overline{C}\to C} \to T^{1,sec}_{C}$ maps the class of $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in (\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial x} \oplus (\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial y}$ to the class of $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$ mod $\mathfrak{m}J$ and similar for the second sequence.

(3) *(i)* $\dim_{\mathbb{C}} T^{1,sec}_{\overline{C}\to C} = \dim_{\mathbb{C}} T^1_{\overline{C}\to C} + \dim_{\mathbb{C}} J/\mathfrak{m}J - r$,
(ii) $\dim_{\mathbb{C}} T^{1,sec}_{C} = \dim_{\mathbb{C}} T^1_{C} + \dim_{\mathbb{C}} J/\mathfrak{m}J - 1$,
(iii) $\dim_{\mathbb{C}} T^{1,sec}_{\overline{C}/C} = \dim_{\mathbb{C}} T^1_{\overline{C}/C} = \mathrm{mt} - r$.
Moreover, if $(C,\mathbf{0})$ *is not smooth, then* $\dim_{\mathbb{C}} T^{1,sec}_{\overline{C}\to C} = \tau - \delta - r + 2$ *and* $\dim_{\mathbb{C}} T^{1,sec}_{C} = \tau + 1$.

Proof. (1) Since $T^{1,sec}_{\overline{C}\to C} \cong T^{1,sec}_{\overline{C}\to \mathbb{C}^2}$ (Proposition 2.23), the first isomorphism follows from Proposition 2.27. The proof of Proposition 1.25 shows that $T^{1,sec}_{C} \cong \mathfrak{m}/\mathfrak{m}J$. The third isomorphism follows in the same way as the isomorphism in Proposition 2.30 (8)(i) (or from the exact sequence in statement (2)).

(2) The exactness at the first three places is given by the exact sequence $\dashrightarrow$ in the braid of Figure 2.14 on page 311 (for deformations with, resp. without, section). The statement about the map $T^{1,sec}_{\overline{C}\to C} \to T^{1,sec}_{C}$ was proved in Lemma 2.33, the cokernel being obviously $\mathfrak{m}/\overline{\mathfrak{m}}J$. The same argument works for $T^1_{\overline{C}\to C} \to T^1_{C}$.

(3) The formula for the dimension of $T^{1,sec}_{\overline{C}/C}$ follows from Proposition 2.30 and using that multiplication with $(t_1,\dots,t_r) \in \overline{\mathfrak{m}}$ induces an isomorphism $T^1_{\overline{C}/C} \cong T^{1,sec}_{\overline{C}/C}$. Since $T^1_C \cong \mathcal{O}/J$, the dimension formula for $T^{1,sec}_{C}$ follows from the inclusions $\mathfrak{m}J \subset J \subset \mathfrak{m} \subset \mathcal{O}$ (for a singular germ $(C,\mathbf{0})$).

To prove the formula in (i), we use the exact sequence in (2). Using that $\dim_{\mathbb{C}} T^1_{\overline{C}\to C} = \tau - \delta$ by Proposition 2.30 and using the exact sequence for deformations without sections, we get $\dim_{\mathbb{C}} \mathcal{O}/\overline{\mathcal{O}}J = \delta + \mathrm{mt} - r$ and, hence, $\dim_{\mathbb{C}} \mathfrak{m}/\overline{\mathfrak{m}}J = \delta + \mathrm{mt} - 1$. Taking into account the dimension formulas for $T^{1,sec}_{\overline{C}/C}$ and for $T^{1,sec}_{C}$, we obtain the formula for $T^{1,sec}_{\overline{C}\to C}$.

To show that $\dim_{\mathbb{C}} J/\mathfrak{m}J = 2$ if $(C,\mathbf{0})$ is singular, we assume to the contrary that $\dim_{\mathbb{C}} J/\mathfrak{m}J = 1$. Then the Tjurina ideal $\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle \subset \mathbb{C}\{x,y\}$ can be generated by f and some $\mathbb{C}\{x,y\}$-linear combination $a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$ of the partials. But then the definition of the intersection multiplicity together with Propositions I.3.12 and I.3.38 imply that

$$\tau(f) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\} \Big/ \Big\langle f,\, a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\Big\rangle \geq \kappa(f) = \mu(f) + \mathrm{mt}(f) - 1\,.$$

But this is impossible if $\mathrm{mt}(f) > 1$.

Corollary 2.35. *The composed map* $T^{1,sec}_{\overline{C}\to C} \to T^1_{\overline{C}\to C} \to T^1_C$ *sending an element* $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in \overline{\mathfrak{m}}\frac{\partial}{\partial x} + \overline{\mathfrak{m}}\frac{\partial}{\partial y}$ *to* $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$ *is injective on the vector subspace*

$$T^{1,em}_{\overline{C}\to C} = T^{1,em}_{\overline{C}\to \mathbb{C}^2} = \left\{ \boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \;\middle|\; \begin{array}{c} \min\{\mathrm{ord}_{t_i} a_i, \mathrm{ord}_{t_i} b_i\} \geq \mathrm{mt}\, f_i \\ \text{for each } i = 1,\dots,r \end{array} \right\}.$$

Proof. If $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in T^{1,em}_{\overline{C}\to C}$ is mapped to zero, then the exact sequence (2) together with (1)(iii) of Proposition 2.34 implies that, for some $h_i \in \mathbb{C}\{t_i\}$,

$$a_i\frac{\partial}{\partial x} + b_i\frac{\partial}{\partial y} = h_i t_i^{-m_i+1}\left(\dot{x}_i\frac{\partial}{\partial x} + \dot{y}_i\frac{\partial}{\partial y}\right) \bmod \overline{\mathfrak{m}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right).$$

By the equimultiplicity assumption, $\operatorname{ord}_{t_i}(h_i t_i^{-m_i+1}) \geq 1$. This shows that $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y}$ is an element of $\overline{\mathfrak{m}}(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y})$ which is zero in $T^{1,sec}_{\overline{C}/C} \subset T^{1,sec}_{\overline{C}\to C}$. □

2.5 Equisingular Deformations of the Parametrization

We define now equisingular deformations of the parametrization. In this context, (embedded) equisingular deformations of the plane curve germ $(C,\mathbf{0})$ as defined in Section 2.1 are referred to as *equisingular deformations of the equation*. In contrast to the semiuniversal equisingular deformation of the equation, the semiuniversal equisingular deformation of the parametrization has an easy explicit description. This description shows that its base space is smooth. We use this to give a new proof of the result of Wahl [Wah] that the base space of the semiuniversal equisingular deformation of the equation is smooth. This implies that the μ-constant stratum in the semiuniversal deformation of $(C,\mathbf{0})$ is smooth.

In order to define equisingular deformations of the parametrization

$$\varphi : (\overline{C},\overline{0}) = \coprod_{i=1}^{r}(\overline{C}_i,\overline{0}_i) \to (\mathbb{C}^2,\mathbf{0})$$

of the reduced plane curve singularity $(C,\mathbf{0}) = \bigcup_{i=1}^{r}(C_i,\mathbf{0})$, we fix some notations that will be in force for the rest of this section.

If x,y are local coordinates of $(\mathbb{C}^2,\mathbf{0})$, and if t_i are local coordinates of $(\overline{C}_i,\overline{0}_i)$, then $\varphi = (\varphi_i)_{i=1}^{r}$ is given by

$$t_i \overset{\varphi_i}{\longmapsto} \big(x_i(t_i), y_i(t_i)\big), \quad i = 1,\dots,r,$$

where $x_i, y_i \in \mathbb{C}\{t_i\}$. Let $C \subset M$ be a representative of $(C,\mathbf{0})$, and let $M \subset \mathbb{C}^2$ be an open neighbourhood of $\mathbf{0}$. Let $\pi : \widetilde{M} \to M$ be a finite sequence of point blowing ups, let $\widetilde{C}$, $\widetilde{C}_i$ be the strict transforms of C and C_i, respectively, and let $\widetilde{p} := \widetilde{C} \cap \pi^{-1}(\mathbf{0})$.

Any point $p \in \widetilde{p}$ arising this way, including $\mathbf{0} \in C$, is called an *infinitely near point belonging to* $(C,\mathbf{0})$. For $p \in \widetilde{p}$, we set

$$\begin{aligned}
\Lambda_p &:= \big\{i \,\big|\, 1 \leq i \leq r,\ \widetilde{C}_i \text{ passes through } p\big\},\\
(C_p,\mathbf{0}) &:= \bigcup_{i\in\Lambda_p}(C_i,\mathbf{0}),\ \text{ the corresponding subgerm of } C \text{ at } \mathbf{0},\\
(\widetilde{C},p) &:= \bigcup_{i\in\Lambda_p}\big(\widetilde{C}_i,p\big),\ \text{ the germ of } \widetilde{C} \text{ at } p,\\
(\overline{C},\overline{p}) &:= \coprod_{i\in\Lambda_p}(\overline{C}_i,\overline{0}_i),\ \text{ the multigerm of } \overline{C} \text{ at } \overline{p}.
\end{aligned}$$

Of course, $\{\Lambda_p \mid p \in \widetilde{p}\}$, is a partition of $\{1,\dots,r\}$. $\big(\widetilde{M},\widetilde{p}\big)$ denotes the multigerm $\coprod_{p\in\widetilde{p}}\big(\widetilde{M},p\big)$, and $\big(\widetilde{C},\widetilde{p}\big)$ denotes the multigerm $\coprod_{p\in\widetilde{p}}\big(\widetilde{C},p\big)$. The restriction of φ,

$$\varphi_p : \big(\overline{C},\overline{p}\big) \longrightarrow \big(\mathbb{C}^2,\mathbf{0}\big)$$

is a parametrization of $(C_p,\mathbf{0})$. Since $(C_p,\mathbf{0})$ and $\big(\widetilde{C},p\big)$ have the same normalization, φ_p factors through $\big(\widetilde{M},p\big)$. The induced map

$$\widetilde{\varphi}_p : \big(\overline{C},\overline{p}\big) \longrightarrow \big(\widetilde{M},p\big)$$

is a parametrization of $(\widetilde{C},p)$. Furthermore, $\pi_p : \big(\widetilde{M},p\big) \to (M,\mathbf{0})$ denotes the germ of π at p.

Let, for the moment, $\pi : \widetilde{M} \to M$ be the single blowing up of the point $\mathbf{0} \in M$. Then we identify $\pi^{-1}(\mathbf{0})$, the *first infinitely near neighbourhood* of $\mathbf{0}$, with $\mathbb{P}^1$, and we have for a point $p = (\beta:\alpha) \in \mathbb{P}^1$

$$\Lambda_p = \big\{ i \;\big|\; 1 \le i \le r\,,\ (C_i,\mathbf{0}) \text{ has tangent direction } p = (\beta:\alpha) \big\}\,.$$

We want to describe $\widetilde{\varphi}_p$ for p belonging to the first infinitely near neighbourhood of $(C,\mathbf{0})$, in terms of local coordinates u,v for $\big(\widetilde{M},p\big)$. We can assume that π_p is given by

$$\pi_p(u,v) = \begin{cases} \big(u, u(v+\alpha)\big) & \text{if } p = (1:\alpha)\,, \\ \big(uv, v\big) & \text{if } p = (0:1)\,, \end{cases} \tag{2.5.17}$$

(see Remark I.3.16.1) and that $\widetilde{\varphi}_p$ is given by

$$\widetilde{\varphi}_i(t_i) = \big(u_i(t_i), v_i(t_i)\big)\,, \quad i \in \Lambda_p\,,$$

for some $u_i, v_i \in t_i\mathbb{C}\{t_i\}$. As $\varphi_p = \pi_p \circ \widetilde{\varphi}_p$, we get, for all $i \in \Lambda_p$,

$$(x_i,y_i) = \begin{cases} \big(u_i, u_i(v_i+\alpha)\big) & \text{if } p = (1:\alpha)\,, \\ \big(u_iv_i, v_i\big) & \text{if } p = (0:1)\,. \end{cases} \tag{2.5.18}$$

Now, consider a deformation $\phi : \big(\overline{\mathscr{C}},\overline{0}\big) \to (\mathscr{M},\mathbf{0})$ of φ over $(T,\mathbf{0})$, with compatible sections $\sigma : (T,\mathbf{0}) \to (\mathscr{M},\mathbf{0})$ and $\overline{\sigma} = (\overline{\sigma}_i)_{i=1}^r : (T,\mathbf{0}) \to \big(\overline{\mathscr{C}},\overline{0}\big)$. For an arbitrary infinitely near point $p \in \widetilde{p}$ consider the restriction of ϕ,

$$\phi_p : \big(\overline{\mathscr{C}},\overline{p}\big) := \coprod_{i\in\Lambda_p} \big(\overline{\mathscr{C}}_i,\overline{0}_i\big) \longrightarrow (\mathscr{M},\mathbf{0})\,,$$

given by

$$t_i \longmapsto \big(X_i(t_i), Y_i(t_i)\big)\,, \quad i \in \Lambda_p\,,$$

$X_i, Y_i \in \mathcal{O}_{\overline{\mathscr{C}}_i,\overline{0}_i} = \mathcal{O}_{T,\mathbf{0}}\{t_i\}$. Together with σ and $\overline{\sigma}_p = \big(\overline{\sigma}_i\big)_{i\in\Lambda_p}$, ϕ_p is a deformation with compatible sections of φ_p over $(T,\mathbf{0})$.

Let T be a representative of $(T,\mathbf{0})$, and let $\mathscr{M} = M \times T$. Assume that $\pi : \widetilde{\mathscr{M}} \to \mathscr{M}$ is a finite sequence of blowing ups of sections over T such that the restriction over $M \times \{\mathbf{0}\}$ induces the blowing up $\widetilde{M} \to M$ considered before (and which was denoted by the same letter π).

For equisingularity, we require that ϕ_p factors through $\big(\widetilde{\mathscr{M}},p\big)$, that is, there exists

$$\widetilde{\phi}_p : \big(\overline{\mathscr{C}},\overline{p}\big) \longrightarrow \big(\widetilde{\mathscr{M}},p\big)\,, \quad p \in \widetilde{p}\,,$$

such that $\phi_p = \pi_p \circ \widetilde{\phi}_p$, $\pi_p : \big(\widetilde{\mathscr{M}},p\big) \to (\mathscr{M},\mathbf{0})$ being the germ of π at p.

The existence of $\widetilde{\phi}_p$ is in general not sufficient, since it is not necessarily a deformation of the parametrization of $\big(\widetilde{C},p\big)$. In fact, the special fibre of $\big(\widetilde{\mathscr{C}},p\big) \to (T,\mathbf{0})$ is in general the union of $\big(\widetilde{C},p\big)$ with some exceptional divisors. This will be clear from the following considerations.

Let $\pi : \widetilde{M} \to M$ be the blowing up of $\mathbf{0} \in M$, and let $\pi : \widetilde{\mathscr{M}} \to \mathscr{M}$ be the blowing up of the trivial section $\{\mathbf{0}\} \times T$ in $\mathscr{M}$. The above coordinates u, v of $\big(\widetilde{M},p\big)$ induce an isomorphism $\big(\widetilde{\mathscr{M}},p\big) \cong (\mathbb{C}^2,\mathbf{0}) \times (T,\mathbf{0})$, and with respect to these coordinates, $\widetilde{\phi}_p : \big(\overline{\mathscr{C}},\overline{p}\big) \to \big(\widetilde{\mathscr{M}},p\big)$ is given by

$$\widetilde{\phi}_p : t_i \longmapsto \big(U_i(t_i), V_i(t_i)\big)\,, \quad i \in \Lambda_p\,,$$

with $U_i, V_i \in \mathcal{O}_{\overline{\mathscr{C}}_i,\overline{0}_i} = \mathcal{O}_{T,\mathbf{0}}\{t_i\}$. Moreover, for all $i \in \Lambda_p$, we have the relation

$$(X_i, Y_i) = \begin{cases} \big(U_i, U_i(V_i+\alpha)\big) & \text{if } p = (1:\alpha)\,,\\ \big(U_iV_i, V_i\big) & \text{if } p = (0:1)\,,\end{cases}$$

where X_i, Y_i define $\phi_p : \big(\overline{\mathscr{C}},\overline{p}\big) \to (\mathscr{M},\mathbf{0})$. Now, let $f \in \mathbb{C}\{x,y\}$ define $(C,\mathbf{0})$, let $F \in \mathcal{O}_{T,\mathbf{0}}\{x,y\}$ define $(\mathscr{C},\mathbf{0}) \subset (\mathscr{M},\mathbf{0})$, and let $\widetilde{F} \in \mathcal{O}_{T,\mathbf{0}}\{u,v\}$ define $\big(\widetilde{\mathscr{C}},p\big) \subset \big(\widetilde{\mathscr{M}},p\big)$.

If the (x,y)-order of F is not constant, that is, if $\operatorname{ord}_{x,y} F = \operatorname{ord}_{x,y} f - n$ for some n, then $\big(\widetilde{F} \bmod \mathfrak{m}_{T,\mathbf{0}}\big) \in \mathbb{C}\{u,v\}$ and $\widetilde{f}$, defining the strict transform $\big(\widetilde{C},p\big)$, satisfy the relation $(\widetilde{F} \bmod \mathfrak{m}_{T,\mathbf{0}}) = e^n \widetilde{f}$ with $e \in \mathbb{C}\{u,v\}$ defining the exceptional divisor of π_p ($e = u$ if $p = (1:\alpha)$, and $e = v$ if $p = (0:1)$). That is, the special fibre of $\big(\widetilde{\mathscr{C}},p\big) \to (T,\mathbf{0})$ is given by the germ of $\{e^n = 0\} \cup \widetilde{C}$ at p.

The definition below forbids this for each infinitely near point belonging to $(C,\mathbf{0})$ if the deformation is equisingular.

Definition 2.36. A deformation $(\phi,\overline{\sigma},\sigma) \in \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ of the parametrization $\varphi : \big(\overline{C},\overline{0}\big) \to (\mathbb{C}^2,\mathbf{0})$,

$$\begin{tikzcd}
(\overline{C},\overline{0}) \arrow[r, hook] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & (\overline{\mathscr{C}},\overline{0}) \arrow[d, "\widetilde{\phi}_p"] \\
(\mathbb{C}^2,\mathbf{0}) = (M,\mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{M},\mathbf{0}) \arrow[d] \\
\{\mathbf{0}\} \arrow[r, hook] & (T,\mathbf{0}) \arrow[u, bend right, "\sigma"'] \arrow[uu, bend right=60, "\overline{\sigma}"']
\end{tikzcd}$$

is called *equisingular* if it is equimultiple and if the following holds

(i) For each infinitely near point $p \in \widetilde{M}$ belonging to $(C,\mathbf{0})$, there exists a germ $\big(\widetilde{\mathscr{M}},p\big)$ and morphisms $\widetilde{\phi}_p$, σ_p, fitting in the commutative diagram with Cartesian squares:

$$\begin{tikzcd}
(\overline{C},\overline{p}) \arrow[r, hook] \arrow[d, "\widetilde{\varphi}_p"'] \arrow[dr, phantom, "\square"] & (\overline{\mathscr{C}},\overline{p}) \arrow[d, "\widetilde{\phi}_p"] \\
\big(\widetilde{M},p\big) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \big(\widetilde{\mathscr{M}},p\big) \arrow[d] \\
(M,\mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{M},\mathbf{0}) \arrow[d] \\
\{\mathbf{0}\} \arrow[r, hook] & (T,\mathbf{0}) \arrow[u, bend right, "\sigma"'] \arrow[uu, bend right=60, "\sigma_p"'] \arrow[uuu, bend right=80, "\overline{\sigma}_p"']
\end{tikzcd}$$

such that $\big(\widetilde{\phi}_p,\overline{\sigma}_p,\sigma_p\big)$ is an equimultiple deformation of the parametrization φ_p of $\big(\widetilde{C},p\big)$ over $(T,\mathbf{0})$, with compatible sections $\overline{\sigma}_p$, σ_p.

(ii) The system of such diagrams is compatible: if the germ $\big(\widetilde{M}',q\big)$ dominates $\big(\widetilde{M},p\big)$ (that is, if there is a morphism $\big(\widetilde{M}',q\big) \to \big(\widetilde{M},p\big)$ with dense image), then there exists a morphism $\big(\widetilde{\mathscr{M}}',q\big) \to \big(\widetilde{\mathscr{M}},p\big)$ such that the obvious diagram commutes.

(iii) If $\big(\widetilde{\mathscr{M}}',q\big)$ is consecutive to $\big(\widetilde{\mathscr{M}},p\big)$ (that is, if there is no infinitely near point between the dominating relation) then $\big(\widetilde{\mathscr{M}}',q\big)$ is the blow up of $\big(\widetilde{\mathscr{M}},p\big)$ along the section σ_p.

Remark 2.36.1. (1) In order to check equisingularity of a deformation of the parametrization $\varphi : (\overline{C},\overline{0}) \to (\mathbb{C}^2,\mathbf{0})$, we need only consider infinitely near points appearing in a minimal embedded resolution of $(C,\mathbf{0})$. Since, if $\pi' : (M',p') \to (\mathbb{C}^2,\mathbf{0})$ is any infinitely near neighbourhood of $(C,\mathbf{0})$, then there is an isomorphism $(M',p') \xrightarrow{\cong} (\widetilde{M},p)$ commuting with π' and π, where $\pi : (\widetilde{M},p) \to (\mathbb{C}^2,\mathbf{0})$ is an infinitely near neighbourhood of $(C,\mathbf{0})$ belonging to the minimal embedded resolution of $(C,\mathbf{0})$.

(2) If $(C,\mathbf{0})$ is an ordinary singularity, then $(\widetilde{C},p)$ is smooth for each infinitely near point $p\neq\mathbf{0}$ belonging to $(C,\mathbf{0})$. Then $\widetilde{\varphi}_p:(\overline{C},\overline{p})\to(\widetilde{C},p)$ is an isomorphism and it follows that $(\phi,\overline{\sigma},\sigma)$ is equisingular iff it is equimultiple.

We denote by $\mathcal{D}ef^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ the *category of equisingular deformations of the parametrization* $\varphi:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$, and by $\underline{\mathcal{D}ef}^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ the corresponding functor of isomorphism classes. Moreover, we introduce

$$T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}:=\underline{\mathcal{D}ef}^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T_\varepsilon)\,,$$

the tangent space to this functor.

Note that $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ is a subspace of $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ for each vector $\boldsymbol{m}$ satisfying $1\leq m_i\leq\operatorname{ord}\varphi_i$ for all i.

Recall the notation $\varphi=(\varphi_i)_{i=1}^r$, with $\varphi_i(t_i)=\big(x_i(t_i),y_i(t_i)\big)$, and

$$\dot\varphi=\left(\frac{\partial x_i}{\partial t_i}\right)_{i=1}^r\frac{\partial}{\partial x}+\left(\frac{\partial y_i}{\partial t_i}\right)_{i=1}^r\frac{\partial}{\partial y}$$

In view of Proposition 2.27, p. 305, we obviously have the following statement:

Lemma 2.37. *There is an isomorphism of* $\mathbb{C}$*-vector spaces,*

$$T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}\cong I^{es}_\varphi\Big/\left(\dot\varphi\cdot\mathfrak{m}_{\overline{C},\overline{0}}+\varphi^\sharp(\mathfrak{m}_{\mathbb{C}^2,\mathbf{0}})\frac{\partial}{\partial x}\oplus\varphi^\sharp(\mathfrak{m}_{\mathbb{C}^2,\mathbf{0}})\frac{\partial}{\partial y}\right)\,,$$

where $I^{es}_\varphi:=I^{es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ *denotes the set of all elements*

$$\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix}\cdot\frac{\partial}{\partial x}+\begin{pmatrix}b_1\\ \vdots\\ b_r\end{pmatrix}\cdot\frac{\partial}{\partial y}\in\mathfrak{m}_{\overline{C},\overline{0}}\cdot\frac{\partial}{\partial x}\oplus\mathfrak{m}_{\overline{C},\overline{0}}\cdot\frac{\partial}{\partial y}$$

such that $\Big\{\big(x_i(t_i)+\varepsilon a_i(t_i),y_i(t_i)+\varepsilon b_i(t_i)\big)\;\Big|\;i=1,\dots,r\Big\}$ *is an equisingular deformation of the parametrization* $\varphi:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$ *along the trivial sections over* T_ε.

We call I^{es}_φ the *equisingularity module of the parametrization* of $(C,\mathbf{0})$. It is an $\mathcal{O}_{C,\mathbf{0}}$-submodule of $\varphi^*\Theta_{\mathbb{C}^2,\mathbf{0}}=\mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial x}\oplus\mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}$, as will be shown in Proposition 2.40. Here, $\Theta_{\mathbb{C}^2,\mathbf{0}}=\operatorname{Der}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big)$.

The natural map $\Theta_{\overline{C},\overline{0}}\to\varphi^*\Theta_{\mathbb{C}^2,\mathbf{0}}$ maps $\frac{\partial}{\partial t_i}$ to $\dot x_i\frac{\partial}{\partial x}+\dot y_i\frac{\partial}{\partial y}$. Hence, in invariant terms, we see that I^{es}_φ is a submodule of

$$\varphi^*\Theta_{\mathbb{C}^2,\mathbf{0}}\big/\big(\mathfrak{m}_{\overline{C},\overline{0}}\Theta_{\overline{C},\overline{0}}+\varphi^{-1}(\mathfrak{m}_{\mathbb{C}^2,\mathbf{0}}\Theta_{\mathbb{C}^2,\mathbf{0}})\big)\,.$$

Remark 2.37.1. (1) If $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ is smooth, then each deformation $(\phi,\overline{\sigma},\sigma)\in\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ is equisingular. This follows as each deformation is equimultiple and the lifting to the blow up of σ is a deformation of the strict transform by the considerations before Definition 2.36. As the strict transform is again smooth, we can continue, and the conditions of Definition 2.36 are fulfilled. It follows that, for a smooth germ $(C,\mathbf{0})$, $I^{es}_{\varphi} = \mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}$ and $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} = \{0\}$.

(2) If $(T,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ and if ϕ is given by $X_i(t_i), Y_i(t_i) \in \mathcal{O}_{T,\mathbf{0}}\{t_i\}$, then we can lift the non-zero coefficients of X_i and Y_i to $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$, getting in this way $\widetilde{X}_i(t_i)$, $\widetilde{Y}_i(t_i) \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}\{t_i\}$ having the same t_i-order as X_i, Y_i. The same holds after blowing up the trivial section. Hence, as there is no flatness requirement (Remark 2.21.1), we can extend $\boldsymbol{m}$-multiple, respectively equisingular, deformations over $(\mathbb{C}^n,\mathbf{0})$. In particular, when considering $\boldsymbol{m}$-multiple, respectively equisingular, deformations of the parametrization, we may always assume that the base $(T,\mathbf{0})$ is smooth.

Example 2.37.2. (Continuation of Example 2.24.1.) The deformation of the parametrization $(t,s) \mapsto (t^3 - s^2t, t^2 - s^2, s)$, $s \in (\mathbb{C},0)$, of the cusp to a node is not equisingular along any section, since it is not equimultiple for any choice of compatible sections $(\overline{\sigma},\sigma)$ (note that $\overline{\sigma}$ must be a single section, not a multisection, since the cusp is unibranch).

The first order deformation of the parametrization

$$(t,\varepsilon) \mapsto (t^3 - \varepsilon t, t^2 - \varepsilon, \varepsilon)\,, \quad \varepsilon^2 = 0\,,$$

is also not equisingular. However, the corresponding deformation of the equation, given by $x^2 - y^3 - \varepsilon y^2$, is equisingular (along the section σ with $I_\sigma = \langle x, y + \frac{\varepsilon}{3}\rangle$). The same deformation of the equation is induced by the equisingular deformation of the parametrization $(t,\varepsilon) \mapsto (t^3, t^2 - \frac{\varepsilon}{3}, \varepsilon)$.

This shows that an equisingular deformation of the equation (over T_ε) can be induced by several deformations of the parametrization. Exactly one of the inducing deformations of the parametrization is equisingular. This example illustrates the existence, resp. uniqueness, statements of Proposition 2.23 and Theorem 2.64.

The following theorem shows that $\underline{\mathcal{D}ef}^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ is a "linear" subfunctor of $\underline{\mathcal{D}ef}^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$. As such, it is already completely determined by its tangent space. We use the notation

$$\boldsymbol{a}^j = \begin{pmatrix} a_1^j \\ \vdots \\ a_r^j \end{pmatrix}, \quad \boldsymbol{b}^j = \begin{pmatrix} b_1^j \\ \vdots \\ b_r^j \end{pmatrix} \in \bigoplus_{i=1}^r \mathbb{C}\{t_i\}\,, \quad j = 1,\dots,k\,.$$

Theorem 2.38. *Let $\varphi : \big(\overline{C},\overline{0}\big) \to (\mathbb{C}^2,\mathbf{0})$ be a parametrization of the reduced plane curve singularity $(C,\mathbf{0})$, and let $\boldsymbol{s} = (s_1,\dots,s_k)$ be local coordinates of $(\mathbb{C}^k,\mathbf{0})$. Then the following holds:*

(1) Let $\phi : (\overline{C},\overline{0}) \times (\mathbb{C}^k,\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0}) \times (\mathbb{C}^k,\mathbf{0})$ be a deformation of φ with trivial sections over $(\mathbb{C}^k,\mathbf{0})$, given by $\phi_i = (X_i, Y_i, \boldsymbol{s})$ with

$$X_i(t_i, \boldsymbol{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i) s_j\,, \quad a_i^j \in t_i\mathbb{C}\{t_i\}\,,$$

$$Y_i(t_i, \boldsymbol{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i) s_j\,, \quad b_i^j \in t_i\mathbb{C}\{t_i\}\,,$$

$i = 1, \dots, r$. *Then ϕ is equisingular iff $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in I^{es}_{\varphi}$ for all $j = 1, \dots, k$.*

(2) Let $\phi = \{(X_i, Y_i, \boldsymbol{s}) \mid i = 1, \dots, r\}$, $X_i, Y_i \in \mathcal{O}_{\mathbb{C}^k,\mathbf{0}}\{t_i\}$, be an equisingular deformation of φ with trivial sections over $(\mathbb{C}^k, \mathbf{0})$. Then ϕ is a versal (respectively semiuniversal) object of $\mathcal{D}ef^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ iff

$$\begin{pmatrix} \frac{\partial X_1}{\partial s_j}(t_1, \mathbf{0}) \\ \vdots \\ \frac{\partial X_r}{\partial s_j}(t_r, \mathbf{0}) \end{pmatrix} \cdot \frac{\partial}{\partial x} + \begin{pmatrix} \frac{\partial Y_1}{\partial s_j}(t_1, \mathbf{0}) \\ \vdots \\ \frac{\partial Y_r}{\partial s_j}(t_r, \mathbf{0}) \end{pmatrix} \cdot \frac{\partial}{\partial y}\,, \quad j = 1, \dots, k\,,$$

represent a system of generators (respectively a basis) of the $\mathbb{C}$-vector space $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$.

(3) Let $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in I^{es}_{\varphi}$, $j = 1, \dots, k$, represent a basis (respectively a system of generators) of $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$. Then $\phi = \{(X_i, Y_i, \boldsymbol{s}) \mid i = 1, \dots, r\}$ with

$$X_i(t_i, \boldsymbol{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i) s_j\,,$$

$$Y_i(t_i, \boldsymbol{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i) s_j\,,$$

$\boldsymbol{s} = (s_1, \dots, s_k) \in (\mathbb{C}^k, \mathbf{0})$, *is a semiuniversal (respectively versal) equisingular deformation of φ with trivial sections over $(\mathbb{C}^k, \mathbf{0})$. In particular, equisingular deformations of the parametrization are unobstructed, and the semiuniversal deformation has a smooth base space of dimension* $\dim_{\mathbb{C}} T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$.

For the proof, we need some preparations. We fix local coordinates x, y of $(\mathbb{C}^2, \mathbf{0})$ and t_i of $\big(\overline{C}_i, \overline{0}_i\big)$. Assume that

$$(\phi, \overline{\sigma}, \sigma) = \{(\phi_i, \overline{\sigma}_i, \sigma) \mid i = 1, \dots, r\} \in \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$$

is given as

$$\big(\overline{\mathscr{C}}, \overline{0}\big) = \big(\overline{C} \times T, \overline{0}\big) \xrightarrow{\phi} \big(\mathbb{C}^2 \times T, \mathbf{0}\big) \xrightarrow{\mathrm{pr}} (T, \mathbf{0})\,, \quad \phi = \big(\phi_i, \mathrm{id}_T\big)_{i=1}^{r},$$

(see Remark 2.21.1), with $\phi_i = (X_i, Y_i)$, $X_i, Y_i \in \mathcal{O}_{\overline{C}_i \times T, (\overline{0}_i, \mathbf{0})}$, and with σ, $\overline{\sigma} = \big(\overline{\sigma}_i\big)_{i=1}^{r}$ the trivial sections.

We have to consider *small extensions* $(T', \mathbf{0}) \subset (T, \mathbf{0})$ of base spaces, that is, we assume that the surjective map $\mathcal{O}_{T,\mathbf{0}} \to \mathcal{O}_{T',\mathbf{0}}$ has one-dimensional kernel (whose generator is denoted by ε). To shorten notation, we set

$$A := \mathcal{O}_{T,\mathbf{0}}\,, \quad A' := \mathcal{O}_{T',\mathbf{0}}\,.$$

Then we have the analytic A-algebras (respectively analytic A'-algebras)

$$\begin{aligned} A\{t_i\} &= \mathcal{O}_{\overline{C}_i \times T, (\overline{0}_i, \mathbf{0})}\,, \quad & A'\{t_i\} &= \mathcal{O}_{\overline{C}_i \times T', (\overline{0}_i, \mathbf{0})}\,, \\ A\{x,y\} &= \mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, \mathbf{0})}\,, \quad & A'\{x,y\} &= \mathcal{O}_{\mathbb{C}^2 \times T', (\mathbf{0}, \mathbf{0})}\,. \end{aligned}$$

Note that, as complex vector spaces, $A = A' \oplus \varepsilon\mathbb{C}$, and that $\varepsilon\mathfrak{m}_A = 0$. The deformation $(\phi, \overline{\sigma}, \sigma)$ over $(T, \mathbf{0})$ is given by

$$X_i(t_i) = X_i'(t_i) + \varepsilon a_i\,, \quad Y_i(t_i) = Y_i'(t_i) + \varepsilon b_i\,,$$

with $X_i, Y_i \in A\{t_i\}$ and $a_i, b_i \in \mathbb{C}\{t_i\}$, where $X_i', Y_i' \in A'\{t_i\}$ define a deformation of the parametrization with compatible sections $\overline{\sigma}', \sigma'$ over $(T', \mathbf{0})$. On the ring level, ϕ_i is given by

$$\phi_i^\sharp : A\{x,y\} \to A\{t_i\}\,, \quad x \mapsto X_i\,, \;\; y \mapsto Y_i\,, \quad i = 1, \dots, r\,.$$

Furthermore, the residue classes $x_i(t_i)$, respectively $y_i(t_i)$, of $X_i(t_i)$, respectively $Y_i(t_i)$ modulo $\mathfrak{m}_A$ define the parametrization of the i-th branch $(C_i, \mathbf{0})$, $i = 1, \dots, r$.

Proposition 2.39. *Consider the diagram with given solid arrows*

$$\begin{tikzcd}
(\overline{\mathscr{C}}', \overline{0}) \arrow[r, hook, dotted] \arrow[d, "\widetilde{\phi}'"] \arrow[dr, phantom, "\square"] & (\overline{\mathscr{C}}, \overline{0}) \arrow[d, dotted, "\widetilde{\phi}"'] \\
(\widetilde{\mathscr{M}}', \widetilde{p}) \arrow[r, hook, dotted] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & (\widetilde{\mathscr{M}}, \widetilde{p}) \arrow[d, dotted, "\pi"'] \\
(\mathscr{M}', \mathbf{0}) \arrow[r, hook, dotted] \arrow[d, shift left] \arrow[dr, phantom, "\square"] & (\mathscr{M}, \mathbf{0}) \arrow[d, dotted, shift right] \\
(T', \mathbf{0}) \arrow[r, hook] \arrow[u, shift left, "\sigma'"] \arrow[uuu, bend left=60, "\overline{\sigma}'"] \arrow[uu, bend left=40, "\widetilde{\sigma}'"] & (T, \mathbf{0})\,. \arrow[u, dotted, shift right, "\sigma"'] \arrow[uuu, dotted, bend right=60, "\overline{\sigma}"'] \arrow[uu, dotted, bend right=40, "\widetilde{\sigma}"']
\end{tikzcd}$$

where $(T', \mathbf{0}) \hookrightarrow (T, \mathbf{0})$ is a small extension of complex germs. Assume that

(i)' $\left(\phi' = \pi' \circ \widetilde{\phi}', \overline{\sigma}', \sigma'\right) \in \mathcal{D}ef^{\,em}_{(\overline{C},\overline{0}) \to (\mathbb{C}^2, \mathbf{0})}(T', \mathbf{0})$.

(ii)' $\pi' : \widetilde{\mathscr{M}}' \to \mathscr{M}'$ is the blowing up of the section σ'. Let $\left(\widetilde{\mathscr{M}}', \widetilde{p}\right)$ be the multigerm at the set $\widetilde{p}$ of infinitely near points belonging to $(C, \mathbf{0})$ in the blow up $\widetilde{M}$ of $\mathbf{0} \in M$.

(iii)' $\widetilde{\sigma}' = \{\widetilde{\sigma}'_p \mid p \in \widetilde{p}\}$ *is a (multi-)section such that* $\big(\widetilde{\phi}', \overline{\sigma}', \widetilde{\sigma}'\big)$ *is an object of* $\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\widetilde{M},\widetilde{p})}(T',\mathbf{0})$.

Then the following holds: The data given in (i)'–(iii)' can be extended over $(T,\mathbf{0})$ *as indicated in the diagram. More precisely, there exist dotted arrows making the above diagram commutative, respectively Cartesian, such that*

(i) $\big(\phi = \pi \circ \widetilde{\phi}, \overline{\sigma}, \sigma\big) \in \mathcal{D}ef^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$,

(ii) $\pi : \widetilde{\mathscr{M}} \to \mathscr{M}$ *is the blowing up of* σ,

(iii) $\widetilde{\sigma} = \big\{\widetilde{\sigma}_p \,\big|\, p \in \widetilde{p}\big\}$ *is a (multi-)section such that* $\big(\widetilde{\phi}, \overline{\sigma}, \widetilde{\sigma}\big)$ *is an element of* $\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\widetilde{M},\widetilde{p})}(T,\mathbf{0})$.

Furthermore, $\big(\widetilde{\phi}, \widetilde{\sigma}\big)$ *satisfying (iii) is uniquely determined by* $(\phi, \overline{\sigma}, \sigma)$ *and* $\big(\widetilde{\phi}', \widetilde{\sigma}'\big)$.

Proof. We use the notations introduced above. Since we consider (multi-) germs at $\mathbf{0}, \overline{0}$ and $\widetilde{p}$, we may assume that all sections σ', $\overline{\sigma}'_i$, $i = 1, \dots, r$, and $\widetilde{\sigma}'_p$, $p \in \widetilde{p}$ are trivial. Let $\varphi : \big(\overline{C}, \overline{0}\big) \to (\mathbb{C}^2, \mathbf{0})$ be given by $x_i, y_i \in t_i\mathbb{C}\{t_i\}$, and let ϕ' be given by X'_i, Y'_i which are elements of $t_iA'\{t_i\}$ as the sections are trivial.

Step 1: Uniqueness. Assume we have extensions $\widetilde{\phi}, \sigma, \overline{\sigma}, \widetilde{\sigma}$ over $(T,\mathbf{0})$ as claimed, with $\sigma, \overline{\sigma}$ the trivial sections. Then $\phi_p : \big(\overline{\mathscr{C}}, \overline{0}\big) \to (\mathscr{M}, \mathbf{0})$, $p \in \widetilde{p}$, is given, on the ring level, by a map

$$\phi_p^\sharp : A\{x,y\} \to \bigoplus_{i\in\Lambda_p} A\{t_i\}\,, \quad x \mapsto (X_i)_{i\in\Lambda_p}\,, \quad y \mapsto (Y_i)_{i\in\Lambda_p}\,,$$

where

$$X_i = X'_i + \varepsilon a_i\,, \quad Y_i = Y'_i + \varepsilon b_i\,, \quad a_i, b_i \in t_i\mathbb{C}\{t_i\}\,.$$

Further, $\widetilde{\phi}_p : \big(\overline{\mathscr{C}}, \overline{p}\big) \to \big(\widetilde{\mathscr{M}}, p\big)$, $p \in \widetilde{p}$, is given by a map

$$\widetilde{\phi}_p^\sharp : A\{u,v\} \to \bigoplus_{i\in\Lambda_p} A\{t_i\}\,, \quad u \mapsto (U_i)_{i\in\Lambda_p}\,, \quad v \mapsto (V_i)_{i\in\Lambda_p}\,,$$

where

$$U_i = U'_i + \varepsilon \widetilde{a}_i\,, \quad V_i = V'_i + \varepsilon \widetilde{b}_i\,,$$

$\widetilde{a}_i, \widetilde{b}_i \in \mathbb{C}\{t_i\}$, and $\widetilde{\phi}'_p$ is given by U'_i, V'_i, $i \in \Lambda_p$. Since σ is the trivial section, the blowing up of σ, $\pi : (\widetilde{\mathscr{M}}, \widetilde{p}) \to (\mathscr{M}, \mathbf{0})$, is given by

$$\pi_p^\sharp : A\{x,y\} \to A\{u,v\}\,, \quad p \in \widetilde{p}\,,$$

with $\pi_p^\sharp(a) = a$ for $a \in A$ and $(u,v) \mapsto \big(u, u(v+\alpha)\big)$ if $p = (1:\alpha)$, respectively $(u,v) \mapsto (uv, v)$ if $p = (0:1)$, where, as usually, we identify the exceptional divisor in $\widetilde{M}$ with $\mathbb{P}^1$. The condition $\phi_p = \pi_p \circ \widetilde{\phi}_p$ implies $X_i = U_i$, $Y_i = U_i(V_i + \alpha)$, hence,

$$X_i' + \varepsilon a_i = U_i' + \varepsilon \widetilde{a}_i\,, \quad Y_i' + \varepsilon b_i = \big(U_i' + \varepsilon \widetilde{a}_i\big)\big(V_i' + \varepsilon \widetilde{b}_i + \alpha\big)$$

for $p = (1 : \alpha)$. Moreover, $X_i = U_i V_i$, $Y_i = V_i$, hence

$$X_i' + \varepsilon a_i = \big(U_i' + \varepsilon \widetilde{a}_i\big)\big(V_i' + \varepsilon \widetilde{b}_i\big)\,, \quad Y_i' + \varepsilon b_i = V_i' + \varepsilon \widetilde{b}_i$$

for $p = (0 : 1)$.

Comparing the coefficients of ε, we obtain for $p = (1 : \alpha)$

$$a_i = \widetilde{a}_i \quad b_i = \widetilde{b}_i u_i + \widetilde{a}_i (v_i + \alpha)\,,$$

where $u_i = (U_i' \bmod \mathfrak{m}_{A'})$ and $v_i = (V_i' \bmod \mathfrak{m}_{A'})$ (recall that $\varepsilon \cdot \mathfrak{m}_{A'} = 0$). Equivalently, since $x_i = u_i$,

$$\widetilde{a}_i = a_i\,, \quad \widetilde{b}_i = \frac{b_i - a_i(v_i + \alpha)}{x_i}\,, \quad i \in \Lambda_p\,. \tag{2.5.19}$$

For $p = (0 : 1)$, we get

$$a_i = \widetilde{a}_i v_i + \widetilde{b}_i u_i\,, \quad b_i = \widetilde{b}_i\,,$$

or, equivalently $(y_i = v_i)$,

$$\widetilde{b}_i = b_i\,, \quad \widetilde{a}_i = \frac{a_i - b_i u_i}{y_i}\,, \quad i \in \Lambda_p\,. \tag{2.5.20}$$

In particular, $\widetilde{\phi}$ is uniquely determined by ϕ, $\overline{\sigma}$ and $\widetilde{\phi}'$.

The condition $\widetilde{\sigma} = \widetilde{\phi} \circ \overline{\sigma}$ implies that $\widetilde{\sigma}$ is uniquely determined with

$$\widetilde{\sigma}_p^\sharp(u) = \overline{\sigma}_i^\sharp(U_i' + \varepsilon \widetilde{a}_i) = (\widetilde{\sigma}_p')^\sharp(u) + \varepsilon \overline{\sigma}_i^\sharp(\widetilde{a}_i) \;\text{ for } i \in \Lambda_p\,.$$

As $\widetilde{\sigma}_p'$ and $\overline{\sigma}_i$ are trivial sections, the right-hand side equals $\varepsilon \widetilde{a}_i(0)$, where $\widetilde{a}_i(0)$ is the constant term of $\widetilde{a}_i$. In the same way, we have $\widetilde{\sigma}_p^\sharp(v) = \varepsilon \widetilde{b}_i(0)$ for all $i \in \Lambda_p$. In particular, we get the equalities

$$\big(\widetilde{a}_i(0), \widetilde{b}_i(0)\big) = \big(\widetilde{a}_j(0), \widetilde{b}_j(0)\big)\,, \;\text{ for all } i, j \in \Lambda_p\,, p \in \widetilde{p}\,, \tag{2.5.21}$$

which is a necessary and sufficient condition for the (multi-)sections $\widetilde{\sigma}$ and $\overline{\sigma}$ to be compatible. Moreover, $\widetilde{\sigma}$ is trivial iff $\widetilde{a}_i(0) = \widetilde{b}_i(0) = 0$ for all $i = 1, \ldots, r$.

Step 2: Existence. We can define the extensions $\phi, \widetilde{\phi}, \sigma, \overline{\sigma}, \widetilde{\sigma}$ over $(T, \mathbf{0})$ using the above conditions. We choose σ and $\overline{\sigma}$ as trivial sections, and we define ϕ by

$$X_i := X_i' + \varepsilon a_i\,, \quad Y_i := Y_i' + \varepsilon b_i\,,$$

with $a_i, b_i \in \mathbb{C}\{t_i\}$ satisfying the following conditions:

$$\mathrm{ord}_{t_i}(a_i), \mathrm{ord}_{t_i}(b_i) \geq \mathrm{mt}(C_i, \mathbf{0})\,, \tag{2.5.22}$$

and, if $p=(1:\alpha)$,

$$\frac{b_i}{x_i}(0)-\alpha\frac{a_i}{x_i}(0)=\frac{b_j}{x_j}(0)-\alpha\frac{a_j}{x_j}(0)\,,\quad \text{for all } i,j\in\Lambda_p\,, \tag{2.5.23}$$

while for $p=(0:1)$

$$\frac{a_i}{y_i}(0)=\frac{a_j}{y_j}(0)\,,\quad \text{for all } i,j\in\Lambda_p\,. \tag{2.5.24}$$

Note that for $p=(1:\alpha)$ and $i\in\Lambda_p$, we have $\mathrm{ord}_{t_i}(x_i)=\mathrm{mt}(C_i,\mathbf{0})$, while for $p=(0:1)$ and $i\in\Lambda_p$, we have $\mathrm{ord}_{t_i}(y_i)=\mathrm{mt}(C_i,\mathbf{0})$, showing that $\frac{b_i}{x_i}$ and $\frac{a_i}{y_i}$ are power series.

By (2.5.22), $(\phi,\overline{\sigma},\sigma)$ is equimultiple and, defining $\widetilde{a}_i,\widetilde{b}_i$ as in (2.5.19), respectively as in (2.5.20), they are well-defined power series in $\mathbb{C}\{t_i\}$. We define $\widetilde{\phi}_p$ by $U_i=U_i'+\varepsilon\widetilde{a}_i$, $V_i=V_i'+\varepsilon\widetilde{b}_i$. Then, using (2.5.19) and (2.5.23), the condition (2.5.21) is satisfied, since for $p=(1:\alpha)$ and $i\in\Lambda_p$ we have $\widetilde{a}_i(0)=0$ and $v_i(0)=0$. For $p=(0:1)$, we can argue similarly using condition (2.5.24). Hence, we can define a section $\widetilde{\sigma}_p$ satisfying $\widetilde{\sigma}_p=\widetilde{\phi}\circ\overline{\sigma}_p$ by setting

$$\widetilde{\sigma}_p^\sharp(u):=\varepsilon\widetilde{a}_i(0)\,,\quad \widetilde{\sigma}_p^\sharp(v):=\varepsilon\widetilde{b}_i(0)\,,$$

for some $i\in\Lambda_p$. The condition $\sigma=\pi\circ\widetilde{\sigma}$ is automatically fulfilled. □

Remark 2.39.1. (1) Note that for $p\neq q\in\widetilde{p}$ and for $i\in\Lambda_p$, $j\in\Lambda_q$ there is no relation between (a_i,b_i) and (a_j,b_j).

(2) If $\sigma,\overline{\sigma}$ and $\widetilde{\sigma}'$ are the trivial sections, then the extension $\widetilde{\sigma}$ is trivial iff, for all $p\in\widetilde{p}$ and all $i\in\Lambda_p$, we have $\frac{b_i}{x_i}(0)=\alpha\frac{a_i}{x_i}(0)$ if $p=(1:\alpha)$ and $\frac{a_i}{y_i}(0)=0$ if $p=(0:1)$.

(3) The extension σ of σ' in Proposition 2.39 has only to satisfy (2.5.22)–(2.5.24). Hence, it is not unique.

We describe now the behaviour of the equisingularity module I^{es}_φ under blowing up.

Let $\varphi:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$ be a parametrization of $(C,\mathbf{0})=\bigcup_{i=1}^r(C_i,\mathbf{0})$, let $\pi:\big(\widetilde{M},\widetilde{p}\big)\to(\mathbb{C}^2,\mathbf{0})$ be the blowing up of $\mathbf{0}$, let $\big(\widetilde{C},\widetilde{p}\big)=\coprod_{p\in\widetilde{p}}\big(\widetilde{C},p\big)$ be the strict transform of $(C,\mathbf{0})$, and let $\widetilde{\varphi}:\big(\overline{C},\overline{0}\big)\to\big(\widetilde{C},\widetilde{p}\big)$ be the induced parametrization of $\big(\widetilde{C},\widetilde{p}\big)$. Further, let x,y be local coordinates for $(\mathbb{C}^2,\mathbf{0})$, and let u,v be local coordinates for $\big(\widetilde{M},p\big)$, satisfying $\pi(u,v)=\big(u,u(v+\alpha)\big)$ if $p=(1:\alpha)\in\pi^{-1}(\mathbf{0})=\mathbb{P}^1$, and $\pi(u,v)=\big(uv,v\big)$ if $p=(0:1)$.

Recall that, for $p\in\widetilde{p}$, we have $i\in\Lambda_p$ iff the strict transform $\widetilde{C}_i$ of C_i passes through p, and that Λ_p, $p\in\widetilde{p}$, is a partition of $\{1,\dots,r\}$.

Then $\widetilde{\varphi}$ is a multigerm $\big(\widetilde{\varphi}_p\big)_{p\in\widetilde{p}}$, with

$$\widetilde{\varphi}_p:\big(\overline{C},\overline{p}\big)=\coprod_{i\in\Lambda_p}\big(\overline{C}_i,\overline{0}_i\big)\longrightarrow\big(\widetilde{M},p\big)\,,\quad t_i\longmapsto\big(u_i(t_i),v_i(t_i)\big)\,,$$

a parametrization of the germ $\big(\widetilde{C},p\big)$. Furthermore, for $\widetilde{a}_i,\widetilde{b}_i,a_i,b_i\in\mathbb{C}\{t_i\}$, we set

$$(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}})=\left(\begin{pmatrix}\widetilde{a}_1\\ \vdots\\ \widetilde{a}_r\end{pmatrix},\begin{pmatrix}\widetilde{b}_1\\ \vdots\\ \widetilde{b}_r\end{pmatrix}\right),\quad(\boldsymbol{a},\boldsymbol{b})=\left(\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix},\begin{pmatrix}b_1\\ \vdots\\ b_r\end{pmatrix}\right).$$

Proposition 2.40. *With the above notations, the following holds:*

(1) Let $(\widetilde{a}_i,\widetilde{b}_i)\in t_i\mathbb{C}\{t_i\}\oplus t_i\mathbb{C}\{t_i\}$, $i=1,\dots,r$, *be given. For* $i\in\Lambda_p$ *set*

$$(a_i,b_i)=\begin{cases}\big(\widetilde{a}_i,\widetilde{b}_iu_i+\widetilde{a}_i(v_i+\alpha)\big) & \text{if } p=(1:\alpha)\,,\\ \big(\widetilde{a}_iv_i+\widetilde{b}_iu_i,\widetilde{b}_i\big) & \text{if } p=(0:1)\,.\end{cases}$$

Then $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$ *iff* $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial v}\in I^{es}_{\widetilde{\varphi}}$ *and* $\min\{\operatorname{ord}_{t_i}a_i,\operatorname{ord}_{t_i}b_i\}\geq\operatorname{mt}(C_i,\mathbf{0})$ *for each* $i=1,\dots,r$.

(2) Given $a_i,b_i\in t_i\mathbb{C}\{t_i\}$ *such that* $\min\{\operatorname{ord}_{t_i}a_i,\operatorname{ord}_{t_i}b_i\}\geq\operatorname{mt}(C_i,\mathbf{0})$ *for each* $i=1,\dots,r$. *For* $i\in\Lambda_p$ *set*

$$(\widetilde{a}_i,\widetilde{b}_i)=\begin{cases}\left(a_i,\dfrac{b_i-a_i(v_i+\alpha)}{x_i}-\dfrac{b_i-a_i}{x_i}(0)\right) & \text{if } p=(1:\alpha)\,,\\ \left(\dfrac{a_i}{y_i}-\dfrac{a_i}{y_i}(0),b_i\right) & \text{if } p=(0:1)\,.\end{cases}$$

Then $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial v}\in I^{es}_{\widetilde{\varphi}}$ *iff* $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$.

(3) I^{es}_{φ} *is an* $\mathcal{O}_{C,\mathbf{0}}$*-submodule of* $\mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial x}\oplus\mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}$.

Proof. (1) By definition, $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$ iff $x_i(t_i)+\varepsilon a_i(t_i)$, $y_i(t_i)+\varepsilon b_i(t_i)$ defines an equisingular deformation ϕ of $\varphi:t_i\mapsto\big(x_i(t_i),y_i(t_i)\big)$ over T_ε along the trivial sections σ, $\overline{\sigma}_i$. Similarly for $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial v}\in I^{es}_{\widetilde{\varphi}}=\bigoplus_{p\in\widetilde{p}}I^{es}_{\widetilde{\varphi}_p}$, where $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}=\big(\widetilde{a}_p\frac{\partial}{\partial u}\big)_{p\in\widetilde{p}}$ and $\big(\widetilde{a}_p\frac{\partial}{\partial u}\big)=(a_i)_{i\in\Lambda_p}\frac{\partial}{\partial u}$.

We apply (the proof of) Proposition 2.39 with $T'=\{0\}$, $T=T_\varepsilon$ and with ϕ given by $x_i+\varepsilon a_i$, $y_i+\varepsilon b_i$. If ϕ is equisingular, it is equimultiple. Then, after blowing up σ, the induced deformation $\widetilde{\phi}_p$ of $\widetilde{\varphi}_p$ over T_ε along the trivial section is given by $\widetilde{u}_i+\varepsilon\widetilde{a}_i$, $\widetilde{v}_i+\varepsilon\widetilde{b}_i$ (see (2.5.19), (2.5.20)).

Since any infinitely near point belonging to $\big(\widetilde{C},p\big)$ belongs also to $(C,\mathbf{0})$, we get: If ϕ is equisingular, then blowing up σ induces, by definition, an equisingular deformation $\widetilde{\phi}_p$ of $\widetilde{\varphi}_p$, for each $p\in\widetilde{p}$. Conversely, if, for each $p\in\widetilde{p}$, $\widetilde{\phi}_p$ is equisingular, and if ϕ is equimultiple, then ϕ is equisingular, too. Thus, $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$ iff ϕ is equimultiple and $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial \boldsymbol{u}}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial \boldsymbol{v}}\in I^{es}_{\widetilde{\varphi}}$.

(2) Given a_i,b_i, we can argue as in (1) if the section $\widetilde{\sigma}$ in the proof of Proposition 2.39 is trivial. The result follows by applying (2.5.19), respectively (2.5.20).

(3) To see that I^{es}_{φ} is an $\mathcal{O}_{C,\mathbf{0}}$-module, let $\boldsymbol{g}=(g_i)_{i=1}^r\in\mathcal{O}_{C,\mathbf{0}}\subset\mathcal{O}_{(\overline{C},\overline{0})}$, and let $(\boldsymbol{a},\boldsymbol{b})$ define an element of I^{es}_{φ}. We argue by induction on the number of blowing ups needed to resolve the singularity $(C,\mathbf{0})$. We start with a smooth germ. Remark 2.37.1 gives $I^{es}_{\varphi}=\mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial x}+\mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}$, which is an $\mathcal{O}_{C,\mathbf{0}}$-module. By induction hypothesis, we may assume that $I^{es}_{\widetilde{\varphi}}$ containing $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial \boldsymbol{u}}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial \boldsymbol{v}}$ is an $\mathcal{O}_{C,\mathbf{0}}$-module. That is, $\boldsymbol{g}\cdot(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}})\in I^{es}_{\widetilde{\varphi}}$, where $\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}}$ are defined as in (2). We notice that $\big(\widetilde{g_ia_i},\widetilde{g_ib_i}\big)-\big(g_i\widetilde{a}_i,g_i\widetilde{b}_i\big)$ equals $\big(0,g_i\big(\frac{b_i-a_i}{x_i}(0)\big)-\frac{g_i(a_i-b_i)}{x_i}(0)\big)$ if $p=(1:\alpha)$, respectively $\big(g_i\big(\frac{a_i}{y_i}(0)\big)-\frac{g_ia_i}{y_i}(0),0\big)$ if $p=(0:1)$. In any case, it has no constant term. It follows that

$$\left(\widetilde{\boldsymbol{g}\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{g}\boldsymbol{b}}\frac{\partial}{\partial v}\right)-\boldsymbol{g}\left(\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial v}\right)\in\mathfrak{m}_{\widetilde{C},\widetilde{p}}\frac{\partial}{\partial u}\oplus\mathfrak{m}_{\widetilde{C},\widetilde{p}}\frac{\partial}{\partial v}\subset I^{es}_{\widetilde{\varphi}}$$

and, hence, $\widetilde{\boldsymbol{g}\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{g}\boldsymbol{b}}\frac{\partial}{\partial v}\in I^{es}_{\widetilde{\varphi}}$. By (1), we conclude $\boldsymbol{g}\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{g}\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$ which proves the claim. □

Lemma 2.41. *Let $(T',\mathbf{0})\subset(T,\mathbf{0})$ be a small extension of germs with ε a vector space generator of* $\ker(\mathcal{O}_{T,\mathbf{0}}\twoheadrightarrow\mathcal{O}_{T',\mathbf{0}})$. *Let*

$$(\phi',\overline{\sigma}',\sigma')\in\mathcal{D}ef^{es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T',\mathbf{0}),$$

with $\overline{\sigma}',\sigma'$ the trivial sections, and with ϕ' given by $X_i',Y_i'\in t_i\mathcal{O}_{T',\mathbf{0}}\{t_i\}$, $i=1,\dots,r$. Furthermore, let $(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{m}_{\overline{C},\overline{0}}\oplus\mathfrak{m}_{\overline{C},\overline{0}}$, and let ϕ be the deformation over $(T,\mathbf{0})$ given by $X_i=X_i'+\varepsilon a_i$, $Y_i=Y_i'+\varepsilon b_i$, with trivial sections $\overline{\sigma},\sigma$. Then

$$(\phi,\overline{\sigma},\sigma)\in\mathcal{D}ef^{es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})\iff\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}.$$

Proof. Let $(\phi,\overline{\sigma},\sigma)$ be equisingular, let $p\in\widetilde{M}$ be an infinitely near point belonging to $(C,\mathbf{0})$, and let $\widetilde{\phi}_p:\big(\overline{\mathscr{C}},\overline{p}\big)\to\big(\widetilde{\mathscr{M}},p\big)$, $\overline{\sigma}_p,\sigma_p$ be as in Definition 2.36. With respect to local coordinates of $\big(\widetilde{\mathscr{M}},p\big)$, $\widetilde{\phi}_p$ is given by $U_i'+\varepsilon\widetilde{a}_i$, $V_i'+\varepsilon\widetilde{b}_i$, and its restriction to $(T',\mathbf{0})$, $\widetilde{\phi}_p'$, is given by U_i',V_i'.

Then U_i',V_i' is equimultiple and, hence, $\operatorname{ord}\widetilde{a}_i,\operatorname{ord}\widetilde{b}_i\geq\min\{\operatorname{ord}u_i,\operatorname{ord}v_i\}$ with u_i,v_i a parametrization of $\big(\widetilde{C},p\big)$, that is, $u_i+\varepsilon\widetilde{a}_i$, $v_i+\varepsilon\widetilde{b}_i$ is equimultiple over T_ε. It follows that $x_i+\varepsilon a_i$, $y_i+\varepsilon b_i$ is equisingular over T_ε. Hence, $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$.

Conversely, let $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$. We argue again by induction on the number of blowing ups needed to resolve $(C,\mathbf{0})$, the case of a smooth germ $(C,\mathbf{0})$ being trivial. As X_i',Y_i' is equisingular over $(T',\mathbf{0})$, it is equimultiple, hence, X_i,Y_i is equimultiple, too. Blowing up the trivial section, we get that

$$U_i=U_i'+\varepsilon\widetilde{a}_i\,,\quad V_i=V_i'+\varepsilon\widetilde{b}_i\,,$$

with (a_i, b_i) and $\big(\widetilde{a}_i, \widetilde{b}_i\big)$ related as in Proposition 2.40, defines a deformation of $\big(\widetilde{C}, p\big)$ over $(T, \mathbf{0})$ by (2.5.19), respectively (2.5.20), in the proof of Proposition 2.39. By induction, this deformation is equisingular and, hence, as $\operatorname{ord} a_i, \operatorname{ord} b_i \geq \min\{\operatorname{ord} x_i, \operatorname{ord} y_i\}$, X_i, Y_i define an equisingular deformation of $(C, \mathbf{0})$. $\square$

Lemma 2.42. *Let* $(\phi, \overline{\sigma}, \sigma) \in \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$*, and let* T_N *denote the fat point given by* $\mathcal{O}_{T,\mathbf{0}}/\mathfrak{m}_{T,\mathbf{0}}^{N+1}$*,* $N \geq 0$*. Then* $(\phi, \overline{\sigma}, \sigma)$ *is equisingular iff it is* formally equisingular*, that is, iff, for each* $N \geq 1$*, the restriction to* T_N*,* $(\phi_N, \overline{\sigma}_N, \sigma_N)$ *is equisingular over* T_N*.*

Proof. Since the necessity is obvious, let $(\phi_N, \overline{\sigma}_N, \sigma_N)$ be equisingular over T_N, for all $N \geq 0$. It follows that, for all $N \geq 0$, ϕ_N is equimultiple along σ_N, hence ϕ itself is equimultiple along σ. Therefore, we can blow up σ and obtain, for each point p in the first infinitely near neighbourhood of $\mathbf{0}$ belonging to $(C, \mathbf{0})$, a deformation $\widetilde{\phi}_p : \big(\overline{\mathscr{C}}, \overline{p}\big) \to \big(\widetilde{\mathscr{M}}, p\big)$ of the strict transform $\big(\widetilde{C}, p\big)$ of $(C, \mathbf{0})$ at p along the sections $\overline{\sigma}_p, \sigma_p$ (see the considerations before Definition 2.36).

The restriction to T_N, $\big(\widetilde{\phi}_{p,N}, \overline{\sigma}_{p,N}, \sigma_{p,N}\big)$, is equisingular, hence equimultiple for all $N \geq 0$. Hence, $\widetilde{\phi}_p$ is equimultiple along σ_p, and we can continue in the same manner. Since an arbitrary infinitely near point belonging to $(C, \mathbf{0})$ is obtained by a finite number of blowing ups, the result follows by induction on this number. $\square$

Proof of Theorem 2.38. (1) Since X_i, Y_i mod $\langle s_1, \ldots, s_j^2, \ldots, s_k\rangle$, $j = 1, \ldots, k$, define an equisingular deformation over T_ε, and since we can apply Lemma 2.41, the necessity is obvious.

For the sufficiency, let $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in I^{es}_{\varphi}$. Since each extension of Artinian local rings factors through small extensions, it follows from Lemma 2.41 that ϕ mod $\langle \boldsymbol{s}\rangle^{N+1}$ is equisingular over the fat point $T_N = \big(\{\mathbf{0}\}, \mathcal{O}_{T,\mathbf{0}}/\langle \boldsymbol{s}\rangle^{N+1}\big)$. Now, apply Lemma 2.42.

As (3) is an immediate consequence of (2), it remains to prove (2): Let ϕ be versal (respectively semiuniversal), and let $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in I^{es}_{\varphi}$. Then the equisingular deformation $(x_i + \varepsilon a_i, y_i + \varepsilon b_i)_{i=1}^r$ can be induced (respectively uniquely induced) from ϕ. Hence, the class of $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y}$ in $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,0)}$ is a linear combination (respectively a unique linear combination) of $\big(\frac{\partial X_i}{\partial s_j}(t_i, \mathbf{0})\big)_{i=1}^r \frac{\partial}{\partial x} + \big(\frac{\partial Y_i}{\partial s_j}(t_i, \mathbf{0})\big)_{i=1}^r \frac{\partial}{\partial y}$, $j = 1, \ldots, r$. This shows that the condition is indeed necessary.

For the other direction, we have only to show that $(\phi, \overline{\sigma}, \sigma)$ with $\sigma, \overline{\sigma}$ denoting the trivial sections, is formally versal by [Fle1, Satz 5.2] (see also Theorem 1.13).

Thus, it is sufficient to consider a small extension $(Z', \mathbf{0}) \subset (Z, \mathbf{0})$, with $\varepsilon\mathbb{C}$ being the kernel of $A = \mathcal{O}_{Z,\mathbf{0}} \twoheadrightarrow \mathcal{O}_{Z',\mathbf{0}} =: A'$.

Let $(\psi, \overline{\tau}, \tau) \in \mathcal{D}ef^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(Z, \mathbf{0})$ with trivial sections $\tau, \overline{\tau}$, such that the restriction $(\psi', \overline{\tau}', \tau') \in \mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(Z', \mathbf{0})$ is induced from $(\psi, \overline{\tau}, \tau)$ by some morphism $\eta' : (Z', \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$. We have to show that $(\psi, \overline{\tau}, \tau)$ is isomorphic to the pull-back of $(\phi, \overline{\sigma}, \sigma)$ by some morphism $\eta : (Z, \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ extending η'. By Remark 2.37.1 (2), we may assume that $(Z', \mathbf{0})$ is smooth, that is, we may assume that $A' = \mathbb{C}\{\boldsymbol{z}\}$, $\boldsymbol{z} = (z_1, \ldots, z_n)$, and that

$$A = \mathbb{C}\{\boldsymbol{z}, \varepsilon\}/\langle z_1\varepsilon, \ldots, z_n\varepsilon, \varepsilon^2\rangle \,.$$

The pull-back map $\eta'^*\phi : \big(\overline{C} \times Z', \mathbf{0}\big) \to (\mathbb{C}^2 \times Z', \mathbf{0})$ is then given by the power series $X_i\big(t_i, \eta'(\boldsymbol{z})\big)$, $Y_i\big(t_i, \eta'(\boldsymbol{z})\big)$.

Let ψ' be given by $U_i'\big(t_i, \boldsymbol{z}\big), V_i'\big(t_i, \boldsymbol{z}\big) \in A'\{t_i\}$, and let ψ be given by

$$U_i = U_i' + \varepsilon u_i \,, \quad V_i = V_i' + \varepsilon v_i \in A\{t_i\} \,, \quad u_i, v_i \in \mathbb{C}\{t_i\} \,,$$

with

$$\big(U_i(t_i), V_i(t_i)\big) \equiv \big(x_i(t_i), y_i(t_i)\big) \bmod \mathfrak{m}_A \,.$$

The morphism $\eta' : (Z', \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ is given by $\eta' = (\eta_1, \ldots, \eta_k)$, $\eta_i \in \mathbb{C}\{\boldsymbol{z}\}$, and the extension $\eta : (Z, \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ is then given by

$$\eta = \eta' + \varepsilon\eta^0 \,, \quad \eta^0 = (\eta_1^0, \ldots, \eta_k^0) \in \mathbb{C}^k \,.$$

The assumption says that there is

- an A'-automorphism H' of $A'\{x, y\} = \mathbb{C}\{x, y, \boldsymbol{z}\}$, $x \mapsto H_1'$, $y \mapsto H_2'$, with $H_1', H_2' \in \langle x, y\rangle A'\{x, y\}$, and
- an A'-automorphism h' of $\bigoplus_{i=1}^r A'\{t_i\}$, $t_i \mapsto h_i' \in t_iA'\{t_i\} = t_i\mathbb{C}\{t_i, \boldsymbol{z}\}$,

with H' and h' being the identity modulo $\mathfrak{m}_{A'}$, such that the following holds for $i = 1, \ldots, r$:

$$X_i(t_i, \eta') = H_1'\big(U_i'(h_i'), V_i'(h_i')\big) \,, \quad Y_i(t_i, \eta') = H_2'\big(U_i'(h_i'), V_i'(h_i')\big) \,. \tag{2.5.25}$$

We have to extend η', H' and h' over $(Z, \mathbf{0})$ such that these equations extend, too. That is, we have to show the existence of $\eta^0 = (\eta_1^0, \ldots, \eta_k^0) \in \mathbb{C}^k$, $H_1^0, H_2^0 \in \langle x, y\rangle\mathbb{C}\{x, y\}$, $h^0 = (h_1^0, \ldots h_r^0) \in \bigoplus_{i=1}^r t_i\mathbb{C}\{t_i\}$, such that

$$X_i(t_i, \eta' + \varepsilon\eta^0) = (H_1' + \varepsilon H_1^0)\big(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0)\big) \,, \tag{2.5.26}$$

$$Y_i(t_i, \eta' + \varepsilon\eta^0) = (H_2' + \varepsilon H_2^0)\big(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0)\big) \,. \tag{2.5.27}$$

Applying Taylor's formula, and using that $\varepsilon\mathfrak{m}_A = 0$, we obtain

$$\begin{aligned} X_i(t_i, \eta' + \varepsilon\eta^0) &= X_i(t_i, \eta') + \varepsilon\sum_{j=1}^k \frac{\partial X_i}{\partial s_j}(t_i, \eta') \cdot \eta_j^0 \\ &= X_i'(t_i, \eta') + \varepsilon\sum_{j=1}^k \frac{\partial X_i}{\partial s_j}(t_i, \mathbf{0}) \cdot \eta_j^0 \,, \end{aligned} \tag{2.5.28}$$

$$Y_i(t_i, \eta' + \varepsilon\eta^0) = Y_i'(t_i, \eta') + \varepsilon\sum_{j=1}^k \frac{\partial Y_i}{\partial s_j}(t_i, \mathbf{0}) \cdot \eta_j^0 \,.$$

Moreover, with $\dot{}$ denoting the derivative with respect to t_i,

$$\begin{aligned} U_i(h'_i+\varepsilon h^0_i) &= U_i(h'_i)+\varepsilon\dot{U}_i(h'_i)\cdot h^0_i \;=\; U'_i(h'_i)+\varepsilon\big(\dot{x}_i h^0_i+u_i\big)\,,\\ V_i(h'_i+\varepsilon h^0_i) &= V'_i(h'_i)+\varepsilon\big(\dot{y}_i h^0_i+v_i\big)\,. \end{aligned}$$

Since H' is the identity mod $\mathfrak{m}_{A'}$, we have

$$\frac{\partial H'_1}{\partial x}=1 \bmod \mathfrak{m}_{A'}\,, \qquad \frac{\partial H'_1}{\partial y}\in \mathfrak{m}_{A'}A'\{x,y\}\,.$$

In particular, $\varepsilon\cdot\frac{\partial H'_1}{\partial y}=0$. Applying again Taylor's formula, and using that $h'=\mathrm{id} \bmod \mathfrak{m}_{A'}$, the right-hand side of (2.5.26) equals

$$\begin{aligned} &(H'_1+\varepsilon H^0_1)\Big(U'_i(h'_i)+\varepsilon\big(\dot{x}_i h^0_i+u_i\big),\, V'_i(h'_i)+\varepsilon\big(\dot{y}_i h^0_i+v_i\big)\Big)\\ &\quad= H'_1\big(U'_i(h'_i),V'_i(h'_i)\big)+\varepsilon\left(H^0_1\big(U'_i(h'_i),V'_i(h'_i)\big)+1\cdot\big(\dot{x}_i h^0_i+u_i\big)\right)\\ &\quad= H'_1\big(U'_i(h'_i),V'_i(h'_i)\big)+\varepsilon\big(H^0_1(x_i,y_i)+\dot{x}_i h^0_i+u_i\big)\,, \end{aligned} \tag{2.5.29}$$

and similar for the right-hand side of (2.5.27).

Using (2.5.25), (2.5.28) and (2.5.29), we have to find $(\eta^0_1,\dots,\eta^0_k)\in\mathbb{C}^k$, $H^0_1,H^0_2\in\langle x,y\rangle\mathbb{C}\{x,y\}$, and $h^0_i\in t_i\mathbb{C}\{t_i\}$, such that

$$\begin{aligned} \big(u_i(t_i),v_i(t_i)\big) &= \sum_{j=1}^{k}\eta^0_j\cdot\left(\frac{\partial X_i}{\partial s_j}(t_i,\mathbf{0}),\frac{\partial Y_i}{\partial s_j}(t_i,\mathbf{0})\right)-h^0_i(t_i)\cdot\Big(\dot{x}_i(t_i),\dot{y}_i(t_i)\Big)\\ &\qquad -\Big(H^0_1\big(x_i(t_i),y_i(t_i)\big),\,H^0_2\big(x_i(t_i),y_i(t_i)\big)\Big)\,. \end{aligned} \tag{2.5.30}$$

Since $(\psi,\overline{\tau},\tau)$, with ψ given by $U'_i+\varepsilon u_i$, $V'_i+\varepsilon v_i$, is equisingular, Lemma 2.41 gives that $(u_i)_{i=1}^r\frac{\partial}{\partial x}+(v_i)_{i=1}^r\frac{\partial}{\partial y}\in I^{es}_{\varphi}$. But then the assumption implies that (2.5.30) can be solved (respectively solved with unique $\eta^0_1,\dots,\eta^0_k$). This proves that $(\phi,\overline{\sigma},\sigma)$ is versal (respectively semiuniversal). $\square$

The fact that I^{es}_{φ} is a module provides an easy proof of the *openness of versality* for equisingular deformations. Consider an *equisingular family of parametrizations* of reduced plane curve singularities over some complex space S. That is, we have morphisms of complex spaces

$$\overline{\mathscr{C}}\xrightarrow{\ \phi\ }\mathscr{M}\xrightarrow{\ \mathrm{pr}\ }S$$

with pr and $\mathrm{pr}\circ\phi$ being flat, ϕ being finite, together with a section $\sigma:S\to\mathscr{M}$ and a multisection $\overline{\sigma}=\big(\overline{\sigma}_i\big)_{i=1}^r:S\to\overline{\mathscr{C}}$, such that, for each $s\in S$, and $\mathscr{M}_s:=pr^{-1}(s)$, $\overline{\mathscr{C}}_s:=(pr\circ\phi)^{-1}(s)$, the following holds:

- $\big(\mathscr{M}_s,\sigma(s)\big)\cong(\mathbb{C}^2,\mathbf{0})\,,$
- $\phi\big(\overline{\sigma}_i(s)\big)=\sigma(s)$, $\big(\overline{\mathscr{C}}_s,\overline{\sigma}_i(s)\big)\cong(\mathbb{C},0)$, $i=1,\dots,r\,,$

- the restriction $\phi_s : (\overline{\mathscr{C}}_s, \overline{\sigma}(s)) = \coprod_{i=1}^r (\overline{\mathscr{C}}_s, \overline{\sigma}_i(s)) \to (\mathscr{M}_s, \sigma(s))$ is the parametrization of a reduced plane curve singularity $(\mathscr{C}_s, \sigma(s))$ with r branches, and
- $\phi : (\overline{\mathscr{C}}, \overline{\sigma}(s)) \to (\mathscr{M}, \sigma(s))$ is an equisingular deformation of the parametrization ϕ_s.

We say that a family $\overline{\mathscr{C}} \xrightarrow{\phi} \mathscr{M} \xrightarrow{\mathrm{pr}} S$ as above is *equisingular* (resp. *equisingular-versal*) along $(\sigma, \overline{\sigma})$ at s, if $\phi : (\overline{\mathscr{C}}, \overline{\sigma}(s)) \to (\mathscr{M}, \sigma(s))$, together with the germs of $\overline{\sigma}$ and σ, is an equisingular (resp. a versal equisingular) deformation of ϕ_s.

More generally, let $\sigma = (\sigma^{(1)}, \dots, \sigma^{(\ell)})$ be a finite set of disjoint sections, $\sigma^{(i)} : S \to \mathscr{M}$, and $\overline{\sigma} = (\overline{\sigma}^{(1)}, \dots, \overline{\sigma}^{(\ell)})$ be disjoint multisections, $\overline{\sigma}^{(i)} : S \to \overline{\mathscr{C}}$, $\overline{\sigma}^{(i)} = (\overline{\sigma}_j^{(i)})_{j=1}^{r_i}$. If $\overline{\mathscr{C}} \to \mathscr{M} \to S$ is equisingular (resp. equisingular-versal) along $(\sigma^{(i)}, \overline{\sigma}^{(i)})$ for $i = 1, \dots, \ell$ at each $s \in S$, then we say that $\overline{\mathscr{C}} \to \mathscr{M} \to S$ is an *equisingular* (resp. *equisingular-versal*) *family of parametrizations* of reduced plane curve singularities (along $(\sigma, \overline{\sigma})$).

Theorem 2.43. *Let $\overline{\mathscr{C}} \xrightarrow{\phi} \mathscr{M} \xrightarrow{\mathrm{pr}} S$ be an equisingular family of parametrizations of reduced plane curve singularities over some complex space S. Then the set of points $s \in S$ such that the family is equisingular-versal at s is analytically open in S.*

Proof. Since the set in question for several sections is the intersection of the corresponding sets for each section, we may assume that σ is just one section. Let $I_{\overline{\sigma}} \subset \mathcal{O}_{\overline{\mathscr{C}}}$ denote the ideal sheaf of the section $\overline{\sigma}$. Then we define a subsheaf $\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ of $I_{\overline{\sigma}} \cdot \phi^* \operatorname{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathscr{M}}, \mathcal{O}_{\mathscr{M}}) = I_{\overline{\sigma}} \cdot \phi^* \Theta_{\mathscr{M}/S}$, as follows: For $s \in S$ and local coordinates x, y of $\mathscr{M}_s$ at $\sigma(s)$ and t_i of $\overline{\mathscr{C}}_s$ at $\overline{\sigma}_i(s)$, ϕ is given near $\overline{\sigma}(s)$ by $X_i, Y_i \in \mathcal{O}_{S,s}\{t_i\}$. Moreover, a local section of $I_{\overline{\sigma}} \cdot \phi^* \operatorname{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathscr{M}}, \mathcal{O}_{\mathscr{M}})$ is given by $(a_i)_{i=1}^r \frac{\partial}{\partial x} + (b_i)_{i=1}^r \frac{\partial}{\partial y}$, $a_i, b_i \in \mathcal{O}_{S,s}\{t_i\}$. The local sections of the sheaf $\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ are, by definition, those local sections of $I_{\overline{\sigma}} \cdot \phi^* \operatorname{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathscr{M}}, \mathcal{O}_{\mathscr{M}})$, for which $X_i + a_i, Y_i + b_i$ defines an equisingular deformation of ϕ_s over the germ (S, s). Since equimultiplicity in the infinitely near points of $(\mathscr{M}, \sigma(s))$ belonging to $(\mathscr{C}_s, \sigma(s))$ is preserved near s, $X_i + a_i, Y_i + b_i$ also define an equisingular deformation of $\phi_{s'}$ over (S, s') for s' in some open neighbourhood of s. It follows that $\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ is, indeed, a sheaf, and that the stalk at s generates the stalks at s' close to s. Hence, $\phi_* \mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ is a coherent $\mathcal{O}_{\mathscr{M}}$-module by Proposition 2.40 and A.7.

Consider the quotient sheaf

$$\mathcal{T}^{1,es}_{\overline{\mathscr{C}} \to \mathscr{M}} = (\mathrm{pr} \circ \phi)_* \Big(\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}} \Big/ \big(I_{\overline{\sigma}} \Theta_{\overline{\mathscr{C}}/S} + \phi^{-1}(I_{\overline{\sigma}} \Theta_{\mathscr{M}/S}) \big) \Big) ,$$

which is a coherent $\mathcal{O}_S$-sheaf, since the support of the sheaf to which $(\mathrm{pr} \circ \phi)_*$ is applied is finite over S. In local coordinates x, y and t_i, the image of $(\mathrm{pr} \circ \phi)_* \Theta_{\overline{\mathscr{C}}/S} = (\mathrm{pr} \circ \phi)_* \bigoplus_{i=1}^r \mathcal{O}_{\overline{\mathscr{C}}} \frac{\partial}{\partial t_i}$ in $(\mathrm{pr} \circ \phi)_* \phi^* \Theta_{\mathscr{M}/S}$ is generated by $\big(\dot{X}_i \frac{\partial}{\partial x} + \dot{Y}_i \frac{\partial}{\partial y}\big)_{i=1}^r$. Hence, the stalk at s of $\mathcal{T}^{1,es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ equals $T^{1,es}_{(\overline{\mathscr{C}}_s, \overline{\sigma}(s)) \to (\mathscr{M}, \sigma(s))}$.

Moreover, we have the *"Kodaira-Spencer map"*

$$\Theta_S \longrightarrow \mathcal{T}^{1,es}_{\overline{\mathscr{C}}\to\mathscr{M}}\,,$$

which maps $\delta \in \Theta_{S,s}$ to $\left(\delta(X_i)\frac{\partial}{\partial x} + \delta(X_i)\frac{\partial}{\partial y}\right)_{i=1}^{r}$ in local coordinates. Theorem 2.38 (2) implies that the cokernel of this map has support at points $s \in S$, where ϕ is not equisingular-versal. But since the cokernel is coherent, this support is analytically closed, which proves the theorem. □

To compute a semiuniversal equisingular deformation of $\varphi : (\overline{C}, \overline{0}) \to (\mathbb{C}^2, \mathbf{0})$, we only need to compute a basis of $T^{1,es}_{\varphi}$ by Theorem 2.38. Moreover, if all branches of $(C, \mathbf{0})$ have different tangents, Remark 2.11.1 gives

$$T^{1,es}_{\varphi} = \bigoplus_{i=1}^{r} T^{1,es}_{\varphi_i}\,,$$

where φ_i is the parametrization of the i-th branch of $(C, \mathbf{0})$. In general, $T^{1,es}_{\varphi}$ can be computed, as a subspace of M^{em}_{φ}, by following the lines of the proof of Proposition 2.39.

Example 2.43.1. (1) Consider the parametrization $\varphi : t \mapsto (t^2, t^7)$ of an A_6-singularity. By Example 2.27.1, M^{em}_{φ} has the basis $\left\{t^3\frac{\partial}{\partial y}, t^5\frac{\partial}{\partial y}\right\}$. Blowing up the trivial section of $X(t, \boldsymbol{s}) = t^2$, $Y(t, \boldsymbol{s}) = t^7 + s_1t^3 + s_2t^5$, we get

$$U(t, \boldsymbol{s}) = t^2\,, \quad V(t, \boldsymbol{s}) = \frac{Y(t, \boldsymbol{s})}{X(t, \boldsymbol{s})} = t^5 + s_1t + s_2t^3\,,$$

which is equimultiple along the trivial section iff $s_1 = 0$. Blowing up once more, we get the necessary condition $s_2 = 0$ for equisingularity. Hence, $T^{1,es}_{\varphi} = 0$, as it should be, since A_6 is a simple singularity.

(2) For $\varphi : t \mapsto (t^3, t^7)$, a basis for the $\mathbb{C}$-vector space M^{em}_{φ} is given by $\left\{t^4\frac{\partial}{\partial y}, t^5\frac{\partial}{\partial y}, t^8\frac{\partial}{\partial y}\right\}$, respectively by $\left\{t^4\frac{\partial}{\partial x}, t^4\frac{\partial}{\partial y}, t^5\frac{\partial}{\partial y}\right\}$. Blowing up the trivial section, only $t^8\frac{\partial}{\partial y}$, respectively $t^4\frac{\partial}{\partial x}$, survives for an equimultiple deformation. It also survives in further blowing ups. Hence, $X(t, s) = t^3$, $Y(t, s) = t^5 + st^8$ (respectively $X(t, s) = t^3 + st^4$, $Y(t, s) = t^5$) is a semiuniversal equisingular deformation of φ.

(3) Reconsider the parametrization given in Example 2.27.1 (2):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 \end{pmatrix}, \quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}.$$

A semiuniversal equimultiple deformation is given by

$$\begin{pmatrix} X_1(t, \boldsymbol{s}) \\ X_2(t, \boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 + s_1t^3 + s_2t^4 + s_3t^6 + s_4t^9 \end{pmatrix}, \quad \begin{pmatrix} Y_1(t, \boldsymbol{s}) \\ Y_2(t, \boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}.$$

Blowing up the trivial section shows that only the parameters s_3 and s_4 survive. These survive also in subsequent blowing up steps. Hence,

$$\begin{pmatrix} X_1(t,\boldsymbol{s}) \\ X_2(t,\boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 + s_3 t^6 + s_4 t^9 \end{pmatrix}, \quad \begin{pmatrix} Y_1(t,\boldsymbol{s}) \\ Y_2(t,\boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}.$$

is a semiuniversal equisingular deformation.

2.6 Equinormalizable Deformations

We show in this section that each deformation of the normalization of a reduced plane curve singularity $(\overline{C},\overline{0}) \to (C,\mathbf{0})$ induces a δ-constant deformation of $(C,\mathbf{0})$. Conversely, if the base space of a δ-constant deformation of $(C,\mathbf{0})$ is normal, then the deformation is equinormalizable, that is, it lifts to a deformation of the normalization $(\overline{C},\overline{0}) \to (C,\mathbf{0})$ and, as such, it induces a simultaneous normalization of each fibre. Hence, over a normal base space, a deformation of $(C,\mathbf{0})$ admits a simultaneous normalization of each of its fibres iff the total δ-invariant of the fibres is constant.

The study of equinormalizable deformations has been initiated by Teissier in the 1970's. The main results of this section are Theorem 2.54 and Theorem 2.56 due to Teissier, resp. to Teissier and Raynaud [Tei]. A generalization to families with (projective) fibres of arbitrary positive dimension was recently given by Chiang-Hsieh and Lipman [ChL]. They also give a complete treatment for families of curve singularities, clarifying some points in the proof given in [Tei]. We follow closely the presentation in [ChL] which is basically algebraic. By working directly in the complex analytic setting, we can avoid technical complications that appear when working with general schemes.

We consider first arbitrary morphisms $f: X \to S$ of complex spaces. Such a morphism f is called *reduced* (resp. *normal*) if it is flat and if all non-empty fibres are reduced (resp. normal).

Definition 2.44. Let $f: X \to S$ be a reduced morphism of complex spaces.

A *simultaneous normalization* of f is a finite morphism $\nu: Z \to X$ of complex spaces such that $\overline{f} = f \circ \nu : Z \to S$ is normal and that, for each $s \in f(X)$, the induced map on the fibres $\nu_s : Z_s = \overline{f}^{-1}(s) \to X_s = f^{-1}(s)$ is the normalization of X_s.

We say that an arbitrary morphism $f: X \to S$ of complex spaces *admits a simultaneous normalization* if it is reduced and if there exists a simultaneous normalization of f.

The morphism f is called *equinormalizable* if X is reduced and if the normalization $\nu: \overline{X} \to X$ of X is a simultaneous normalization of f. We call f *equinormalizable at* $x \in X$, if $\overline{f} = f \circ \nu : \overline{X} \to S$ is flat at each point of the fibre $\nu^{-1}(x)$ and if, for $s = f(x)$, the induced map $\nu_s : \overline{f}^{-1}(s) =: \overline{X}_s \to X_s$ is the normalization. A morphism $(X,x) \to (S,s)$ of complex space germs is *equinormalizable* if it has a representative which is equinormalizable at x. We shall show below that, under some mild assumptions, equinormalizability is an open property.

Remark 2.44.1. If $\nu : Z \to S$ is a simultaneous normalization of f, then, for each $s \in S$ and $x \in X_s = f^{-1}(s)$, the diagram

$$\begin{array}{ccc} (Z_s, \overline{z}) & \hookrightarrow & (Z, \overline{z}) \\ {\scriptstyle \nu_s}\downarrow & & \downarrow{\scriptstyle \nu} \\ (X_s, x) & \hookrightarrow & (X, x) \\ \downarrow & & \downarrow{\scriptstyle f} \\ \{s\} & \hookrightarrow & (S, s) \end{array}$$

is a deformation of the normalization map $\nu_s : (Z_s, \overline{z}) \to (X_s, x)$, that is, an object of $\mathcal{D}ef_{(Z_s,\overline{z})\to(X_s,x)}(S, s)$ in the sense of Definition 2.20. Here, $(Z, \overline{z})$ denotes the multigerm of Z at $\overline{z} = \nu^{-1}(x)$.

First, we show that a simultaneous normalization does not modify X at normal points of the fibres.

Lemma 2.45. *Let $\nu : Z \to X$ be a simultaneous normalization of the reduced morphism $f : X \to S$, and let*

$$\mathrm{NNor}(f) := \{x \in X \mid x \textit{ is a non-normal point of the fibre } f^{-1}(f(x))\}\,.$$

Then $N := \mathrm{NNor}(f)$ is analytic and nowhere dense in X, and the restriction $\nu : Z \setminus \nu^{-1}(N) \to X \setminus N$ is biholomorphic.

Proof. Since f is flat, $\mathrm{NNor}(f)$ is the set of non-normal points of f, which is analytic by Theorem I.1.100. Since the fibres of f are reduced, hence generically smooth, every component of X contains points outside of N. It follows that N is nowhere dense in X.

For $x \in X$, $s = f(x)$, the restriction $\nu_s : Z_s \to X_s$ is the normalization of the fibre X_s by assumption. For $x \notin N$, the germ (X_s, x) is normal and, therefore, $\nu^{-1}(x)$ consists of exactly one point $z \in Z$ and $\nu_s : (Z_s, z) \to (X_s, x)$ is an isomorphism of germs. Since $f \circ \nu$ is flat, $\nu : (Z, z) \to (X, x)$ is an isomorphism, too, by Lemma I.1.86. This shows that $\nu : Z \setminus \nu^{-1}(N) \to X \setminus N$ is bijective and locally an isomorphism, hence biholomorphic. □

Proposition 2.46. *Let $f : X \to S$ be a morphism of complex spaces.*

(1) If f is reduced, then X is reduced iff S is reduced at each point of the image $f(X)$.
(2) If f is normal, then X is normal iff S is normal at each point of the image $f(X)$.

Proof. Since each reduced (resp. normal) morphism is flat, the statement follows immediately from Theorem B.8.20. □

In particular, if f admits a simultaneous normalization $\nu : Z \to X$, then Z is normal iff S is normal at each point of $f(X)$ (apply Proposition 2.46 to $f \circ \nu$).

Corollary 2.47. *Let $f : X \to S$ be a reduced morphism of complex spaces. If f is equinormalizable at $x \in X$, then S is normal at $f(x)$.*

The corollary implies that a normal morphism need not be equinormalizable: If $f : X \to S$ is a normal morphism, then id_X is a simultaneous normalization of f, independent of S. However, if S is not normal at some point $f(x)$, then X is not normal at x. If $\nu : \overline{X} \to X$ is the normalization then $f \circ \nu$ is not normal by Proposition 2.46 and, hence, f is not equinormalizable at x. For example, by Theorem I.1.100, each small representative $f : X \to S$ of a deformation of a normal singularity X_0 (e.g., an isolated hypersurface singularity of dimension at least 2) is a normal morphism and, hence, equinormalizable if and only if S is normal.

The following example shows that for a non-normal base space S strange things can happen:

Example 2.47.1. Let $F(x, y, u, v) = x^3 + y^2 + ux + v$ be the semiuniversal unfolding of the cusp $C = \{x^3 + y^2 = 0\}$, let $D = 4u^3 + 27v^2$ be the discriminant equation of the projection $\pi : V(F) \to \mathbb{C}^2$, $(x, y, u, v) \mapsto (u, v)$, and consider

$$f : X = V(F, D) \to \varDelta = V(D) \subset \mathbb{C}^2 .$$

That is, f is the restriction of the semiuniversal deformation π of $(C, \mathbf{0})$ over the discriminant $\varDelta$ which is, in this case, the δ-constant stratum (see page 355).

Then f is reduced, but f is not equinormalizable because otherwise $\varDelta$ has to be normal by Proposition 2.46. To see what happens, note that the normalization map is $\nu : \overline{X} = \mathbb{C}^2 \to X \subset \mathbb{C}^4$, given by

$$\big(T_1, T_2\big) \stackrel{\nu}{\longmapsto} (x, y, u, v) = \left(-\tfrac{1}{81}T_2^2 - \tfrac{4}{3}T_1,\ \tfrac{1}{729}T_2^2 + \tfrac{2}{27}T_1T_2,\ -\tfrac{4}{27}T_1^2,\ \tfrac{16}{729}T_1^3\right).$$

The map $\overline{f} = f \circ \nu$, given by the last two components of ν, has $V(T_1^2)$ as special fibre, which is not reduced. The morphism $\overline{f}$ is also not flat, since $\mathcal{O}_{\overline{X},\mathbf{0}} \otimes_{\mathcal{O}_{\varDelta,\mathbf{0}}} \mathfrak{m}_{\varDelta,\mathbf{0}} \to \mathcal{O}_{\overline{X},\mathbf{0}} = \mathbb{C}\{T_1, T_2\}$ is not injective (the non-zero element $\frac{27}{4}T_1 \otimes u + \frac{729}{16} \otimes v$ is mapped to zero).

This family does not even admit a simultaneous normalization $\nu : Z \to X$ with Z non-normal. Otherwise, the corresponding deformation of the normalization of the cusp (see Remark 2.44.1) could be induced by a morphism $(\varDelta, \mathbf{0}) \to (\mathbb{C}, 0)$, where $(\mathbb{C}, 0)$ is the base space of the semiuniversal deformation of the normalization of $\varDelta$ (see Proposition 2.27). By the semiuniversality of the deformations, the tangent map of the composition $(\varDelta, \mathbf{0}) \to (\mathbb{C}, 0) \to (\varDelta, \mathbf{0})$ must be the identity, contradicting the fact that the tangent map of the normalization $(\mathbb{C}, 0) \to (\varDelta, \mathbf{0})$ is the zero map.

Exercise 2.6.1. Recompute Example 2.47.1 by using SINGULAR. First compute the singular locus, then the discriminant by eliminating x and y, and finally the normalization of X using the library `normal.lib`.

We start by studying the equinormalizability condition locally.

Lemma 2.48. *Let $f:(X,x)\to(S,s)$ be a flat morphism of complex space germs with reduced fibre (X_s,x) and with reduced base (S,s). Further, let $(\overline{X},\overline{x})\xrightarrow{\nu}(X,x)$ be the normalization of (X,x), let $(\overline{X}_s,\overline{x})$ be the fibre of $\overline{f}=f\circ\nu$, let $\nu_s:(\overline{X}_s,\overline{x})\to(X_s,x)$ be the restriction of ν, and let $n:\big(\widetilde{X}_s,\widetilde{x}\big)\to(X_s,x)$ be the normalization of (X_s,x). Set*

$$\begin{aligned}\mathcal{O}&:=\mathcal{O}_{X,x}\,, & \overline{\mathcal{O}}&:=\nu_*\mathcal{O}_{\overline{X},\overline{x}}\,,\\ \mathcal{O}_s&:=\mathcal{O}_{X_s,x}\,, & \overline{\mathcal{O}}_s&:=\nu_{s*}\mathcal{O}_{\overline{X}_s,\overline{x}}\,, \quad \widetilde{\mathcal{O}}_s:=n_*\mathcal{O}_{\widetilde{X}_s,\widetilde{x}}\,.\end{aligned}$$

Then there is an $h\in\mathcal{O}$ such that the following holds:

(1) h is a non-zerodivisor of $\mathcal{O}$, $\overline{\mathcal{O}}$, $\mathcal{O}_s$, $\widetilde{\mathcal{O}}_s$ and $h\widetilde{\mathcal{O}}_s\subset\mathcal{O}_s$. Moreover, $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat.

(2) If $(\overline{X}_s,\overline{x})$ is reduced, then n factors as $n:(\widetilde{X}_s,\widetilde{x})\xrightarrow{\overline{n}}(\overline{X}_s,\overline{x})\xrightarrow{\nu_s}(X_s,x)$ where $\overline{n}$ is the normalization of the multigerm $(\overline{X}_s,\overline{x})$. Hence, there are inclusions $\mathcal{O}_s\hookrightarrow\overline{\mathcal{O}}_s\hookrightarrow\widetilde{\mathcal{O}}_s$ and h is a non-zerodivisor of $\overline{\mathcal{O}}_s$.

(3) If (S,s) is normal, then $h\overline{\mathcal{O}}\subset\mathcal{O}$ and the $\mathcal{O}_{S,s}$-module $\mathcal{O}/h\overline{\mathcal{O}}$ is torsion free.

Proof. Let $(N,x)\subset(X,x)$ denote the non-normal locus of f, which is an analytic subgerm by Theorem I.1.100. Since f is flat, the intersection $(N\cap X_s,x)$ is the non-normal locus of (X_s,x), which is nowhere dense as the fibre (X_s,x) is reduced. Again by Theorem I.1.100, the nearby fibres of f are also reduced, hence (N,x) is nowhere dense in (X,x).

Therefore, there exists some $h\in\mathcal{O}$ which vanishes along (N,x) but not along any irreducible component of (X,x) or of (X_s,x). Thus, h is a non-zerodivisor of $\mathcal{O}$ and of $\mathcal{O}_s$. Since h is invertible in the total ring of fractions of $\mathcal{O}$ and of $\mathcal{O}_s$, it is a non-zerodivisor of $\overline{\mathcal{O}}$ and of $\widetilde{\mathcal{O}}_s$. Further, since h vanishes on the support $(N\cap X_s,x)$ of the conductor $I_s^{cd}=\mathrm{Ann}_{\mathcal{O}_s}(\widetilde{\mathcal{O}}_s/\mathcal{O}_s)$, some power of h is contained in I_s^{cd} by the Hilbert-Rückert Nullstellensatz. Replacing h by some power of h, we get the first part of (1). Applying Proposition B.5.3 to $\mathcal{O}_{S,s}\hookrightarrow\mathcal{O}$ and $h:\mathcal{O}\to\mathcal{O}$, it follows that $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat.

A similar argument shows that (X_s,x) and $(\overline{X}_s,\overline{x})$ have the same normalization if $(\overline{X},\overline{x})$ is reduced, which shows (2).

Finally, we prove (3). Since (S,s) is normal, the non-normal locus of (X,x) is contained in the non-normal locus of f. Thus, h vanishes along the non-normal locus of (X,x). As above, it follows that some power of h is contained in $\mathrm{Ann}_{\mathcal{O}}(\overline{\mathcal{O}}/\mathcal{O})$. Replacing h by an appropriate power, h satisfies $h\overline{\mathcal{O}}\subset\mathcal{O}$.

To show that $\mathcal{O}/h\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-torsion free, we have to show that each non-zero element of $\mathcal{O}_{S,s}$ is a non-zerodivisor of $\mathcal{O}/h\overline{\mathcal{O}}$, that is, $\mathcal{O}_{S,s}\cap P=\{0\}$

for each associated prime P of the $\mathcal{O}$-ideal $h\overline{\mathcal{O}}$ (see Appendix B.1). Since $\overline{\mathcal{O}}$ is the integral closure of $\mathcal{O}$ in $\mathrm{Quot}(\mathcal{O})$, the ideal $h\overline{\mathcal{O}}$ is the integral closure of the ideal $h\mathcal{O}$ in $\mathrm{Quot}(\mathcal{O})$. Hence, $h\mathcal{O} \subset h\overline{\mathcal{O}} \subset \sqrt{h\mathcal{O}}$ and $h\mathcal{O}$ and $h\overline{\mathcal{O}}$ have the same associated prime ideals. Since $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat by (1), $\mathcal{O}_{S,s} \cap P = \{0\}$ as required. □

Proposition 2.49. *Let $f : (X,x) \to (S,s)$ and $\nu : (\overline{X},\overline{x}) \to (X,x)$ be as in Lemma 2.48. Denote by $(\overline{X}_s,\overline{x})$ the fibre of $\overline{f} = f \circ \nu$, and by $\nu_s : (\overline{X}_s,\overline{x}) \to (X_s,x)$ the restriction of ν to this fibre.*

(1) If $(\overline{X}_s,\overline{x})$ is reduced, then $\overline{f} : (\overline{X},\overline{x}) \to (S,s)$ is flat.
(2) Let (S,s) be normal. If $(\overline{X}_s,\overline{x})$ is normal and if (X,x) is equidimensional, then ν_s is the normalization of (X_s,x) and f is equinormalizable.

Note that, under the assumptions of Proposition 2.49, (X,x) is equidimensional iff there exists a representative $f : X \to S$ such that every fibre of f is equidimensional. This follows from Proposition B.8.13 since f is flat and (S,s) is normal (hence, equidimensional).

Proof. (1) We use the notations of Lemma 2.48. The element h is invertible in the total ring of fractions of $\mathcal{O}$, and we have inclusions $\overline{\mathcal{O}} \hookrightarrow h^{-1}\mathcal{O}$ and $\widetilde{\mathcal{O}}_s \hookrightarrow h^{-1}\mathcal{O}_s$. Tensoring $\overline{\mathcal{O}} \hookrightarrow h^{-1}\mathcal{O}$ with $\mathbb{C}$, we get a long exact Tor-sequence

$$\begin{aligned} \ldots \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\overline{\mathcal{O}},\mathbb{C}) \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O},\mathbb{C}) \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}},\mathbb{C}) \\ \longrightarrow \overline{\mathcal{O}} \otimes_{\mathcal{O}_{S,s}} \mathbb{C} \longrightarrow h^{-1}\mathcal{O} \otimes_{\mathcal{O}_{S,s}} \mathbb{C}\,. \end{aligned}$$

Since $h^{-1}\mathcal{O} \cong \mathcal{O}$ is flat over $\mathcal{O}_{S,s}$, we have $\mathrm{Tor}_i^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O},\mathbb{C}) = 0$ for each $i \geq 1$ (Proposition B.3.2). Further, by assumption, $\overline{\mathcal{O}} \otimes_{\mathcal{O}_{S,s}} \mathbb{C} = \overline{\mathcal{O}}_s$ is reduced and, hence, injects into $\widetilde{\mathcal{O}}_s$ by Lemma 2.48. Thus, the last arrow displayed in the above sequence is injective. Altogether, this shows that $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}},\mathbb{C}) = 0$, and the local criterion of flatness (Theorem B.5.1) implies that $h^{-1}\mathcal{O}/\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-flat and that $\mathrm{Tor}_i^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}},\mathbb{C}) = 0$ for $i \geq 1$. From the Tor-sequence, we read that $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\overline{\mathcal{O}},\mathbb{C}) = 0$, whence $\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-flat by the local criterion of flatness.

For (2), we choose sufficiently small representatives of the involved morphisms and spaces. Let N_s be the (analytic) set of non-normal points of $X_s = f^{-1}(s)$. If $x' \in X_s \setminus N_s$, then X is normal at x' by Proposition 2.46 (since (S,s) is normal). Hence, the fibre $\nu^{-1}(x')$ consists of a unique point $z' \in \overline{X}$, and $\nu : (\overline{X},z') \to (X,x')$ is an isomorphism. It follows that $\nu_s : (\overline{X}_s,z') \to (X_s,x')$ is an isomorphism, too. Thus, $\nu_s : \overline{X}_s \setminus \nu_s^{-1}(N_s) \to X_s \setminus N_s$ is bijective and locally an isomorphism, hence biholomorphic. To show that $\nu_s : \overline{X}_s \to X_s$ is the normalization, it suffices to show that $\nu_s^{-1}(N_s)$ is nowhere dense in $\overline{X}_s$.

Choose $z \in \overline{x} = \nu^{-1}(x)$. Then $\nu : (\overline{X},z) \to (X,x)$ normalizes some component of (X,x) and, since (X,x) is equidimensional, $\dim(\overline{X},z) = \dim(X,x)$. Since the germ $(\overline{X},z)$ is normal, it is irreducible. Applying Theorem B.8.13 to $\overline{f}$ and to f, we get

$$\begin{aligned}\dim(\overline{X}_s, z) &\geq \dim(\overline{X}, z) - \dim(S, s)\\ &= \dim(X, x) - \dim(S, s) = \dim(X_s, x)\,.\end{aligned}$$

Since ν_s is a finite morphism, $\nu_s(\overline{X}_s, z)$ is an analytic subgerm of (X_s, x) of dimension $\dim(\overline{X}_s, z)$. It follows that $\dim \nu(\overline{X}_s, z)$ must be equal to $\dim(X_s, z)$ and, therefore, $\nu(\overline{X}_s, z)$ is an irreducible component of (X_s, x). As N_s is nowhere dense in X_s, it follows that $\nu^{-1}(N_s)$ is nowhere dense in $\overline{X}_s$. □

The following corollary shows that equinormalizability of $f : X \to S$ at a point $x \in X$ is an open property, provided that X is equidimensional at x and S is normal at $f(x)$.

Corollary 2.50. *Let $f : (X, x) \to (S, s)$ be a flat morphism of complex space germs, where (X, x) is equidimensional, (S, s) is normal, and the fibre $(X_s, x) = (f^{-1}(s), x)$ is reduced. Let $(\overline{X}, \overline{x}) \xrightarrow{\nu} (X, x)$ be the normalization of (X, x), and assume that the fibre $(\overline{X}_s, \overline{x})$ of $\overline{f} = f \circ \nu$ is normal. Then there exists a representative $f : X \to S$ which is equinormalizable.*

Proof. We may choose sufficiently small representatives $f : X \to S$ such that f is reduced (Theorem I.1.100), S is normal and X is reduced (Proposition I.1.93) and equidimensional. Let $\nu : \overline{X} \to X$ be the normalization. Then we may assume that the special fibre $\overline{X}_s$ of $\overline{f} = f \circ \nu$ is normal at each point $z \in \overline{x} = \nu^{-1}(x)$. By Proposition 2.49 (1), $\overline{f}$ is flat, hence normal at each point $z \in \nu^{-1}(x)$. By Theorem I.1.100, the set of normal points of $\overline{f}$ is open, hence we may assume (after shrinking X and $\overline{X}$ if necessary) that $\overline{f}$ is normal. Since X is equidimensional at each point, we can apply Proposition 2.49 (2) to every non-empty fibre of f which shows that ν normalizes every fibre of f. Hence, $\nu : \overline{X} \to X$ is a simultaneous normalization of $f : X \to S$. □

We turn back to global morphisms and show, in particular, that a reduced morphism $f : X \to S$ with X equidimensional and S normal is equinormalizable iff all non-empty fibres of $\overline{f} = f \circ \nu$ are normal.

Theorem 2.51. *Let $f : X \to S$ be a reduced morphism of complex spaces, where S is normal.*

(1) If f admits a simultaneous normalization $\nu : Z \to X$, then ν is necessarily the normalization of X.

(2) Let $\nu : \overline{X} \to X$ be the normalization of X and $\overline{f} = f \circ \nu$. Then the following holds:

(i) ν is a simultaneous normalization of f iff for each $s \in f(X)$ the map $\nu_s : \overline{f}^{-1}(s) \to f^{-1}(s)$ is the normalization of the fibre $f^{-1}(s)$.

(ii) If X is locally equidimensional, then ν is a simultaneous normalization of f iff for each $s \in f(X)$, the fibre $\overline{f}^{-1}(s)$ is normal.

Proof. (1) Since S is normal, hence reduced, X is also reduced (Proposition 2.46 (1)) and the normalization of X exists. Moreover, since $\overline{f} = f \circ \nu$ is normal by assumption and, since S is normal, Z is normal, too (Proposition

2.46 (2)). To show that $\nu : Z \to X$ is the normalization, it suffices (since ν is finite and surjective by assumption) that $\nu : Z \setminus \nu^{-1}(N) \to X \setminus N$ is biholomorphic, where N denotes the set of non-normal points of f. But this was shown in Lemma 2.45.

(2) If ν is a simultaneous normalization, all non-empty fibres of $\overline{f}$ are normal and ν induces a normalization of all non-empty fibres of f by definition. The converse is a direct consequence of Proposition 2.49 (1), resp. Corollary 2.50. $\square$

Next, we consider families of curves and prove the δ-constant criterion for equinormalizability. In order to shorten notation, we introduce the following notion:

Definition 2.52. A morphism $f : \mathscr{C} \to S$ of complex spaces is a *family of reduced curves* if f is reduced, if the restriction $f : \mathrm{Sing}(f) \to S$ is finite and if all non-empty fibres $\mathscr{C}_s = f^{-1}(s)$ are purely one-dimensional.

Recall that for a reduced curve singularity (C, x) the δ-invariant is defined as $\delta(C, x) = \dim_{\mathbb{C}}\big(n_*\mathcal{O}_{\widetilde{C},\widetilde{x}}/\mathcal{O}_{C,x}\big)$, where $n : (\widetilde{C}, \widetilde{x}) \to (C, x)$ is the normalization of (C, x). For a family of reduced curves $f : \mathscr{C} \to S$ and $s \in S$, we define

$$\delta(\mathscr{C}_s) := \sum_{x \in \mathscr{C}_s} \delta(\mathscr{C}_s, x)\,.$$

This is a finite number, since the fibre $\mathscr{C}_s$ has only finitely many singularities and since $\delta(\mathscr{C}_s, x)$ is zero if (and only if) $(\mathscr{C}_s, x)$ is smooth. The family $f : \mathscr{C} \to S$ is called *(locally) δ-constant* if the function $s \mapsto \delta(\mathscr{C}_s)$ is (locally) constant on S.

If $f : (\mathscr{C}, x) \to (S, s)$ is a flat map germ with reduced and one-dimensional fibre $(\mathscr{C}_s, x)$, then there exists a representative $f : \mathscr{C} \to S$ which is a family of reduced curves such that $\mathscr{C}_s \setminus \{x\}$ is smooth. If there exists such a representative which is δ-constant, we call the germ $f : (\mathscr{C}, x) \to (S, s)$ *δ-constant* or a *δ-constant deformation* of $(\mathscr{C}_s, x)$.

Lemma 2.53. *Let $f : \mathscr{C} \to S$ be a family of reduced curves with reduced base S. If f is equinormalizable, then f is locally δ-constant.*

Proof. Let $\nu : \overline{\mathscr{C}} \to \mathscr{C}$ be the normalization. By assumption, the composition $\overline{f} = f \circ \nu$ is normal, hence flat, and the direct image sheaf $\nu_*\mathcal{O}_{\overline{\mathscr{C}}}$ is also flat over $\mathcal{O}_S$. Moreover, since $\nu_s : \overline{\mathscr{C}}_s = \overline{f}^{-1}(s) \to \mathscr{C}_s$ is the normalization, the induced map $\mathcal{O}_{\mathscr{C}_s} \to \nu_*\mathcal{O}_{\overline{\mathscr{C}}_s}$ is injective. Thus, Proposition B.5.3 gives that the quotient $\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}}$ is a flat $\mathcal{O}_S$-module. Since this quotient is concentrated on $\mathrm{Sing}(f)$, which is finite over S, the direct image $f_*(\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}})$ is locally free on S. Since ν normalizes the fibres, we get that

$$\dim_{\mathbb{C}}\big(f_*\big(\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}}\big) \otimes_{\mathcal{O}_{S,s}} \mathbb{C}\big) = \dim_{\mathbb{C}}\big(\nu_{s*}\mathcal{O}_{\overline{\mathscr{C}}_s}/\mathcal{O}_{\mathscr{C}_s}\big) = \delta(\mathscr{C}_s)$$

is locally constant on S. $\square$

We want to show the converse implication under the assumption that S is normal. We start with the case that S is a smooth curve:

Theorem 2.54 (Teissier). *Let $f : (\mathscr{C}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a flat morphism such that the fibre $(\mathscr{C}_0, \mathbf{0})$ is a reduced curve singularity. If $\nu : (\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{C}, \mathbf{0})$ is the normalization and $\overline{f} = f \circ \nu$, then the fibre $(\overline{\mathscr{C}}_0, \overline{0}) = (\overline{f}^{-1}(0), \overline{0})$ is reduced. Moreover:*

(1) For each sufficiently small representative $f : \mathscr{C} \to S \subset \mathbb{C}$, we have

$$\delta(\mathscr{C}_0, \mathbf{0}) = \delta(\mathscr{C}_s) + \delta(\overline{\mathscr{C}}_0, \overline{0}) \ \textit{ for each } \ s \in S \setminus \{0\}\,.$$

In particular, δ is upper semicontinuous on S.

(2) $f : (\mathscr{C}, \mathbf{0}) \to (\mathbb{C}, 0)$ is equinormalizable iff it is δ-constant.

Proof. Note that $(\overline{\mathscr{C}}, \overline{0})$ has only isolated singularities since the germ $(\mathscr{C}, \mathbf{0})$ is purely two-dimensional. Moreover, by Remark B.8.10.1 (2), $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$ is Cohen-Macaulay, thus $\operatorname{depth}(\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}) = 2$ and $\overline{f}$ is a non-zerodivisor of $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$. The latter shows that $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$ is flat over $\mathcal{O}_{\mathbb{C},0}$ (Theorem B.8.11). Since $\mathcal{O}_{\mathscr{C},\mathbf{0}}$ is also $\mathcal{O}_{\mathbb{C},0}$-flat, and since the fibre $(\mathscr{C}_0, \mathbf{0})$ is reduced, the quotient $\nu_* \mathcal{O}_{\overline{\mathscr{C}},\overline{0}} / \mathcal{O}_{\mathscr{C},\mathbf{0}}$ is $\mathcal{O}_{\mathbb{C},0}$-flat (Proposition B.5.3), hence free. This shows that there exists a sufficiently small representative $\overline{f} : \overline{\mathscr{C}} \xrightarrow{\nu} \mathscr{C} \xrightarrow{f} S \subset \mathbb{C}$ such that

$$\overline{\delta}(\mathscr{C}_s) := \dim_{\mathbb{C}} \left(f_* \left(\nu_* \mathcal{O}_{\overline{\mathscr{C}}} / \mathcal{O}_{\mathscr{C}} \right) \otimes_{\mathcal{O}_{S,s}} \mathbb{C} \right) = \dim_{\mathbb{C}} \left(\nu_{s*} \mathcal{O}_{\overline{\mathscr{C}}_s} / \mathcal{O}_{\mathscr{C}_s} \right)$$

is constant on S.

Since $\overline{f}$ is a non-zerodivisor of $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$, $\operatorname{depth} \mathcal{O}_{\overline{\mathscr{C}}_0,\overline{0}} = 1$ and the fibre $\overline{\mathscr{C}}_0$ is reduced at $\overline{0}$. After shrinking the chosen representatives, we may assume that each fibre $\overline{\mathscr{C}}_s$, $s \in S$, is reduced at each of its points. Hence, $\mathscr{C}_s$ and $\overline{\mathscr{C}}_s$ have the same normalization $\widetilde{\mathscr{C}}_s$.

By Proposition 2.55 below, we may assume that $0 \in S$ is the only critical value of $\overline{f}$. Therefore, $\overline{\mathscr{C}}_s$ is smooth, that is, $\widetilde{\mathscr{C}}_s = \overline{\mathscr{C}}_s$ for $s \neq 0$, which implies that

$$\delta(\mathscr{C}_s) = \overline{\delta}(\mathscr{C}_s) \ \text{ for } \ s \neq 0\,.$$

For $s = 0$, we have inclusions $\mathcal{O}_{\mathscr{C}_0} \hookrightarrow \mathcal{O}_{\overline{\mathscr{C}}_0} \hookrightarrow \mathcal{O}_{\widetilde{\mathscr{C}}_0}$, hence

$$\delta(\mathscr{C}_0) = \overline{\delta}(\mathscr{C}_0) + \delta(\overline{\mathscr{C}}_0)\,,$$

which proves (1), because $\overline{\delta}(\mathscr{C}_0) = \overline{\delta}(\mathscr{C}_s)$ for $s \neq 0$.

For (2), note that if f is δ-constant, then $\delta(\mathscr{C}_0, \mathbf{0}) = \delta(\mathscr{C}_s)$ for each s and, by (1), $\delta(\overline{\mathscr{C}}_0, \overline{0}) = 0$, which shows that $(\overline{\mathscr{C}}_0, \overline{0})$ is smooth. Hence, f is equinormalizable by Proposition 2.49 (2). The converse implication was shown in Lemma 2.53. $\square$

Proposition 2.55. *Let $f : \mathscr{C} \to S$ be a family of reduced curves with S reduced. Then there is an analytically open dense subset $U \subset S$ such that the restriction $f : f^{-1}(U) \to U$ is equinormalizable.*

Proof. Since S is reduced, $S \setminus \mathrm{Sing}(S)$ is open and dense in S and, replacing S by $S \setminus \mathrm{Sing}(S)$, we may assume that S is smooth.

Let $\nu : \overline{\mathscr{C}} \to \mathscr{C}$ be the normalization of $\mathscr{C}$ and $\overline{f} := f \circ \nu$. For a point $x \in \mathscr{C} \setminus \mathrm{Sing}(f)$, we have $(\mathscr{C}, x) \cong (\mathscr{C}_s, x) \times (S, s)$ with f being the projection to the second factor under this isomorphism (Theorem I.1.115). Since (S, s) is smooth and $\overline{f}$ (in particular, $\overline{\mathscr{C}}$) is smooth at $\overline{x} = \nu^{-1}(x)$, it follows that $\overline{x}$ consists of one point and that $\nu : (\overline{\mathscr{C}}, \overline{x}) \xrightarrow{\cong} (\mathscr{C}, x)$ is an isomorphism. Thus, $\nu(\mathrm{Sing}(\overline{f})) \subset \mathrm{Sing}(f)$, the restriction $\overline{f} : \mathrm{Sing}(\overline{f}) \to S$ is finite as composition of the finite maps ν and $f|_{\mathrm{Sing}(f)}$, and the set of critical values $\Sigma := \overline{f}(\mathrm{Sing}(\overline{f})) \subset S$ is a closed analytic subset of S.

Since the fibres $\overline{f}^{-1}(s)$, $s \in U := S \setminus \Sigma$, are smooth, the restriction $\nu : \overline{f}^{-1}(U) \to f^{-1}(U)$ is a simultaneous normalization of $f^{-1}(U)$ (Theorem 2.51 (2)(i)). We have to show that U is dense in S.

Since $\overline{\mathscr{C}}$ is normal, the singular locus $\mathrm{Sing}(\overline{\mathscr{C}})$ has codimension at least 2 in $\overline{\mathscr{C}}$. Since f is flat and its fibres have dimension 1, the image $\overline{f}(\mathrm{Sing}(\overline{\mathscr{C}})) = f(\nu(\mathrm{Sing}(\overline{\mathscr{C}}))) \subset \Sigma$ has codimension at least 1 in S, too (Theorem I.B.8.13). Hence, $\overline{f} : \overline{f}^{-1}(U) \to U$ is a morphism of complex manifolds and $\mathrm{Sing}(\overline{f}|_{\overline{f}^{-1}(U)})$ is nowhere dense in U (Sard's Theorem I.1.103). Therefore, Σ is nowhere dense in S and its complement U is open and dense in S. $\square$

We turn now to the general theorem due to Teissier and Raynaud [Tei] (see the proof given by Chiang-Hsieh and Lipman [ChL]):

Theorem 2.56 (Teissier, Raynaud). *Let $f : \mathscr{C} \to S$ be a family of reduced curves with normal base S. Then f is equinormalizable iff f is locally δ-constant.*

Proof. By Lemma 2.53, it suffices to show that f equinormalizable implies that f is locally δ-constant.

Step 1. Let $\nu : \overline{\mathscr{C}} \to \mathscr{C}$ be the normalization and $\overline{f} = f \circ \nu$. For each $s \in S$, let $\mathscr{C}_s = f^{-1}(s)$ and $\overline{\mathscr{C}}_s = \overline{f}^{-1}(s)$. By Theorem 2.51, we have to show that, for every fixed $s \in S$, the fibre $\overline{\mathscr{C}}_s$ is normal. Hence, the problem is local on S and we may (and will) replace $f : \mathscr{C} \to S$ by the restriction over a sufficiently small (connected) neighbourhood of s in S. Let $n : \widetilde{\mathscr{C}}_s \to \mathscr{C}_s$ be the normalization of $\mathscr{C}_s$ and denote by $\nu_s : \overline{\mathscr{C}}_s \to \mathscr{C}_s$ the restriction of ν. By the universal property of the normalization, the map $\widetilde{\mathscr{C}}_s \xrightarrow{n} \mathscr{C}_s \hookrightarrow \mathscr{C}$ factors through ν and, hence, n factors through ν_s,

$$n : \widetilde{\mathscr{C}}_s \to \overline{\mathscr{C}}_s \xrightarrow{\nu_s} \mathscr{C}_s\,.$$

For $x \in \mathscr{C}_s$, let $\overline{x} := \nu_s^{-1}(x)$ and $\widetilde{x} := n^{-1}(x)$. We have, on the ring level, morphisms of (semilocal) algebras

$$\mathcal{O}_s := \mathcal{O}_{\mathscr{C}_s,x} \to \overline{\mathcal{O}}_s := \mathcal{O}_{\overline{\mathscr{C}}_s,\overline{x}} \to \widetilde{\mathcal{O}}_s := \mathcal{O}_{\widetilde{\mathscr{C}}_s,\widetilde{x}}\,,$$

where the composition $\mathcal{O}_s \to \widetilde{\mathcal{O}}_s$ is injective.

We have to show that, for $s \in S$ and $x \in \mathscr{C}_s$, the map $\overline{\mathcal{O}}_s \to \widetilde{\mathcal{O}}_s$ is an isomorphism.

Step 2. We show that $\overline{\mathcal{O}}_s \cong \widetilde{\mathcal{O}}_s$ holds for s outside a one-codimensional analytic subset of S.

Since normalization is a local operation, we have $\overline{f}^{-1}(U) = \overline{f^{-1}(U)}$ for each open subset $U \subset S$. Hence, the claim follows from Proposition 2.55. That is, there is a closed analytic subset $\Sigma \subset S$ of codimension at least 1 containing the set of critical values $\overline{f}(\mathrm{Sing}(\overline{f}))$. Then

$$\overline{f} : \overline{f}^{-1}(U) \xrightarrow{\nu} f^{-1}(U) \xrightarrow{f} U := S \setminus \Sigma .$$

is smooth and $\nu : \overline{f}^{-1}(U) \to f^{-1}(U)$ is a simultaneous normalization, hence $\overline{\mathcal{O}}_s \cong \widetilde{\mathcal{O}}_s$ for $s \in U$.

Step 3. Let $s \in S$ be arbitrary, $x \in \mathscr{C}_s$, $\overline{x} = \nu^{-1}(x)$, and set

$$\mathcal{O} := \mathcal{O}_{\mathscr{C},x}, \quad \overline{\mathcal{O}} = \nu_* \mathcal{O}_{\overline{\mathscr{C}},\overline{x}} .$$

We choose $h \in \mathcal{O}$ as in Lemma 2.48. Since h is a non-zerodivisor of $\mathcal{O}_s$, and since $\mathcal{O}_s$ has dimension 1, the quotient $\mathcal{O}_s/h\mathcal{O}_s$ is Artinian. Therefore, $\mathcal{O}/h\mathcal{O}$ is a quasifinite, hence finite, $\mathcal{O}_{S,s}$-module. By Lemma 2.48 (1), $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat, hence free of some rank d. Since h is invertible in the total ring of fractions of $\mathcal{O}$, $h^{-1}\mathcal{O}/\mathcal{O} \cong \mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-free of rank d.

The question whether $\overline{\mathcal{O}}_s \to \widetilde{\mathcal{O}}_s$ is an isomorphism is local in x and s. Thus, we fix x and $s = f(x)$ and we can assume that $\mathscr{C}$ and S are sufficiently small neighbourhoods of x and s such that h is a global section of $\mathcal{O}_{\mathscr{C}}$ and such that

$$\mathcal{E} := f_*(h^{-1}\mathcal{O}_{\mathscr{C}}/\mathcal{O}_{\mathscr{C}})$$

is a locally free $\mathcal{O}_S$-sheaf of rank d. Moreover, $f : \mathscr{C} \to S$ is δ-constant with $\delta := \delta(\mathscr{C}_s, x)$. Since the quotient $\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}}$ is concentrated on $\mathrm{Sing}(f)$ (by Step 2), which is finite over S, we get that

$$\mathcal{L} := f_*(\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}})$$

is a coherent $\mathcal{O}_S$-module. Since $h\overline{\mathcal{O}} \subset \mathcal{O}$, we have $\nu_*\mathcal{O}_{\overline{\mathscr{C}}} \subset h^{-1}\mathcal{O}_{\mathscr{C}}$, which induces an exact sequence

$$0 \to \nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}} \to h^{-1}\mathcal{O}_{\mathscr{C}}/\mathcal{O}_{\mathscr{C}} \to h^{-1}\mathcal{O}_{\mathscr{C}}/\nu_*\mathcal{O}_{\overline{\mathscr{C}}} \to 0$$

of coherent $\mathcal{O}_{\mathscr{C}}$-modules whose support is finite over S. Hence, applying f_*, we obtain an exact sequence of coherent $\mathcal{O}_S$-modules

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L} \to 0 \tag{2.6.31}$$

with $\mathcal{E}$ being locally free of rank d.

Let U be as in Step 2. Then, for $s' \in U \subset S$, we have $\nu_{s*}\mathcal{O}_{\overline{\mathscr{C}}_{s'}} \cong n_*\mathcal{O}_{\widetilde{\mathscr{C}}_{s'}}$ and

$$\mathcal{L} \otimes_{\mathcal{O}_{S,s'}} \mathbb{C} \cong n_*\mathcal{O}_{\widetilde{\mathscr{C}}_{s'}}/\mathcal{O}_{\mathscr{C}_{s'}} = \bigoplus_{y \in \operatorname{Sing}(\mathscr{C}_{s'})} \big(n_*\mathcal{O}_{\widetilde{\mathscr{C}}_{s'}}\big)_y/\mathcal{O}_{\mathscr{C}_{s'},y}$$

has complex dimension $\delta(\mathscr{C}_{s'})$, which coincides with δ since f is δ-constant. Therefore, $\mathcal{L}|_U$ is locally free of rank δ.

By Lemma 2.48, $\widetilde{\mathcal{O}}_{s'} \subset h^{-1}\mathcal{O}_{s'}$ and, hence, $\overline{\mathcal{O}}_{s'}/\mathcal{O}_{s'} \to h^{-1}\mathcal{O}_{s'}/\mathcal{O}_{s'}$ is injective. It follows that the sequence (2.6.31) stays exact if we tensor it with $\mathbb{C}$ over $\mathcal{O}_{S,s'}$, $s' \in U$. As a consequence, the restriction $\mathcal{E}/\mathcal{L}|_U$ is locally free of rank $d-\delta$.

Step 4. Assume for the moment that the quotient $\mathcal{E}/\mathcal{L}$ is everywhere locally free on S. Then $\operatorname{Tor}_1^{\mathcal{O}_{S,s}}(\mathcal{E}/\mathcal{L},\mathbb{C}) = \operatorname{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}}) = 0$ and, applying $\otimes_{\mathcal{O}_{S,s}}\mathbb{C}$ to the exact sequence $0 \to \overline{\mathcal{O}} \to h^{-1}\mathcal{O} \to h^{-1}\mathcal{O}/\overline{\mathcal{O}} \to 0$, we get that $\overline{\mathcal{O}}_s \to h^{-1}\mathcal{O}_s$ is injective. Hence, $\overline{\mathcal{O}}_s$ is reduced and we have inclusions $\mathcal{O}_s \hookrightarrow \overline{\mathcal{O}}_s \hookrightarrow \widetilde{\mathcal{O}}_s$.

Since $\mathcal{E}/\mathcal{L}$ is locally free of rank $d-\delta$, $\mathcal{L}$ is locally free of rank δ and, hence, $\overline{\mathcal{O}}_s/\mathcal{O}_s \cong \mathcal{L} \otimes_{\mathcal{O}_{S,s}} \mathbb{C}$ has complex dimension δ. This proves that $\overline{\mathcal{O}}_s = \widetilde{\mathcal{O}}_s$ which, by Step 1, implies that f is equinormalizable.

Hence, it remains to show that $\mathcal{E}/\mathcal{L}$ is locally free on S.

Step 5. Assume that there exists a coherent subsheaf $\widetilde{\mathcal{L}}$ of $\mathcal{E}$ with $\widetilde{\mathcal{L}}|_U = \mathcal{L}|_U$ for some open dense subset $U \subset S$, such that the quotient $\mathcal{E}/\widetilde{\mathcal{L}}$ is locally free on S. We show that $\widetilde{\mathcal{L}} \cong \mathcal{L}$.

By Lemma 2.48 (3), we know that $h^{-1}\mathcal{O}/\overline{\mathcal{O}} \cong \mathcal{O}/h\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-torsion free. Hence, the quotient $\mathcal{E}/\mathcal{L}$ is torsion free for S sufficiently small. Consider the subsheaf $\mathcal{L} + \widetilde{\mathcal{L}}$ of $\mathcal{E}$, which coincides with $\mathcal{L}$ on U. Thus, the quotient $(\mathcal{L}+\widetilde{\mathcal{L}})/\mathcal{L}$ is a torsion subsheaf of the torsion free sheaf $\mathcal{E}/\mathcal{L}$. It follows that $(\mathcal{L}+\widetilde{\mathcal{L}})/\mathcal{L}$ is the zero sheaf, that is, $\widetilde{\mathcal{L}} \subset \mathcal{L}$. Similarly, $\mathcal{L} \subset \widetilde{\mathcal{L}}$.

Thus, it remains to show that the sheaf $\mathcal{L}|_U$ has an extension to a coherent subsheaf $\widetilde{\mathcal{L}}$ of $\mathcal{E}$ on S such that the quotient $\mathcal{E}/\widetilde{\mathcal{L}}$ is a locally free $\mathcal{O}_S$-sheaf.

Step 6. To show the existence of $\widetilde{\mathcal{L}}$, we use the quot scheme of $\mathcal{E}$. Let $\mathscr{G} = \operatorname{Grass}_{d-\delta}(\mathcal{E})$ be the Grassmannian of locally free $\mathcal{O}_S$-module quotients of $\mathcal{E}$ of rank $d-\delta$ (see [GrD, 9.7]). That is, $\mathscr{G}$ is a complex space, projective over[15] S, such that, for each complex spaces T over S, there is a bijection

$$\operatorname{Mor}_S(T,\mathscr{G}) \longleftrightarrow \left\{ \begin{array}{c} \mathcal{O}_T\text{-subsheaves } \mathcal{F} \text{ of } \mathcal{E}_T \text{ such that } \mathcal{E}_T/\mathcal{F} \text{ is} \\ \text{a locally free } \mathcal{O}_T\text{-module of rank } d-\delta \end{array} \right\}.$$

which is functorial in T. Here, for an $\mathcal{O}_S$-module $\mathscr{M}$, we denote by $\mathscr{M}_T$ the pull-back to T.

[15] We say that X is *projective over* S if the morphism $X \to S$ is *projective*, that is, if it factors through a closed immersion $X \hookrightarrow \mathbb{P}^N \times S$ for some N, followed by the projection $\mathbb{P}^N \times S \to S$.

By Step 3, the quotient $\mathcal{E}_U/\mathcal{L}_U$ is a locally free $\mathcal{O}_U$-module of rank $d-\delta$. Hence, under the above bijection, $\mathcal{L}_U$ corresponds to an S-morphism $\psi : U \to \mathscr{G}$. Any extension $\widetilde{\psi} : S \to \mathscr{G}$ of ψ corresponds to an $\mathcal{O}_S$-submodule $\widetilde{\mathcal{L}} \subset \mathcal{E}$ such that $\widetilde{\mathcal{L}}_U = \mathcal{L}_U$ and $\mathcal{E}/\widetilde{\mathcal{L}}$ is a locally free $\mathcal{O}_S$-module of rank $d-\delta$.

To see that ψ has such an extension $\widetilde{\psi} : S \to \mathscr{G}$, consider the graph of ψ, $\Gamma_\psi \subset U \times \mathscr{G}$, and let $\overline{\Gamma_\psi}$ be the analytic closure[16] of Γ_ψ in $S \times \mathscr{G}$. We have to show that in the commutative diagram

$$\begin{tikzcd}
S & & \overline{\Gamma_\psi} \subset S \times \mathscr{G} \arrow[ll, "p"'] \arrow[dd, "q"] \\
U \arrow[u, hook] \arrow[rrd, "\psi"'] & \Gamma_\psi \arrow[l, "\cong", "p_0"'] \arrow[ur, hook] \arrow[dr, "q_0"] & \\
 & & \mathscr{G}
\end{tikzcd}$$

the map p is an isomorphism. Here, p, p_0, resp. q, q_0 are induced by the projections to the first, resp. second, factor.

Step 7. Since S is normal and the restriction of p to a dense open subset of $\overline{\Gamma_\psi}$ is an isomorphism onto a dense open subset of S, we only have to show that p is a homeomorphism (Theorem 1.102). Since $\mathscr{G} \to S$ is projective and since $\overline{\Gamma_\psi}$ is closed in $S \times \mathscr{G}$, the projection p is projective, hence closed. It follows that p is surjective and that p^{-1} is continuous if it exists (that is, if p is injective). Thus, it remains to show that, for each $s \in S \setminus U$, the fibre $p^{-1}(s)$ consists of only one point.

Let $z = (s, L) \in p^{-1}(s) \subset \overline{\Gamma_\psi}$ be any point. Then $z \in \overline{\Gamma_\psi} \setminus \Gamma_\psi$, where, by our assumptions, $\overline{\Gamma_\psi} \setminus \Gamma_\psi$ is of codimension at least 1 in $\overline{\Gamma_\psi}$ and Γ_ψ is smooth. Thus, we can choose an irreducible germ of a curve C in $\overline{\Gamma_\psi}$ such that $C \cap (\overline{\Gamma_\psi} \setminus \Gamma_\psi) = \{z\}$ and $C \setminus \{z\}$ is smooth (we may, for instance, intersect $(\overline{\Gamma_\psi}, z)$ after a local embedding in some $(\mathbb{C}^N, \mathbf{0})$ with a general linear subspace of dimension $\dim(\overline{\Gamma_\psi}, z) - 1$ and taking an irreducible component if necessary). Then the image $p(C)$ is an irreducible curve in S such that $p(C) \cap (S \setminus U) = \{s\}$ and $p(C) \setminus \{s\}$ is smooth. Consider the normalization of $p(C)$, $\varphi : D \to p(C) \subset S$, $\varphi(0) = s$, where $D \subset \mathbb{C}$ is a small disc with centre 0. The map $\psi \circ (\varphi|_{D\setminus\{0\}}) : D \setminus \{0\} \to \mathscr{G}$ corresponds to a subsheaf $\mathcal{L}_{D\setminus\{0\}} = \varphi^*(\mathcal{L}_U)$ of $\mathcal{E}_{D\setminus\{0\}} = \varphi^*(\mathcal{E}_U)$ such that $\mathcal{E}_{D\setminus\{0\}}/\mathcal{L}_{D\setminus\{0\}}$ is locally free of rank $d-\delta$.

By Theorem 2.54, the submodule $\mathcal{L}_{D\setminus\{0\}} \subset \mathcal{E}_{D\setminus\{0\}}$ extends over D to a submodule $\mathcal{L}' \subset \mathcal{E}_D := \varphi^*\mathcal{E}$ such that $\mathcal{E}_D/\mathcal{L}'$ is locally free of rank $d-\delta$ and $\mathcal{L}' \otimes_{\mathcal{O}_{D,0}} \mathbb{C} = \widetilde{\mathcal{O}}_s/\mathcal{O}_s \subset \mathcal{E}_D \otimes_{\mathcal{O}_{D,0}} \mathbb{C} = h^{-1}\mathcal{O}_s/\mathcal{O}_s$. The $\mathcal{O}_D$-module $\mathcal{L}'$ corresponds to an extension $\chi : D \to \mathscr{G}$ of $\psi \circ (\varphi|_{D\setminus\{0\}})$ and $\chi(0)$ corresponds to the vector subspace $\mathcal{O}_s$ of $\widetilde{\mathcal{O}}_s$.

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[16] The *analytic closure* of a set M in a complex space X is the intersection of all closed analytic subsets of X containing M.

The graph $\Gamma_D \subset D \times \mathscr{G}$ is mapped under $\varphi \times \mathrm{id}$ onto $C \times \mathscr{G} \subset \overline{\Gamma_\psi} \times \mathscr{G}$ such that $(0, \mathcal{O}_s)$ is mapped to (s, L). Hence $(s, \mathcal{O}_s)$ is the unique point of the fibre $p^{-1}(s)$. $\square$

2.7 δ-Constant and μ-Constant Stratum

In the previous sections, we considered equisingular, respectively equinormalizable deformations. Here, we study arbitrary deformations of a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ and we analyse the maximal strata in the base space such that the restriction to these strata is equisingular, resp. equinormalizable (possibly after base change). Recall that $\delta(C, \mathbf{0}) = \dim_{\mathbb{C}} n_* \mathcal{O}_{(\overline{C}, \overline{0})} / \mathcal{O}_{C,\mathbf{0}}$, where $n : (\overline{C}, \overline{0}) \to (C, \mathbf{0})$ is the normalization, and that $\mu(C, \mathbf{0}) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} / \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$, where $f = 0$ is a local equation of $(C, \mathbf{0})$.

Let $F : \mathscr{C} \to S$ be a family of reduced curves (see Definition 2.52). If F is equinormalizable, then F is locally δ-constant by Lemma 2.53. We show now that, for each given k, the set of points $s \in S$ such that $\delta(\mathscr{C}_s) = k$ is a locally closed analytic subset of S. Here, $\mathscr{C}_s = F^{-1}(s)$ and $\delta(\mathscr{C}_s) = \sum_{x \in \mathscr{C}_s} \delta(\mathscr{C}_s, x)$.

Let us introduce the notation

$$\Delta := F\bigl(\mathrm{Sing}(F)\bigr) \subset S\,,$$

the set of *critical values* of F, also called the *discriminant* of F. Since $F : \mathrm{Sing}(F) \to S$ is finite, the discriminant is a closed analytic subset of S (by the finite mapping Theorem I.1.68). We endow Δ with the Fitting structure of Definition I.1.45.

For $k \geq 0$, we define

$$\begin{aligned} \Delta^\delta_F(k) &:= \Delta^\delta(k) := \{ s \in S \mid \delta(\mathscr{C}_s) \geq k \}\,, \\ \Delta^\mu_F(k) &:= \Delta^\mu(k) := \{ s \in S \mid \mu(\mathscr{C}_s) \geq k \}\,, \end{aligned}$$

where $\mu(\mathscr{C}_s) = \sum_{x \in \mathscr{C}_s} \mu(\mathscr{C}_s, x) < \infty$ (since $\mu(\mathscr{C}_s, x) = 0$ for x a smooth point of $\mathscr{C}_s$). We show below that $\Delta^\delta(k)$ and $\Delta^\mu(k)$ are closed analytic subsets of S (Proposition 2.57) which we endow with its reduced structure. In particular, $\Delta^\delta(0) = \Delta^\mu(0) = S_{red}$ and $\Delta^\delta(1) = \Delta^\mu(1) = \Delta_{red}$.

If $T \to S$ is any morphism, we use the notation

$$F_T : \mathscr{C}_T \to T$$

to denote the pull-back of $F : \mathscr{C} \to S$ to T.

Proposition 2.57. *Let $F : \mathscr{C} \to S$ be a family of reduced curves and let k be a non-negative integer. Then $\Delta^\delta(k)$ and $\Delta^\mu(k)$ are closed analytic subsets of S.*

Proof. Since $\Delta^\delta(k)$ and $\Delta^\mu(k)$ are defined set-theoretically and since the induced map $F_{S_{red}} : \mathscr{C}_{S_{red}} \to S_{red}$ is a family of reduced curves, too, we may assume that S is reduced.

We start with $\Delta^\delta(k)$ for some fixed k. Since the question is local in S, we may shrink S if necessary. If $\Delta^\delta(k) = S$, we are done. Otherwise, there exists an irreducible component S' of S such that $\delta(\mathscr{C}_{s_0}) < k$ for at least one $s_0 \in S'$. By Proposition 2.55 and Lemma 2.53, there exists an analytically open dense subset $U' \subset S'$ such that $\delta(\mathscr{C}_s)$ is constant, say k', for $s \in U'$ and some integer $k' \leq k$ (U' is connected since S' is irreducible).

We claim that $k' \leq \delta(\mathscr{C}_{s_0}) < k$. Indeed, if $k' \neq \delta(\mathscr{C}_{s_0})$, choose a curve germ $(D, s_0) \subset (S, s_0)$ which meets $S' \setminus U'$ only in s_0 and apply Teissier's Theorem 2.54 to the pull-back of $F_{(D,s_0)}$ to the normalization of (D, s_0) (see Step 7 in the proof of Theorem 2.56) to obtain that $k' \leq \delta(\mathscr{C}_{s_0})$. Hence, $\Delta^\delta(k) \cap S' \subset S' \setminus U'$ which is closed in S.

We see that if $\Delta^\delta(k) \neq S$ then there exists a closed analytic subset $S_1 \subsetneq S$ such that $\Delta^\delta(k) \subset S_1$. Applying the same argument to F_{S_1}, we get that either $\Delta^\delta(k) = S_1$ or there exists a closed analytic subset $S_2 \subsetneq S_1$ such that $\Delta^\delta(k) \subsetneq S_2$, etc.. In this way, we obtain a sequence $S \supsetneq S_1 \supsetneq S_2 \supsetneq \ldots$ of closed analytic subsets containing $\Delta^\delta(k)$. This sequence cannot be infinite, since the intersection of all the S_i is locally finite. Hence, $\Delta^\delta(k) = S_\ell$ for some ℓ which proves the proposition.

For $\Delta^\mu(k)$ we may argue similarly, using Theorem I.2.6 and Remark I.2.7.1 (and its proof), to show the existence of U' as above such that $\mu(\mathscr{C}_s) < k$ if $\Delta^\mu(k) \subsetneq S$. □

Exercise 2.7.1. Call a morphism $F : \mathscr{X} \to S$ of complex spaces a *family of hypersurfaces with isolated singularities* if F is reduced, if the restriction $F : \mathrm{Sing}(F) \to S$ is finite and all non-empty fibres $\mathscr{X}_s = F^{-1}(s)$ are pure dimensional and satisfy $\mathrm{edim}(\mathscr{X}_s, x) = \dim(\mathscr{X}_s, x) + 1$ for each $x \in \mathscr{X}_s$. Show that, locally, $\mathscr{X}_s$ is isomorphic to a hypersurface in some $\mathbb{C}^n$ having only isolated singularities. Moreover, show that the sets

$$\Delta^\mu_F(k) := \{ s \in S \mid \mu(\mathscr{X}_s) \geq k \} ,$$
$$\Delta^\tau_F(k) := \{ s \in S \mid \tau(\mathscr{X}_s) \geq k \} ,$$

are closed analytic subsets of S. Here, $\tau(\mathscr{X}_s) = \sum_{x \in \mathscr{X}_s} \tau(\mathscr{X}_s, x)$ is the total Tjurina number of $\mathscr{X}_s$.

HINT: For $\Delta^\mu_F(k)$ you may proceed as in the proof of Proposition 2.57 and for $\Delta^\tau_F(k)$ as in Theorem I.2.6.

We continue by studying in more detail the relation between deformations of the normalization $(\overline{C}, \overline{0}) \to (C, \mathbf{0})$ and deformations of the equation of $(C, \mathbf{0})$. To simplify notations, we omit the base points of the germs, resp. multigerms, in the notation and work with sufficiently small representatives.

This understood, let $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C} \to C}$ denote the semiuniversal deformation of the normalization $\overline{C} \to C$, and let $\mathscr{D} \to B_C$ be the semiuniversal deformation of C. By versality of $\mathscr{D} \to B_C$, there exists a morphism

$$\alpha : (B_{\overline{C}\to C}, \mathbf{0}) \to (B_C, \mathbf{0})$$

such that the pull-back of $\mathscr{D} \to B_C$ via α is isomorphic to $\mathscr{C} \to B_{\overline{C}\to C}$. A priori, α is not unique (only its tangent map is unique due the semiuniversality of $\mathscr{D} \to B_C$). However, in our situation, α itself is unique (see Theorem 2.59). The statements about the δ-constant stratum, resp. the μ-constant stratum, of $(C, \mathbf{0})$ in $(B_C, \mathbf{0})$ established below then follow from properties of α.

We first study deformations of $\overline{C}/C$, that is, deformations of the normalization which fix C. In terms of the notation introduced in Definition 1.21, we study objects of $\mathcal{D}ef_{(\overline{C},\overline{0})/(C,\mathbf{0})}$, resp. their isomorphism classes. Recall that $\mathrm{mt} := \mathrm{mt}(C, \mathbf{0})$ denotes the multiplicity, $r := r(C, \mathbf{0})$ the number of branches and $\delta := \delta(C, \mathbf{0})$ the δ-invariant of $(C, \mathbf{0})$.

Proposition 2.58. *With the above notations, the following holds:*

(1) The restriction of $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ *to* $\alpha^{-1}(\mathbf{0})$ *represents a semiuniversal deformation of* $\overline{C}/C$.
(2) The map α *is finite; it is a closed embedding iff* $\mathrm{mt} = r$.
(3) In particular, the functor $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})/(C,\mathbf{0})}$ *has a semiuniversal deformation whose base space* $B_{\overline{C}/C}$ *consists of a single point of embedding dimension* $\mathrm{mt} - r$. *This point is reduced iff* $(C, \mathbf{0})$ *consists of* r *smooth branches.*

Proof. (1) Since each object in $\mathcal{D}ef_{\overline{C}\to C}(S)$, S any complex space germ, maps to the trivial deformation in $\mathcal{D}ef_C(S)$ iff it is an object of $\mathcal{D}ef_{\overline{C}/C}(S)$, we get that the restriction of the semiuniversal deformation of the normalization to $B_{\overline{C}/C} := \alpha^{-1}(\mathbf{0})$ is a versal element of $\mathcal{D}ef_{\overline{C}/C}$. By Lemma 2.28 (1), the map $T^0_{\overline{C}\to C} \to T^0_C$ induced by α is an isomorphism. From the braid for the normalization (see Figure 2.14 on page 311), we get an exact sequence

$$0 \to T^1_{\overline{C}/C} \to T^1_{\overline{C}\to C} \to T^1_C\,. \tag{2.7.32}$$

Thus, the pull-back of the semiuniversal object of $\mathcal{D}ef_{\overline{C}\to C}$ to $B_{\overline{C}/C}$ satisfies the uniqueness condition on the tangent level to ensure that it is a semiuniversal object for $\mathcal{D}ef_{\overline{C}/C}$.

(2) Assume to the contrary that α is not finite, that is, $\dim B_{\overline{C}/C} > 0$. Then there exists a reduced curve germ $(D, \mathbf{0}) \subset (B_{\overline{C}/C}, \mathbf{0})$ such that, for each $s \in D \setminus \{\mathbf{0}\}$, the germ of D at s is smooth (D sufficiently small). The restriction of $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ to (D, s) is a family $(\overline{\mathscr{C}}_D, \overline{x}) \to (\mathscr{C}_D, x) \to (D, s)$ such that $(\mathscr{C}_D, x) \cong (C, \mathbf{0}) \times (D, s) \to (D, s)$ is the projection (since $\mathscr{C}_D \to D$ is trivial). By Theorem 2.51 (1), $(\overline{\mathscr{C}}_D, \overline{x}) \to (\mathscr{C}_D, x)$ is the normalization of $(\mathscr{C}_D, x)$. Hence, $(\overline{\mathscr{C}}_D, \overline{x}) \cong (\overline{C}, \overline{0}) \times (D, s)$ and $(\overline{\mathscr{C}}_D, \overline{x}) \to (\mathscr{C}_D, x) \to (D, s)$ is a trivial deformation of the normalization $\overline{C} \to C$.

By openness of versality [Fle1, Satz 4.3], $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ is versal over $(B_{\overline{C}\to C}, s)$. However, it is not semiuniversal as it contains the trivial subfamily over (D, s).

By Propositions 2.30 and Theorem 2.38, $B_{\overline{C}\to C}$ is a smooth complex space germ of dimension $\tau(C,\mathbf{0})-\delta(C,\mathbf{0})$. Hence, $\dim(B_{\overline{C}\to C},s)=\tau(C,\mathbf{0})-\delta(C,\mathbf{0})$. Because $(B_{\overline{C}\to C},s)$ is a versal, but not a semiuniversal base space for the deformation of the normalization of the fibre $(\mathscr{C}_s,x)$, its dimension is bigger than $\tau(\mathscr{C}_s,x)-\delta(\mathscr{C}_s,x)$. However, $(\mathscr{C}_s,x)\cong(C,\mathbf{0})$ and, therefore, $\tau(\mathscr{C}_s,x)=\tau(C,\mathbf{0})$ and $\delta(\mathscr{C}_s,x)=\delta(C,\mathbf{0})$, which is a contradiction.

This shows that α is finite and, hence, $\alpha^{-1}(\mathbf{0})$ is a single point, which is of embedding dimension $\dim_{\mathbb{C}} T^1_{\overline{C}/C}=\mathrm{mt}-r$ by Proposition 2.30.

Thus, we proved statement (2) and, at the same time, (3) since $\mathrm{mt}=r$ iff $(C,\mathbf{0})$ has r smooth branches. □

The next theorem relates deformations of the parametrization to the δ-constant stratum in the base space of the semiuniversal deformation of the reduced plane curve singularity $(C,\mathbf{0})$.

Let $\Psi:\mathscr{D}\to B_C$ denote a sufficiently small representative of the semiuniversal deformation of $(C,\mathbf{0})$, $\mathscr{D}_s=\Psi^{-1}(s)$ the fibre over s, and call

$$\Delta^\delta := \{s\in B_C \mid \delta(\mathscr{D}_s)=\delta(C,\mathbf{0})\}\,,$$

respectively the germ $(\Delta^\delta,\mathbf{0})\subset(B_C,\mathbf{0})$, the *$\delta$-constant stratum* of Ψ. Since $\delta(\mathscr{D}_s)\le\delta(C,\mathbf{0})$ by Theorem 2.54, $\Delta^\delta=\Delta^\delta_\Psi(\delta(C,\mathbf{0}))$ and $\Delta^\delta\subset B_C$ is a closed analytic subset (Proposition 2.57). We set $\delta:=\delta(C,\mathbf{0})$ and $\tau:=\tau(C,\mathbf{0})$. Using these notations, we have the following theorem:

Theorem 2.59. *Let $\Psi:\mathscr{D}\to B_C$, resp. $\overline{\mathscr{C}}\to\mathscr{C}\to B_{\overline{C}\to C}$, be sufficiently small representatives of the semiuniversal deformation of $(C,\mathbf{0})$, resp. of the semiuniversal deformation of the normalization $(\overline{C},\overline{0})\to(C,\mathbf{0})$. Then the following holds:*

(0) B_C, resp. $B_{\overline{C}\to C}$ are smooth of dimension τ, resp. $\tau-\delta$.
(1) The δ-constant stratum $\Delta^\delta\subset B_C$ has the following properties:
  (a) Δ^δ is irreducible of dimension $\tau-\delta$.
  (b) $s\in\Delta^\delta$ is a smooth point of Δ^δ iff each singularity of the fibre $\mathscr{D}_s=\Psi^{-1}(s)$ has only smooth branches.
  (c) There exists an open dense set $U\subset\Delta^\delta$ such that each fibre $\mathscr{D}_s$, $s\in U$, of Ψ has only ordinary nodes as singularities.
(2) Each map $\alpha:B_{\overline{C}\to C}\to B_C$ induced by versality of Ψ satisfies:
  (a) $\alpha(B_{\overline{C}\to C})=\Delta^\delta$.
  (b) $\alpha:B_{\overline{C}\to C}\to\Delta^\delta$ is the normalization of Δ^δ, hence unique.
  (c) The pull-back of $\Psi:\mathscr{D}\to B_C$ to $B_{\overline{C}\to C}$ via α is isomorphic to $\mathscr{C}\to B_{\overline{C}\to C}$ and, hence, lifts to the semiuniversal deformation $\overline{\mathscr{C}}\to\mathscr{C}\to B_{\overline{C}\to C}$ of the normalization.

Corollary 2.60 (Diaz, Harris). *The δ-constant stratum $(\Delta^\delta,\mathbf{0})$ has a smooth normalization. It is smooth iff $(C,\mathbf{0})$ is the union of smooth branches.*

Proof of Theorem 2.59. Recall that we work with a sufficiently small representative Ψ of the semiuniversal deformation of $(C,\mathbf{0})$.

(0) follows from Corollary 1.17, p. 239, resp. from Theorem 2.38, p. 327, and Proposition 2.30, p. 312.

(1) Let s be any point of Δ^δ. Then, by openness of versality, the restriction of Ψ over a sufficiently small neighbourhood of s in B_C is a joint versal deformation of all singular points of $\mathscr{D}_s$. It is known ([Gus, Lemma 1], [ACa, Théorème 1], [Tei1, Proposition II.5.2.1, Lemma II.5.2.8]) that each reduced plane curve singularity can be deformed, with total δ-invariant being constant, to a plane curve with only nodes as singularities. Hence, arbitrarily close to s, there exists $s' \in \Delta^\delta$ such that the fibre $\mathscr{D}_{s'}$ has only nodes as singularities. For a node, the δ-constant stratum consists of a (reduced) point in the one-dimensional base space of the semiuniversal deformation. Since Ψ induces over some neighbourhood of s' a versal deformation of the nodal curve $\mathscr{D}_{s'}$ with δ nodes, it follows that (Δ^δ, s') is smooth of codimension δ in (B_C, s').

It follows that the set $U \subset \Delta^\delta$ of all $s \in \Delta^\delta$ such that $\mathscr{D}_s$ is a nodal curve is open and dense in Δ^δ of dimension $\tau(C,\mathbf{0}) - \delta(C,\mathbf{0})$.

To show the irreducibility of the δ-constant stratum, we have to prove that U is connected (see Remark (B) on page 62). Let $s_0, s_1 \in U$ be two points. Although the fibres $\mathscr{D}_{s_i}$ are not germs, they appear as fibres in a δ-constant deformation of $(C,\mathbf{0})$ and, hence, can be parametrized: if Δ_i^δ is the irreducible component of Δ^δ to which s_i belongs, let $\widetilde{\Delta}_i^\delta \to \Delta_i^\delta$ be the normalization and apply Theorem 2.56 to the pull-back of Ψ to $\widetilde{\Delta}_i^\delta$.

In particular, there exist parametrizations

$$\varphi^{(i)} = \big(\varphi_j^{(i)}\big)_{j=1}^r : \coprod_{j=1}^r D_j \to \mathscr{D}_{s_i} \subset B\,, \quad i = 0,1\,,$$

of $\mathscr{D}_{s_i}$, where the $D_j \subset \mathbb{C}$ are small discs, $B \subset \mathbb{C}^2$ is a small ball, and where r is the number of branches of $(C,\mathbf{0})$.

Now, join the two parametrizations by the family

$$\phi = \big(\phi_j\big)_{j=1}^r : \coprod_{j=1}^r D_j \times D \to B \times D\,,$$

where $D \subset \mathbb{C}$ is a disc containing 0 and 1, and where

$$\phi_j : (t_j, s) \mapsto (1-s)\varphi_j^{(0)}(t_j) + s\varphi_j^{(1)}(t_j)\,, \quad j = 1,\dots,r\,.$$

Being a nodal curve is an open property. Hence, for almost all s, ϕ parametrizes a nodal curve. That is, there is an open set $V \subset D$, being the complement of finitely many points, such that $0,1 \in V$ and the restriction $\phi' : \coprod_{j=1}^r D_j \times V \to B \times V$ is finite and $\phi(D_j \times \{s\})$ is a nodal curve for $s \in V$. Applying Proposition 2.9 to ϕ', we see that the image of ϕ' defines a family of nodal curves which is δ-constant (being induced by a deformation of the normalization) and which connects $\mathscr{D}_{s_0}$ and $\mathscr{D}_{s_1}$. This show that U is connected.

To complete the proof of (1), we have to show that (Δ^δ, s) is smooth iff the singularities of $\mathscr{D}_s$ have only smooth branches. By openness of versality, (B_C, s) is the base space of a versal deformation for each of the singularities of $\mathscr{D}_s$. By Proposition 1.14, (Δ^δ, s) is smooth if the germs of the δ-constant strata in the semiuniversal deformations of all singularities of $\mathscr{D}_s$ are smooth. Hence, it suffices to show that $(\Delta^\delta, \mathbf{0})$ is smooth iff $(C, \mathbf{0})$ has only smooth branches. This is shown now when we prove (2).

(2) By (0), $B_{\overline{C}\to C}$ is smooth of dimension $\tau(C, \mathbf{0}) - \delta(C, \mathbf{0})$. Its image under α is contained in Δ^δ by Lemma 2.53.

Since Δ^δ is irreducible and of the same dimension (as shown in part (1) above), α surjects onto Δ^δ. Let $n : \widetilde{\Delta}^\delta \to \Delta^\delta$ denote the normalization of Δ^δ. By the universal property of the normalization, α factors as $\alpha = n \circ \widetilde{\alpha}$ for a unique morphism $\widetilde{\alpha} : B_{\overline{C}\to C} \to \widetilde{\Delta}^\delta$. By the first part of the proof, we know already that (Δ^δ, s) is a smooth germ (hence, $\widetilde{\Delta}^\delta \cong \Delta^\delta$ locally at s) for $\mathscr{D}_s$ a nodal curve. Further, α is finite and bijective over the locus of nodal fibres by Proposition 2.58. Hence, $\widetilde{\alpha} : B_{\overline{C}\to C} \to \widetilde{\Delta}^\delta$ is surjective and finite and an isomorphism outside a nowhere dense analytic subset. It follows that $\widetilde{\alpha}$ is the normalization of $\widetilde{\Delta}^\delta$ (Remark I.1.94.1) and, hence, an isomorphism (since $\widetilde{\Delta}^\delta$ is normal).

Finally, we show the smoothness statement of (1)(b). The epimorphism Theorem I.1.20 implies that $\alpha : (B_{\overline{C}\to C}, \mathbf{0}) \to (B_C, \mathbf{0})$ is a closed embedding (hence, an isomorphism onto $(\Delta^\delta, \mathbf{0})$) iff the induced map of the cotangent spaces is surjective, that is, iff the dual map $T^1_{\overline{C}\to C} \to T^1_C$ is injective. However, from the exact sequence (2.7.32), we know that the kernel of this map is $T^1_{\overline{C}/C}$ which has dimension $m - r$ by Proposition 2.30. This shows that $(\Delta^\delta, \mathbf{0})$ is smooth iff $m = r$, which means that $(C, \mathbf{0})$ has only smooth branches. □

We turn now to the μ-constant stratum. As before, let $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$, resp. $\mathscr{D} \to B_C$, denote the semiuniversal deformation of the normalization, resp. of the equation, of $(C, \mathbf{0})$. Moreover, let the right vertical sequence of the diagram

$$\begin{tikzcd}
\overline{\mathscr{C}}^{es} \arrow[r, hook] \arrow[d] & \overline{\mathscr{C}}^{sec} \arrow[d] \\
\mathscr{C}^{es} \arrow[r, hook] \arrow[d] & \mathscr{C}^{sec} \arrow[d, shift right] \\
B^{es}_{\overline{C}\to C} \arrow[r, hook] & B^{sec}_{\overline{C}\to C} \arrow[u, shift right, "\sigma"'] \arrow[uu, bend right, "\overline{\sigma}"']
\end{tikzcd}$$

be the semiuniversal deformation with section of the normalization which contains as a subfamily the *semiuniversal equisingular deformation (with section) of the normalization*, given by the left vertical sequence.

Here and in what follows, we identify the semiuniversal deformations (with section) of the normalization and of the parametrization according to Proposition 2.23.

Forgetting the sections, we get a (non-unique) morphism $B^{sec}_{\overline{C}\to C} \to B_{\overline{C}\to C}$ which we compose with the map α defined above to obtain a morphism

$$\alpha^{sec} : B^{sec}_{\overline{C}\to C} \to B_{\overline{C}\to C} \xrightarrow{\alpha} B_C \,.$$

We can formulate now the main result about the μ-constant stratum:

Theorem 2.61. *Let $B^{sec}_{\overline{C}\to C}$, resp. B_C, be sufficiently small representatives of the base spaces of the semiuniversal deformation with section of the normalization, resp. of the semiuniversal deformation of the equation, of the reduced plane curve singularity $(C, \mathbf{0})$. Let $\alpha^{sec} : B^{sec}_{\overline{C}\to C} \to B_C$ be any morphism induced by versality as above. Then the following holds:*

(1) The tangent map of α^{sec} restricted to the tangent space of $B^{es}_{\overline{C}\to C}$ is injective.
(2) α^{sec} maps the base space $B^{es}_{\overline{C}\to C}$ of the semiuniversal equisingular deformation of the normalization isomorphically onto the μ-constant stratum $\Delta^\mu \subset B_C$.
(3) In particular, Δ^μ is smooth of dimension $\dim_{\mathbb{C}} T^{1,es}_{\overline{C}\to\mathbb{C}^2}$.

Before giving the proof of this theorem, we recall the explicit description of the maps $\overline{\mathscr{C}}^{sec} \to B^{sec}_{\overline{C}\to\mathbb{C}^2}$ and $\overline{\mathscr{C}}^{es} \to B^{es}_{\overline{C}\to\mathbb{C}^2}$ from Proposition 2.27 and Theorem 2.38: Let $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in \overline{\mathfrak{m}}\frac{\partial}{\partial x} \oplus \overline{\mathfrak{m}}\frac{\partial}{\partial y}$, $j = 1, \dots, k$, represent a basis of

$$T^{1,sec}_{\overline{C}\to\mathbb{C}^2} = \overline{\mathfrak{m}}\frac{\partial}{\partial x} \oplus \overline{\mathfrak{m}}\frac{\partial}{\partial y} \Big/ \left(\overline{\mathfrak{m}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y} \right) + \left(\mathfrak{m}\frac{\partial}{\partial x} \oplus \mathfrak{m}\frac{\partial}{\partial y} \right) \right) .$$

Then the deformation

$$\begin{aligned} X_i(t_i, \boldsymbol{s}) &= x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i) s_j \,, \\ Y_i(t_i, \boldsymbol{s}) &= y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i) s_j \,, \end{aligned} \tag{2.7.33}$$

represents a semiuniversal deformation of the normalization over

$$(B^{sec}_{\overline{C}\to C}, \mathbf{0}) \cong (T^{1,sec}_{\overline{C}\to\mathbb{C}^2}, \mathbf{0}) \cong (\mathbb{C}^k, \mathbf{0}) \,.$$

If the $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y}$, $j = 1, \dots, \ell$, $\ell \le k$, are chosen from $I^{es}_{\overline{C}\to\mathbb{C}^2}$ such that they represent a basis of the vector subspace

$$T^{1,es}_{\overline{C}\to\mathbb{C}^2} = I^{es}_{\overline{C}\to\mathbb{C}^2} \Big/ \left(\overline{\mathfrak{m}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y} \right) + \left(\mathfrak{m}\frac{\partial}{\partial x} \oplus \mathfrak{m}\frac{\partial}{\partial y} \right) \right)$$

of $T^{1,sec}_{\overline{C}\to\mathbb{C}^2}$, then (2.7.33) with k replaced by ℓ represents a semiuniversal equisingular deformation of the normalization over

$$(B^{es}_{\overline{C}\to C},\mathbf{0})\cong(T^{1,es}_{\overline{C}\to\mathbb{C}^2},\mathbf{0})\subset(T^{1,sec}_{\overline{C}\to\mathbb{C}^2},\mathbf{0})\cong(B^{sec}_{\overline{C}\to C},\mathbf{0})\,.$$

For the proof of Theorem 2.61, we make now use of the following results of Lazzeri, Lê and Teissier:

Proposition 2.62. *Let $\phi:\mathscr{C}\to S$ be a sufficiently small representative of an arbitrary deformation of the reduced plane curve singularity $(C,\mathbf{0})$ with S reduced. Then the following holds:*

(1) If $\mu(\mathscr{C}_s)=\mu(C,\mathbf{0})$ for each $s\in S$, then there exists a unique section $\sigma:S\to\mathscr{C}$ of ϕ such that $\mathscr{C}_s\setminus\{\sigma(s)\}$ is smooth and, hence, $\mu(\mathscr{C}_s)=\mu(\mathscr{C}_s,\sigma(s))$ for each $s\in S$.
(2) Let $\sigma:S\to\mathscr{C}$ be a section of ϕ. Then $\mu(\mathscr{C}_s,\sigma(s))$ is independent of $s\in S$ iff $\delta(\mathscr{C}_s,\sigma(s))$ and $r(\mathscr{C}_s,\sigma(s))$ are independent of $s\in S$.
(3) If $\sigma:S\to\mathscr{C}$ is a section of ϕ such that $\mu(\mathscr{C}_s,\sigma(s))$ is independent of $s\in S$ then the multiplicity $\mathrm{mt}(\mathscr{C}_s,\sigma(s))$ is independent of $s\in S$.

Proof. (1) is due to C. Has Bey [Has] and Lazzeri [Laz] (for arbitrary isolated hypersurface singularities); the existence of the section to Teissier [Tei]. For a proof of (2), see e.g. [Tei]. (3) is due to Lê [Le, LeR]. □

Proof of Theorem 2.61. (1) The tangent map of $\alpha^{sec}:B^{sec}_{\overline{C}\to C}\to B_C$ is the map $\alpha':T^{1,sec}_{\overline{C}\to\mathbb{C}^2}\to T^1_C$ described in Lemma 2.33. By Corollary 2.35, we know that $\alpha'|_{T^{1,es}_{\overline{C}\to\mathbb{C}^2}}$ is injective since $T^{1,es}_{\overline{C}\to\mathbb{C}^2}\subset T^{1,em}_{\overline{C}\to\mathbb{C}^2}$ by construction.

This proves already (applying the epimorphism Theorem I.1.20) that $\alpha^{sec}|_{B^{es}_{\overline{C}\to C}}$ is a closed embedding mapping $B^{es}_{\overline{C}\to C}$ isomorphically onto a smooth closed analytic subset $\Delta^{es}\subset B_C$ (for sufficiently small representatives).

(2) We prove that $\Delta^{es}=\Delta^{\mu}$. For the inclusion $\Delta^{es}\subset\Delta^{\mu}$ note that the deformation $\mathscr{C}^{es}\to B^{es}_{\overline{C}\to C}$ is δ-constant along the given section $\sigma:B^{es}_{\overline{C}\to C}\to\mathscr{C}^{es}$ since it has a simultaneous normalization $\overline{\mathscr{C}}^{es}\to\mathscr{C}^{es}$ (see Lemma 2.53).

Moreover, we claim that $r(\mathscr{C}_s,\sigma(s))=r(C,\mathbf{0})$ for all $s\in S$. If we assume the contrary, then $r(\mathscr{C}^{es}_s,\sigma(s))>r(C,\mathbf{0})=:r$ for $s\in U\setminus\{\mathbf{0}\}$, U some open neighbourhood of $\mathbf{0}\in S$ (since r branches are given by the parametrization). The extra branches of $(\mathscr{C}^{es}_s,\sigma(s))$ split off in some strict transform $\mathscr{C}'_s$ obtained by successively blowing up equimultiple sections. Hence, they are not in the image of $(\overline{\mathscr{C}}^{es}_s,\overline{0})$. This implies that the deformation of the parametrization of $(\mathscr{C}^{es}_s,\sigma(s))$ is not equimultiple (see Example 2.26.1), contradicting the definition of equisingularity.

From the relation $\mu=2\delta-r+1$ (Proposition I.3.35), we get that the Milnor number $\mu(\mathscr{C}^{es}_s,\sigma(s))$ is constant in s and, hence, $\Delta^{es}\subset\Delta^{\mu}$.

To show the opposite inclusion, $\Delta^{es} \supset \Delta^{\mu}$, we apply Proposition 2.62. It yields the existence of a section $\rho : \Delta^{\mu} \to \mathscr{D}^{\mu}$ of the restriction of $\mathscr{D} \to B_C$ to Δ^{μ} such that $\delta(\mathscr{D}_s, \rho(s))$, $r(\mathscr{D}_s, \rho(s))$ and $\mu(\mathscr{D}_s, \rho(s))$ are constant for $s \in \Delta^{\mu}$.

Hence, $\Delta^{\mu} \subset \Delta^{\delta}$ and, therefore, Δ^{μ} is in the image of $\alpha : B_{\overline{C}\to C} \to B_C$. Moreover, being r-constant and mt-constant implies as in the proof of (1) that the restriction of $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ to $\alpha^{-1}(\Delta^{\mu})$ admits uniquely determined compatible sections $\sigma : \alpha^{-1}(\Delta^{\mu}) \to \mathscr{C}$ and $\overline{\sigma}_i : \alpha^{-1}(\Delta^{\mu}) \to \overline{\mathscr{C}}$, $i = 1, \ldots, r = r(C, \mathbf{0})$, such that the deformation of the parametrization $\coprod_{i=1}^{r} \big(\overline{\mathscr{C}}, \overline{\sigma}_i(s)\big) \to \big(\mathscr{C}, \sigma(s)\big)$ is equimultiple for $s \in \alpha^{-1}(\Delta^{\mu})$. This shows that $\alpha^{-1}(\Delta^{\mu})$ is in the image of the morphism $B^{em}_{\overline{C}\to C} \hookrightarrow B^{sec}_{\overline{C}\to C} \to B_{\overline{C}\to C}$, that is, $(\alpha^{sec})^{-1}(\Delta^{\mu}) \subset B^{em}_{\overline{C}\to C}$.

Now, we blow up $\mathscr{C}$ along the equimultiple section $\sigma : \alpha^{-1}(\Delta^{\mu}) \to \mathscr{C}$ to get a family $\mathscr{C}' = \coprod_{i=1}^{r'} \big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$ of (multi)germs. Since $r\big(\mathscr{C}_s, \sigma(s)\big)$ is constant, the number of branches of $\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$ is constant for $i = 1, \ldots, r'$. By Proposition I.3.34, we have

$$\delta\big(\mathscr{C}, \sigma(s)\big) = \delta(\mathscr{C}') + \frac{\mathrm{mt}(\mathrm{mt} - 1)}{2},$$

where $\mathrm{mt} = \mathrm{mt}\big(\mathscr{C}, \sigma(s)\big) = \mathrm{mt}(C, \mathbf{0})$. Hence, for each $i = 1, \ldots, r'$, the map germ $\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big) \to (B_{\overline{C}\to C}, s)$, is a δ-constant family. Applying, again, the relation $\mu = 2\delta - r + 1$, we get that $\mu\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$ and, hence, $\mathrm{mt}\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$, is constant for $s \in \alpha^{-1}(\Delta^{\mu})$. Therefore, we can argue by induction on the number of blowing ups needed to resolve $(C, \mathbf{0})$, to show that after blowing up there exist always equimultiple sections. We conclude that $(\alpha^{sec})^{-1}(\Delta^{\mu}) \subset B^{es}_{\overline{C}\to C}$. □

2.8 Comparison of Equisingular Deformations

The main purpose of this section is to prove the equivalence of the functors of equisingular deformations of the parametrization and of equisingular deformations of the equation. Moreover, we discuss related deformations.

We start by reconsidering the constructions and results of this chapter, describe their relation and discuss computational aspects.

In Section 2.1, we introduced equisingular deformations of $(C, \mathbf{0})$, also denoted equisingular deformations of the equation, and proved that equisingular deformations of $(C, \mathbf{0})$ induce equisingular deformations of the branches (Proposition 2.11). We defined (Definition 2.7) the equisingular deformation functor $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}$ as a subfunctor of $\underline{\mathcal{D}ef}_{(C,\mathbf{0})}$, where we required the existence of an equimultiple section $\sigma = \sigma^{(0)}$ and of equimultiple sections $\sigma^{(\ell)}$ through the infinitely near points of successive blow ups of $(C, \mathbf{0})$. By Proposition 2.8, these sections are unique if $(C, \mathbf{0})$ is singular (which we assume in this discussion).

We can also consider equisingular deformations as deformations with section, where the section σ is part of the data (Definition 2.6). The set of isomorphism classes of equisingular deformations with section over (T, t_0) is denoted by $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}(T, t_0)$ and the functor

$$\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})} : (\text{complex germs}) \to (\text{sets})\,, \quad (T, t_0) \mapsto \underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}(T, t_0)$$

is called the *equisingular deformation functor with section*. By definition, $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}$ is a subfunctor of $\underline{\mathcal{D}ef}^{sec}_{(C,\mathbf{0})}$.

Since σ is uniquely determined by Proposition 2.8, $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}$ and $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}$ are isomorphic functors, but they are not equal. In particular, in concrete calculations, we have to distinguish them carefully.

In Section 2.2, we defined the equisingularity ideals $I^{es}(f)$, $I^{es}_{fix}(f)$ and gave explicit descriptions for semiquasihomogeneous and Newton non-degenerate singularities. For Newton-degenerate singularities, these ideals are quite complicated and no other description, besides their definition, is available.

We show now how $I^{es}(f)$ and $I^{es}_{fix}(f)$ are related to the functors $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}$ and $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}$.

Proposition 2.63. *Let* $T^{1,es}_{(C,\mathbf{0})} = \underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}(T_\varepsilon)$, *resp.* $T^{1,es,sec}_{(C,\mathbf{0})} = \underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}(T_\varepsilon)$ *be the vector spaces of infinitesimal equisingular deformations (resp. with section) of* $(C, \mathbf{0})$. *Then we have*

$$T^{1,es}_{(C,\mathbf{0})} \cong I^{es}(f)/\langle f, j(f)\rangle \subset \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}/\langle f, j(f)\rangle = T^1_{(C,\mathbf{0})}\,,$$
$$T^{1,es,sec}_{(C,\mathbf{0})} \cong I^{es}_{fix}(f)/\langle f, \mathfrak{m}j(f)\rangle \subset \mathfrak{m}/\langle f, \mathfrak{m}j(f)\rangle = T^{1,sec}_{(C,\mathbf{0})}\,,$$

where $\mathfrak{m} = \mathfrak{m}_{\mathbb{C}^2,\mathbf{0}}$.

The statement follows from Proposition 2.14, noting that the ideals $\langle f, j(f)\rangle$ (resp. $\langle f, \mathfrak{m}j(f)\rangle$) describe the infinitesimally trivial (embedded) deformations (resp. with trivial section) of $(C, \mathbf{0})$ (see Remark 1.25.1 and Corollary 2.3).

For Newton degenerate singularities, the vector spaces $T^{1,es}_{(C,\mathbf{0})}$ and $T^{1,es,sec}_{(C,\mathbf{0})}$ cannot be easily described. In particular, they are, in general, not generated by monomials (see the example below). However, in [CGL1], an algorithm to compute both vector spaces is given. This algorithm is implemented in Singular and can be used to compute explicit examples:

Example 2.63.1 (Continuation of Example 2.17.2). The following Singular session computes a list `Ies` whose first entry is the ideal $I^{es}(f)$ (given by a list of generators), whose second entry is the ideal $I^{es}_{fix}(f)$, and whose third entry is the ideal $\langle j(f), I^s\rangle$:

```
LIB "equising.lib";
ring R = 0,(x,y),ds;
poly f = (x-2y)^2*(x-y)^2*x2y2+x9+y9;
list Ies = esIdeal(f,1);
```

We make SINGULAR display ideal generators for the quotient $I^{es}(f)/\langle f, j(f)\rangle$ and for $I^{es}_{fix}(f)/\langle f, \mathfrak{m}j(f)\rangle$:

```
ideal J = f,jacob(f);
ideal IesQ = reduce(Ies[1],std(J));
simplify(IesQ,11);
//-> _[1]=x3y5-3x2y6+2xy7
//-> _[2]=y9
//-> _[3]=x2y7
//-> _[4]=xy8
ideal mJ = f,maxideal(1)*jacob(f);
ideal IesfixQ = reduce(Ies[2],std(mJ));
simplify(IesfixQ,11);
//-> _[1]=x3y5-xy7+9/4y9
//-> _[2]=x2y7-y9
//-> _[3]=xy8+y9
//-> _[4]=y10
```

From the output, we read that $I^{es}(f)$ is generated as an ideal by the Tjurina ideal $\langle f, j(f)\rangle$ and the polynomials $x^3y^5 - 3x^2y^6 + 2xy^7$, y^9, x^2y^7 and xy^8, and similarly for $I^{es}_{fix}(f)$. Finally, we check that

$$x^5y^3 - 6x^4y^4 + 13x^3y^5 - 12x^2y^6 + 4xy^7 \in I^{es}(f) \setminus \langle f, j(f), I^s(f)\rangle$$

as claimed in Example 2.17.2:

```
poly g=x5y3-6x4y4+13x3y5-12x2y6+4xy7;
reduce(g,std(Ies[1]));
//-> 0
reduce(g,std(Ies[3]));
//-> 1/3x3y5-x2y6+2/3xy7
```

In order to prove properties of equisingular deformations of $(C, \mathbf{0})$, we introduced in Section 2.3 (equimultiple) deformations of the parametrization $\varphi : (\overline{C}, \overline{0}) \to (\mathbb{C}^2, \mathbf{0})$, and we computed the vector spaces T^1 and T^2 for several related deformation functors in Section 2.4. In Section 2.5, we defined equisingular deformations of φ and showed that they have a rather simple description. In particular, the functor of equisingular deformations of φ is a linear subfunctor of the functor of (arbitrary) deformations with section of φ and, thus, each versal equisingular deformation of φ has a smooth base (Theorem 2.38).

The link between deformations of the parametrization and deformations of the equation is given in Proposition 2.23 which is based on Proposition 2.9. It says that each deformation of the parametrization induces a unique (up to isomorphism) deformation of the equation. By Lemma 2.53, such deformations of $(C, \mathbf{0})$ are δ-constant. Conversely, if the base space $(T, \mathbf{0})$ is normal, then a δ-constant deformation of $(C, \mathbf{0})$ over $(T, \mathbf{0})$ is induced by a deformation of the parametrization (Theorem 2.56).

If $(B_C, \mathbf{0})$, resp. $(B_{\overline{C}\to C}, \mathbf{0})$, is the base space of the semiuniversal deformation of $(C, \mathbf{0})$, resp. of the parametrization φ of $(C, \mathbf{0})$, then $(B_C, \mathbf{0})$ and $(B_{\overline{C}\to C}, \mathbf{0})$ are smooth, the natural map $\alpha : (B_{\overline{C}\to C}, \mathbf{0}) \to (B_C, \mathbf{0})$ maps $(B_{\overline{C}\to C}, \mathbf{0})$ onto the δ-constant stratum and $(B_{\overline{C}\to C}, \mathbf{0}) \to (\Delta^\delta, \mathbf{0})$ is the normalization (Theorem 2.59, using that deformations of the parametrization and of the normalization coincide by Proposition 2.23).

The base space $(B^{es}_{\overline{C}\to\mathbb{C}^2}, \mathbf{0})$ of the semiuniversal equisingular deformation of the parametrization is a subspace of the base space $(B^{sec}_{\overline{C}\to\mathbb{C}^2}, \mathbf{0})$ of the semiuniversal deformation of the parametrization with section. Theorem 2.38 yields that $(B^{es}_{\overline{C}\to\mathbb{C}^2}, \mathbf{0})$ is smooth. Moreover, the natural map $\alpha^{sec} : (B^{sec}_{\overline{C}\to\mathbb{C}^2}, \mathbf{0}) \to (B_C, \mathbf{0})$ takes $(B^{es}_{\overline{C}\to\mathbb{C}^2}, \mathbf{0})$ isomorphically onto the μ-constant stratum $(\Delta^\mu, \mathbf{0}) \subset (B_C, \mathbf{0})$ (Theorem 2.61).

It still remains to complete the relation between equisingular deformations of the parametrization (Definition 2.36) and equisingular deformations of the equation (Definition 2.7):

Theorem 2.64. *Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve singularity.*

(1) Every equisingular deformation of the parametrization of $(C, \mathbf{0})$ induces a unique equisingular deformation of the equation, providing a functor $\mathcal{D}ef^{es}_{\overline{C}\to\mathbb{C}^2} \to \mathcal{D}ef^{es}_C$.

(2) Every equisingular embedded deformation of the equation of $(C, \mathbf{0})$ comes from an equisingular deformation of the parametrization (which is induced by the equisingular deformation of the resolution); that is, $\mathcal{D}ef^{es}_{\overline{C}\to\mathbb{C}^2} \to \mathcal{D}ef^{es}_C$ is surjective.

(3) The functor $\mathcal{D}ef^{es}_{\overline{C}\to\mathbb{C}^2} \to \mathcal{D}ef^{es}_C$ induces a natural equivalence between the functors $\underline{\mathcal{D}ef}^{es}_{\overline{C}\to\mathbb{C}^2}$ and $\underline{\mathcal{D}ef}^{es}_C$.

The proof of this theorem is less evident than one might think, in particular for non-reduced base spaces.

Before giving the proof, we need some preparations. If

$$\psi : (\mathscr{C}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$$

is an embedded equisingular deformation of the reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ along a section $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, \mathbf{0})$, then we consider the associated equisingular deformation of the resolution (see Definition 2.6, p. 271, and Remark 2.6.1 (6)),

$$\begin{tikzcd}
(\mathscr{C}^{(N)}, p^{(N)}) \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & (\mathscr{C}^{(1)}, p^{(1)}) \arrow[r] \arrow[d, hook] & (\mathscr{C}, \mathbf{0}) \arrow[d, hook] \arrow[dr, "\psi"] & \\
 & & & & (T, \mathbf{0}) \\
(\mathscr{M}^{(N)}, p^{(N)}) \arrow[r, "\pi_N"] & \cdots \arrow[r, "\pi_2"] & (\mathscr{M}^{(1)}, p^{(1)}) \arrow[r, "\pi_1"] & (\mathscr{M}, \mathbf{0}) \arrow[ur] &
\end{tikzcd} \tag{2.8.34}$$

with (multi-)sections $\sigma^{(\ell)} : (T, \mathbf{0}) \to (\mathscr{C}^{(\ell)}, p^{(\ell)})$, $\ell = 1, \ldots, N$, which are unique by Proposition 2.8, p. 275.

If we restrict the diagram to $\{\mathbf{0}\} \subset T$, we obtain an embedded (minimal) resolution of $(C, \mathbf{0})$ with $(C^{(\ell)}, p^{(\ell)}) \subset (M^{(\ell)}, p^{(\ell)})$ the strict transform of $(C, \mathbf{0})$. We denote by $E^{(\ell)} \subset M^{(\ell)}$, resp. $\mathscr{E}^{(\ell)} \subset \mathscr{M}^{(\ell)}$, the exceptional divisor of the successive blowing ups of the points $\mathbf{0}$, $p^{(i)}$, resp. of the (multi-)sections σ and $\sigma^{(j)}$, $j < \ell$, such that $C^{(\ell)} \cup E^{(\ell)} \subset M^{(\ell)}$, resp. $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)} \subset \mathscr{M}^{(\ell)}$, are the reduced total transforms of $(C, \mathbf{0})$, resp. the deformations of the reduced total transforms.

The composition $(\overline{\mathscr{C}}, \overline{0}) := (\mathscr{C}^{(N)}, p^{(N)}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ together with the section $(T, \mathbf{0}) \xrightarrow{\sigma} (\mathscr{M}, \mathbf{0}) \hookrightarrow (T, \mathbf{0})$, which we denote also by σ, and the (multi-)section $\sigma^{(N)}$, which we denote by $\overline{\sigma}$, is a deformation of the parametrization. The deformation $(\phi, \overline{\sigma}, \sigma) \in \mathcal{D}ef^{\,sec}_{\overline{C} \to C}(T, \mathbf{0})$ is uniquely determined (up to isomorphism) by $(\psi, \sigma) \in \mathcal{D}ef^{\,es}_{C}(T, \mathbf{0})$. We call it the *deformation of the parametrization induced by the equisingular deformation of the resolution of* (ψ, σ).

Theorem 2.64 implies that $(\phi, \overline{\sigma}, \sigma)$ is equisingular, that is, an object of $\mathcal{D}ef_{\overline{C} \to C}(T, \mathbf{0})$.

We have to generalize the concept of constant intersection multiplicity (see page 281) to families with non-reduced base spaces.

Let (M, p) be a germ of a two-dimensional complex manifold and let $(C, p) \subset (M, p)$ be a reduced curve singularity given by $f \in \mathcal{O}_{M,p}$. Consider an embedded deformation

$$\psi : (\mathscr{C}, p) \hookrightarrow (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$$

of (C, p) with section $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, p)$. Then $(\mathscr{C}, p) \subset (\mathscr{M}, p)$ is defined by a holomorphic germ $F \in \mathcal{O}_{\mathscr{M},p}$.

Consider a second reduced curve singularity $(D, p) \subset (M, p)$ given by a parametrization $\varphi : (\overline{D}, \overline{0}) \to (M, p)$ such that (D, p) and (C, p) have no common component. Let

$$\phi : (\overline{\mathscr{D}}, \overline{0}) \hookrightarrow (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$$

be a deformation of φ with compatible sections $\overline{\sigma} : (T, \mathbf{0}) \to (\overline{\mathscr{D}}, \overline{0})$ and $\sigma : (T, \mathbf{0}) \to (\mathscr{M}, p)$. We assume that the section σ coincides with the composition $(T, \mathbf{0}) \xrightarrow{\sigma} (\mathscr{C}, p) \hookrightarrow (\mathscr{M}, p)$, where $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, p)$ is the section for the embedded deformation $(\mathscr{C}, p) \to (T, \mathbf{0})$ from above.

If (D, p) has r branches (D_i, p), $i = 1, \ldots, r$, then $(\overline{\mathscr{D}}, \overline{0}) = \coprod_{i=1}^{r} (\overline{\mathscr{D}}_i, \overline{0}_i)$ and $\overline{\sigma} = (\overline{\sigma}_i)_{i=1..r}$. We may (and do) assume that $(\overline{\mathscr{D}}_i, \overline{0}_i) = (\mathbb{C} \times T, \mathbf{0})$, $i = 1, \ldots, r$, that $(\mathscr{M}, p) = (\mathbb{C}^2 \times T, \mathbf{0})$, and that $\overline{\sigma}_i$ and σ are the trivial sections. Then the deformation ϕ is given by maps $\phi_i : \mathbb{C} \times T \to \mathbb{C}^2$, $i = 1, \ldots, r$,

$$\begin{aligned} (\overline{\mathscr{D}}_i, \overline{0}_i) = (\mathbb{C} \times T, \mathbf{0}) &\longrightarrow (\mathbb{C}^2 \times T, \mathbf{0}) = (\mathscr{M}, p)\,, \\ (t_i, s) &\longmapsto \big(\phi_i(t_i, s), s\big)\,. \end{aligned}$$

Definition 2.65. With the above notations, we say that the deformation of the equation $\psi : (\mathscr{C}, p) \hookrightarrow (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$ of $(C, \mathbf{0})$ with section σ and the deformation of the parametrization $(\overline{\mathscr{D}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$ of (D, p) with compatible sections $\overline{\sigma}$ and σ are *equiintersectional (along σ)* if

$$\mathrm{ord}_{t_i}(F \circ \phi_i) = \mathrm{ord}_{t_i}(f \circ \varphi_i), \quad i = 1, \dots, r\,.$$

We call $\mathrm{ord}_{t_i}(F \circ \phi_i)$ the *intersection multiplicity of the deformations* (ψ, σ) and $(\phi, \overline{\sigma}, \sigma)$.

Remark 2.65.1. Let the base space $(T, \mathbf{0})$ be reduced. Then, for sufficiently small representatives,

$$\mathrm{ord}_{t_i}\big(F \circ \phi_i(t_i, s)\big) = i_{\sigma(s)}(\mathscr{C}_s, \mathscr{D}_{i,s})\,, \quad s \in T\,.$$

Here, $\mathscr{C}_s = \psi^{-1}(s)$ and $\mathscr{D}_{i,s} = \phi_i(\overline{\mathscr{D}}_i \cap (\mathbb{C} \times \{s\})$ are the fibres of $\psi : \mathscr{C} \to T$ and $\mathscr{D} \to T$ over s, where $\mathscr{D} = \phi(\overline{\mathscr{D}}) \to T$ is the induced deformation of the equation (Corollary 2.24).

Hence, for reduced base spaces, equiintersectional along σ means that the intersection number of $\mathscr{C}_s$ and $\mathscr{D}_{i,s}$ at $\sigma(s)$ is independent of $s \in T$ for $i = 1, \dots, r$.

Proposition 2.66. *Let $(D, \mathbf{0})$, $(L, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be reduced curve singularities with $(L, \mathbf{0})$ smooth and not a component of $(D, \mathbf{0})$. Let*

$$\chi : (\mathscr{D}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) = (\mathbb{C}^2 \times T, \mathbf{0}) \to (T, \mathbf{0})$$

be an equisingular deformation of the equation of $(D, \mathbf{0})$ along the trivial section σ and let $(\overline{\mathscr{D}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be the deformation of the parametrization of $(D, \mathbf{0})$ with trivial (multi-)section $\overline{\sigma} : (T, \mathbf{0}) \to (\overline{\mathscr{D}}, \overline{0})$ induced by the equisingular deformation of the resolution of $(D, \mathbf{0})$ associated to (χ, σ). Assume that $(\phi, \overline{\sigma}, \sigma)$ is equisingular as deformation of the parametrization. Further, denote by $\psi : (\mathscr{L}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ the (trivial) deformation of $(L, \mathbf{0})$ along σ and by

$$\chi_L : (\mathscr{C}, \mathbf{0}) := (\mathscr{D}, \mathbf{0}) \cup (\mathscr{L}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$$

the induced deformation of the equation of $(C, \mathbf{0}) := (D, \mathbf{0}) \cup (L, \mathbf{0})$ along σ.

Then (χ_L, σ) is equisingular iff (ψ, σ) and $(\phi, \overline{\sigma}, \sigma)$ are equiintersectional along σ.

Proof. (1) Let (χ_L, σ) be equisingular. Since the statement is about the branches of $(D, \mathbf{0})$, we may assume that $(D, \mathbf{0})$ is irreducible. Choosing local analytic coordinates x, y of $(\mathbb{C}^2, \mathbf{0})$ and t of $(\mathbb{C}, 0)$, the map $\phi : (\overline{\mathscr{C}}, \overline{0}) = (\mathbb{C} \times T, \mathbf{0}) \to (\mathbb{C}^2 \times T, \mathbf{0})$ is given by

$$t \mapsto \big(X(t), Y(t)\big) \;\text{ with }\; X, Y \in \mathcal{O}_{T,\mathbf{0}}\{t\}$$

such that $\big(x(t),y(t)\big) := \big(X(t),Y(t)\big)$ mod $\mathfrak{m}_{T,\mathbf{0}}$ parametrize $(D,\mathbf{0})$.

Since $(\phi,\overline{\sigma},\sigma)$ is equisingular, it is equimultiple along σ, that is,

$$\min\big\{\mathrm{ord}_t\,X(t),\mathrm{ord}_t\,Y(t)\big\} = m\,,$$

where $m = \min\big\{\mathrm{ord}_t\,x(t),\mathrm{ord}_t\,y(t)\big\}$ is the multiplicity of $(D,\mathbf{0})$.

We may choose the coordinates such that $x=0$ is an equation for $(L,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$. Then $\mathrm{ord}_t\,X(t)$ is the intersection multiplicity of (ψ,σ) and $(\phi,\overline{\sigma},\sigma)$ and we have to show that $\mathrm{ord}_t\,X(t) = \mathrm{ord}_t\,x(t)$.

We prove this by induction on the number n of blowing ups needed to separate $(D,\mathbf{0})$ and $(L,\mathbf{0})$.

If $n=1$, then the germs $(D,\mathbf{0})$ and $(L,\mathbf{0})$ intersect transversally so that $m = \mathrm{mt}(D,\mathbf{0}) = i_{\mathbf{0}}(D,L) = \mathrm{ord}_t\,x(t)$. Since $(\phi,\overline{\sigma},\sigma)$ is equimultiple along σ, $\mathrm{ord}_t\,X(t) \geq m$ and, hence, $\mathrm{ord}_t\,X(t) = m = \mathrm{ord}_t\,x(t)$.

Now, let $n>1$ and consider the blowing up $\mathscr{M}^{(1)} \to \mathscr{M}$ of the trivial section σ (for a small representative $\mathscr{M}$ of $(\mathscr{M},\mathbf{0})$). Since $n>1$, there is a unique point $p = p^{(1)} \in \mathscr{M}^{(1)}$ belonging to $D \cap L$, and $(\mathscr{M}^{(1)},p) \cong (M^{(1)},p) \times (T,\mathbf{0})$ in the notation introduced right after Theorem 2.64.

We choose local coordinates u,v identifying $(M^{(1)},p)$ with $(\mathbb{C}^2,\mathbf{0})$. Then the (germ of the) blowing up

$$(\mathbb{C}^2 \times T,\mathbf{0}) \cong (\mathscr{M}^{(1)},p) \xrightarrow{\pi} (\mathscr{M},\mathbf{0}) = (\mathbb{C}^2 \times T,\mathbf{0})$$

is given by $(x,y) = (uv,v)$ and the identity on $(T,\mathbf{0})$. We assume again that $\sigma^{(1)}\colon (T,\mathbf{0}) \to (\mathscr{M}^{(1)},p)$ is the trivial section.

Let $(C^{(1)},p) = (D^{(1)},p) \cup (L^{(1)},p)$ be the strict transform of $(C,\mathbf{0})$. Then, by Remark 2.6.1 (5), p. 273,

$$(\mathscr{C}^{(1)},p) = (\mathscr{D}^{(1)} \cup \mathscr{L}^{(1)},p) \hookrightarrow (\mathscr{M}^{(1)},p) \to (T,\mathbf{0})$$

is an equisingular embedded deformation of $(C^{(1)},p)$ along $\sigma^{(1)}$. The induced deformation $\psi^{(1)}\colon (\mathscr{L}^{(1)},p) \hookrightarrow (\mathscr{M}^{(1)},p) \to (T,\mathbf{0})$ of $(L^{(1)},p)$ along $\sigma^{(1)}$ is denoted by $(\psi^{(1)},\sigma^{(1)})$.

Since $(\overline{\mathscr{D}},\overline{0}) \xrightarrow{\phi} (\mathscr{M},\mathbf{0}) \to (T,\mathbf{0})$ is induced by the equisingular deformation of the resolution, we have an induced map

$$(\overline{\mathscr{D}},\overline{0}) \xrightarrow{\phi^{(1)}} (\mathscr{M}^{(1)},p) \to (T,\mathbf{0})$$

such that $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ is an equisingular deformation of the parametrization of $(D^{(1)},p)$. Using the coordinates u,v, the map $\phi^{(1)}\colon (\mathbb{C} \times T,\mathbf{0}) \to (\mathbb{C}^2 \times T,\mathbf{0})$ is given by

$$t \mapsto \big(U(t),V(t)\big) \;\text{ with }\; U,V \in \mathcal{O}_{T,\mathbf{0}}\{t\}$$

such that $\big(u(t),v(t)\big) := \big(U(t),V(t)\big)$ mod $\mathfrak{m}_{T,\mathbf{0}}$ parametrize $(D^{(1)},p)$.

Now, $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ satisfy the assumptions of the lemma and $(\mathscr{D}^{(1)},p)$ and $(\mathscr{L}^{(1)},p)$ are separated after $n-1$ blowing ups. Hence, by induction assumption, $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ are equiintersectional along $\sigma^{(1)}$.

Since we assumed that L is given as $x=0$, we get that $L^{(1)}$ and $D^{(1)}$ meet in the chart given by $(x,y)=(uv,v)$ and $L^{(1)}$ is given by $u=0$. Moreover, the exceptional divisor $\mathscr{E}^{(1)}$ in $(\mathbb{C}^2\times T,\mathbf{0})$ is given by $v=0$ and we have

$$X(t)=U(t)V(t)\,,\quad Y(t)=V(t)\,.$$

The assumption $n>1$ implies that $i_{\mathbf{0}}(L,D)=\operatorname{ord}_t x(t)>\operatorname{ord}_t y(t)=m$. Thus, $\operatorname{ord}_t V(t)=\operatorname{ord}_t Y(t)=m$. Since $\operatorname{ord}_t X(t)=\operatorname{ord}_t U(t)+\operatorname{ord}_t V(t)$ and $\operatorname{ord}_t x(t)=\operatorname{ord}_t u(t)+\operatorname{ord}_t v(t)$, we have to show that $\operatorname{ord}_t U(t)=\operatorname{ord}_t u(t)$.

Since $L^{(1)}$ is given by $u=0$, the intersection multiplicity of $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ is $\operatorname{ord}_t U(t)$. Since $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ are equiintersectional along $\sigma^{(1)}$, we have $\operatorname{ord}_t U(t)=\operatorname{ord}_t u(t)$ as claimed.

(2) Let (ψ,σ) and $(\phi,\overline{\sigma},\sigma)$ be equiintersectional. Since $(L,\mathbf{0})$ is smooth, (ψ,σ) is equimultiple. Since (χ,σ) is equisingular, (χ_L,σ) is equimultiple, too.

Consider the equisingular deformation of the minimal embedded resolution of $(D,\mathbf{0})$ associated to (χ,σ). Then the deformation

$$(\mathscr{D}^{(\ell)}\cup\mathscr{E}^{(\ell)},p^{(\ell)})\hookrightarrow(\mathscr{M}^{(\ell)},p^{(\ell)})\to(T,\mathbf{0})$$

of the reduced total transform $(D^{(\ell)}\cup E^{(\ell)},p^{(\ell)})$ of $(D,\mathbf{0})$ is equimultiple along the (multi-)section $\sigma^{(\ell)}$. It remains to show that the deformation

$$(\mathscr{D}^{(\ell)}\cup\mathscr{L}^{(\ell)}\cup\mathscr{E}^{(\ell)},p^{(\ell)})\hookrightarrow(\mathscr{M}^{(\ell)},p^{(\ell)})\to(T,\mathbf{0}) \tag{2.8.35}$$

of the reduced total transform $(D^{(\ell)}\cup L^{(\ell)}\cup E^{(\ell)},p^{(\ell)})$ of $(D\cup L,\mathbf{0})$ is equimultiple along $\sigma^{(\ell)}$ for $\ell\geq 1$.

We prove this claim again by induction on n, the number of blowing ups needed to separate $(D,\mathbf{0})$ and $(L,\mathbf{0})$.

If $n=1$, then $D^{(\ell)}$ and $L^{(\ell)}$ do not meet in $M^{(\ell)}$ for $\ell\geq 1$ and the claim is trivially true.

Let $n>1$ and $p\in M^{(1)}$ the unique intersection point of $L^{(1)}$ and $E^{(1)}$. Denote by $\Lambda_p\subset\{1,\dots,r\}$ the set of indices such that the strict transform $D_i^{(1)}$ of the i-th branch $(D_i,\mathbf{0})$ of $(D,\mathbf{0})$ passes through p for $i\in\Lambda_p$. Set

$$(D_p,\mathbf{0}):=\bigcup_{i\in\Lambda_p}(D_i,\mathbf{0})\,,$$

and let $(D^{(1)},p)$ be the strict transform of $(D_p,\mathbf{0})$.

Choose coordinates x,y of $(\mathbb{C}^2,\mathbf{0})$ and u,v of $(M^{(1)},p)$, and let $x=0$ be the equation of $(L,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$. Since $n>1$, we have the relation $x=uv$ and $y=v$ and $(L^{(1)},p)\subset(M^{(1)},p)$ is given by $u=0$.

Let $\phi_i:(\overline{\mathscr{D}}_i,\overline{0}_i)\to(\mathscr{M},\mathbf{0})$, resp. $\phi_i^{(1)}:(\overline{\mathscr{D}}_i,\overline{0}_i)\to(\mathscr{M}^{(1)},p)$, be the deformation of the parametrization of $(D_i,\mathbf{0})$, resp. $(D_i^{(1)},p)$ (along the trivial section $\sigma_p^{(1)}$), $i\in\Lambda_p$, given by $t_i\mapsto\big(X_i(t_i),Y_i(t_i)\big)$, resp. $t_i\mapsto\big(U_i(t_i),V_i(t_i)\big)$. We have the relations

$$X_i(t_i) = U_i(t_i)V_i(t_i)\,,\quad Y_i(t_i) = V_i(t_i)\,,\quad i \in \Lambda_p\,,$$

and the same for the reductions mod $\mathfrak{m}_{(T,\mathbf{0})}$, $\big(x_i(t_i), y_i(t_i)\big)$, resp. $\big(u_i(t_i), v_i(t_i)\big)$, which are the paramerizations of $(D_i, \mathbf{0})$, resp. $(D_i^{(1)}, p)$.

For $i \in \Lambda_p$, the smooth germ $(L, \mathbf{0})$ is tangent to $(D_i, \mathbf{0})$ and, hence,

$$i_{\mathbf{0}}(L, D_i) = \mathrm{ord}_{t_i}\, x_i(t_i) > \mathrm{ord}_{t_i}\, y_i(t_i) =: m_i = \mathrm{mt}(D_i, \mathbf{0})\,.$$

Since ϕ_i is equimultiple along σ, $\mathrm{ord}_{t_i} Y_i(t_i) = m_i$ and, since (ψ, σ) and $(\phi, \overline{\sigma}, \sigma)$ are equiintersectional, $\mathrm{ord}_{t_i} X_i(t_i) = \mathrm{ord}_{t_i} x_i(t_i)$. Since, by the above relations, $\mathrm{ord}_{t_i} X_i(t_i) = \mathrm{ord}_{t_i} U_i(t_i) + m_i$ and $\mathrm{ord}_{t_i} x_i(t_i) = \mathrm{ord}_{t_i} u_i(t_i) + m_i$, we get $\mathrm{ord}_{t_i} U_i(t_i) = \mathrm{ord}_{t_i} u_i(t_i)$ for all $i \in \Lambda_p$.

Since $u = 0$ is the equation of the trivial deformation

$$\psi^{(1)} \colon (\mathscr{L}^{(1)}, p) \hookrightarrow (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})\,,$$

it follows that $(\psi^{(1)}, \sigma_p^{(1)})$ and $\big(\phi_p^{(1)} = \coprod_{i \in \Lambda_p} \phi_i^{(1)}, \overline{\sigma}_p, \sigma_p^{(1)}\big)$ are equiintersectional. Hence, we can apply the induction hypothesis to $(D^{(1)} \cup L^{(1)}, p)$ and it follows that the deformation (2.8.35) is equimultiple along $\sigma^{(\ell)}$ for all $\ell \geq 1$ as claimed. $\square$

Proof of Theorem 2.64. (2) Let $\psi : (\mathscr{C}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be an embedded equisingular deformation of the equation along σ, and let

$$(\overline{\mathscr{C}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$$

be the deformation of the parametrization induced by the equisingular deformation of the resolution along sections $\overline{\sigma}$, σ.

We prove that $(\phi, \overline{\sigma}, \sigma)$ is equisingular by induction on the number $N = N(C, \mathbf{0})$ of blowing ups needed to obtain a minimal embedded resolution of $(C, \mathbf{0})$.

If $N = 0$, then $(C, \mathbf{0})$ is smooth and every deformation is equisingular. Thus, let $N > 0$.

We consider the blowing up $(\mathscr{M}^{(1)}, p^{(1)}) \to (\mathscr{M}, \mathbf{0})$ of $(\mathscr{M}, \mathbf{0})$ along σ, with the uniquely determined equimultiple sections $\sigma_p^{(1)} : (T, \mathbf{0}) \to (\mathscr{M}^{(1)}, p)$ for each $p \in p^{(1)}$ (see Proposition 2.8, p. 275). Let $(\mathscr{C}^{(1)}, p)$, resp. $(\mathscr{E}^{(1)}, p)$, denote the strict transform of $(\mathscr{C}, \mathbf{0})$, resp. the exceptional divisor. By Remark 2.6.1 (5), $(\mathscr{C}^{(1)} \cup \mathscr{E}^{(1)}, p) \hookrightarrow (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})$ is an equisingular embedded deformation of the reduced total transform $(C^{(1)} \cup E^{(1)}, p)$ of $(C, \mathbf{0})$.

Moreover, by induction hypothesis ($N(C^{(1)}, p) < N$), the induced map

$$(\overline{\mathscr{C}}, \overline{p}) \xrightarrow{\phi^{(1)}} (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})$$

together with the sections $\overline{\sigma}_p$ and $\sigma_p^{(1)}$ defines an equisingular deformation of the parametrization of $(C^{(1)}, p)$. To show that $(\phi, \overline{\sigma}, \sigma)$ is equisingular, it remains to show that $\phi = (\phi_i)_{i=1..r}$ is equimultiple along $\overline{\sigma}, \sigma$ (see Remark 2.36.1 (1)).

Choosing coordinates and using the notations as in the proof of Proposition 2.66 with all sections trivial,

$$\phi_i : (\mathbb{C} \times T, \mathbf{0}) \cong (\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{M}, \mathbf{0}) \cong (\mathbb{C}^2 \times T, \mathbf{0})$$

is given by $X_i(t_i)$, $Y_i(t_i)$ and we have to show that

$$\min\{\operatorname{ord}_{t_i} X_i(t_i), \operatorname{ord}_{t_i} Y_i(t_i)\} = \min\{\operatorname{ord}_{t_i} x_i(t_i), \operatorname{ord}_{t_i} y_i(t_i)\} =: m_i .$$

Let $(C_i^{(1)}, p_i) \subset (M^{(1)}, p_i)$ be the strict transform of $(C_i, \mathbf{0})$.

Choosing coordinates u, v of $(M^{(1)}, p_i) \cong (\mathbb{C}^2, \mathbf{0})$,

$$\phi_i^{(1)} : (\mathbb{C} \times T, \mathbf{0}) = (\overline{\mathscr{C}}_i, \overline{0}_i) \to (\mathscr{M}^{(1)}, p_i) \cong (\mathbb{C}^2 \times T, \mathbf{0})$$

is given by $U_i(t_i), V_i(t_i) \in \mathcal{O}_{T,\mathbf{0}}\{t_i\}$ defining an equimultiple deformation of the parametrization of $(C_i^{(1)}, p_i)$.

In the two charts covering $\mathscr{M}^{(1)}$, we have $(x, y) = (u, uv)$, resp. $(x, y) = (uv, v)$, depending on $p_i \in E^{(1)} = \mathbb{P}^1$. We may assume that $\{x = 0\}$ is tangent to the branch $(C_i, \mathbf{0})$. Then $\{y = 0\}$ is transversal to $(C_i, \mathbf{0})$, hence $m_i = \operatorname{ord}_{t_i} y_i(t_i)$. Since $\{x = 0\}$ and $(C_i, \mathbf{0})$ are not separated by blowing up $\mathbf{0}$ in $(\mathbb{C}^2, \mathbf{0})$, $p_i = \mathbf{0}$ in the chart given by $(x, y) = (uv, v)$ and $\mathscr{E}^{(1)}$ is given by $v = 0$ in $(\mathbb{C}^2 \times T, \mathbf{0})$.

Now, we apply Proposition 2.66 with $L = E^{(1)}$ to the deformation of the parametrization $(\phi_i^{(1)}, \overline{\sigma}_i, \sigma_i^{(1)})$ of $(C_i^{(1)}, p_i)$ and to the embedded deformation

$$(\mathscr{C}_i^{(1)} \cup \mathscr{E}^{(1)}, p) \hookrightarrow (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})$$

of $(C_i^{(1)} \cup E^{(1)}, p)$ and get that they are equiintersectional.

Since $\mathscr{E}^{(1)}$ is defined by v, equiintersectional means that

$$\operatorname{ord}_{t_i} V_i(t_i) = \operatorname{ord}_{t_i} v_i(t_i) .$$

Since we have the relations $x_i(t_i) = u_i(t_i)v_i(t_i)$, $y_i(t_i) = v_i(t_i)$, and

$$X_i(t_i) = U_i(t_i)V_i(t_i) , \quad Y_i(t_i) = V_i(t_i) ,$$

we get $m_i = \operatorname{ord}_{t_i} y_i(t_i) = \operatorname{ord}_{t_i} v_i(t_i) = \operatorname{ord}_{t_i} Y_i(t_i) \le \operatorname{ord}_{t_i} X_i(t_i)$. This proves that ϕ_i is equimultiple along $\overline{\sigma}_i$, $\sigma_i^{(1)}$, $i = 1, \dots, r$, which had to be shown.

(1) Let $(\overline{\mathscr{C}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be an equisingular deformation of the parametrization with section $\overline{\sigma}, \sigma$ and $(\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{C}, \mathbf{0}) \xrightarrow{\psi} (T, \mathbf{0})$ the induced deformation of the normalization, which yields a functor (by Proposition 2.23, p. 301). We have to show that $\psi : (\mathscr{C}, \mathbf{0}) \to (T, \mathbf{0})$ together with the section σ is equisingular in the sense of Definition 2.6.

We may assume that $\overline{\sigma}$ and σ are trivial sections. We argue by induction on the number of blowing ups needed to resolve the singularity $(C,\mathbf{0})$. The case $(C,\mathbf{0})$ being smooth is trivial. In the general case, Lemma 2.26, p. 303 yields that $(\mathscr{C},\mathbf{0}) \to (T,\mathbf{0})$ is equimultiple and we may consider the blowing up of $(\mathscr{M},\mathbf{0})$ along σ,

$$\begin{tikzcd}
(\widetilde{\mathscr{C}},\widetilde{p}) \arrow[r, "\pi"] \arrow[d, hook] & (\mathscr{C},\mathbf{0}) \arrow[d, hook] \arrow[dr] & \\
& & (T,\mathbf{0}), \\
(\widetilde{\mathscr{M}},\widetilde{p}) \arrow[r] & (\mathscr{M},\mathbf{0}) \arrow[ur] &
\end{tikzcd}$$

where $(\widetilde{\mathscr{C}},\widetilde{p})$ is the (multi)germ of the strict transform of $(\mathscr{C},\mathbf{0})$. By Definition 2.36 and Proposition 2.23, there is a morphism $\widetilde{\phi} : (\overline{\mathscr{C}},\overline{0}) \to (\widetilde{\mathscr{C}},\widetilde{p}) \to (T,\mathbf{0})$ and a (multi)section $\widetilde{\sigma} : (T,\mathbf{0}) \to (\widetilde{\mathscr{C}},\widetilde{p})$ such that $(\widetilde{\phi},\overline{\sigma},\widetilde{\sigma})$ is an equisingular deformation of the parametrization of $(\widetilde{\mathscr{C}},\widetilde{p})$. By induction hypothesis, for every $p \in \widetilde{p} = \pi^{-1}(\mathbf{0})$, $(\widetilde{\mathscr{C}},p) \to (T,\mathbf{0})$ is an equisingular deformation of the equation of the strict transform $(\widetilde{C},p)$ of $(C,\mathbf{0})$ along $\widetilde{\sigma}$ by Lemma 2.26.

Let $\mathscr{E} \subset \widetilde{\mathscr{M}}$ be the exceptional divisor. We have to show that, for each $p \in \widetilde{p}$,

$$(\widetilde{\mathscr{C}} \cup \mathscr{E},p) \hookrightarrow (\widetilde{\mathscr{M}},p) \to (T,\mathbf{0})$$

is an equisingular embedded deformation of the reduced total transform $(\widetilde{C} \cup E,p)$ along $\widetilde{\sigma}_p : (T,\mathbf{0}) \to (\widetilde{\mathscr{M}},p)$. By Proposition 2.66, we have to show that the deformations $(\widetilde{\phi},\overline{\sigma},\sigma)$ and (ψ,σ) with $\psi : (\mathscr{E},p) \hookrightarrow (\widetilde{\mathscr{M}},p) \to (T,\mathbf{0})$ are equiintersectional along σ.

We choose coordinates x,y of $(M,\mathbf{0}) = (\mathbb{C}^2,\mathbf{0})$ and u,v of $(\widetilde{M},p) = (\mathbb{C}^2,\mathbf{0})$ as in the proof of (2) and consider a branch $(\widetilde{C}_i,p_i)$ of $(\widetilde{C},p)$. Assuming that $\{x=0\}$ is tangent to $(C_i,\mathbf{0})$, we have $m_i := \mathrm{mt}(C_i,\mathbf{0}) = \mathrm{ord}_{t_i} y_i(t_i)$. As in the proof of (2), we have that the deformations of the parametrization $\widetilde{\phi}_i$, resp. ϕ_i, of $(\widetilde{C}_i,p_i)$, resp. $(C_i,\mathbf{0})$, are given by $U_i(t_i)$, $V_i(t_i)$, resp. by $X_i(t_i)$, $Y_i(t_i)$, satisfying the relations $X_i(t_i) = U_i(t_i)V_i(t_i)$ and $Y_i(t_i) = V_i(t_i)$. Since ϕ_i is equimultiple along the trivial sections $\overline{\sigma}_i, \sigma$, we have

$$\mathrm{ord}_{t_i} Y_i(t_i) = \mathrm{ord}_{t_i} y_i(t_i) = \mathrm{ord}_{t_i} v_i(t_i) = \mathrm{ord}_{t_i} V_i(t_i)\,.$$

This proves that (ψ,σ) and $(\widetilde{\phi}_i,\overline{\sigma}_i,\sigma_i)$ are equiintersectional along σ and hence (1).

(3) By (1), we have a natural transformation $\underline{\mathcal{D}ef}^{\,es}_{\overline{C}\to C} \to \underline{\mathcal{D}ef}^{\,es}_{C}$. It is easy to see that the equisingular deformation of the parametrization in (2) is unique up to isomorphism. This proves the claim. $\square$

Corollary 2.67. *Let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve singularity and let $i : (\Delta^\mu,\mathbf{0}) \hookrightarrow (B_C,\mathbf{0})$ be the inclusion of the μ-constant stratum in the base space of the semiuniversal deformation of $(C,\mathbf{0})$. Then the restriction of* *the semiuniversal deformation* $(\mathscr{C},\mathbf{0}) \to (B_C,\mathbf{0})$ *to* $(\Delta^\mu,\mathbf{0})$ *is an equisingular semiuniversal deformation of* $(C,\mathbf{0})$*, that is,* $i^*(\mathscr{C},\mathbf{0}) \to (\Delta^\mu,\mathbf{0})$ *is isomorphic to* $(\mathscr{C}_C^{es},\mathbf{0}) \to (B_C^{es},\mathbf{0})$*.*

Proof. By Theorem 2.61, $i^*(\mathscr{C},\mathbf{0}) \to (\Delta^\mu,\mathbf{0})$ lifts to a semiuniversal equisingular deformation of the parametrization $(\overline{\mathscr{C}}^{es},\overline{0}) \to i^*(\mathscr{C},\mathbf{0}) \to (\Delta^\mu,\mathbf{0})$ and, therefore, the result follows from Theorem 2.64. □

As an immediate consequence, we obtain:

Corollary 2.68. *A deformation of the equation of* $(C,\mathbf{0})$ *over a reduced base* $(T,\mathbf{0})$ *is equisingular iff, for sufficiently small representatives, the Milnor number is constant (along the unique singular section).*

For a reduced plane curve singularity $(C,\mathbf{0})$ with local equation $f \in \mathbb{C}\{x,y\}$, we introduce

$$\tau^{es}(C,\mathbf{0}) := \tau(C,\mathbf{0}) - \dim_{\mathbb{C}} T^{1,es}(C,\mathbf{0}) = \dim_{\mathbb{C}}(\mathbb{C}\{x,y\}/I^{es}(f))\,,$$

which is equal to the codimension of the μ-constant stratum $(\Delta^\mu,\mathbf{0})$ in the base of the semiuniversal deformation of $(C,\mathbf{0})$ (Theorem 2.64 and Proposition 2.63).

One of the reasons why equisingular deformations of the parametrization are so easy is that they form a linear subspace in the base space of the semiuniversal deformation of the parametrization (Theorem 2.38). This is in general not the case for equisingular deformation of the equation (see Example 2.71.1 below). Hence, the question arises whether there are singularities for which the μ-constant stratum is linear. The answer was given in [Wah]:

Proposition 2.69 (Wahl). *Let* $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ *be a reduced plane curve singularity with local equation* f*. Then the following are equivalent:*

(a) There are $\tau' = \tau(C,\mathbf{0}) - \tau^{es}(C,\mathbf{0})$ *elements* $g_1,\dots,g_{\tau'} \in I^{es}(f)$ *such that*

$$\varphi^{es} : V\Big(f + \sum\nolimits_i t_i g_i\Big) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'},\mathbf{0}) \xrightarrow{\mathrm{pr}} (\mathbb{C}^{\tau'},\mathbf{0})$$

is a semiuniversal equisingular deformation for $(C,\mathbf{0})$*.*

(b) Let $g_1,\dots,g_{\tau'} \in I^{es}(f)$ *induce a basis for* $I^{es}(f)/\langle f, j(f)\rangle$*. Then*

$$\varphi^{es} : V\Big(f + \sum\nolimits_i t_i g_i\Big) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'},\mathbf{0}) \xrightarrow{\mathrm{pr}} (\mathbb{C}^{\tau'},\mathbf{0})\,,$$

is a semiuniversal equisingular deformation for $(C,\mathbf{0})$*.*

(c) Each equisingular deformation of $(C,\mathbf{0})$ *is isomorphic to an equisingular deformation where all the equimultiple sections* $\sigma_j^{(\ell)}$ *through non-nodes of*

the reduced total transform $C^{(\ell)} \cup E^{(\ell)} \subset M^{(\ell)}$, $\ell = 1, \dots, N$, *of* $(C, \mathbf{0})$ *are globally trivial sections.*[17]

(d) *Each locally trivial deformation of the reduced exceptional divisor* E *of a minimal embedded resolution of* $(C, \mathbf{0}) \subset (\mathbb{C}^2 \times \{\mathbf{0}\}, \mathbf{0})$ *is trivial.*

(e) $I^{es}(f) = \langle f, j(f), I^s(f) \rangle$.[18]

Our construction implies the following "openness of versality" result for equisingular deformations: Call a flat morphism $\phi : \mathscr{C} \to S$ of complex spaces a *family of reduced plane curve singularities* if the restriction of ϕ to $\mathrm{Sing}(\phi)$ is finite and if, for each $s \in S$ and each $x \in \mathscr{C}_s := \phi^{-1}(s)$, there is an isomorphism of germs $(\mathscr{C}, x) \cong (\mathbb{C}^2, \mathbf{0})$ mapping $(\mathscr{C}_s, x)$ to the germ of a reduced plane curve singularity in $(\mathbb{C}^2, \mathbf{0})$.

If $\sigma = (\sigma^{(1)}, \dots, \sigma^{(\ell)})$ is a system of disjoint sections $\sigma^{(i)} : S \to \mathscr{C}$ of ϕ, then we call the family ϕ *equisingular* (resp. *equisingular-versal*) *at* $s \in S$ *along* σ if the induced morphism of germs $\phi : \big(\mathscr{C}, \sigma^{(i)}(s)\big) \to (S, s)$ is an equisingular (resp. equisingular-versal) deformation of $\big(\mathscr{C}_s, \sigma^{(i)}(s)\big)$ for $i = 1, \dots, \ell$.

Combining Theorems 2.64 and 2.43, we get openness of equisingular-versality:

Theorem 2.70. *Let* $\phi : \mathscr{C} \to S$ *be a family of reduced plane curve singularities which is an equisingular family at* s *along* σ *for all* $s \in S$. *Then the set of points* $s \in S$ *such that* ϕ *is equisingular-versal at* s *is analytically open in* S.

Let us conclude with formulating explicitely how equisingular deformations look like for semiquasihomogeneous and Newton non-degenerate singularities. In fact, Propositions 2.17 and 2.69 imply that for semiquasihomogeneous and for Newton non-degenerate plane curve singularities, the semiuniversal equisingular deformation of the equation is completely determined by its tangent space:

Corollary 2.71. *Let* $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ *be a reduced plane curve singularity with local equation* $f \in \mathbb{C}\{x, y\}$, *and let* $\tau' = \tau(C, \mathbf{0}) - \tau^{es}(C, \mathbf{0})$.

(a) *If* $f = f_0 + f'$ *is semiquasihomogeneous with principal part* f_0 *being quasihomogeneous of type* $(w_1, w_2; d)$, *then a semiuniversal equisingular deformation for* $(C, \mathbf{0})$ *is given by*

$$\varphi^{es} : V\left(f + \sum\nolimits_{i=1}^{\tau'} t_i g_i\right) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'}, \mathbf{0}) \xrightarrow{\mathrm{pr}} (\mathbb{C}^{\tau'}, \mathbf{0}),$$

[17] See Definition 2.6, p. 271, for notations. Since the reduced total transform contains the (compact) exceptional divisors, there are obstructions against the global trivialization (that is, by an isomorphism of a neighbourhood of the exceptional divisors) of the sections, for instance by the cross-ratio of more than three sections through one exceptional component.

[18] For the definition of $I^s(f)$, see Remark 2.17.1, p. 288.

where $g_1,\dots,g_{\tau'}$ represent a $\mathbb{C}$-basis for the quotient

$$\langle j(f),\, x^\alpha y^\beta \mid w_1\alpha + w_2\beta \geq d\rangle / j(f)\,.$$

(b) If f is Newton non-degenerate with Newton diagram $\Gamma(f,0)$ at the origin, then a semiuniversal equisingular deformation for $(C,\mathbf{0})$ is given by

$$\varphi^{es} : V\left(f + \sum\nolimits_{i=1}^{\tau'} t_i g_i\right) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'}, \mathbf{0}) \xrightarrow{\ \mathrm{pr}\ } (\mathbb{C}^{\tau'}, \mathbf{0})\,,$$

where $g_1,\dots,g_{\tau'}$ represent a $\mathbb{C}$-basis for the quotient

$$\langle j(f),\, x^\alpha y^\beta \mid x^\alpha y^\beta \textit{ has Newton order } \geq 1\rangle / j(f)\,.$$

Moreover, in both cases each equisingular deformation of $(C,0)$ is isomorphic to an equisingular deformation where all the equimultiple sections through non-nodes of the reduced total transform of $(C,0)$ are trivial sections.

A.N. Varchenko proved that the last statement holds for equisingular deformations of isolated semiquasihomogeneous hypersurface singularities of arbitrary dimension [Var, Thm. 2]. In particular, if $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1,\dots,x_n\}$ is a convenient semiquasihomogeneous power series, then Varchenko's result says that all fibres of a μ-constant deformation of the singularity defined by f are semiquasihomogeneous of the same type (see [Var]). The analogous statement for Newton non-degenerate hypersurface singularities does not hold for $n \geq 3$. In fact, the Newton diagram of the fibres may vary in a μ-constant deformation of a Newton non-degenerate singularity (see, for instance, [Dim, Example 2.14]).

Remarks and Exercises

Using Gabrielov's result ([Gab1]) which states that the *modality* of the function f with respect to right equivalence is equal to the dimension of the μ-constant stratum of f in the (μ-dimensional) semiuniversal unfolding of f, we get

$$\tau^{es}(C,\mathbf{0}) = \mu(C,\mathbf{0}) - \mathrm{modality}(f)\,.$$

In fact, the semiuniversal unfolding of f being a versal deformation of $(C,\mathbf{0})$, this formula follows from Theorem 2.64, since the codimension of the μ-constant stratum in any versal deformation of $(C,\mathbf{0})$ is the same.

Alternatively, in terms of the minimal free resolution of $(C,\mathbf{0})$, the codimension $\tau^{es}(C,\mathbf{0})$ can be computed as

$$\tau^{es}(C,\mathbf{0}) = \sum_q \frac{m_q(m_q+1)}{2} - \#\{q \mid q \text{ is a free point}\} - 1\,, \qquad (2.8.36)$$

where the sum extends over all infinitely near points to $\mathbf{0}$ belonging to $(C,\mathbf{0})$ which appear when resolving the plane curve singularity $(C,\mathbf{0})$, and m_q denotes the multiplicity of the strict transform of $(C,\mathbf{0})$ at q. Here, an infinitely near point is called *free* if it lies on at most one component of the exceptional divisor. The computation of $\tau^{es}(C,\mathbf{0})$ by Formula (2.8.36) is implemented in SINGULAR and accessible via the `tau_es` command. For instance, continuing the SINGULAR session of Example 2.63.1, we get:

```
tau_es(f);            // compute tau^es by the formula
//-> 38
vdim(std(Ies[1])); // compute tau^es as codimension of I^es(f)
//-> 38
```

Comparing the equisingularity ideal $I^{es}(f)$ with the equiclassical ideal $I^{ec}(f)$ and the equigeneric ideal $I^{eg}(f)$ (see [DiH]), we can give an estimate for $\tau^{es}(C,\mathbf{0})$ in terms of the "classical" invariants δ and κ:

$$\kappa(C,\mathbf{0})-\delta(C,\mathbf{0}) \,\le\, \tau^{es}(C,\mathbf{0}) \,\le\, \kappa(C,\mathbf{0}) \,\le\, 2\tau^{es}(C,\mathbf{0})\,.$$

In fact, the vector spaces $I^{ec}(f)/\langle f,j(f)\rangle$ and $I^{eg}(f)/\langle f,j(f)\rangle$ are isomorphic to the tangent cones of the germs of the (κ,δ)-constant stratum (that is, the stratum where κ and δ are both constant), and the δ-constant stratum, respectively. Since equisingular deformations preserve the multiplicities of the successive strict transforms, δ and $\kappa=\mu-\mathrm{mt}+1$ (Propositions I.3.34 and I.3.38) are constant under such deformations. Therefore, the equisingularity stratum is contained in the equiclassical stratum and the same holds for the tangent cones. For a smooth germ, the tangent cone is the same as the tangent space and therefore we have

$$j(f)\subset\langle f,j(f)\rangle\subset I^{es}(f)\subset I^{ec}(f)\subset I^{eg}(f)\,. \tag{2.8.37}$$

The above estimate follows then from the dimension formulas

$$\begin{aligned}\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/I^{es}(f)&=\tau^{es}(C,0)\,,\\ \dim_{\mathbb{C}}\mathbb{C}\{x,y\}/I^{ec}(f)&=\kappa(C,0)-\delta(C,0)\,,\\ \dim_{\mathbb{C}}\mathbb{C}\{x,y\}/I^{eg}(f)&=\delta(C,0)\,.\end{aligned}$$

Exercise 2.8.1. Compute the Milnor number, Tjurina number, τ^{es}, and modality for the singularities at the origin of $\{x^m+y^n=0\}$, $\{x^my+y^n=0\}$, resp. $\{x^my+xy^n=0\}$, $m,n\ge 2$.

In [CGL1], we give an algorithm which, given a deformation with section of a reduced plane curve singularity, computes equations for the equisingularity stratum (that is, the μ-constant stratum in characteristic 0) in the parameter space of the deformation. The algorithm works for any, not necessarily reduced, parameter space and for algebroid curve singularities C defined over an algebraically closed field of characteristic 0 (or of characteristic $p>\mathrm{ord}(C)$). It has been implemented in the SINGULAR library `equising.lib`. The following example shows the implemented algorithm at work.

Example 2.71.1. Consider the reduced (Newton degenerate) plane curve singularity with local equation $f = (y^4 - x^4)^2 - x^{10}$. We compute equations for the μ-constant stratum in the base space of the semiuniversal deformation with section of $(C, \mathbf{0})$ where the section is trivialized (for more details see [CGL1]):

```
LIB "equising.lib";      //loads deform.lib, sing.lib, too
ring R = 0, (x,y), ls;
poly f = (y4-x4)^2 - x10;
ideal J = f, maxideal(1)*jacob(f);
ideal KbJ = kbase(std(J));
int N = size(KbJ);
N;                       //number of deformation parameters
//-> 50
ring Px = 0, (a(1..N),x,y), ls;
matrix A[N][1] = a(1..N);
poly F = imap(R,f)+(matrix(imap(R,KbJ))*A)[1,1];
list M = esStratum(F);   //compute the stratum of equisingularity
                         //along the trivial section
def ESSring = M[1]; setring ESSring;
option(redSB);
ES = std(ES);
size(ES);                //number of equations for ES stratum
//-> 44
```

Inspecting the elements of ES, we see that 42 of the 50 deformation parameters must vanish. Additionally, there are two non-linear equations, showing that the equisingularity (μ-constant) stratum is smooth (of dimension 6) but not linear:

```
ES[9];
//-> 8*a(42)+a(2)*a(24)-a(2)^2
ES[26];
//-> 8*a(24)+8*a(2)+a(2)^3
```

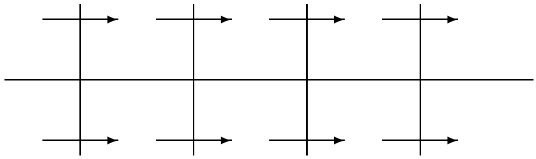
The correctness of the computed equations can be checked by choosing a random point $\boldsymbol{p}$ satisfying the equations and computing the system of Hamburger-Noether expansions for the evaluation of F at $\boldsymbol{s} = \boldsymbol{p}$. From the system of Hamburger-Noether expansions, we can read a complete set of numerical invariants of the equisingularity type (such as the Puiseux pairs and the intersection numbers) which have to coincide with the respective invariants of f. In characteristic 0, it suffices to compare the two Milnor numbers. To do this, we reduce F by ES and evaluate the result at a random point satisfying the above two non-linear conditions:

```
poly F = reduce(imap(Px,F),ES);  //a(2),a(24) both appear in F
poly g = subst(F, a(24), -a(2)-(1/8)*a(2)^3);
for (int ii=1; ii<=44; ii++){ g = subst(g,a(ii),random(1,100)); }
setring R;
```

```
milnor(f);                        //Milnor number of f
//-> 57
milnor(imap(ESSring,g));          //Milnor number of g
//-> 57
```

Finally, we show that for the reduced plane curve singularity with local equation $f = (y^4 - x^4)^2 - x^{10}$ none of the properties (a) – (e) of Proposition 2.69 is satisfied.

Its reduced total transform has the form



(lines and arrows indicating components of the exceptional divisor and the strict transform, respectively). In particular, since the cross-ratio of the 4 intersection points of components of the exceptional divisor E is preserved by a trivial deformation, (d) is not satisfied.

To see the failure of (c), consider the equisingular deformation

$$F = \left(y^4 - x^4 + t \cdot x^2 y^2\right)^2 - x^{10}\,.$$

Since F induces a locally trivial deformation of E which varies the cross-ratio of the four intersection points, it cannot be isomorphic to an equisingular deformation with trivial equimultiple sections $\sigma_j^{(i)}$.

Property (e) fails, too:

```
LIB "equising.lib";
ring R = 0,(x,y),ds;
poly f = (y4-x4)^2-x10;
list Ies = esIdeal(f,1);
Ies[3];                         // the ideal <f,j(f),I^s(f)>
//-> _[1]=x3y7
//-> _[2]=x2y8
//-> _[3]=xy9
//-> _[4]=y10
//-> _[5]=x8-2x4y4+y8-x10
//-> _[6]=8x7-8x3y4-10x9
//-> _[7]=-8x4y3+8y7
ideal J = std(Ies[3]);          // compute standard basis
size(reduce(maxideal(10),J));   // m^10 in <f,j(f),I^s(f)>?
//-> 0
vdim(J);                        // dim_C C{x,y}/<f,j(f),I^s(f)>
//-> 43
vdim(std(Ies[1]));              // dim_C C{x,y}/<f,j(f),I^es(f)>
//-> 42
simplify(reduce(Ies[1],J),10);
//-> _[1]=x6y2-x2y6
```

From the output, we read that $\langle f, j(f), I^s(f)\rangle = \langle f, j(f), \mathfrak{m}^{10}\rangle$, while, as complex vector space, the equisingularity ideal is generated by $\langle f, j(f), \mathfrak{m}^{10}\rangle$ and the polynomial $x^2y^2(y^4 - x^4) \notin \langle f, j(f), \mathfrak{m}^{10}\rangle$.

A

Sheaves

For the convenience of the reader, we collect the relevant properties of sheaves and sheaf cohomology. Full proofs of the statements can be found in [Ser1, God, Ive, GrR2].

A.1 Presheaves and Sheaves

Let X be a topological space. A *presheaf* of Abelian groups on X is a contravariant functor $\mathcal{F}$ from the category of open subsets of X to the category of Abelian groups.[1] A *morphism of presheaves* is a natural transformation of functors. Hence, giving $\mathcal{F}$ is the same as giving, for each open set $U \subset X$ an Abelian group $\mathcal{F}(U)$, where $\mathcal{F}(\emptyset) = 0$, and, for each inclusion $V \subset U$ of open sets a restriction morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$, $s \mapsto \rho_V^U(s) =: s|_V$, such that $\rho_U^U = \mathrm{id}_U$ and $\rho_W^V \circ \rho_V^U = \rho_W^U$ for $W \subset V \subset U$. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is the same as a system $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ of morphisms of Abelian groups, compatible with the restriction maps. Such a morphism φ is called *injective* if all φ_U are injective. Furthermore, a presheaf $\mathcal{F}$ is called a *sub-presheaf* of a presheaf $\mathcal{G}$ if, for each open set $U \subset X$, $\mathcal{F}(U)$ is a subgroup of $\mathcal{G}(U)$ and if the inclusion maps $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ define a morphism of presheaves.

A presheaf $\mathcal{F}$ is called a *sheaf* if for each open set $U \subset X$ and for each open covering $(U_i)_{i \in I}$ of U the following two axioms hold:

(S1) Given $s_1, s_2 \in \mathcal{F}(U)$ such that $s_1|_{U_i} = s_2|_{U_i}$ in $\mathcal{F}(U_i)$ for all $i \in I$ then $s_1 = s_2$.

(S2) Given $s_i \in \mathcal{F}(U_i)$, $i \in I$, such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists an $s \in U$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A *morphism of sheaves* is, by definition, a morphism of presheaves and a sheaf $\mathcal{F}$ is called a *subsheaf* of a sheaf $\mathcal{G}$ if it is a sub-presheaf. Elements of $\mathcal{F}(U)$

[1] More generally, a presheaf of sets on a set X is a contravariant functor from the category of subsets of X to the category of sets. However, in this book we need only (pre-)sheaves with an algebraic structure.

are called *sections* of $\mathcal{F}$ over U. Hence, (S2) says that we can glue sections s_i if they coincide on common intersections, and (S1) implies that this gluing is unique. For $x \in X$ we define

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)\,.$$

Here, if $A \subset X$ is any subset, we define the (*direct* or *inductive*) *limit*

$$\varinjlim_{U \supset A} \mathcal{F}(U)$$

as $\coprod_{U \supset A} \mathcal{F}(U)/\sim$ where U runs over the open neighbourhoods of A and $s_1 \sim s_2$ for $s_1 \in \mathcal{F}(U)$, $s_2 \in \mathcal{F}(V)$ if there exists an open neighbourhood W of A contained in $U \cap V$ such that $s_1|_W = s_2|_W$. $\mathcal{F}_x$ is called the *stalk* of $\mathcal{F}$ at x, and elements of $\mathcal{F}_x$ are called *germs of sections* of $\mathcal{F}$ at x. $\mathcal{F}_x$ is naturally an Abelian group. For any section $s \in \mathcal{F}(U)$ and $x \in U$, s_x denotes the germ of s at x, that is, the canonical image of s in $\mathcal{F}_x$, and s is called a *representative* of s_x. Note that two sections $s_1, s_2 \in \mathcal{F}(U)$ with $s_{1,x} = s_{2,x}$ for all $x \in U$ necessarily coincide if $\mathcal{F}$ satisfies (S1), in particular, if $\mathcal{F}$ is a sheaf.

Typical examples of presheaves which are sheaves are given by $\mathcal{F}$ with $\mathcal{F}(U)$ being a set of functions or maps defined by certain local properties. For instance, the presheaf defined by $\mathscr{C}_X(U) := \{f : U \to \mathbb{C} \text{ continuous}\}$ and the natural restriction maps is a sheaf $\mathscr{C}_X$, the *sheaf of continuous complex valued functions on* X.

Presheaves which are not sheaves naturally occur when we consider Abelian groups $\mathcal{F}(U)$ consisting of equivalence classes and if there is no obvious gluing by choosing appropriate representatives (cf. the quotient sheaf in A.3).

For a presheaf $\mathcal{F}$, consider the presheaf $\mathcal{W}(\mathcal{F})$ defined by

$$\mathcal{W}(\mathcal{F})(U) := \prod_{x \in U} \mathcal{F}_x = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x \;\middle|\; s(x) \in \mathcal{F}_x \text{ for all } x \in U \right\}.$$

As can easily be seen, $\mathcal{W}(\mathcal{F})$ is a sheaf. It is sometimes called the *sheaf of discontinuous sections of* $\mathcal{F}$. The canonical map $j : \mathcal{F} \to \mathcal{W}(\mathcal{F})$ mapping $s \in \mathcal{F}(U)$ to $(s_x)_{x \in U} \in \mathcal{W}(\mathcal{F})(U)$ is an injective morphism of presheaves iff $\mathcal{F}$ satisfies (S1). The induced morphisms of stalks are clearly injective.

There is a general procedure to pass from a presheaf to sheaf: for any presheaf $\mathcal{F}$ there is an *associated sheaf* $\widehat{\mathcal{F}} \subset \mathcal{W}(\mathcal{F})$ constructed as follows. Define $\widehat{\mathcal{F}}(U)$ to be the set of functions $s : U \to \coprod_{x \in U} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ and, for each $x \in U$, there is a neighbourhood V of x and a section $t \in \mathcal{F}(V)$ satisfying $t_y = s(y)$ for all $y \in V$. It is almost a tautology to see that $\widehat{\mathcal{F}}$, together with the natural restrictions, is a sheaf. A morphism of presheaves induces a unique morphism of the associated sheaves.

The sheaf $\widehat{\mathcal{F}}$ contains $j(\mathcal{F})$ as a sub-presheaf and we denote by $\theta : \mathcal{F} \to \widehat{\mathcal{F}}$ the induced morphism of presheaves given by $\theta_U(s) : x \mapsto s_x$. The sheaf $\widehat{\mathcal{F}}$ may be considered as the smallest subsheaf of $\mathcal{W}(\mathcal{F})$ containing $j(\mathcal{F})$ as a sub-presheaf. It is a formal exercise to show that the sheaf $\widehat{\mathcal{F}}$ satisfies (and up to isomorphism is uniquely determined by) the following *universal property*: for each sheaf $\mathcal{G}$ on X and each morphism $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\psi : \widehat{\mathcal{F}} \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. Note that $\mathcal{F}$ is a sheaf iff θ is an isomorphism. If the latter is the case, we identify $\mathcal{F}$ and $\widehat{\mathcal{F}}$ via θ.

We set $\Gamma(U, \mathcal{F}) := \widehat{\mathcal{F}}(U)$ where $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ if $\mathcal{F}$ is a sheaf. For a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we denote φ_U also by $\Gamma(U, \varphi)$ or just by $\varphi : \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$.

Finally, note that for any presheaf $\mathcal{F}$ the stalks $\mathcal{F}_x$ and $\widehat{\mathcal{F}}_x$ coincide. We usually define a sheaf by giving its presheaf data $\mathcal{F}(U)$ and then passing directly to the associated sheaf which will be also denoted by $\mathcal{F}$.

A.2 Gluing Sheaves

We can not only glue sections in a sheaf, but we can also *glue sheaves* if they satisfy the "cocycle condition". Let $(U_i)_{i \in I}$ be an open covering of X and $\mathcal{F}_i$ a sheaf on U_i, $i \in I$, and assume that for each (i, j) such that $U_{ij} := U_i \cap U_j \neq \emptyset$ there is an isomorphism $\varphi_{ij} : \mathcal{F}_j|_{U_{ij}} \xrightarrow{\cong} \mathcal{F}_i|_{U_{ij}}$ such that the *cocycle condition*

$$\varphi_{ii} = \mathrm{id}\,, \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$$

holds for all i, j, k with $U_i \cap U_j \cap U_k \neq \emptyset$. Then we can glue the sheaves $\mathcal{F}_i$, $i \in I$, via the isomorphisms φ_{ij} to get a sheaf $\mathcal{F}$ on X with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\cong} \mathcal{F}_i$, $i \in I$, such that $\psi_i \circ \psi_j^{-1} = \varphi_{ij}$ on U_{ij}. $\mathcal{F}$ is unique up to isomorphism.

It is worthwile to look where the cocycle condition appears: $\mathcal{F}$ is defined by setting

$$\mathcal{F}(U) := \left\{ (s_i)_{i \in I} \in \coprod_{i \in I} \mathcal{F}_i(U \cap U_i) \;\middle|\; s_i\big|_{U \cap U_{ij}} = \varphi_{ij}\big(s_j\big|_{U \cap U_{ij}}\big) \text{ for each } i, j \right\}.$$

The cocycle condition guarantees that "nothing bad happens on triple intersections". That is, gluing $\mathcal{F}_k$ with $\mathcal{F}_i$ gives, over $U_i \cap U_j \cap U_k$, the same result as gluing first $\mathcal{F}_k$ with $\mathcal{F}_j$ and then the result with $\mathcal{F}_i$. The fact that the cocycle condition does not always hold is the reason for the existence of cohomology theory. Indeed, the failure of the cocycle condition is measured by the first cohomology group of a certain sheaf of first order automorphisms.

Gluing of sheaves is used, for instance, when we have defined sheaves $\mathcal{F}_i$ locally, satisfying a universal property. In this case, the cocycle condition is automatically satisfied, and we can glue.

A.3 Sheaves of Rings and Modules

Let $\mathcal{A}$ be a sheaf of Abelian groups on X. $\mathcal{A}$ is called a *sheaf of rings* if $\Gamma(U, \mathcal{A})$ is a ring, and if the restrictions ρ_V^U are morphisms of rings for all open sets U, V of X. For example, the sheaf $\mathscr{C}_X$ of complex valued continuous functions on X is a sheaf of rings.

If $\mathcal{A}$ is a sheaf of rings on X, then a sheaf $\mathcal{F}$ on X is called a *sheaf of* $\mathcal{A}$*-modules* or just an $\mathcal{A}$*-module* if the $\Gamma(U, \mathcal{F})$ are $\Gamma(U, \mathcal{A})$-modules, and if the restriction maps ρ_V^U are morphisms of $\Gamma(U, \mathcal{A})$-modules. A morphism φ of sheaves with algebraic structure (rings, $\mathcal{A}$-modules, etc.) is a morphism of sheaves such that the φ_U are morphisms of the considered structure.

Let $\mathcal{F}_1, \mathcal{F}_2$ be $\mathcal{A}$-modules on X. $\mathcal{F}_1$ is a *subsheaf* (or an $\mathcal{A}$*-submodule*) of $\mathcal{F}_2$ if $\Gamma(U, \mathcal{F}_1) \subset \Gamma(U, \mathcal{F}_2)$ is a submodule for $U \subset X$ open, and if the restriction maps of $\mathcal{F}_1$ are induced by those of $\mathcal{F}_2$.

$\mathcal{A}$-submodules of $\mathcal{A}$ itself are called *sheaves of ideals*, or *ideals in* $\mathcal{A}$.

The *direct sum* $\bigoplus_{i \in I} \mathcal{F}_i$ of sheaves (I any index set) is defined by the presheaf (which is a sheaf if I is finite)

$$U \longmapsto \bigoplus_{i \in I} \Gamma(U, \mathcal{F}_i)\,, \qquad U \subset X \text{ open}\,.$$

The $\mathcal{A}$-module $\mathcal{F}$ is called *free* or a *free* $\mathcal{A}$*-module* if $\mathcal{F} \cong \bigoplus_{i \in I} \mathcal{A}_i$; if I is finite, its number of elements is called the *rank* of $\mathcal{F}$. More generally, $\mathcal{F}$ is called *locally free* if each $x \in X$ has a neighbourhood U such that $\mathcal{F}|_U$ is a free $\mathcal{A}|_U$-module. If, for each such U, $\mathcal{F}|_U$ is free of rank n, then $\mathcal{F}$ is called *locally free of rank* n. Here, the *restriction* $\mathcal{F}|_U$ is the sheaf on U defined by $\Gamma(V, \mathcal{F}|_U) = \Gamma(V, \mathcal{F})$ for $V \subset U$ open.

The *quotient sheaf* $\mathcal{F}_2/\mathcal{F}_1$ is the sheaf associated to the presheaf (which, in general, is not a sheaf)

$$U \longmapsto \Gamma(U, \mathcal{F}_2)\big/\Gamma(U, \mathcal{F}_1)\,, \qquad U \subset X \text{ open}\,.$$

We have $\left(\bigoplus_{i \in I} \mathcal{F}_i\right)_x = \bigoplus_{i \in I} \mathcal{F}_{i,x}$ and $(\mathcal{F}_2/\mathcal{F}_1)_x = \mathcal{F}_{2,x}/\mathcal{F}_{1,x}$ for $x \in X$.

Each sheaf $\mathcal{F}$ of Abelian groups has a unique zero section $0 \in \Gamma(U, \mathcal{F})$, $x \mapsto 0_x$, and we define the *support* of the section $s \in \Gamma(U, \mathcal{F})$, respectively of the sheaf $\mathcal{F}$ as

$$\operatorname{supp}(s) := \{x \in U \mid s_x \neq 0_x\}\,, \qquad \operatorname{supp}(\mathcal{F}) := \big\{x \in X \,\big|\, \mathcal{F}_x \neq 0\big\}\,.$$

We assume in this book that a sheaf of rings $\mathcal{A}$ has a global unit section $1 \in \Gamma(X, \mathcal{A})$ and that all $\mathcal{A}$-modules $\mathcal{F}$ are unitary. Then each stalk $\mathcal{A}_x$ is a ring with $1 = 1_x$, and $\mathcal{F}_x$ is a unitary $\mathcal{A}_x$-module for each $x \in X$.

Note that, for any section $s \in \Gamma(U, \mathcal{F})$, $\operatorname{supp}(s)$ is closed in U, while the support of a sheaf is not necessarily closed (unless $\mathcal{F}$ is of finite type, see A.7). We say that $\mathcal{F}$ is *concentrated on* A if $\operatorname{supp}(\mathcal{F}) = A$.

A.4 Image and Preimage Sheaf

If $f : X \to Y$ is a continuous map of topological spaces and $\mathcal{F}$ a sheaf on X then we define the *direct image* $f_*\mathcal{F}$ to be the sheaf associated to the presheaf, which is actually a sheaf,

$$V \longmapsto \Gamma\big(f^{-1}(V), \mathcal{F}\big)\,, \qquad V \subset Y \text{ open}\,.$$

If $\mathcal{A}$ is a sheaf of rings on X and $\mathcal{F}$ an $\mathcal{A}$-module then $f_*\mathcal{A}$ is a sheaf of rings and $f_*\mathcal{F}$ a sheaf of $f_*\mathcal{A}$-modules on Y. One easily verifies that for $g : Y \to Z$ continuous, we obtain $g_*(f_*\mathcal{F}) = (g \circ f)_*\mathcal{F}$. Note that for $x \in X$, we have a natural morphism

$$(f_*\mathcal{F})_{f(x)} = \varinjlim_{V \ni f(x)} \Gamma\big(f^{-1}(V), \mathcal{F}\big) \longrightarrow \mathcal{F}_x$$

the limit being taken over all open neighbourhoods of $f(x)$.

Moreover, for a sheaf $\mathcal{G}$ on Y we define the *topological preimage*[2] sheaf $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \longmapsto \varinjlim_{V \supset f(U)} \Gamma(V, \mathcal{G})\,, \qquad U \subset X \text{ open}\,,$$

the limit being taken over all open sets $V \subset Y$ containing $f(U)$. If $\mathcal{B}$ is a sheaf of rings on Y and $\mathcal{G}$ a $\mathcal{B}$-module then $f^{-1}\mathcal{B}$ is a sheaf of rings on X and $f^{-1}\mathcal{G}$ a $f^{-1}\mathcal{B}$-module. For $x \in X$ we have, obviously, $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.

For $g : Y \to Z$ continuous and $\mathcal{G}$ a sheaf on Z we have a canonical map $f^{-1}(g^{-1}\mathcal{G}) \to (g \circ f)^{-1}\mathcal{G}$ which is stalkwise an isomorphism, hence, an isomorphism of sheaves (cf. A.5).

If $i : X \hookrightarrow Y$ is the inclusion map of a subspace X of Y then $\mathcal{G}|_X = i^{-1}\mathcal{G}$ is called the *(topological) restriction* of $\mathcal{G}$ to X. If X is closed in Y, then $i_*\mathcal{F}$ is called the *trivial extension of* $\mathcal{F}$ *to* Y; it satisfies $(i_*\mathcal{F})_x = \mathcal{F}_x$ for $x \in X$ and $(i_*\mathcal{F})_y = 0$ for $y \in Y \setminus X$.

Let $\varphi : \mathcal{F} \to \mathcal{F}'$, respectively $\psi : \mathcal{G} \to \mathcal{G}'$ are morphisms of sheaves on X, respectively on Y. Then we have obvious morphisms $f_*\varphi : f_*\mathcal{F} \to f_*\mathcal{F}'$ on Y, respectively $f^{-1}\psi : f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}'$ on X. Moreover, there are canonical maps $\alpha : f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\beta : \mathcal{G} \to f_*f^{-1}\mathcal{G}$, establishing a bijection of sets

$$\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \xleftrightarrow{1:1} \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})\,,$$

via $\varphi \longmapsto (f_*\varphi) \circ \beta$ and $\alpha \circ (f^{-1}\psi) \longleftarrow\!\shortmid \psi$.

[2] not to be confused with the analytic preimage sheaf $f^*\mathcal{G}$ (A.6)

A.5 Algebraic Operations on Sheaves

We restrict our attention to sheaves $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ of $\mathcal{A}$-modules ($\mathcal{A}$ a sheaf of rings), the construction for other structures is analogous.

Given an $\mathcal{A}$-module morphism $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$, the *kernel* of φ, *cokernel* of φ and *image* of φ are defined by the presheaves

$$U \longmapsto \mathrm{Ker}(\varphi_U)\,, \quad U \longmapsto \mathrm{Coker}(\varphi_U)\,, \quad U \longmapsto \mathrm{Im}(\varphi_U)\,.$$

The associated sheaves are denoted by $\mathcal{K}er\,(\varphi)$, $\mathcal{C}oker\,(\varphi)$ and $\mathcal{I}m\,(\varphi)$ which are again $\mathcal{A}$-modules. $\mathcal{K}er\,(\varphi)$, respectively $\mathcal{I}m\,(\varphi)$, are submodules of $\mathcal{F}_1$, respectively $\mathcal{F}_2$. Note that $\Gamma\big(U, \mathcal{K}er\,(\varphi)\big) = \mathrm{Ker}\big(\Gamma(U,\varphi)\big)$, but, in general, $\Gamma\big(U, \mathcal{C}oker\,(\varphi)\big) \neq \mathrm{Coker}\big(\Gamma(U,\varphi)\big)$ and $\Gamma\big(U, \mathcal{I}m\,(\varphi)\big) \neq \mathrm{Im}\big(\Gamma(U,\varphi)\big)$. However, we have for each $x \in X$

$$\mathcal{K}er\,(\varphi)_x = \mathrm{Ker}(\varphi_x)\,, \quad \mathcal{C}oker\,(\varphi)_x = \mathrm{Coker}(\varphi_x)\,, \quad \mathcal{I}m\,(\varphi)_x = \mathrm{Im}(\varphi_x)\,.$$

If $\mathcal{F}_1$, $\mathcal{F}_2$ are subsheaves of $\mathcal{F}$ then we can define, in an obvious way, the *sum* $\mathcal{F}_1 + \mathcal{F}_2$ and the *intersection* $\mathcal{F}_1 \cap \mathcal{F}_2$, which are submodules of $\mathcal{F}$ and satisfy $(\mathcal{F}_1 + \mathcal{F}_2)_x = \mathcal{F}_{1,x} + \mathcal{F}_{2,x}$ and $(\mathcal{F}_1 \cap \mathcal{F}_2)_x = \mathcal{F}_{1,x} \cap \mathcal{F}_{2,x}$.

If $\mathcal{I} \subset \mathcal{A}$ is an ideal sheaf then the *product* $\mathcal{I} \cdot \mathcal{F}$ is a submodule of $\mathcal{F}$ with stalks $(\mathcal{I} \cdot \mathcal{F})_x = \mathcal{I}_x \cdot \mathcal{F}_x$. We define the *radical* $\sqrt{\mathcal{I}}$ *of* $\mathcal{I}$ to be the sheaf associated to the presheaf

$$U \longmapsto \sqrt{\Gamma(U,\mathcal{I})} = \big\{ f \in \Gamma(U,\mathcal{A}) \,\big|\, \exists\, m \in \mathbb{N} \text{ such that } f^m \in \Gamma(U,\mathcal{I}) \big\}\,,$$

and we say that $\mathcal{I}$ is a *radical sheaf* if $\mathcal{I} = \sqrt{\mathcal{I}}$. For $x \in X$, we have

$$\big(\sqrt{\mathcal{I}}\big)_x = \sqrt{\mathcal{I}_x} = \big\{ f \in \mathcal{A}_x \,\big|\, \exists\, m \in \mathbb{N} \text{ such that } f_x^m \in \mathcal{I}_x \big\}\,.$$

The radical of the 0-ideal sheaf is called the *nilradical* of $\mathcal{A}$,

$$\mathcal{N}il\,(\mathcal{A}) = \sqrt{0} \subset \mathcal{A}\,,$$

or the *sheaf of nilpotent elements* of $\mathcal{A}$. The *torsion submodule* of $\mathcal{F}$ is the $\mathcal{A}$-submodule $\mathscr{T}ors\,(\mathcal{F}) \subset \mathcal{F}$ associated to the presheaf defined by

$$U \longmapsto \mathrm{Tors}\big(\Gamma(U,\mathcal{F})\big) := \left\{ m \in \Gamma(U,\mathcal{F}) \,\middle|\, \begin{array}{l} f \cdot m = 0 \text{ for some non-zero-} \\ \text{divisor } 0 \neq f \in \Gamma(U,\mathcal{A}) \end{array} \right\}.$$

It satisfies $\mathscr{T}ors\,(\mathcal{F})_x = \mathrm{Tors}(\mathcal{F}_x)$ for all $x \in X$. The $\mathcal{A}$-module $\mathcal{F}$ is called *torsion free* (resp. *torsion free at* x) if $\mathscr{T}ors\,(\mathcal{F}) = 0$ (resp. $\mathrm{Tors}(\mathcal{F}_x) = 0$). It is called a *torsion sheaf* if $\mathscr{T}ors\,(\mathcal{F}) = \mathcal{F}$.

The *tensor product* $\mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2$ is the sheaf associated to the presheaf

$$U \longmapsto \Gamma(U,\mathcal{F}_1) \otimes_{\Gamma(U,\mathcal{A})} \Gamma(U,\mathcal{F}_2)\,, \qquad U \subset X \text{ open}\,,$$

with the obvious restriction maps. This generalizes inductively to finitely many factors. We have for $x \in X$

$$(\mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2)_x \;=\; \mathcal{F}_{1,x} \otimes_{\mathcal{A}_x} \mathcal{F}_{2,x}\,.$$

Similarly, the *exterior product* $\Lambda^r\mathcal{F}$ is the sheaf associated to $U \mapsto \Lambda^r\Gamma(U,\mathcal{F})$. It is an $\mathcal{A}$-module satisfying $(\Lambda^r\mathcal{F})_x = \Lambda^r(\mathcal{F}_x)$.

The definition of the $\mathscr{H}\!om$-sheaves is a little more tricky. Denote by $\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{F}_1|_U, \mathcal{F}_2|_U)$ the $\Gamma(U,\mathcal{A})$-module of all $\mathcal{A}|_U$-module homomorphisms from $\mathcal{F}_1|_U$ to $\mathcal{F}_2|_U$. The sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{\mathcal{A}|_U}\big(\mathcal{F}_1\big|_U, \mathcal{F}_2\big|_U\big)\,, \qquad U \subset X \text{ open}\,,$$

is called the *sheaf of $\mathcal{A}$-homomorphisms* from $\mathcal{F}_1$ to $\mathcal{F}_2$ and denoted by $\mathscr{H}\!om_{\mathcal{A}}(\mathcal{F}_1,\mathcal{F}_2)$. It is an $\mathcal{A}$-module. The sheaf $\mathcal{F}^* := \mathscr{H}\!om_{\mathcal{A}}(\mathcal{F},\mathcal{A})$ is called the ($\mathcal{A}$-) *dual sheaf.*

The natural morphism

$$\mathscr{H}\!om_{\mathcal{A}}(\mathcal{F}_1,\mathcal{F}_2)_x \to \mathscr{H}\!om_{\mathcal{A}_x}(\mathcal{F}_{1,x},\mathcal{F}_{2,x})$$

is, in general, neither injective nor surjective. However, if $\mathcal{F}_1$ is of finite type, resp. coherent (cf. A.7), then this morphism is injective, resp. an isomorphism. A morphism $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is *injective*, respectively *surjective*, if $\mathcal{K}er(\varphi) = 0$, respectively $\mathcal{C}oker(\varphi) = 0$. A sheaf *isomorphism* φ is a morphism which has a two-sided inverse, or, equivalently, which is bijective, that is, injective and surjective.

A sequence of $\mathcal{A}$-modules and morphisms

$$\ldots \longrightarrow \mathcal{F}_{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{F}_i \xrightarrow{\varphi_i} \mathcal{F}_{i+1} \longrightarrow \ldots$$

is called a *complex* (respectively *exact*) if, for each i, $\mathcal{I}m(\varphi_{i-1}) \subset \mathcal{K}er(\varphi_i)$ (respectively $\mathcal{I}m(\varphi_{i-1}) = \mathcal{K}er(\varphi_i)$). Equivalently, if $\mathcal{I}m(\varphi_{i-1})_x \subset \mathcal{K}er(\varphi_i)_x$ (respectively $\mathcal{I}m(\varphi_{i-1})_x = \mathcal{K}er(\varphi_i)_x$) for all $x \in X$. Hence, a sequence of sheaves is a complex (exact) iff it is stalkwise a complex (exact).

A.6 Ringed Spaces

A *ringed space* $(X,\mathcal{A})$ consists of a topological space X and a sheaf of rings $\mathcal{A}$ on X. If there is no doubt about it, we just write X. The sheaf $\mathcal{A}$ is called the *structure sheaf* of the ringed space X. A ringed space X is called a *locally ringed space*, if for every $x \in X$ the stalk $\mathcal{A}_x$ is a local ring, the maximal ideal being denoted by $\mathfrak{m}_x$.

Let K be a field. Then a locally ringed space $(X,\mathcal{A})$ is called a *K-ringed space* iff $\mathcal{A}$ is a sheaf of local K-algebras such that, for all $x \in X$, $K \to \mathcal{A}_x$ induces an isomorphism $K \xrightarrow{\cong} \mathcal{A}_x/\mathfrak{m}_x$.

If all the stalks $\mathcal{A}_x$, $x \in X$, are analytic K-algebras then we call $(X,\mathcal{A})$ a *K-analytic ringed space.* A complex space, as defined in Section I.1.3, is a *$\mathbb{C}$-analytic ringed space.*

A *morphism of K-ringed spaces* is, by definition, a pair of maps

$$\left(f, f^\sharp\right) : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y),$$

where $f : X \to Y$ is a continuous map of the underlying topological spaces, and $f^\sharp : \mathcal{A}_Y \to f_*\mathcal{A}_X$ is a morphism of sheaves of local K-algebras, that is, a morphism of sheaves of rings such that for all $x \in X$ the induced map $f_x^\sharp : \mathcal{A}_{Y,f(x)} \to \mathcal{A}_{X,x}$ is a morphism of local K-algebras (in particular, $f_x^\sharp(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$). Here, $f_*\mathcal{A}_X$ is the direct image sheaf, and $f_x^\sharp$ is the composition

$$f_x^\sharp : \mathcal{A}_{Y,f(x)} \to (f_*\mathcal{A}_X)_{f(x)} = \varinjlim_{U \supset f^{-1}(f(x))} \Gamma(U, \mathcal{A}_X) \to \mathcal{A}_{X,x}.$$

Define the *composition* of two morphisms $\left(f, f^\sharp\right) : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ and $\left(g, g^\sharp\right) : (Y, \mathcal{A}_Y) \to (Z, \mathcal{A}_Z)$ as $\left(g \circ f, (g_* f^\sharp) \circ g^\sharp\right)$ with

$$g_* f^\sharp : g_*\mathcal{A}_Y \longrightarrow g_* f_* \mathcal{A}_X = (g \circ f)_* \mathcal{A}_X.$$

A morphism $\left(f, f^\sharp\right) : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ is an *isomorphism* if f is a homeomorphism and $f^\sharp$ is an isomorphism of sheaves of rings, or, equivalently, if $\left(f, f^\sharp\right)$ has a two-sided inverse.

Usually, we omit $f^\sharp$ and write $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ or just $f : X \to Y$ if there is no doubt about $\mathcal{A}_X$ and $\mathcal{A}_Y$ for a morphism of (K-)ringed spaces. Since there is a canonical bijection (cf. A.4)

$$\mathrm{Hom}(\mathcal{A}_Y, f_*\mathcal{A}_X) \xleftrightarrow{1:1} \mathrm{Hom}(f^{-1}\mathcal{A}_Y, \mathcal{A}_X)$$

a morphism of locally ringed spaces can equivalently be given by a pair

$$\left(f, \widehat{f}\right) : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y),$$

with $\widehat{f} : f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$ a morphism of sheaves of rings (respectively of local K-algebras). It is easy to see that $f_x^\sharp = \widehat{f}_x : \mathcal{A}_{Y,f(x)} \to \mathcal{A}_{X,x}$.

We define the *algebraic* or *analytic preimage* sheaf of an $\mathcal{A}_Y$-module $\mathcal{G}$ as

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{A}_Y} \mathcal{A}_X,$$

which is an $\mathcal{A}_X$-module. It follows easily from A.4 and the properties of the tensor product that

$$f^*(g^*\mathcal{G}) \cong (g \circ f)^*\mathcal{G}$$

for $f : X \to Y$, $g : Y \to Z$ morphisms of ringed spaces.

Complex spaces, as defined in Section I.1.3 are our main examples of $\mathbb{C}$-ringed spaces.

A.7 Coherent Sheaves

Let $(X, \mathcal{A})$ be a ringed space and $\mathcal{F}$ an $\mathcal{A}$-module on X. $\mathcal{F}$ is of *finite type* iff for any $x \in X$ there exists a neighbourhood U of x and $s_1, \ldots, s_q \in \Gamma(U, \mathcal{F})$ such that $\mathcal{F}|_U = s_1\mathcal{A}|_U + \ldots + s_q\mathcal{A}|_U$, that is, $\mathcal{F}$ is locally generated by a finite number of sections.

$\mathcal{F}$ is *relation finite* iff for any open $U \subset X$ and for any $s'_1, \ldots, s'_q \in \Gamma(U, \mathcal{F})$ the relation sheaf on U

$$\mathcal{R}el\big(s'_1, \ldots, s'_q\big) := \mathcal{K}er\left(\begin{array}{c} \mathcal{A}^q\big|_U \to \mathcal{F}\big|_U \\ (a_1, \ldots, a_q) \mapsto \sum a_i s'_i \end{array}\right)$$

is of finite type, or, equivalently, if for any surjection $\varphi : \mathcal{A}^q\big|_U \to \mathcal{F}\big|_U$ the sheaf $\mathcal{K}er(\varphi)$ is of finite type. $\mathcal{F}$ is *coherent* iff it is of finite type and relation finite.

It follows easily from the definitions that the structure sheaf $\mathcal{A}$ is coherent iff it is relation finite and that a subsheaf of a coherent sheaf is coherent iff it is of finite type.

Coherence is a purely local property, that is, $\mathcal{F}$ is coherent iff each point $x \in X$ has a neighbourhood U such that $\mathcal{F}\big|_U$ is coherent. However, $\mathcal{F}$ being coherent is usually stronger than just requiring that all stalks $\mathcal{F}_x$ have a finite presentation.

For a typical non-coherent sheaf, consider the inclusion $i : \{0\} \hookrightarrow \mathbb{C}$. Then $i_* i^{-1}\mathcal{O}_\mathbb{C}$ is a sheaf on $\mathbb{C}$, concentrated in 0 with stalk $\mathcal{O}_{\mathbb{C},0}$. It is not coherent although all stalks are Noetherian (Corollary I.1.74). In fact, the kernel of the canonical surjection $\mathcal{O}_\mathbb{C} \to i_* i^{-1}\mathcal{O}_\mathbb{C}$ cannot be of finite type by Fact 1, below.

If the $\mathcal{A}$-module sheaf $\mathcal{F}$ is coherent then every point $x \in X$ admits a neighbourhood $U = U(x)$ and an exact sequence of $\mathcal{A}_U$-modules

$$\mathcal{A}^q\big|_U \longrightarrow \mathcal{A}^p\big|_U \longrightarrow \mathcal{F}\big|_U \longrightarrow 0\,.$$

If $\mathcal{A}$ is coherent, the converse follows from Fact 3, below.

We list some standard facts for coherent sheaves on a ringed space $(X, \mathcal{A})$, for the proofs we refer to [Ser1, GrR2].

Fact 1. (Support) *Let $\mathcal{F}$ be of finite type. If $s_{1,x}, \ldots, s_{p,x}$ generate $\mathcal{F}_x$ then there exists a neighbourhood U of x and representatives $s_1, \ldots, s_p \in \Gamma(U, \mathcal{F})$ such that $s_{1,y}, \ldots, s_{p,y}$ generate $\mathcal{F}_y$ for all $y \in U$. It follows that the support of $\mathcal{F}$ is closed in X.*

Fact 2. (Three lemma) *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves. If any two of $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$ are coherent then the third is coherent, too. In particular, the finite direct sum of coherent sheaves is coherent.*

Moreover, if $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is a sequence of coherent sheaves, and if $\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$ is exact then there is a neighbourhood U of x such that $\mathcal{F}'|_U \to \mathcal{F}|_U \to \mathcal{F}''|_U$ is exact.

Fact 3. (Kernel, Image, Cokernel) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of coherent sheaves then the sheaves $\mathcal{K}er(\varphi)$, $\mathcal{I}m(\varphi)$ and $\mathcal{C}oker(\varphi)$ are coherent.*

Fact 4. (Hom and Tensor) *If $\mathcal{F}$, $\mathcal{G}$ are coherent then so are $\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})$ and $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$, and we have cononical isomorphisms*

$$\big(\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})\big)_x \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}_x}\big(\mathcal{F}_x,\mathcal{G}_x\big)\,, \qquad (\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G})_x \xrightarrow{\cong} \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x\,.$$

If $\mathcal{F}_1$, $\mathcal{F}_2$ are coherent subsheaves of the coherent sheaf $\mathcal{G}$ then $\mathcal{F}_1 + \mathcal{F}_2$ and $\mathcal{F}_1 \cap \mathcal{F}_2$ are coherent, too.

This follows since $\mathcal{F}_1 + \mathcal{F}_2$ is the image of $\mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{G}$, and $\mathcal{F}_1 \cap \mathcal{F}_2$ is the kernel of $\mathcal{F}_1 \to \mathcal{G}/\mathcal{F}_2$.

Fact 5. (Annihilator) *If $\mathcal{A}$ and $\mathcal{F}$ are coherent, then the* annihilator sheaf

$$\mathcal{A}nn_{\mathcal{A}}(\mathcal{F}) := \mathrm{Ker}\,\big(m : \mathcal{A} \to \mathscr{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{F})\big)$$

with $m(a)(f) = af$, is a coherent ideal sheaf. Moreover, we have

$$\begin{aligned}\big(\mathcal{A}nn_{\mathcal{A}}(\mathcal{F})\big)_x &\cong \mathrm{Ann}_{\mathcal{A}_x}(\mathcal{F}_x) = \big\{a \in \mathcal{A}_x \;\big|\; a\mathcal{F}_x = 0\big\}\,,\\ \mathrm{supp}(\mathcal{F}) &= \mathrm{supp}\big(\mathcal{A}/\mathcal{A}nn_{\mathcal{A}}(\mathcal{F})\big)\,.\end{aligned}$$

Fact 6. (Extension principle) *Let $\mathcal{A}$ be coherent and $\mathcal{J} \subset \mathcal{A}$ an ideal sheaf of finite type. Let $(Y,\mathcal{B})$ be the ringed space with $Y = \mathrm{supp}(\mathcal{A}/\mathcal{J})$ and $\mathcal{B} = (\mathcal{A}/\mathcal{J})|_Y$, and let $i : Y \hookrightarrow X$ be the inclusion map. Then a sheaf $\mathcal{F}$ of $\mathcal{B}$-modules on Y is $\mathcal{B}$-coherent iff the trivial extension $i_*\mathcal{F}$ is $\mathcal{A}$-coherent on X.*

Note that Y is *closed* in X and, hence, $(i_*\mathcal{F})_x = \mathcal{F}_x$ if $x \in Y$ and $(i_*\mathcal{F})_x = 0$ if $x \in X \setminus Y$.

A.8 Sheaf Cohomology

Starting point for sheaf cohomology is the fact that the functor of global sections is *left exact but not right exact.* That is, if

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of sheaves on X, then the sequence of global sections

$$0 \longrightarrow \Gamma(X,\mathcal{F}') \xrightarrow{\Gamma(X,\varphi)} \Gamma(X,\mathcal{F}) \xrightarrow{\Gamma(X,\psi)} \Gamma(X,\mathcal{F}'') \tag{A.8.1}$$

is exact, but the map $\Gamma(X,\psi)$ does not need to be surjective.

If we try to prove the surjectivity of $\Gamma(X,\psi)$, we find, for a given $s'' \in \Gamma(X,\mathcal{F}'')$, an open covering $(U_i)_{i\in I}$ of X and elements $s_i \in \Gamma(U_i,\mathcal{F})$ such that $\psi(s_i) = s''|_{U_i}$ for all i. However, if $\Gamma(X,\psi)$ is not injective then we cannot glue the s_i to a global section $s \in \Gamma(X, \mathcal{F})$, at least not in general. Whether such a gluing is possible depends on the space X and on the sheaves. We shall first consider "nice" (w.r.t. gluing) sheaves.

A sheaf $\mathcal{W}$ on X is called *flabby* if, for any open set $U \subset X$, the restriction map $\Gamma(X, \mathcal{W}) \to \Gamma(U, \mathcal{W})$ is surjective.

If the sheaf $\mathcal{F}'$ is flabby, then we can glue the $(s_i)_{i \in I}$ as above to a section $s \in \Gamma(X, \mathcal{F})$: consider the system $\mathcal{E}$ of all pairs (J, t) with $J \subset I$, $t \in \Gamma\big(\bigcup_{i \in J} U_i, \mathcal{F}\big)$ such that $\psi(t) = s'|_{\bigcup_{i \in J} U_i}$, which is partially ordered by

$$(J_1, t_1) \leq (J_2, t_2) :\Longleftrightarrow J_1 \subset J_2 \text{ and } t_2\big|_{\bigcup_{i \in J_1} U_i} = t_1 .$$

By Zorn's Lemma, $\mathcal{E}$ has a maximal element (U, s) and, using the exactness of (A.8.1), it is easy to see that $U = X$. Hence, we get the following lemma:

Lemma A.8.1. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves with $\mathcal{F}'$ flabby, then the induced sequence of global sections*

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact.

As a consequence, we obtain

Proposition A.8.2. *(1) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves. If $\mathcal{F}'$ and $\mathcal{F}$ flabby then $\mathcal{F}''$ is flabby, too.*
(2) For any exact sequence $0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \dots$ of flabby sheaves the induced sequence $0 \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \Gamma(X, \mathcal{F}^2) \to \dots$ is exact, too.

Let $\mathcal{F}$ be any sheaf of Abelian groups. A *resolution of $\mathcal{F}$* is a complex

$$\mathcal{S}^\bullet : \ \mathcal{S}^0 \xrightarrow{d^0} \mathcal{S}^1 \xrightarrow{d^1} \mathcal{S}^2 \xrightarrow{d^2} \mathcal{S}^3 \xrightarrow{d^3} \dots$$

of sheaves together with a morphism $j : \mathcal{F} \to \mathcal{S}^0$ such that the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{j} \mathcal{S}^0 \xrightarrow{d^0} \mathcal{S}^1 \xrightarrow{d^1} \mathcal{S}^2 \xrightarrow{d^2} \mathcal{S}^3 \xrightarrow{d^3} \dots$$

is exact. A *flabby resolution of $\mathcal{F}$* is a resolution $(\mathcal{S}^\bullet, j)$ as above with all sheaves $\mathcal{S}^i$, $i \geq 0$, being flabby.

We shall construct now a canonical flabby resolution $\mathcal{W}^\bullet(\mathcal{F})$ for any sheaf $\mathcal{F}$ on X: let $\mathcal{W}(\mathcal{F})$ be the sheaf of discontinuous sections (see A.1). It is easily seen to be a flabby sheaf. We call $\mathcal{W}(\mathcal{F})$ the *canonical flabby sheaf of $\mathcal{F}$*. For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, the induced morphisms of stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ induce a morphism $\mathcal{W}(\varphi) : \mathcal{W}(\mathcal{F}) \to \mathcal{W}(\mathcal{G})$. It is obvious that, in this way, we obtain an exact covariant functor $\mathcal{W}(\_)$.

Moreover, we have a canonical inclusion of sheaves $j : \mathcal{F} \to \mathcal{W}(\mathcal{F})$ mapping $s \in \Gamma(U, \mathcal{F})$ to $(s_x)_{x \in U} \in \Gamma(U, \mathcal{W}(\mathcal{F}))$. Now set $\mathcal{W}^0(\mathcal{F}) := \mathcal{W}(\mathcal{F})$,

$$\varphi^0 : \mathcal{W}^0(\mathcal{F}) \twoheadrightarrow \mathcal{W}^0(\mathcal{F})/j(\mathcal{F}) \hookrightarrow \mathcal{W}^1(\mathcal{F}) := \mathcal{W}\big(\mathcal{W}^0(\mathcal{F})/j(\mathcal{F})\big)\,,$$

and observe that $\mathrm{Ker}(\varphi^0) = j(\mathcal{F})$. Define, inductively,

$$\mathcal{W}^{q+1}(\mathcal{F}) := \mathcal{W}\big(\mathcal{W}^q(\mathcal{F})/\varphi^{q-1}\big(\mathcal{W}^{q-1}(\mathcal{F})\big)\big)\,,$$
$$\varphi^q : \mathcal{W}^q(\mathcal{F}) \twoheadrightarrow \mathcal{W}^q(\mathcal{F})/\varphi^{q-1}\big(\mathcal{W}^{q-1}(\mathcal{F})\big) \hookrightarrow \mathcal{W}^{q+1}(\mathcal{F})\,.$$

Clearly, together with the induced morphisms this defines exact functors $\mathcal{W}^q(\_)$, $q \geq 0$. Moreover, by construction, the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{j} \mathcal{W}^0(\mathcal{F}) \xrightarrow{\varphi^0} \mathcal{W}^1(\mathcal{F}) \xrightarrow{\varphi^1} \mathcal{W}^2(\mathcal{F}) \xrightarrow{\varphi^2} \mathcal{W}^3(\mathcal{F}) \xrightarrow{\varphi^3} \ldots$$

is exact with $\mathcal{W}^q(\mathcal{F})$ being flabby for all $q \geq 0$. Hence,

$$\mathcal{W}^\bullet(\mathcal{F}) : \mathcal{W}^0(\mathcal{F}) \xrightarrow{\varphi^0} \mathcal{W}^1(\mathcal{F}) \xrightarrow{\varphi^1} \mathcal{W}^2(\mathcal{F}) \xrightarrow{\varphi^2} \mathcal{W}^3(\mathcal{F}) \xrightarrow{\varphi^3} \ldots$$

is a flabby resolution of $\mathcal{F}$. It is called the *canonical flabby resolution* or the *Godement resolution of* $\mathcal{F}$.

Splitting $\mathcal{W}^\bullet(\mathcal{F})$ into short exact sequences and applying the global section functor $\Gamma(X,\_)$, we obtain from Lemma A.8.1 that

$$\Gamma\big(X, \mathcal{W}^\bullet(\mathcal{F})\big) : \Gamma\big(X, \mathcal{W}^0(\mathcal{F})\big) \xrightarrow{\Gamma(X,\varphi^0)} \Gamma\big(X, \mathcal{W}^1(\mathcal{F})\big) \xrightarrow{\Gamma(X,\varphi^1)} \ldots$$

is a complex. We set, for $q \geq 0$,

$$H^q(X,\mathcal{F}) := H^q\big(\Gamma\big(X, \mathcal{W}^\bullet(\mathcal{F})\big)\big) := \mathrm{Ker}\big(\Gamma(X,\varphi^q)\big)/\,\mathrm{Im}\big(\Gamma(X,\varphi^{q-1})\big)$$

(with φ^{-1} the zero-map) and call it the *q-th cohomology of X with values in $\mathcal{F}$*. If $\mathcal{F}$ is a sheaf of Abelian groups, respectively $\mathcal{A}$-modules, then $\mathcal{W}(\mathcal{F})$ and, hence, also the $H^q(X,\mathcal{F})$, are Abelian groups, respectively $\Gamma(X,\mathcal{A})$-modules. It is easy to see that $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$ and, for each $q \geq 0$, $H^q(X, \mathcal{F} \oplus \mathcal{G}) = H^q(X,\mathcal{F}) \oplus H^q(X,\mathcal{G})$ (since $\mathcal{W}(\mathcal{F} \oplus \mathcal{G}) = \mathcal{W}(\mathcal{F}) \oplus \mathcal{W}(\mathcal{G})$).

The most important tool for applications is certainly the long exact cohomology sequence:

Proposition A.8.3. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves on X. Then there exists an exact sequence of cohomology groups*

$$0 \to H^0(X,\mathcal{F}') \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}'') \xrightarrow{\delta^0} H^1(X,\mathcal{F}') \to \ldots$$
$$\ldots \to H^q(X,\mathcal{F}') \to H^q(X,\mathcal{F}) \to H^q(X,\mathcal{F}'') \xrightarrow{\delta^q} H^{q+1}(X,\mathcal{F}') \to \ldots\,,$$

The latter sequence is called the *long exact cohomology sequence* corresponding to the short exact sequence of sheaves. The homomorphisms δ^q are called *connecting homomorphisms.*

Moreover, the long exact cohomology sequence is functorial, that is, given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows, we obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(X,\mathcal{F}') \arrow[r] \arrow[d] & H^q(X,\mathcal{F}) \arrow[r] \arrow[d] & H^q(X,\mathcal{F}'') \arrow[r, "\delta^q"] \arrow[d] & H^{q+1}(X,\mathcal{F}') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^q(X,\mathcal{G}') \arrow[r] & H^q(X,\mathcal{G}) \arrow[r] & H^q(X,\mathcal{G}'') \arrow[r, "\delta^q"] & H^{q+1}(X,\mathcal{G}') \arrow[r] & \cdots
\end{tikzcd}$$

If the sheaves are sheaves of Abelian groups, respectively of $\mathcal{A}$-modules, then all the morphisms between the cohomology groups in the above diagram are morphisms of Abelian groups, respectively of $\Gamma(X,\mathcal{A})$-modules.

The proof of Proposition A.8.3 is standard yoga in homological algebra. We only recall the construction of δ^q "by ascending stairs" (cf. the diagram in Figure A.8.1): take an element of $H^q(X,\mathcal{F}'')$; it can be represented by some element $\alpha \in \Gamma\big(X,\mathcal{W}^q(\mathcal{F}'')\big)$ mapping to 0 in $\Gamma\big(X,\mathcal{W}^{q+1}(\mathcal{F}'')\big)$. Since $\mathcal{W}^q(\_)$ is an exact functor and since $\Gamma(X,\_)$ is exact on exact sequences of flabby sheaves, α has a preimage $\beta \in \Gamma\big(X,\mathcal{W}^q(\mathcal{F})\big)$. The image γ of β in $\Gamma\big(X,\mathcal{W}^{q+1}(\mathcal{F})\big)$ maps to the image of α in $\Gamma\big(X,\mathcal{W}^{q+1}(\mathcal{F}'')\big)$ which is 0. Hence, γ has a preimage ϵ in $\Gamma\big(X,\mathcal{W}^{q+1}(\mathcal{F}')\big)$. Now, define $\delta^q([\alpha]) := [\epsilon]$, where [ ] denotes cohomology classes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \Gamma\big(X,\mathcal{W}^{q-1}(\mathcal{F}')\big) \arrow[r] \arrow[ddd] & \Gamma\big(X,\mathcal{W}^{q}(\mathcal{F}')\big) \arrow[r] \arrow[ddd] & \Gamma\big(X,\mathcal{W}^{q+1}(\mathcal{F}')\big) \arrow[r] \arrow[ddd] & \cdots \\
 & & & \epsilon \arrow[d, mapsto] & \\
 & & & \gamma & \\
\cdots \arrow[r] & \Gamma\big(X,\mathcal{W}^{q-1}(\mathcal{F})\big) \arrow[r] \arrow[ddd] & \Gamma\big(X,\mathcal{W}^{q}(\mathcal{F})\big) \arrow[r] \arrow[ddd] & \Gamma\big(X,\mathcal{W}^{q+1}(\mathcal{F})\big) \arrow[r] \arrow[ddd] & \cdots \\
 & & \beta \arrow[r, mapsto] \arrow[d, mapsto] & \gamma & \\
 & & \alpha & & \\
\cdots \arrow[r] & \Gamma\big(X,\mathcal{W}^{q-1}(\mathcal{F}'')\big) \arrow[r] \arrow[d] & \Gamma\big(X,\mathcal{W}^{q}(\mathcal{F}'')\big) \arrow[r] \arrow[d] & \Gamma\big(X,\mathcal{W}^{q+1}(\mathcal{F}'')\big) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Fig. A.8.1. Construction of connecting homomorphism.

Hence, if $H^{q+1}(X,\mathcal{F}') = 0$ then $H^q(X,\mathcal{F}) \to H^q(X,\mathcal{F}'')$ is surjective. This is a typical application of Proposition A.8.3.

If $\mathcal{F}$ itself is flabby then $0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{W}^\bullet(\mathcal{F}))$ is exact and, hence, $H^q(X,\mathcal{F}) = 0$ for all $q \geq 1$. More generally, we define a sheaf $\mathcal{F}$ to be *acyclic* if $H^q(X,\mathcal{F}) = 0$ for all $q \geq 1$.

The canonical flabby sheaf $\mathcal{W}(\mathcal{F})$ is very easy to define, but it is useless for any concrete computation, since it is just too large. Therefore, in order to be able to compute cohomology groups we have to use other resolutions which are computable. The basic statement which allows this is the following proposition.

Proposition A.8.4. *Let $\mathcal{L}^\bullet$ be any acyclic resolution of $\mathcal{F}$ (that is, $\mathcal{L}^\bullet$ is a resolution of $\mathcal{F}$ and all $\mathcal{L}^q$, $q \geq 0$, are acyclic). Then there are natural isomorphisms*

$$\tau_q : H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big) \xrightarrow{\cong} H^q\big(X,\mathcal{F}\big)\,, \quad q \geq 0\,,$$

which are compatible with the connecting homomorphisms.

For $q = 0$ the statement is obvious. In general, the proposition can be proved either by induction on q or by using spectral sequences.

In particular, since any flabby sheaf is acyclic, we can compute $H^q(X,\mathcal{F})$ by using an arbitrary acyclic resolution of $\mathcal{F}$.

To give an example, we show that skyscraper sheaves are flabby, while constant sheaves are, in general, not flabby.

Let $S \subset X$ be a finite set of points, which are all closed in X. If $\mathcal{F}$ is a sheaf of Abelian groups on X with $\operatorname{supp}(\mathcal{F}) = S$, then $\mathcal{F}$ is flabby. Indeed, let $s \in \Gamma(U,\mathcal{F})$ with $U \subset X$ open, then there exists a (trivial) extension $\widetilde{s} \in \Gamma(X,\mathcal{F})$, setting $\widetilde{s}(x) := 0$ for $x \notin U \cap S$ and $\widetilde{s}(x) = s_x$ for $x \in U \cap S$. Such sheaves are called *skyscraper sheaves.* It follows that skyscraper sheaves are acyclic.

Now, let G be an arbitrary Abelian group which we endow with the discrete topology. We define the *(locally) constant sheaf* G_X on X by the presheaf of locally constant sections of G, or in other words,

$$G_X(U) := \big\{ s : U \to G \,\big|\, s \text{ is continuous} \big\}\,,$$

which is a sheaf. Hence, for a connected subset $U \subset X$ each section $s \in \Gamma(U, G_X)$ is constant. Let X be connected, $G \neq 0$, and assume that there exist two disjoint non-empty open sets $U_1, U_2 \subset X$. Then a section $s \in \Gamma(U_1 \cup U_2, G_X)$ with $s|_{U_1} = 0$, $s|_{U_2} \neq 0$ has no extension to X. Hence, G_X is not flabby.

A.9 Čech Cohomology and Comparison

Another, more geometric construction of sheaf cohomology is described in the following. For details and much more material see [Ser1, God, GrR1].

Let X be a topological space and $\mathcal{F}$ a sheaf of Abelian groups on X. For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ and $(i_0, \dots, i_q) \in I^{q+1}$ we set

$$U_{i_0,\dots,i_q} := U_{i_0} \cap \dots \cap U_{i_q} .$$

For $q \geq 0$, we set

$$C^q(\mathfrak{U}, \mathcal{F}) := \prod_{(i_0,\dots,i_q)\in I^{q+1}} \Gamma\big(U_{i_0,\dots,i_q}, \mathcal{F}\big) ,$$

which is an Abelian group, the group of *q-cochains of $\mathfrak{U}$ with values in $\mathcal{F}$*. Hence, a *q-cochain* is an element $\boldsymbol{c} = (c_{i_0,\dots,i_q})_{(i_0,\dots,i_q)\in I^{q+1}} \in C^q(\mathfrak{U}, \mathcal{F})$ with $(i_0, \dots, i_q)$-component $c_{i_0,\dots,i_q} \in \Gamma\big(U_{i_0,\dots,i_q}, \mathcal{F}\big)$.

Define a morphism of Abelian groups

$$d^q : \ C^q(\mathfrak{U}, \mathcal{F}) \longrightarrow C^{q+1}(\mathfrak{U}, \mathcal{F})$$

component-wise by

$$d^q(\boldsymbol{c})_{i_0,\dots,i_{q+1}} := \sum_{k=0}^{q+1} (-1)^k c_{i_0,\dots,\widehat{i_k},\dots,i_{q+1}}\big|_{U_{i_0,\dots,i_{q+1}}} ,$$

where $\widehat{\ }$ denotes deletion of the respective term. $d = d^q$ is called a *coboundary map*. One checks by an explicit computation that $d^{q+1} \circ d^q = 0$, hence,

$$C^\bullet(\mathfrak{U}, \mathcal{F}) : \ C^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^1} C^2(\mathfrak{U}, \mathcal{F}) \longrightarrow \dots$$

is a complex of Abelian groups, the *Čech complex of $\mathfrak{U}$ with values in $\mathcal{F}$*.

We define the *Čech cohomology of $\mathfrak{U}$ with values in $\mathcal{F}$* to be the cohomology of the complex $C^\bullet(\mathfrak{U}, \mathcal{F})$, that is,

$$H^q(\mathfrak{U}, \mathcal{F}) := H^q\big(C^\bullet(\mathfrak{U}, \mathcal{F})\big) = \operatorname{Ker}(d^q)/\operatorname{Im}(d^{q-1}) .$$

Note that for $\boldsymbol{c} = (c_i) \in C^0(\mathfrak{U}, \mathcal{F}) = \prod_{i\in I} \Gamma(U_i, \mathcal{F})$, the condition $d^0(\boldsymbol{c}) = 0$ means that $c_i|_{U_i\cap U_j} = c_j|_{U_i\cap U_j}$ for all $(i, j) \in I^2$. Since $\mathcal{F}$ is a sheaf, it follows that there exists a unique (global) section $c \in \Gamma(X, \mathcal{F})$ such that $c|_{U_i} = c_i$. Hence,

$$H^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) .$$

For $q \geq 1$, however, $H^q(\mathfrak{U}, \mathcal{F})$ depends on $\mathfrak{U}$. To make these cohomology groups only depend on X, we have to pass to the limit by finer and finer coverings. An open covering $\mathfrak{V} = \{V_j \mid j \in J\}$ is a *refinement of* $\mathfrak{U}$ if a map $f : J \to I$ is given such that $V_{j_0,\dots,j_q} \subset U_{f(j_0),\dots,f(j_q)}$ for all $(j_0, \dots, j_q) \in J^q$ and all $q \geq 0$. Such a refinement defines (in an obvious way) a morphism of Abelian groups

$$r^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U}, \mathcal{F}) \longrightarrow H^q(\mathfrak{V}, \mathcal{F}) , \quad q \geq 0 ,$$

which commutes with the coboundary maps. $r^{\mathfrak{U}}_{\mathfrak{V}}$ can be checked to depend only on $\mathfrak{U}$ and $\mathfrak{V}$ (and $\mathcal{F}$), but not on the map f.

Two arbitrary coverings $\mathfrak{U} = \{U_i \mid i \in I\}$ and $\mathfrak{V} = \{V_j \mid j \in J\}$ have always a common refinement $\mathfrak{W} = \{U_i \cap V_j \mid (i,j) \in I \times J\}$. This allows us to pass to the limit of the inductive system $\big(H^q(\mathfrak{U}, \mathcal{F})\big)_{\mathfrak{U}}$, where $\mathfrak{U}$ passes through all open coverings of X. (Indexing an element $U \in \mathfrak{U}$ by the subset $U \subset X$, we may assume that all open coverings of X have as indexing set a subset of $\mathcal{P}(X)$; hence, we may speak about the "set of open coverings of X".)

We define the *q-th Čech cohomology group of X with values in $\mathcal{F}$* to be the direct limit

$$\check{H}^q(X, \mathcal{F}) := \varinjlim_{\mathfrak{U}} H^q(\mathfrak{U}, \mathcal{F}) := \coprod_{\mathfrak{U}} H^q(\mathfrak{U}, \mathcal{F}) / \sim \ .$$

Here $\boldsymbol{c} \in H^q(\mathfrak{U}, \mathcal{F})$ and $\boldsymbol{c}' \in H^q(\mathfrak{V}, \mathcal{F})$ are equivalent ($\boldsymbol{c} \sim \boldsymbol{c}'$) if there exists an open covering $\mathfrak{W}$ of X which is a refinement of $\mathfrak{U}$ and of $\mathfrak{V}$ such that $r^{\mathfrak{U}}_{\mathfrak{W}}(\boldsymbol{c}) = r^{\mathfrak{V}}_{\mathfrak{W}}(\boldsymbol{c}')$ in $H^q(\mathfrak{W}, \mathcal{F})$. By associating to $\boldsymbol{c} \in H^q(\mathfrak{U}, \mathcal{F})$ its equivalence class, we obtain natural morphisms of Abelian groups

$$\check{r}^q_{\mathfrak{U}} : H^q(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{H}^q(X, \mathcal{F}) , \quad q \geq 0 ,$$

with $\check{r}^0_{\mathfrak{U}} : \check{H}^0(X, \mathcal{F}) \xrightarrow{\cong} H^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$. Moreover, one can show that $\check{r}^1_{\mathfrak{U}}$ is injective (cf. [Ser1]) but in general $\check{r}^q_{\mathfrak{U}}$ is neither injective nor surjective.

However, we have the following theorem [Ser1, Sect. 29], which is quite useful in algebraic and analytic geometry.

We call a covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of X *$\mathcal{F}$-acyclic*, if $H^q(U_{i_0,\ldots,i_q}, \mathcal{F}) = 0$ for all $(i_0, \ldots, i_q) \in I^{q+1}$ and all $q \geq 1$.

Theorem A.9.1. *Let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an $\mathcal{F}$-acyclic covering of the topological space X, $\mathcal{F}$ a sheaf of Abelian groups on X. Assume there exists a family $\mathfrak{V}^\alpha$, $\alpha \in A$, of open coverings of X which satisfy*

(a) for any open covering $\mathfrak{W}$ of X there exists an $\alpha \in A$ such that $\mathfrak{V}^\alpha$ is a refinement of $\mathfrak{W}$;
(b) $\mathfrak{V}^\alpha$ is $\mathcal{F}$-acyclic for each $\alpha \in A$.

Then $\check{r}^q_{\mathfrak{U}} : H^q(\mathfrak{U}, \mathcal{F}) \to \check{H}^q(X, \mathcal{F})$ is an isomorphism for each $q \geq 0$.

Let us now compare the Čech cohomology groups $\check{H}^q(X, \mathcal{F})$ with the groups $H^q(X, \mathcal{F})$ defined by the canonical flabby resolution of $\mathcal{F}$. First we have [God, II.5.4]:

Theorem A.9.2. *Let X be a topological space and $\mathcal{F}$ a sheaf of Abelian groups on X. Then, for any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of X, there are natural homomorphisms*

$$\check{r}^q_{\mathfrak{U}} : H^q(\mathfrak{U}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}) , \quad q \geq 0 ,$$

which are isomorphisms if the covering $\mathfrak{U}$ is $\mathcal{F}$-acyclic.

Passing to the limit, the morphisms $\check{r}^q_{\mathfrak{U}}$ define natural homomorphisms

$$r^q : \check{H}^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}), \quad q \geq 0,$$

which are bijective for $q = 0$ and injective for $q = 1$. Combining Theorems A.9.1 and A.9.2, we obtain the following result [God, II.5.9.2]:

Theorem A.9.3. *Let $\mathcal{F}$ be a sheaf of Abelian groups on the topological space X. Assume there exists a family $\mathfrak{U}$ of open subsets of X such that*

(a) if $U, U' \in \mathfrak{U}$ then $U \cap U' \in \mathfrak{U}$;
(b) for each $x \in X$, $\mathfrak{U}$ contains arbitrary small neighbourhoods of x;
(c) each $U \in \mathfrak{U}$ satisfies $\check{H}^q(U, \mathcal{F}) = 0$ for $q \geq 1$.

Then the maps $\check{r}^q_{\mathfrak{U}} : H^q(\mathfrak{U}, \mathcal{F}) \to H^q(X, \mathcal{F})$ and $r^q : \check{H}^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$ are bijective for $q \geq 0$.

The above results can be used to show [God, II.5.10.1]:

Theorem A.9.4. *Let X be a paracompact topological space and $\mathcal{F}$ a sheaf of Abelian groups on X. Then*

$$r^q : \check{H}^q(X, \mathcal{F}) \xrightarrow{\cong} H^q(X, \mathcal{F})$$

is an isomorphism for $q \geq 0$.

Hence, if either X is paracompact or we are in the situation of Theorem A.9.3 then we can use, alternatively, flabby or Čech cohomology.

Finally, let us compare, in the case where $\mathcal{F} = G_X$ is a locally constant sheaf on X, sheaf cohomology with values in G_X with *singular cohomology with values in G* (which is usually applied in algebraic topology).

Let G be an Abelian group and G_X the locally constant sheaf on X. We denote by ${}^sH^q(X, G)$ the q-th singular cohomology group on X with values in G. Then we have the following comparison theorem [Bre, Thm. 3.1.1]:

Theorem A.9.5. *Let X be a paracompact, locally contractible topological space. Then, for any Abelian group G, we have*

$$H^q\big(X, G_X\big) \cong {}^sH^q(X, G), \quad q \geq 0.$$

Remark A.9.5.1. (1) Let X be a complex space. Then the family of Stein open subsets satisfy the assumptions of Theorem A.9.3 (see [GrR1]). Moreover, X is paracompact (by definition), locally path-connected and locally contractible (e.g. by the triangularization theorem of Łojasievich [Łoj]). Hence, the conclusions of Theorems A.9.3 and A.9.5 hold for a complex space X.

(2) If X is an algebraic scheme (see [Har]), then the affine open subsets of X satisfy the assumptions of Theorem A.9.3 (see [Har]). Hence, the Čech cohomology and the flabby cohomology coincide. However, algebraic schemes are not paracompact in their Zariski topology, since this topology is too coarse. One usually passes, if necessary, to the étale topology and étale cohomology (see [Mil]).

B

Commutative Algebra

We collect the basic properties of finitely generated modules over Noetherian rings. Throughout this appendix a ring is always meant to be commutative with 1 and a module means a unitary module, that is, multiplication with 1 is the identity map. Excellent references for these topics are [AtM], [Mat2], [AlK], [Eis]; for a constructive approach see [GrP].

A *local ring* A, that is, a ring with unique maximal ideal, is denoted by $(A, \mathfrak{m})$ or $(A, \mathfrak{m}, K)$, where $\mathfrak{m}$ denotes the maximal ideal and $K = A/\mathfrak{m}$ the residue field of A. A morphism between local rings is always assumed to be a *local morphism*, that is, maps the maximal ideal of the source ring to the maximal ideal of the target ring.

B.1 Associated Primes and Primary Decomposition

Let A be a ring and M an A-module. A prime ideal $\mathfrak{p} \subset A$ is called an *associated prime of* M if there exists an $m \in M \setminus \{0\}$ such that

$$\mathfrak{p} = \mathrm{Ann}(m) := \{x \in A \mid xm = 0\}.$$

The set of associated primes is denoted by $\mathrm{Ass}_A(M) = \mathrm{Ass}(M)$. If $I \subset A$ is an ideal, then, by abuse of notation, $\mathrm{Ass}(A/I)$ is usually called the set of associated primes of I. Hence, in this notation $\mathrm{Ass}(A)$ is the same as the set of assiciated primes of $\langle 0 \rangle$. An element $x \in A$ is called a *zerodivisor of* M if there exists an $m \in M \setminus \{0\}$ such that $xm = 0$, otherwise x is a *non-zerodivisor*. It is easy to see that every maximal element of the family of ideals $\{\mathrm{Ann}(m) \mid m \in M \setminus \{0\}\}$ is a prime ideal, hence $\mathrm{Ass}(M) \neq \emptyset$ and

$$Z(M) := \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$$

coincides with the set of zerodivisors of M. If $\mathfrak{p} = \mathrm{Ann}(m) \in \mathrm{Ass}(M)$ then $A/\mathfrak{p}$ is, via $1 \mapsto m$, isomorphic to a submodule of M.

Now let A be Noetherian and M finitely generated. Then $\mathrm{Ass}(M)$ is a finite set, and the minimal elements of $\mathrm{Ass}(M)$ are called the *minimal primes* or *isolated primes* of M. The set of these primes is denoted by $\mathrm{MinAss}(M)$. The non-minimal primes are called *embedded primes.* If $I \subset A$ is an ideal, then $\mathrm{MinAss}(A/I)$ consists precisely of the minimal elements of the set of prime ideals in A which contain I. In particular, the set $\{\mathfrak{p} \subset A \text{ prime} \mid I \subset \mathfrak{p},\ \mathfrak{p} \text{ minimal}\}$ is finite.

A submodule $N \subset M$ is called a *primary submodule* if for all $x \in A$ and $m \in M$ the following holds: $(m \notin N,\ xm \in N \Rightarrow x^n M \subset N)$. A *primary ideal* of A is just a primary submodule of A. A submodule N is primary iff $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$ for some prime ideal $\mathfrak{p}$.

Theorem B.1.1. *Any proper submodule N of a finitely generated module M over a Noetherian ring has a* primary decomposition

$$N = \bigcap_{i=1}^{n} N_i\,, \quad N_i \subset M \textit{ primary,}$$

which is minimal *or* irredundant *(no N_i can be deleted).*

The N_i need not be unique but the associated primes $\mathfrak{p}_i = \mathrm{Ass}(M/N_i)$ satisfy $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$*, hence are uniquely determined. Moreover those N_i belonging to minimal associated primes $\mathfrak{p}_i \in \mathrm{MinAss}(M/N)$ are uniquely determined, too.*

Thus, if $I \subset A$ is an ideal then there exists a minimal primary decomposition

$$I = \bigcap_{i=1}^{n} \mathfrak{q}_i\,, \quad \mathfrak{q}_i \subset A \text{ primary}$$

with $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, $i = 1, \dots, n$, the associated primes of A/I. If $\mathfrak{q}$ is any primary ideal, then $\mathfrak{p} = \sqrt{\mathfrak{q}}$ is a prime ideal and $\mathfrak{q}$ is called $\mathfrak{p}$*-primary.*

We can compute the minimal associated primes, respectively a primary decomposition of $I \subset K[x_1, \dots, x_n]$ in Singular by using the commands `minAssGTZ` or `minAssChar`, respectively `primdecGTZ` or `primdecSY`. For example, the cylinder over an A_3 plane curve singularity and over a D_4 plane curve singularity in $\mathbb{C}^3$ intersect along three space curves. A primary decomposition shows that one of these three curves is the line $y = z = 0$, with a triple structure:

```
LIB "primdec.lib";
ring A = 0,(x,y,z),dp;
ideal I = xz+y4, y*(y2+z2);
primdecGTZ(I);
//-> [1]:                    [2]:              [3]:
//->    [1]:                    [1]:              [1]:
//->      _[1]=y2+z2              _[1]=z            _[1]=y
//->      _[2]=z3+x               _[2]=y3           _[2]=x
```

```
//->    [2]:                [2]:              [2]:
//->       _[1]=y2+z2          _[1]=z            _[1]=y
//->       _[2]=z3+x           _[2]=y            _[2]=x
```

We get a list of three entries, each entry consisting of two ideals, the first one being a primary ideal, the second one the associated prime.

The *absolute primary decomposition* (that is, the primary decomposition over the algebraic closure of the current ground field) is provided by the SINGULAR procedure `absPrimdecGTZ` from `primdec.lib` (see [DeL] for details). Primary decomposition for modules is provided by the SINGULAR library `mprimdec.lib`.

B.2 Dimension Theory

In this section we mention a few results from dimension theory over Noetherian rings.

Let A be a Noetherian ring, then the *(Krull) dimension* of A is the supremum of the lengths of strictly decreasing chains of prime ideals in A,

$$\dim A \;:=\; \sup\big\{\, d \;\big|\; A \supsetneq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \ldots \supsetneq \mathfrak{p}_d \,,\; \mathfrak{p}_i \subset A \text{ prime} \,\big\}\,.$$

If M is a finitely generated A-module we define

$$\dim M := \dim(A/\operatorname{Ann}_A(M))$$

where $\operatorname{Ann}_A(M) = \operatorname{Ann}(M) = \{x \in A \,|\, xM = 0\}$ is the *annihilator* of M in A. Hence, $\dim(A)$ is independent of whether we consider A as a ring or as an A-module. More generally, if B is an A-algebra which is finitely generated as an A-module, then $\dim B$ is independent of whether we consider B as a ring or as an A-module.

If A is the polynomial ring $K[x_1,\ldots,x_n]$, or the (convergent) power series ring $K\langle x_1,\ldots,x_n\rangle$ over a field K, then $\dim A = n$. If A is an arbitrary affine ring $K[x_1,\ldots,x_n]/I$, respectively an analytic K-algebra $K\langle x_1,\ldots,x_n\rangle/I$, then there exists a Noether normalization $K[y_1,\ldots,y_d] \hookrightarrow K[x_1,\ldots,x_n]/I$, respectively $K\langle y_1,\ldots,y_d\rangle \hookrightarrow K\langle x_1,\ldots,x_n\rangle/I$ (Theorem I.1.25) such that $\dim A = d$ (Exercise I.1.3.1). Affine and analytic K-algebras A are *catenary*, that is, if $\mathfrak{p} \subset \mathfrak{q}$ are two prime ideals, then all maximal chains of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r = \mathfrak{q}$$

have the same (finite) length. For details on catenary rings, see [Mat2, §15 and §31].

Let M be a finitely generated module over an arbitrary Noetherian ring A. Then we have

$$\begin{aligned}\dim M &= \max\{\dim(A/\mathfrak{p}) \,|\, \mathfrak{p} \in \operatorname{Ass}(M)\}\\ &= \max\{\dim(A/\mathfrak{p}) \,|\, \mathfrak{p} \in \operatorname{MinAss}(M)\}\,.\end{aligned}$$

M is called *equidimensional* or *pure dimensional* iff $\dim(M) = \dim(A/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{MinAss}(M)$. We see that $\dim A$ is the maximum of $\dim(A/\mathfrak{p})$ where $\mathfrak{p}$ runs through the associated primes of a primary decomposition of $\langle 0 \rangle \subset A$. It follows that

$$\dim A = \dim A_{\mathrm{red}}$$

where $A_{\mathrm{red}} := A/\operatorname{nil}(A)$ is the *reduced ring*, or the *reduction* of A. Here,

$$\operatorname{nil}(A) := \sqrt{\langle 0 \rangle} = \{x \in A \mid x^p = 0 \text{ for some } p \in \mathbb{N}\}$$

is the *nilradical* of A. An element $x \in A$ is called *nilpotent* if $x^p = 0$ for some $p \geq 0$, that is, if $x \in \operatorname{nil}(A)$. A is called *reduced* if $\operatorname{nil}(A) = 0$.

If $\mathfrak{p} \subset A$ is a prime ideal, then the *height* of $\mathfrak{p}$ is the maximal length of a descending chain of prime ideals in A contained in $\mathfrak{p}$. Hence, the ideals of height 0 are just the minimal prime ideals of A and, by Appendix B.1, there are only finitely many of them. By definition, we have the inequality

$$\operatorname{height}(\mathfrak{p}) + \dim(A/\mathfrak{p}) \leq \dim A$$

with equality if A is catenary and equidimensional. It is easy to see that $\operatorname{height}(\mathfrak{p}) = \dim A_{\mathfrak{p}}$ where $A_{\mathfrak{p}}$ is the localization of A in $\mathfrak{p}$. Hence, we get

$$\dim(A) = \sup \dim(A_{\mathfrak{p}}) = \sup \dim(A_{\mathfrak{m}})$$

where $\mathfrak{p}$, respectively $\mathfrak{m}$, run through all the prime, respectively maximal, ideals of A. This reduces dimension theory to dimension theory of local rings.

Now, let $(A, \mathfrak{m}, K)$ be a local Noetherian ring and M a finitely generated A-module. Then there are two more ways to define $\dim(M)$.

Let $\mathfrak{q} \subset \mathfrak{m}$ be any $\mathfrak{m}$-primary ideal, then we define the *Hilbert-Samuel function*

$$H_{\mathfrak{q}}(M, n) := \ell(M/\mathfrak{q}^{n+1}M)$$

where ℓ denotes the length of a composition series. Recall that a *composition series* of an A-module N is a filtration

$$\langle 0 \rangle = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

such that N_i/N_{i-1} is a simple A-module. For a local ring all simple modules are isomorphic to K. By the Jordan-Hölder theorem all composition series of N have the same length. If A contains a field which is isomorphic to K via the natural morphism $A \to A/\mathfrak{m}$, then N is in a natural way a K-vectorspace and $\ell(N) = \dim_K(N)$. This holds, in particular, for analytic K-algebras.

There exists a polynomial $P_{\mathfrak{q}}(M, t) \in \mathbb{Q}[t]$ such that

$$P_{\mathfrak{q}}(M, n) = H_{\mathfrak{q}}(M, n) \text{ for sufficiently large } n \in \mathbb{Z}.$$

$P_{\mathfrak{q}}(M, t)$ is called the *Hilbert-Samuel polynomial* of M and $\mathfrak{q}$. Its degree and leading coefficient are independent of the primary ideal $\mathfrak{q} \subset \mathfrak{m}$ and

$$\dim(M) = \deg P_{\mathfrak{q}}(M,t)\,. \tag{B.2.1}$$

The Hilbert-Samuel polynomial of an analytic local ring A with respect to the maximal ideal $\mathfrak{m}$ of A can effectively be computed (cf. [GrP, Cor. 5.5.5 and S-Exa. 5.5.13]).

Another way to define $\dim(M)$ is by systems of parameters. A sequence of elements $x_1,\dots,x_s \in \mathfrak{m}$ is called a *system of parameters* of M if

$$\ell(M/\langle x_1,\dots,x_s\rangle M) < \infty\,.$$

Hence, $x_1,\dots,x_s$ is a system of parameters of A if $x_1,\dots,x_s$ generate an $\mathfrak{m}$-primary ideal. We have

$$\dim(M) = \min\{s \mid x_1,\dots,x_s \text{ is a system of parameters of } M\} \tag{B.2.2}$$

It follows that $\dim(M)=0$ iff $\ell(M)<\infty$ and that $x_1,\dots,x_s$ is a system of parameters of M iff $\dim(M/\langle x_1,\dots,x_s\rangle M)=0$. Moreover, if A contains a field isomorphic to K, then

$$\dim(M)=0 \iff \dim_K(M)<\infty\,.$$

This holds in particular for analytic K-algebras.

The equality (B.2.2) yields a simple proof of *Krull's principal ideal*[1] *theorem*:

Theorem B.2.1. *If M is a finitely generated A-module over a local Noetherian ring A, then, for each $x \in \mathfrak{m}$, the following holds*

(1) $\dim(M/xM) \geq \dim(M)-1$,
(2) $\dim(M/xM) = \dim(M)-1$ *iff* $x \notin \bigcup_i \mathfrak{p}_i$, *where the union is taken over those primes* $\mathfrak{p}_i$ *in* $\mathrm{Ass}(M)$ *with* $\dim(M)=\dim(A/\mathfrak{p}_i)$.

In particular, if x is a non-zerodivisor of M, that is, $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$, then $\dim(M/xM)=\dim(M)-1$.

Proof. To see (1), note that if $x_1,\dots,x_s$ is a minimal system of parameters for M/xM, then $x,x_1,\dots,x_s$ is a system of parameters for M, hence $\dim(M)\leq \dim(M/xM)+1$ by (B.2.2).

For (2) consider the inclusions

$$\mathfrak{p}_i \subset \mathfrak{p}_i + \langle x\rangle \subset \mathrm{Ann}(M/xM) =: I\,.$$

Since $\dim(A/I)$ is the maximal length of chains of prime ideals in A containing I and since $\mathfrak{p}_i$ is prime and $x_i \notin \mathfrak{p}_i$ we obtain

$$\begin{aligned}\dim(M/xM) &= \dim(A/I) \leq \dim(A/\mathfrak{p}_i + \langle x\rangle)\\ &\leq \dim(A/\mathfrak{p}_i)-1 = \dim(M)-1\,.\end{aligned}$$

Therefore, (2) follows from (1). $\square$

[1] An ideal I in a ring A is called a *principal ideal* if it is generated by one element.

As a consequence we get for a finitely generated module M over a local ring $(A, \mathfrak{m})$ and for $x_1, \dots, x_s \in \mathfrak{m}$,

$$\dim(M/\langle x_1, \dots, x_s\rangle M) \geq \dim(M) - s$$

with equality if $x_1, \dots, x_s$ is an *M-regular sequence*, see Definition B.6.2 (if M is Cohen-Macaulay, the inverse implication holds, too).

If M is given as the cokernel of a matrix C, $P^r \xrightarrow{C} P^s \to M \to 0$, with $P = K[x_1, \dots, x_n]$, respectively $P = K\langle x_1, \dots, x_n\rangle$, then `dim(groebner(C))` is a Singular command that returns the dimension of M (in a ring with global, respectively local monomial ordering).

B.3 Tensor Product and Flatness

In this section we define flatness and derive some elementary properties.

Definition B.3.1. Let A be a ring. An A-module M is said to be *flat*, if for any exact sequence $0 \to N' \to N \to N'' \to 0$ of A-modules, the sequence

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is also exact. A flat module M is called *faithfully flat*, if for any sequence $0 \to N' \to N \to N'' \to 0$ the exactness of

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

implies the exactness of $0 \to N' \to N \to N'' \to 0$.

Free modules and, more generally, projective modules are flat. Recall that an A-module P is called *projective* if for any surjection $\pi : M \to M'$ and any map $f : P \to M'$ there exists a lifting $h : P \to M$ such that $f = \pi \circ h$. Free modules are projective, and projective modules can be characterized as direct summands of a free module (see Exercise I.1.7.1).

If M is not flat, then tensoring an exact sequence $0 \to N' \to N \to N'' \to 0$ with M leads to a *long exact* Tor-*sequence*

$$\begin{gathered} \ldots \to \mathrm{Tor}_i^A(M, N') \to \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N'') \to \mathrm{Tor}_{i-1}^A(M, N') \to \\ \ldots \to \mathrm{Tor}_1^A(M, N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0\,. \end{gathered}$$

Let us recall the definition of Tor. Choose any free resolution $F_\bullet$ of M or $G_\bullet$ of N, then

$$\mathrm{Tor}_i^A(M, N) = H_i(F_\bullet \otimes_A N) = H_i(G_\bullet \otimes_A M)\,.$$

Here, a *free resolution* $F_\bullet$ of M is an exact complex

$$F_\bullet\colon \quad \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0$$

with F_i free A-modules and $\mathrm{Coker}(d_1) = M$. We usually also call the *augmented complex* $F_\bullet \to M \to 0$ a free resolution of M.

We recall some general statements about flatness.

Proposition B.3.2. *Let A be a ring and M an A-module. The following are equivalent:*

(1) M is flat,
(2) $\mathrm{Tor}_1^A(M,N) = 0$ for all A-modules N,
(3) $\mathrm{Tor}_i^A(M,N) = 0$ for all A-modules N and all $i \geq 1$,
(4) $\mathrm{Tor}_1^A(M,A/I) = 0$ for all ideals $I \subset A$,
(5) the canonical surjection $I \otimes_A M \to IM$, $a \otimes m \mapsto am$, is bijective for all ideals $I \subset A$.

Proof. Since any A-module N is the direct limit of its finitely generated submodules and $\mathrm{Tor}_i^A(M, \_)$ commutes with direct limits for all i (which can be seen directly from the definition), we may equivalently require in (2) that N is finitely generated.

(1) $\Rightarrow$ (2). If N is finitely generated then there exists an exact sequence $0 \to K \to A^r \to N \to 0$. Tensoring with M yields an exact sequence

$$\mathrm{Tor}_1^A(M, A^r) \to \mathrm{Tor}_1^A(M,N) \to K \otimes_A M \to A^r \otimes_A M\,.$$

Since $K \otimes M \to A^r \otimes M$ is injective (M is flat) and since $\mathrm{Tor}_1^A(M,A^r) = 0$ (use induction on r), we obtain $\mathrm{Tor}_1^A(M,N) = 0$.

The implication (2) $\Rightarrow$ (1) follows from the long exact Tor-sequence. To show (1) $\Rightarrow$ (3), let $G_\bullet$ be a free presentation of N. Since M is flat, $G_\bullet \otimes_A M$ is exact, hence $\mathrm{Tor}_i^A(M,N) = H_i(G_\bullet \otimes_A M) = 0$ for all $i \geq 1$.

(3) $\Rightarrow$ (4) is trivial, and (4) $\Leftrightarrow$ (5) follows from the exact sequence

$$0 \to \mathrm{Tor}_1^A(M, A/I) \to I \otimes_A M \to IM \to 0\,.$$

(4) $\Rightarrow$ (2). Let N be finitely generated A-module; so there exists a surjection $A^r \twoheadrightarrow N$. Choose $A^{r-1} \subset A^r$ and let $N' \subset N$ be the image of A^{r-1} in N. Then $N/N' \cong A/I$ for some ideal $I \subset A$ and hence we get an exact sequence

$$\mathrm{Tor}_1^A(M,N') \to \mathrm{Tor}_1^A(M,N) \to \mathrm{Tor}_1^A(M,A/I)\,.$$

By induction on r we get $\mathrm{Tor}_1^A(M,N) = 0$. $\square$

Let $\varphi\colon A \to B$ be a morphism of rings and M a B-module. We say that M is *flat over* A or M is A-*flat* if M is flat as A-module (via φ). The morphism φ is called *flat* if B is flat over A.

Proposition B.3.3 (Facts on (faithful) flatness). *Let $\varphi\colon A \to B$ be a morphism of rings, M a B-module and N an A-module.*

(1) **(Flatness of tensor product)** *If M, N are A-flat (respectively faithfully flat), then $M \otimes_A N$ is A-flat (respectively faithfully flat).*

(2) **(Transitivity of flatness)** *If B is A-flat (respectively faithfully flat) and M is B-flat (respectively faithfully flat), then M is A-flat (respectively faithfully flat).*

(3) **(Preservation under base change)** *If N is A-flat (respectively faithfully flat), then $N \otimes_A B$ is B-flat (respectively faithfully flat).*

(4) **(Flatness before base change)** *If B is faithfully flat over A, and if $N \otimes_A B$ be flat (respectively faithfully flat) over B, then N is flat (respectively faithfully flat) over A.*

(5) **(Module test for flatness)** *Let $M \neq 0$ be flat (respectively faithfully flat) over A and faithfully flat over B, then B is flat (respectively faithfully flat) over A.*

(6) **(Localization preserves flatness)** *If $S \subset A$ is a multiplicatively closed set, then the localization $S^{-1}A$ is A-flat. If M is A-flat (respectively faithfully flat), then $S^{-1}M = M \otimes_A S^{-1}A$ is $(S^{-1}A)$-flat (respectively faithfully flat)*

(7) **(Flatness is a local property)** *M is B-flat iff the localization $M_{\mathfrak{p}}$ is $B_{\mathfrak{p}}$-flat for every prime ideal (equivalently, every maximal ideal) $\mathfrak{p}$ of B.*

(8) **(Completion is flat)** *Let A be Noetherian and $I \subset A$ an ideal. Then the I-adic completion $\widehat{A} = \varprojlim A/I^k$ of A is flat over A.*

(9) **(Flat implies faithfully flat for local rings)** *If $\varphi\colon A \to B$ is a local homomorphism of local rings, then B is faithfully flat over A iff B is flat over A.*

Moreover, we have the following

(10) **(Characterization of faithful flatness)**

(i) N is faithfully flat

$\iff$ *N is flat and $N \otimes_A N' \neq 0$ for any A-module $N' \neq 0$*

$\iff$ *N is flat and $\mathfrak{m}N \neq N$ for any maximal ideal $\mathfrak{m} \subset A$.*

(ii) B is faithfully flat over A

$\iff$ *φ is injective and $B/\varphi(A)$ is flat over A*

$\iff$ *for any ideal $I \subset A$, the natural map $I \otimes_A B \to IB$ is bijective and $(IB) \cap A := \varphi^{-1}(IB) = I$.*

The proofs of the above facts are not difficult and left as an exercise (see also [AlK]).

From Proposition B.3.3, we deduce the following useful relations between algebraic and analytic local rings:

Lemma B.3.4. *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$, and let $I \subset K[\boldsymbol{x}]$ be an ideal contained in $\langle \boldsymbol{x} \rangle$. Then all inclusions in the sequence of rings*

$$K[\boldsymbol{x}]/I \subset K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}/IK[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle} \subset K\langle \boldsymbol{x} \rangle/IK\langle \boldsymbol{x} \rangle \subset K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]$$

are flat, all inclusions between the last three rings are faithfully flat

The flatness of $K[\boldsymbol{x}]/I \subset K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}/IK[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle} = (K[\boldsymbol{x}]/I)_{\langle \boldsymbol{x} \rangle}$ follows from Proposition B.3.3 (6), the flatness of $K[\boldsymbol{x}]/I \subset K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]$ from B.3.3 (8). The flatness of the remaining inclusions follows from B.3.3 (5), while faithful flatness is a consequence of B.3.3 (9).

We recall the fundamental flatness criterion for finitely generated modules over local rings.

Proposition B.3.5. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, $K = A/\mathfrak{m}$ and M a finitely generated A-module. Then the following conditions are equivalent:*

(a) M is flat.
(b) M is free.
(c) $\operatorname{Tor}_1^A(M, K) = 0$.
(d) M is faithfully flat.

Proof. The implication (b) $\Rightarrow$ (a) and the equivalence (b) $\Leftrightarrow$ (d) are obvious from the definitions.

To prove (a) $\Rightarrow$ (c), we tensor the exact sequence $0 \to \mathfrak{m} \to A \to K \to 0$ with M and obtain the exact sequence

$$\operatorname{Tor}_1^A(M, A) \to \operatorname{Tor}_1^A(M, K) \to M \otimes_A \mathfrak{m} \to M \to M \otimes_A K \to 0\,.$$

As M is A-flat, $M \otimes_A \mathfrak{m} \to M$ is injective, thus, $\operatorname{Tor}_1^A(M, K) \to M \otimes_A \mathfrak{m}$ is the zero homomorphism. As A is free, $\operatorname{Tor}_1^A(M, A) = 0$, and we obtain $\operatorname{Tor}_1^A(M, K) = 0$ as required.

Finally, we prove (c) $\Rightarrow$ (b). Let $m_1, \ldots, m_r \in M$ be generators of M as A-module, representing a K-basis of $M/\mathfrak{m}M$. Consider the exact sequence

$$0 \to \operatorname{Ker}(\varphi) \to A^r \xrightarrow{\varphi} M \to 0\,, \quad \varphi(a_1, \ldots, a_r) = a_1 m_1 + \ldots + a_r m_r\,.$$

Tensoring it with $K = A/\mathfrak{m}$, we obtain from the choice of the m_i and as $\operatorname{Tor}_1^A(M, K) = 0$,

$$0 \longrightarrow \operatorname{Ker}(\varphi) \otimes_A K \longrightarrow K^r \xrightarrow{\cong} M/\mathfrak{m}M \longrightarrow 0\,,$$

hence $\operatorname{Ker}(\varphi) \otimes_A K = 0$ and, by Nakayama's lemma, $\operatorname{Ker}(\varphi) = 0$. $\square$

The last result is not completely satisfactory since the finiteness assumption on M is too strong for many applications. We give a satisfactory generalization in Theorem B.5.1.

Let us recall the very useful Nakayama Lemma.

Proposition B.3.6 (Nakayama Lemma). *Let $(A, \mathfrak{m}, K)$ be a local ring, M a finitely generated A-module and $N \subset M$ a submodule.*

(1) If $M = \mathfrak{m}M + N$ then $M = N$, in particular, if $M = \mathfrak{m}M$ then $M = 0$.
(2) $m_1, \dots, m_k \in M$ generate the K-vector space $M/\mathfrak{m}M$ iff they generate the A-module M. In particular, $m_1, \dots, m_k$ is a minimal set of generators for M iff it is a basis of the K-vector space $M/\mathfrak{m}M$.

Proof. (1) Passing from M to M/N, we may assume that $N = 0$. Assume $M \neq 0$, and let $m_1, \dots, m_k \in M$ be a system of generators of M, that cannot be shortened. Since $m_k \in M = \mathfrak{m}M$, there are $a_1, \dots, a_k \in \mathfrak{m}$ such that $m_k = a_1 m_1 + \ldots + a_k m_k$. This implies

$$m_k(1 - a_k) = a_1 m_1 + \ldots + a_{k-1} m_{k-1}\,.$$

Since $1 - a_k$ is a unit in the local ring A, the latter contradicts the minimality of the chosen system of generators.

(2) Let $N = \langle m_1, \dots, m_k \rangle \subset M$. Then $m_1, \dots, m_k$ generate the K-vector space $M/\mathfrak{m}M$ iff $\mathfrak{m}M + N = M$ and, by (1), the latter is equivalent to $M = N$. This proves the claim. □

The Tor-groups can be effectively computed by using the `Tor` command from `homolog.lib`. The same SINGULAR library contains procedures to test for flatness, etc.

B.4 Artin-Rees and Krull Intersection Theorem

Let A be a Noetherian ring, $I \subset A$ an ideal, M a finitely generated A-module and $N \subset M$ a submodule. The *I-adic filtration* on A, respectively on M, is given by the ideals $I^n \subset A$, respectively by the submodules $I^n M \subset M$, $n \geq 0$, with $I^0 = A$. Then

$$\mathrm{gr}_I(A) := \bigoplus_{n \geq 0} I^n / I^{n+1}$$

is a graded ring and

$$\mathrm{gr}_I(M) := \bigoplus_{n \geq 0} I^n M / I^{n+1} M$$

is a graded $\mathrm{gr}_I(A)$-module.

On any submodule $N \subset M$ we may consider two filtrations, either the I-adic filtration $\{I^n N \mid n \geq 0\}$ or the filtration induced by the I-adic filtration of M, that is, $\{N \cap I^n M \mid n \geq 0\}$. Of course, $I^n N \subset N \cap I^n M$ for each n. The theorem of Artin-Rees says that for sufficiently large n, the higher terms of the induced filtration are generated by multiplication with I^k.

Theorem B.4.1 (Artin-Rees theorem). *With the notations above, there exists an m such that*

$$I^k(N \cap I^n M) = N \cap I^{n+k} M\,, \text{ for all } k \geq 0, n \geq m\,.$$

*In this situation we say that the induced filtration $\{N \cap I^n M \mid n \geq 0\}$ is I-*good.

Proof. Since "$\subset$" is obvious, we have to show "$\supset$". Set $N_n := N \cap I^n M$. The map

$$N_n/N_{n+1} \longrightarrow I^n M/I^{n+1} M$$

is injective and, hence,

$$\mathrm{gr}(N) := \bigoplus_{n \geq 0} N_n/N_{n+1} \longrightarrow \mathrm{gr}_I(M)$$

is injective. Since $\mathrm{gr}_I(A)$ is generated by I/I^2 as A/I-algebra and A is Noetherian, $\mathrm{gr}_I(A)$ is a finitely generated (A/I)-algebra and hence $\mathrm{gr}_I(A)$ is a Noetherian ring by Hilbert's basis theorem. Moreover, $\mathrm{gr}_I(M)$ is generated by M/IM over $\mathrm{gr}_I(A)$. Since M is finitely generated over A, $\mathrm{gr}_I(M)$ is finitely generated over $\mathrm{gr}_I(A)$, hence Noetherian. Therefore $\mathrm{gr}(N)$ is a finitely generated $\mathrm{gr}_I(A)$-module.

Let $n_1, \ldots, n_r$ be homogeneous generators of $\mathrm{gr}(N)$. Then, for any $m \geq \max\{\deg(n_i)\}$, we obtain an inclusion $N_{m+1} \subset IN_m$, and the result follows. $\square$

Theorem B.4.2 (Krull intersection theorem). *Let A be a local Noetherian ring, $I \subsetneq A$ an ideal and M a finitely generated A-module. Then*

$$\bigcap_{n \geq 0} I^n M = 0\,.$$

Proof. Set $N = \bigcap_n I^n M$. By the theorem of Artin-Rees

$$N \cap I^{n+1} M \subset I(N \cap I^n M)$$

for all sufficiently big n. By definition of N we have $N \subset IN$, hence $N = 0$ by Nakayama's lemma. $\square$

B.5 The Local Criterion of Flatness

The following flatness criterion is used in the theory of deformations. Let $(A, \mathfrak{m}, K)$ be a Noetherian local ring. If M is a finitely generated A-module, we have (Proposition B.3.5)

$$M \text{ is flat} \iff M \text{ is free} \iff \mathrm{Tor}_1^A(M, K) = 0.$$

However, the finiteness assumption on M is not fulfilled in many applications. Fortunately, there is another flatness criterion where this finiteness condition is considerably weakened. (Recall that all morphisms between local rings are necessarily local.)

Theorem B.5.1 (Local criterion of flatness). *Let $\varphi\colon A \to B$ be a morphism of local rings, $I \subset \mathfrak{m}_A$ an A-ideal and $K = A/\mathfrak{m}_A$. Then, for each finitely generated B-module M, the following are equivalent:*

(1) M *is A-flat,*
(2) M/IM *is A/I-flat and* $\operatorname{Tor}_1^A(M, A/I) = 0$,
(3) $\operatorname{Tor}_1^A(M, N) = 0$ *for every A/I-module N,*
(4) $M/I^{k+1}M$ *is A/I^{k+1}-flat for all $k \geq 0$,*
(5) $\operatorname{Tor}_1^A(M, K) = 0$,
(6) $\operatorname{Tor}_i^A(M, K) = 0$ *for all $i \geq 1$.*

The importance of this theorem lies in the fact that we do not require M to be finitely generated over A. For a proof, we refer to [Mat2, Theorem 22.3].

Corollary B.5.2. *Let $\varphi\colon A \to B$ be a morphism of local rings, and let*

$$0 \longrightarrow M' \xrightarrow{\varphi} M \longrightarrow M'' \longrightarrow 0 \qquad \text{(B.5.3)}$$

be an exact sequence of finitely generated B-modules. Then the following implications hold:

(a) If M', M'' are flat A-modules then M is a flat A-module.
(b) If M, M'' are flat A-modules then M' is a flat A-module.
(c) If A is an Artinian local K-algebra ($K = A/\mathfrak{m}_A$) and if M', M are flat A-modules then M'' is a flat A-module.

Note that (c) does not hold in general if we omit the condition "A Artinian".

Proof. Statements (a) and (b) follow from Theorem B.5.1 when considering the long exact Tor-sequence (with $K := A/\mathfrak{m}_A$)

$$\operatorname{Tor}_2^A(M'', K) \to \operatorname{Tor}_1^A(M', K) \to \operatorname{Tor}_1^A(M, K) \to \operatorname{Tor}_1^A(M'', K) \to \dots$$

To prove (c), we proceed by induction on $d := \dim_K(A)$ (which is finite as A is an Artinian K-algebra). $d = 1$ means that $A = K$ and the statement follows since K-vector spaces are K-flat.

Let $d \geq 2$ and $a \in \mathfrak{m}_A \setminus \{0\}$. Then, since a is no unit, $\dim_K(A/aA) > 0$ and we get an exact sequence of A-modules

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\cdot a} A \longrightarrow A/aA \longrightarrow 0$$

with $I := \operatorname{Ann}_A(a) \subset \mathfrak{m}_A$. Comparing dimensions, we get that $\dim_K(I) = \dim_K(A/aA)$, which is positive. Hence, $\overline{A} := A/I$ has a smaller dimension than A, and the induction hypothesis applies to $\overline{A}$. Since M', M are A-flat, $\mathrm{Tor}_1^A(M, \overline{A}) = 0$ and from the long exact Tor-sequence to (B.5.3) we get the exact sequence of $\overline{A}$-modules

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{Tor}_1^A(M'', \overline{A}) \longrightarrow & M' \otimes_A \overline{A} & \longrightarrow & M \otimes_A \overline{A} & \longrightarrow M'' \otimes_A \overline{A} \longrightarrow 0\,. \\ & \| & & \| & \\ & M'/IM' & \xrightarrow{\overline{\varphi}} & M/IM & \end{array}$$

If $\overline{\varphi}$ is injective, then Theorem B.5.1, (2) $\Rightarrow$ (1), gives the A-flatness of M''. Indeed, injectivity of $\overline{\varphi}$ implies $\mathrm{Tor}_1^A(M'', \overline{A}) = 0$ and a short exact sequence of $\overline{A}$-modules

$$0 \longrightarrow M'/IM' \xrightarrow{\varphi} M/IM \longrightarrow M'' \otimes_A \overline{A} \longrightarrow 0\,.$$

Since M'/IM' and M/IM are $\overline{A}$-flat (by Theorem B.5.1 (1) $\Rightarrow$ (2)), and since the induction hypothesis applies to $\overline{A}$-flatness, we obtain that $M'' \otimes_A \overline{A}$ is a flat $\overline{A}$-module.

It remains to show that $\overline{\varphi}$ is injective. Let $m' \in M'$ be such that $\varphi(m') \in IM$. Then $\varphi(am') = a\varphi(m') \in a \cdot IM = 0$, hence $am' = 0 \in M'$. By Remark B.5.2.1 below, this implies $m' \in \mathrm{Ann}_A(a) \cdot M' = IM'$. □

Remark B.5.2.1. Let A be a ring and M a flat A-module. Then, for any $a \in A$, $\mathrm{Ann}_M(a) := \{m \in M \mid am = 0\} = \mathrm{Ann}_A(a) \cdot M$. This follows when tensoring the exact sequence $0 \to \mathrm{Ann}_A(a) \to A \xrightarrow{\cdot a} aA \to 0$ with $\otimes_A M$.

The following proposition looks quite technical but it has many applications, in particular to deformation theory.

Proposition B.5.3. *Let $\varphi\colon A \to B$ be a morphism of local Noetherian rings, $f\colon M \to N$ a morphism of finitely generated B-modules with N flat over A. Then the following are equivalent.*

(1) f is injective and $P :=$ Coker(f) is A-flat,
(2) $f \otimes 1\colon M \otimes_A K \to N \otimes_A K$ is injective, $K = A/\mathfrak{m}_A$.

Moreover, f is an isomorphism iff $f \otimes 1$ is an isomorphism.

Proof. (1) $\Rightarrow$ (2). Tensoring

$$0 \longrightarrow M \xrightarrow{f} N \longrightarrow P \longrightarrow 0$$

with K over A we get an exact sequence

$$\mathrm{Tor}_1^A(P, K) \longrightarrow M \otimes_A K \xrightarrow{f \otimes 1} N \otimes_A K\,.$$

Since P is A-flat, $\mathrm{Tor}_1^A(P, K) = 0$ and $f \otimes 1$ is injective.

(2) $\Rightarrow$ (1). Consider the exact sequence $0 \to f(M) \to N \to P \to 0$, which yields after tensoring with K over A

$$\begin{aligned} \ldots \longrightarrow \operatorname{Tor}_1^A(f(M),K) \longrightarrow \operatorname{Tor}_1^A(N,K) &\longrightarrow \operatorname{Tor}_1^A(P,K) \\ \longrightarrow f(M)\otimes_A K &\longrightarrow N\otimes_A K\,, \end{aligned} \tag{B.5.4}$$

where $\operatorname{Tor}_1^A(N,K)=0$ since N is A-flat.

The assumption (2) implies that $f\otimes 1\colon M\otimes K \to f(M)\otimes K$ is bijective and the inclusion $f(M)\subset N$ induces an injective map $f(M)\otimes K\to N\otimes K$. Hence, $\operatorname{Tor}_1^A(P,K)=0$. By the local criterion of flatness (Theorem B.5.1) we conclude that P is A-flat.

We still have to show that f is injective. From the flatness of P we get that $\operatorname{Tor}_i^A(P,K)=0$ for all $i\geq 1$. Using (B.5.4) we deduce $\operatorname{Tor}_1^A(f(M),K)=0$ and, therefore, that $f(M)$ is A-flat, again by the local criterion of flatness. Tensoring the exact sequence

$$0\longrightarrow \operatorname{Ker}(f)\longrightarrow M\xrightarrow{\;f\;} f(M)\longrightarrow 0$$

with K over A we obtain the exact sequence

$$0\longrightarrow \operatorname{Ker}(f)\otimes_A K\longrightarrow M\otimes_A K\xrightarrow{\;f\otimes 1\;} f(M)\otimes_A K\longrightarrow 0\,.$$

Since $f\otimes 1$ is injective, $\operatorname{Ker}(f)\otimes_A K=\operatorname{Ker}(f)/\mathfrak{m}_A\operatorname{Ker}(f)=0$. Finally, Nakayama's lemma implies $\operatorname{Ker}(f)=0$.

Moreover, if $f\otimes 1$ is an isomorphism, then $P\otimes_A K=P/\mathfrak{m}_A P=0$ and, using Nakayama's lemma, we conclude that $P=0$. This implies the last statement of Proposition B.5.3. $\square$

B.6 The Koszul Complex

Let A be a ring and $\boldsymbol{x}=(x_1,\ldots,x_n)$ a sequence of elements in A. We define the *Koszul complex* $K(\boldsymbol{x})_\bullet$ as follows:

$$\begin{aligned} K(\boldsymbol{x})_0 &:= A\,,\quad \text{and}\\ K(\boldsymbol{x})_p &:= \bigoplus_{1\leq i_1<\cdots<i_p\leq n} Ae_{i_1\ldots i_p}\ \text{ for } p\geq 1 \end{aligned}$$

as the free module of rank $\binom{n}{p}$ with basis $\{e_{i_1\ldots i_p}\mid 1\leq i_1<\cdots<i_p\leq n\}$. The differential $d\colon K(\boldsymbol{x})_p\to K(\boldsymbol{x})_{p-1}$ is defined by

$$d_p(e_{i_1\ldots i_p}) := \sum_{j=1}^{p}(-1)^j x_j e_{i_1\ldots\widehat{i_j}\ldots i_p}\,,$$

where $\widehat{\ }$ denotes deletion of the corresponding index.

A direct calculation shows $d\circ d=0$, hence we get a complex.

For any A-module M, $K(\boldsymbol{x},M)_\bullet := K(\boldsymbol{x})_\bullet\otimes_A M$ is called the *Koszul complex of M and $\boldsymbol{x}$*. This is a complex

$$0 \longrightarrow K(\boldsymbol{x}, M)_n \longrightarrow \ldots \xrightarrow{d_2} K(\boldsymbol{x}, M)_1 \xrightarrow{d_1} K(\boldsymbol{x}, M)_0 \longrightarrow 0\,,$$

with $K(\boldsymbol{x}, M)_n \cong M$, $K(\boldsymbol{x}, M)_1 \cong M^n$, $K(\boldsymbol{x}, M)_0 \cong M$, and the differential $d_1 \colon K(\boldsymbol{x}, M)_1 \to K(\boldsymbol{x}, M)_0$ is given by

$$d_1 \colon (a_1, \ldots, a_n) \longmapsto \sum_{i=1}^{n} x_i a_i$$

via these isomorphisms. The homology of this complex is denoted by

$$H_p(\boldsymbol{x}, M) := H_p(K(\boldsymbol{x}, M)_\bullet)$$

and called the *Koszul homology of M and the sequence $\boldsymbol{x}$*. We have

$$\begin{aligned} H_0(\boldsymbol{x}, M) &\cong M/\boldsymbol{x}M\,,\ \boldsymbol{x}M = \langle x_1, \ldots, x_n \rangle M\,, \\ H_n(\boldsymbol{x}, M) &\cong \{a \in M \mid x_1 a = \ldots = x_n a = 0\}\,, \\ H_i(\boldsymbol{x}, M) &= 0 \ \text{ if } i < 0 \text{ or } i > n\,. \end{aligned} \tag{B.6.5}$$

More generally, we define for any complex $(C_\bullet, d_\bullet)$, $d_p : C_p \to C_{p-1}$ of A-modules the tensor product complex

$$C(\boldsymbol{x})_\bullet := C_\bullet \otimes_A K(\boldsymbol{x})_\bullet\,,$$

with $C(\boldsymbol{x})_n = \bigoplus_{p+q=n} C_p \otimes_A K(\boldsymbol{x})_q$ and differential $d_n : C(\boldsymbol{x})_n \to C(\boldsymbol{x})_{n-1}$, mapping an element $a \otimes b \in C_p \otimes_A K(\boldsymbol{x})_q$ to $d_p(a) \otimes b + (-1)^p a \otimes d_q(b)$.

Note that the Koszul complex of one element x_i is just

$$K(x_i)_\bullet \colon \quad 0 \longrightarrow A \xrightarrow{\cdot x_i} A \longrightarrow 0\,.$$

We can easily check by induction that

$$\begin{aligned} K(\boldsymbol{x}, M)_\bullet &\cong K(\boldsymbol{x}', M)_\bullet \otimes_A K(x_i, M)_\bullet\,, \quad \boldsymbol{x}' = (x_1, \ldots, \widehat{x_i}, \ldots, x_n)\,, \\ &\cong K(x_1, M)_\bullet \otimes_A \cdots \otimes_A K(x_n, M)_\bullet\,. \end{aligned}$$

Since the tensor product is commutative, the formation of $K(\boldsymbol{x}, M)_\bullet$ is, up to isomorphism, invariant under permutation of $x_1, \ldots, x_n$.
Now let $C_\bullet$ be any complex of A-modules. For $x \in A$, we have

$$C(x)_p = (C_\bullet \otimes K(x)_\bullet)_p \cong C_p \oplus C_{p-1}$$

and the differential $d \colon C(x)_p \to C(x)_{p-1}$ satisfies

$$d(a, b) = \big(d_p a + (-1)^{p-1} x b, d_{p-1} b\big)\,.$$

Let $C_\bullet[-1]$ be the shifted complex, that is, $C_p[-1] = C_{p-1}$ together with the shifted differential, then we obtain an exact sequence of complexes

$$0 \to C_\bullet \to C(x)_\bullet \to C_\bullet[-1] \to 0\,.$$

$C(x)_\bullet$ is the *mapping cone* of the map $C_\bullet \to C_\bullet[-1]$ of complexes given by multiplication with x. We get an exact homology sequence

$$\ldots \to H_p(C_\bullet) \to H_p(C(x)_\bullet) \to H_{p-1}(C_\bullet) \xrightarrow{\cdot(-1)^{p-1}x} H_{p-1}(C_\bullet) \to \ldots\,, \tag{B.6.6}$$

and it is not difficult to check that, for all p,

$$x \cdot H_p(C(x)_\bullet) = 0\,. \tag{B.6.7}$$

Lemma B.6.1. *Let A be a ring, $\boldsymbol{x} = (x_1, \ldots, x_n) \in A^n$.*

(1) For any A-module M and all $p \in \mathbb{Z}$ we have $\langle \boldsymbol{x} \rangle H_p(\boldsymbol{x}, M) = 0$.
(2) Let $\boldsymbol{x}' = (x_1, \ldots, \widehat{x_i}, \ldots, x_n)$, then there exists a long exact sequence

$$\begin{gathered}\ldots \to H_p(\boldsymbol{x}', M) \to H_p(\boldsymbol{x}, M) \to H_{p-1}(\boldsymbol{x}', M) \xrightarrow{\cdot(-1)^{p-1}x_i} H_{p-1}(\boldsymbol{x}', M) \to \ldots \\ \ldots \to H_1(\boldsymbol{x}, M) \to M/\langle \boldsymbol{x}' \rangle M \overset{\cdot x_i}{\to} M/\langle \boldsymbol{x}' \rangle M \to M/\langle \boldsymbol{x} \rangle M \to 0\,.\end{gathered}$$

(3) For any short exact sequence $0 \to M' \to M \to M'' \to 0$ of A-modules there exists a long exact sequence

$$\begin{gathered}0 \to H_n(\boldsymbol{x}, M') \to \ldots \to H_p(\boldsymbol{x}, M) \to H_p(\boldsymbol{x}, M'') \to H_{p-1}(\boldsymbol{x}, M') \to \\ \ldots \to H_1(\boldsymbol{x}, M'') \to M'/\langle \boldsymbol{x} \rangle M' \to M/\langle \boldsymbol{x} \rangle M \to M''/\langle \boldsymbol{x} \rangle M'' \to 0\,.\end{gathered}$$

Proof. (1) follows from (B.6.7) and (2) from (B.6.6). Since $K(\boldsymbol{x})_p = K(\boldsymbol{x}, A)_p$ is free, $0 \to M' \otimes_A K(\boldsymbol{x})_\bullet \to M \otimes_A K(\boldsymbol{x})_\bullet \to M'' \otimes_A K(\boldsymbol{x})_\bullet \to 0$ is an exact sequence of complexes for which (3) is the corresponding homology sequence. □

We relate now the homology of the Koszul complex to regular sequences.

Definition B.6.2. Let A be a ring and M an A-module. An ordered sequence of elements $x_1, \ldots, x_n \in A$ is called an *M-regular sequence* iff

(1) $\langle x_1, \ldots, x_n \rangle M \neq M$,
(2) for $i = 1, \ldots, n$, x_i is a non-zerodivisor of $M/\langle x_1, \ldots, x_{i-1} \rangle M$.

If only (2) holds, we say that $x_1, \ldots, x_n$ is a *non-zerodivisor sequence* of M, that is, $x_i \neq 0$ and the multiplication with x_i is an injective map on $M/\langle x_1, \ldots, x_{i-1} \rangle$.

Theorem B.6.3. *Let A be a ring and M an A-module.*

(1) $x \in A \setminus \{0\}$ is a non-zerodivisor of M iff $H_1(x, M) \neq 0$.
(2) If $x_1, \ldots, x_n \in A$ is an M-regular sequence, then $H_p(\boldsymbol{x}, M) = 0$ for $p \geq 1$.
(3) Let $\varphi\colon A \to B$ be a morphism of local Noetherian rings, $x_1, \ldots, x_n \in \mathfrak{m}_A$, and let $M \neq 0$ be a finitely generated B-module. If $H_1(\boldsymbol{x}, M) = 0$, then $x_1, \ldots, x_n$ is an M-regular sequence. In particular, $H_1(\boldsymbol{x}, M) = 0$ implies $H_p(\boldsymbol{x}, M) = 0$ for all $p \geq 1$ and, moreover, being an M-sequence is independent of the order of the x_i.

Proof. (1) Since $H_1(x, M) = \{a \in M \mid xa = 0\}$ by (B.6.5), this is clear.

(2) By induction on n; the case $n = 1$ follows from (1). For $n > 1$ we set $\boldsymbol{x}' = (x_1, \dots, x_{n-1})$, and we can read off the result from the exact sequence in Lemma B.6.1 (2).

(3) Let $y_i = \varphi(x_i) \in \mathfrak{m}_B$, then $H_p(\boldsymbol{x}, M) = H_p(\boldsymbol{y}, M)$. Again we use induction on n and look at the exact sequence in Lemma B.6.1 (2),

$$\dots \to H_1(\boldsymbol{y}', M) \xrightarrow{-y_n} H_1(\boldsymbol{y}', M) \to H_1(\boldsymbol{y}, M) = 0\,,$$

$\boldsymbol{y}' = (y_1, \dots, y_{n-1})$. Since M is finite over B, $H_1(\boldsymbol{y}', M)$ is finite over B and, hence, $H_1(\boldsymbol{y}', M) = 0$ by Nakayama's lemma. By induction we obtain that $y_1, \dots, y_{n-1}$ is M-regular. Using the last piece of the exact sequence in Lemma B.6.1 (2) again, we get that y_n is $(M/\langle \boldsymbol{y}' \rangle M)$-regular. Hence, $x_1, \dots, x_n$ is M-regular. □

Remark B.6.3.1. For non-local rings, being an M-sequence may depend on the order: for instance, the sequence $xy, y-1, yz$ is $K[x,y,z]$-regular but $xy, yz, y-1$ is not.

In a local ring, this cannot happen by the above theorem. However, being a non-zerodivisor sequence may depend on the order even in local rings: in $K[[x,y,z]]$, $xy, y-1, yz$ is a non-zerodivisor sequence while $xy, yz, y-1$ is not. Of course, the sequence is not $K[[x,y,z]]$-regular in any order since it contains a unit.

The SINGULAR procedures `is_regs` from `sing.lib`, respectively `isReg` from `homolog.lib`, check whether the (ordered) generators of an ideal are a non-zerodivisor sequence, respectively a regular sequence.

```
LIB "homolog.lib";
ring A = 0,(x,y,z),ds;
poly f = x3+y4+z5+xyz;
ideal J = jacob(f);
J;
//-> J[1]=3x2+yz
//-> J[2]=xz+4y3
//-> J[3]=xy+5z4
module M = jacob(J);     // the module generated by the columns of
                         // Jacobian matrix of J
isReg(J,M);
//-> 0
ideal I = J[1..2];
isReg(I,M);
//-> 1
```

Hence, `J[1],J[2]` is an M-regular sequence, while `J[1],J[2],J[3]` is not M-regular.

B.7 Regular Sequences and Depth

We use the Koszul complex to analyse regular sequences.

Definition B.7.1. Let A be a Noetherian ring, $I \subset A$ an ideal and M an A-module. If $IM \neq M$ we call the maximal length of an M-regular sequence contained in I the *I-depth of M* and denote it by $\operatorname{depth}_A(I, M)$ or, simply, $\operatorname{depth}(I, M)$. If $IM = M$ we set $\operatorname{depth}(I, M) = \infty$.

If A is local with maximal ideal $\mathfrak{m}$, then $\operatorname{depth}(\mathfrak{m}, M)$ is called the *depth of M* and denoted by $\operatorname{depth}_A(M) = \operatorname{depth}(M)$.

From the definition it is not a priori clear that any maximal (that is, not extendable) M-regular sequence has the same length (but it must be finite since A is Noetherian). However, this is true, and follows from the "depth-sensitivity" of the Koszul complex. We start with the case of I-depth 0.

Lemma B.7.2. *With the notations of B.7.1, the following are equivalent.*

(1) $\operatorname{depth}(I, M) = 0$,
(2) there is a $\mathfrak{p} \in \operatorname{Ass}(M)$ *such that* $I \subset \mathfrak{p}$,
(3) $H_k(y_1, \dots, y_k, M) \neq 0$ *for every sequence* $y_1, \dots, y_k \in I$, $k \geq 1$,
(4) $H_1(y, M) \neq 0$ *for each* $y \in I$.

Proof. Since the set of zerodivisors of M is the union of the associated prime ideal $\mathfrak{p}_i \in \operatorname{Ass}(M)$, (1) is equivalent to $I \subset \bigcup_{\mathfrak{p}_i \in \operatorname{Ass}(M)} \mathfrak{p}_i$, but this is equivalent to (2) since the $\mathfrak{p}_i$ are prime ideals.

Moreover, any $\mathfrak{p} \in \operatorname{Ass}(M)$ is of the form $\operatorname{Ann}(m) = \{x \in A \mid xm = 0\}$ for some $m \in M \setminus \{0\}$. Hence, we get $\langle y_1, \dots, y_k \rangle \cdot m = 0$ for $y_1, \dots, y_k \in I$. This means that $0 \neq m \in H_k(\boldsymbol{y}, M) = \{a \in M \mid ay_1 = \ldots = ay_k = 0\}$, which shows the implication (2) $\Rightarrow$ (3).

Taking $y_1, \dots, y_k$ a set of generators of I, we see that (3) implies $I \cdot m = 0$ for some $m \in M \setminus \{0\}$, whence (2).

Finally, since $H_k(y_1, \dots, y_k, M) = H_1(y_1, M) \otimes \cdots \otimes H_1(y_k, M)$, we get (3) $\Leftrightarrow$ (4). $\square$

Theorem B.7.3. *Let A be a ring, M an A-module, $I \subset A$ an ideal such that $IM \neq M$, and let $y_1, \dots, y_k$ be any finite set of generators of I. Assume that for some $r \geq 0$, $H_r(\boldsymbol{y}, M) \neq 0$ and $H_i(\boldsymbol{y}, M) = 0$ for $i > r$.*

Then every maximal M-regular sequence in I has length $k - r$. In particular, $\operatorname{depth}(I, M) = k - r$.

Proof. Let $x_1, \dots, x_n$ be any maximal M-sequence in I. We show by induction that $n = k - r$. The case $n = 0$ follows from Lemma B.7.2.

For $n > 0$ consider the long exact sequence in Lemma B.6.1 (3) for the exact sequence
$$0 \longrightarrow M \xrightarrow{\cdot x_1} M \longrightarrow M/\langle x_1 \rangle M \longrightarrow 0\,.$$
Since, for all p, $\langle \boldsymbol{y} \rangle \cdot H_p(\boldsymbol{y}, M)$ vanishes by Lemma B.6.1 (1), and since $x_1 \in I = \langle \boldsymbol{y} \rangle$, this yields an exact sequence

$$0 \longrightarrow H_p(\boldsymbol{y}, M) \longrightarrow H_p(\boldsymbol{y}, M/x_1M) \longrightarrow H_{p-1}(\boldsymbol{y}, M) \longrightarrow 0\,.$$

The definition of r implies $H_{r+1}(\boldsymbol{y}, M/x_1M) \neq 0$ and $H_{r+i}(\boldsymbol{y}, M/x_1M) = 0$ for $i > 1$. Since $x_2, \ldots, x_n$ is a maximal (M/x_1M)-sequence, we have by induction $n - 1 = k - (r+1)$, that is, $n = k - r$. □

Corollary B.7.4. *Let $\boldsymbol{x} = (x_1, \ldots, x_s)$ be an M-regular sequence in I. Then*

(1) $\boldsymbol{x}$ can be extended to an M-regular sequence of length $\operatorname{depth}(I, M)$*,*
(2) $\operatorname{depth}(I, M) = \operatorname{depth}(I, M/\langle \boldsymbol{x} \rangle M) + s$*.*

Proof. Choose any maximal $(M/\langle \boldsymbol{x} \rangle M)$-regular sequence $y_1, \ldots, y_r \in I$, then $x_1, \ldots, x_s, y_1, \ldots, y_r$ is a maximal M-regular sequence. Now (1) and (2) follow from Theorem B.7.3. □

Lemma B.7.5. *Let A be a local Noetherian ring and $M \neq 0$ a finitely generated A-module. Then* $\operatorname{depth}(M) \leq \operatorname{depth}(A/\mathfrak{p})$ *for any* $\mathfrak{p} \in \operatorname{Ass}(M)$*. In particular,* $\operatorname{depth}(M) \leq \dim(M)$*.*

Proof. If $\operatorname{depth} M > 0$ then there exists a $y \in \mathfrak{m}_A$, which is a non-zerodivisor of M, i.e. $y \notin \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Ass}(M)$. Choose $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\mathfrak{p} = \operatorname{Ann}(a)$ for some $a \in M \setminus \{0\}$. Since $\bigcap_{i \geq 0} y^i A = 0$ by Krull's intersection theorem, there is an $n \geq 0$ with $a \in y^n M$, $a \notin y^{n+1}M$. Since $\operatorname{Ann}(y^n b) = \operatorname{Ann}(b)$ we may assume $a \notin yM$. Hence, $\underline{a} \neq 0$, where $\underline{a}$ is the class of a in M/yM. Since $(\mathfrak{p} + \langle y \rangle) \cdot \underline{a} = 0$, it follows $\mathfrak{p} + \langle y \rangle \in \operatorname{Ann}(\underline{a})$ and there is a $\mathfrak{p}' \in \operatorname{Ass}(M/yM)$ such that $\mathfrak{p} + \langle y \rangle \subset \mathfrak{p}'$. Hence, we have

$$\dim(A/\mathfrak{p}') \leq \dim(A/\mathfrak{p} + \langle y \rangle) = \dim(A/\mathfrak{p}) - 1$$

by Krull's principal ideal theorem (Theorem B.2.1).

Since $\operatorname{depth}(M/yM) = \operatorname{depth}(M) - 1$ by Corollary B.7.4, the result follows by induction on $\operatorname{depth}(M)$. □

The inequality $\operatorname{depth}(A) \leq \dim(A/\mathfrak{p})$ for any associated prime of A means geometrically that $\operatorname{depth}(A)$ is less or equal to the minimum of the dimensions of the irreducible components of $\operatorname{Spec}(A)$.

Lemma B.7.6. *Let $\varphi \colon A \to B$ be a morphism of local Noetherian rings and M a B-module, which is finitely generated over A. Then*

$$\operatorname{depth}_A(M) = \operatorname{depth}_B(M)\,.$$

Proof. Let $x_1, \ldots, x_n \in \mathfrak{m}_A$ be a maximal M-regular sequence. Then the sequence $\varphi(x_1), \ldots, \varphi(x_n) \in \mathfrak{m}_B$ is M-regular and

$$\operatorname{depth}_A(M/\langle x \rangle M) = \operatorname{depth}_A(M) - n = 0$$

and $\operatorname{depth}_B(M/\langle \boldsymbol{x} \rangle M) = \operatorname{depth}_B(M) - n$ by Corollary B.7.4.

Hence, we may assume $\operatorname{depth}_A(M) = 0$. By Lemma B.7.2 this is equivalent to $\mathfrak{m}_A \in \operatorname{Ass}_A(M)$, that is, $\mathfrak{m}_A = \operatorname{Ann}(a)$ for some $a \in M \setminus \{0\}$. Setting $N = Ba \subset M$, N is a finitely generated A-module, and since $\mathfrak{m}_A N = 0$, we know that $\dim_{A/\mathfrak{m}_A}(N) < \infty$. Since $\dim_{B/\mathfrak{m}_B}(N) \leq \dim_{A/\mathfrak{m}_A}(N)$, we obtain $\mathfrak{m}_B = \operatorname{Ass}_B(N) \subset \operatorname{Ass}_B(M)$ and, therefore, $\operatorname{depth}_B(M) = 0$, by Lemma B.7.2. $\square$

Lemma B.7.6 has many applications. For example, if M is a finitely generated A/I-module, then $\operatorname{depth}_A(M) = \operatorname{depth}_{A/I}(M)$. In particular, any analytic algebra B is of the form $K\langle \boldsymbol{x} \rangle / I$ and then $\operatorname{depth}_B M = \operatorname{depth}_{K\langle \boldsymbol{x}\rangle} M$, that is, we can compute the depth over a regular local ring. For this we can for example use the Auslander-Buchsbaum formula B.9.3.

B.8 Cohen-Macaulay, Flatness and Fibres

Recall that each A-module M satisfies $\operatorname{depth} M \leq \dim M$ (Lemma B.7.5). Those modules with $\operatorname{depth} M = \dim M$ have special geometric properties and, therefore, a special name:

Definition B.8.1. Let A be a local Noetherian ring and M a finitely generated A-module.

(1) M is called *Cohen-Macaulay* or a *CM-module* if $\operatorname{depth}(M) = \dim(M)$.
(2) M is called a *maximal Cohen-Macaulay module* or *MCM-module* if $\operatorname{depth}(M) = \dim(A)$.
(3) A is called a *Cohen-Macaulay ring* if it is a CM A-module.

Proposition B.8.2. *Let A be a local Noetherian ring and M a CM A-module. Then*

(1) M is equidimensional and without embedded primes.
(2) If $x \in \mathfrak{m}_A$ satisfies $\dim M/xM = \dim M - 1$, then x is M-regular and M/xM is Cohen-Macaulay.

Proof. By Lemma B.7.5, $\operatorname{depth}(M) \leq \min\{\dim(A/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(M)\}$. On the other hand, $\dim(M) = \max\{\dim(A/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(M)\}$, hence (1).

(2) Since $\dim(M/xM) = \dim(M) - 1$ iff $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$ with $\dim(A/\mathfrak{p}) = \dim(M)$, it follows from (1) that $x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$, hence x is M-regular. Then M/xM is CM by Corollary B.7.4. $\square$

Corollary B.8.3. *Let A be a local Noetherian ring, $x_1, \dots, x_n \in \mathfrak{m}_A$ and M a finitely generated A-module.*

(1) If $x_1, \dots, x_n$ is M-regular then M is Cohen-Macaulay iff $M/\langle x_1, \dots, x_n\rangle$ is Cohen-Macaulay.
(2) Let M be Cohen-Macaulay. Then the sequence $x_1, \dots, x_n$ is M-regular iff $\dim(M/\langle x_1, \dots, x_n\rangle) = \dim(M) - n$.

Proof. Use Proposition B.8.2 and Corollary B.7.4. □

A sequence $x_1, \dots, x_d \in \mathfrak{m}_A$ with $d = \dim(M)$ and $\dim(M/\langle x_1, \dots, x_d\rangle) = 0$ is called a *system of parameters* for M. Note that systems of parameters do always exist by Theorem B.2.1.

Corollary B.8.4. *Let A be a local Noetherian ring and M a finitely generated A-module. The following are equivalent:*

(1) M is a CM-module.
(2) There exists a system of parameters for M, which is an M-regular sequence.
(3) Every system of parameters for M is an M-regular sequence.

Moreover, if M is Cohen-Macaulay then the M-regular sequences are just the partial systems of parameters.

We shall now consider regular local rings A.

Definition B.8.5. A Noetherian local ring $(A, \mathfrak{m}, K)$ is called *regular* if $\dim A = \operatorname{edim} A$, where $\operatorname{edim} A := \dim_K \mathfrak{m}/\mathfrak{m}^2$ is the *embedding dimension* of A.

Then the maximal ideal $\mathfrak{m}_A$ is minimally generated by $\dim(A)$ elements which form a system of parameters for A.

Proposition B.8.6. *Let A be a local Noetherian ring of dimension d and $I \subset \mathfrak{m}$ an ideal. Then the following are equivalent*

(1) A is regular and I is generated by k elements, which are K-linearly independent in $\mathfrak{m}/\mathfrak{m}^2$,
(2) $B = A/I$ is regular of dimension $d - k$ and I is generated by k elements.

In particular, if $A/\langle f\rangle$ is regular of dimension $\dim(A) - 1$*, then A is regular.*

Proof. Set $\mathfrak{n} = \mathfrak{m}_B = \mathfrak{m} + I/I$. The equivalence follows easily from the exact sequence of $K = (A/\mathfrak{m})$-vector spaces

$$0 \longrightarrow \mathfrak{m}^2 + I/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2 \longrightarrow 0\,,$$

noting that $\dim_K(\mathfrak{m}^2 + I/\mathfrak{m}^2) = \dim_K(I/I \cap \mathfrak{m}^2) \leq \dim_K(I/\mathfrak{m}I) = k$ with equality iff I is generated by k regular parameters, and $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim(A)$ with equality iff A is regular, and similarly for $B, \mathfrak{n}$. □

Proposition B.8.7. *Any regular Noetherian local ring is an integral domain.*

Proof. Let A be regular with maximal ideal $\mathfrak{m}$. If $\dim(A) = 0$ then $\mathfrak{m} = 0$ and A is a field, in particular an integral domain.

Let $\dim(A) = d > 0$, then $\mathfrak{m}^2 \neq \mathfrak{m}$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of A (these are just the minimal associated primes, hence there are only finitely many of them, cf. B.1). Since height$(\mathfrak{m}) = d > 0$, $\mathfrak{p}_i \neq \mathfrak{m}$ for all i and hence there is an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, $x \notin \mathfrak{p}_i$ for $i = 1, \ldots, r$ (prime avoidance, see, e.g., [GrP, Lemma 1.3.12]).

By Proposition B.8.6, $A/\langle x \rangle$ is regular of dimension $d-1$ and, by induction on d, we may assume that $A/\langle x \rangle$ is an integral domain. Therefore, $\langle x \rangle \subset A$ is a prime ideal, which must contain one of the minimal primes $\mathfrak{p}_i$. If $a \in \mathfrak{p}_i$ is arbitrary, then $a = xb$ for some $b \in A$ and, since $x \notin \mathfrak{p}_i$, we know $b \in \mathfrak{p}_i$ ($\mathfrak{p}_i$ is prime). This implies $\mathfrak{p}_i \subset x\mathfrak{p}_i$, hence $\mathfrak{p}_i = \langle 0 \rangle$ by Nakayama's lemma. That is, $\langle 0 \rangle$ is a prime ideal, which means that A is an integral domain. □

Corollary B.8.8. *Any regular local ring* $(A, \mathfrak{m})$ *is Cohen-Macaulay. Any minimal set of generators of* $\mathfrak{m}$ *is a maximal* A*-regular sequence.*

Proof. Let $x_1, \ldots, x_d \in \mathfrak{m}$ be a basis of $\mathfrak{m}/\mathfrak{m}^2$. Then, by Proposition B.8.6, the quotient $A/\langle x_1, \ldots, x_i \rangle$ is regular for each i and, by Proposition B.8.7, it is an integral domain. Hence, x_{i+1} is $A/\langle x_1, \ldots, x_i \rangle$-regular which implies $\dim(A) = d = \operatorname{depth}(A)$. □

Any minimal set of generators of $\mathfrak{m}$ in a regular local ring A is an A-regular sequence and a system of parameters. We call it a *regular system of parameters*. The elements of a minimal set of generators of $\mathfrak{m}$ are called *regular parameters.*

Definition B.8.9. Let A be a regular local ring and $I \subset A$ an ideal. Then A/I is called a *complete intersection ring* if I is generated by $\dim(A) - \dim(A/I)$ elements.

Consequently, any minimal set of generators $x_1, \ldots, x_k$ of I is an A-regular sequence. Hence, $\operatorname{depth}(A/I) = \dim(A) - k = \dim(A/I)$, that is,

Corollary B.8.10. *Any complete intersection is Cohen-Macaulay.*

In particular, a *hypersurface ring* $A/\langle f \rangle$ (i.e., A regular and $f \in \mathfrak{m} \setminus \{0\}$) is Cohen-Macaulay.

Remark B.8.10.1. Let A be a local Noetherian ring. Then the following hold:

(1) If $\dim(A) = 1$ and A is reduced then A is Cohen-Macaulay (that is, reduced curve singularities are Cohen-Macaulay),
(2) If $\dim(A) = 2$ and A is normal, then A is Cohen-Macaulay (normal surface singularities are Cohen-Macaulay).

The proof of these statements is left as an exercise. For the proof of (2), Serre's conditions for normality (see [Mat2, Thm. 23.8]) are helpful.

The following theorem is also used when proving the continuity of the Milnor number (see Theorem I.2.6, p. 114). It is indispensable for the study of complete intersection singularities.

Theorem B.8.11. *Let $\varphi\colon A \to B$ be a morphism of local rings with A regular and M a finitely generated B-module. Let $x_1, \dots, x_d$ be a minimal set of generators of $\mathfrak{m}_A$ and $f_i = \varphi(x_i)$, $i = 1, \dots, d$. Then the following are equivalent.*

(1) M is A-flat,
(2) $\operatorname{depth}_A(M) = d$ or, equivalently, $\operatorname{depth}_B(\mathfrak{m}_A B, M) = d$,
(3) $f_1, \dots, f_d$ is an M-regular sequence.

In particular, if B is Cohen-Macaulay, then φ is flat if and only if

$$\dim B = \dim A + \dim B/\mathfrak{m}_A B\,.$$

Corollary B.8.12. *Let M be a finitely generated module over a regular local ring A. Then M is Cohen-Macaulay iff M is free.*

Proof. Since flat and free are the same for finitely generated modules over local rings, the statement follows from the equivalence of (1) and (2) in Theorem B.8.11. □

Before proving the theorem let us establish a connection between the Koszul homology and Tor.

Let A be a ring, M an A-module and $x_1, \dots, x_n \in A$ an A-regular sequence. Then the Koszul complex of $\boldsymbol{x} = x_1, \dots, x_n$ is exact in degree > 0, and hence $0 \to K_n(\boldsymbol{x}) \to \ldots \to K_2(\boldsymbol{x}) \to K_1(\boldsymbol{x}) \to A/\langle x_1, \dots, x_n\rangle \to 0$ is a free resolution of $A/\langle \boldsymbol{x}\rangle$. Since $K(\boldsymbol{x}, M)_\bullet = M \otimes_A K(\boldsymbol{x})_\bullet$ we get from the definition of Tor that $\operatorname{Tor}_i(M, A/\langle \boldsymbol{x}\rangle) = H_i(\boldsymbol{x}, M)$ for $i \geq 0$.

In particular, if A is regular and if $x_1, \dots, x_d$ is a regular system of parameters, then $H_i(\boldsymbol{x}, M) = \operatorname{Tor}_i(M, K)$, with $K = A/\mathfrak{m}_A$, for any A-module M.

Proof of Theorem B.8.11. By the local criterion for flatness (Theorem B.5.1), M is A-flat iff $\operatorname{Tor}_1(M, K) = 0$, or, equivalently, $H_1(\boldsymbol{x}, M) = 0$. By Theorem B.7.3 this is equivalent to (2), noting that $\operatorname{depth}_A(M) = \operatorname{depth}_B(\mathfrak{m}_A B, M)$. By Theorem B.6.3, $H_1(\boldsymbol{x}, M) = 0$ iff $x_1, \dots, x_d$ is an M-regular sequence. Since x_i acts on M via φ as f_i, the equivalence of (2) and (3) follows. The last statement is a consequence of Corollary B.8.3. □

We collect now several useful facts about flat morphisms. For the proofs, we mainly refer to [Mat2].

Theorem B.8.13. *Let $\varphi : A \to B$ be a local morphism of Noetherian rings, let $\mathfrak{p} \subset B$ be a prime ideal of B, and let $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$. Then*

$$\dim(B_\mathfrak{p}) \leq \dim(A_\mathfrak{q}) + \dim(B_\mathfrak{p}/\mathfrak{q}B_\mathfrak{p})$$

with equality if φ is flat.

Proof. See [Mat2, Theorem 15.1]. □

If A and B are local rings, and if φ is a local morphism, then we get $\dim(B) \leq \dim(A) + \dim(B/\mathfrak{m}_A B)$, which geometrically says that the dimension of the total space is at most the dimension of the base space plus the dimension of the fibre. The equality $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$ is sometimes called the *"additivity of dimension"*. Due to Theorem B.8.13, it holds for flat maps. Conversely, if A is regular and B is Cohen-Macaulay, additivity of dimension implies flatness (Theorem B.8.11).

Theorem B.8.14. *Let $\varphi : A \to B$ be a local morphism of local Noetherian rings, M a finitely generated A-module, and N a finitely generated B-module. If N is flat over A then*

$$\operatorname{depth}_B(M \otimes_A N) = \operatorname{depth}_A(M) + \operatorname{depth}(N/\mathfrak{m}_A N)\,.$$

Proof. See [Mat2, Theorem 23.3]. □

Theorem B.8.15. *If $\varphi : A \to B$ is a flat local morphism of local Noetherian rings, then the following hold:*

(1) $\operatorname{depth}(B) = \operatorname{depth}(A) + \operatorname{depth}(B/\mathfrak{m}_A B)\,.$
(2) B is Cohen-Macaulay iff A and $B/\mathfrak{m}_A B$ are both Cohen-Macaulay.
(3) B is Gorenstein[2] iff A and $B/\mathfrak{m}_A B$ are both Gorenstein.
(4) B is a complete intersection iff A and $B/\mathfrak{m}_A B$ are both complete intersections.

Proof. (1) is a corollary of Theorem B.8.14, setting $M = A$, $N = B$. Statement (2) follows from (1), using Theorem B.8.13. For the proof of (3) and (4), we refer to [Mat2, Theorem 23.4, Remark p. 182]. □

Theorem B.8.16. *Let $A = K\langle \boldsymbol{x}\rangle/I$ be an analytic K-algebra, and let B be a free power series algebra over A. Then*

(1) B is A-flat.
(2) If A is Cohen-Macaulay (respectively Gorenstein, respectively a complete intersection), then so is B.

Proof. (1) is proved for $K = \mathbb{C}$ and convergent power series in Section I.1.8, Corollary I.1.88. The general case can be proved along the same lines. For (2), we refer to [Mat2, Theorem 23.5, Remark p. 182]. □

Theorem B.8.17. *If $\varphi : A \to B$ is a flat local morphism of local Noetherian rings, then the following hold:*

(1) If B is regular, then A is regular, too.
(2) If A and $B/\mathfrak{m}_A B$ are regular, then so is B.

[2] A local Noetherian ring A is called *Gorenstein* if A is Cohen-Macaulay and if $\operatorname{Ext}^d_A(A/\mathfrak{m}_A, A) \cong A$ where $d = \dim(A)$.

Proof. To prove (1), we apply Serre's criterion (Theorem B.9.2). Let $F_\bullet$ be a minimal A-free resolution of the residue field K of A. Since A is B-flat, $F_\bullet \otimes_A B$ is a minimal B-free resolution of $B/\mathfrak{m}_A B$. Hence, $F_i \otimes_A B = 0$ for $i > \dim(B)$ by Theorem B.9.1. Since B is faithfully flat over A, this implies $F_i = 0$ for $i > \dim(B)$ (Proposition B.3.3 (9),(10)), and the result follows.

Since A is regular, the maximal ideal $\mathfrak{m}_A$ is generated by $\dim(A)$ elements, hence $\mathfrak{m}_A B$ is generated by $\dim(A)$ elements, too. By Theorem B.8.13, $\dim(B/\mathfrak{m}_A B) = \dim(B) - \dim(A)$ and the result follows from Proposition B.8.6. □

Lemma B.8.18. *Let $\varphi : A \to B$ be a flat local morphism of local rings. Then the $a \in A$ is a non-zerodivisor of A iff $\varphi(a)$ is a non-zero-divisor of B.*

Proof. This holds, since B is faithfully flat over A (Proposition B.3.3 (9)), and since the sequence $0 \to B \xrightarrow{\cdot\varphi(a)} B$ is obtained from $0 \to A \xrightarrow{\cdot a} A$ by applying $\otimes_A B$. □

Theorem B.8.19. *For a flat local morphism $\varphi : A \to B$ of local Noetherian ring, the following holds*

(1) If B is reduced (resp. normal) then so is A.
(2) If A and $B \otimes_A \kappa(\mathfrak{p})$ are reduced (resp. normal) for every prime ideal $\mathfrak{p}$ of A, then so is B.

Here, $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \mathrm{Quot}(A/\mathfrak{p})$ denotes the residue field of the "generic point" of the component $\mathrm{Spec}(A/\mathfrak{p})$ of $\mathrm{Spec}(A)$.

Note that A is reduced if B is. This follows since φ is injective by Proposition B.3.3 (9), (10). For the remaining statements see [Mat2, Corollary to Theorem 2.3.9].

Unfortunately, it is in general not sufficient to require in (2) that A and $B \otimes_A \kappa(\mathfrak{m}_A)$ are reduced (resp. normal). However, under some extra conditions it is sufficient. To formulate these conditions, we need some notations.

Recall that a ring A is *normal* iff $A_\mathfrak{p}$ is an integrally closed domain for all $\mathfrak{p} \in \mathrm{Spec}(A)$. A Noetherian k-algebra is *geometrically reduced* (resp. *geometrically normal*) if $A \otimes_k k'$ is a reduced (resp. normal) ring for all field extensions $k \subset k'$. It actually suffices to consider only purely inseparable field extensions. If $\mathrm{char}(k) = 0$ or, more generally, if k is perfect, then reduced (resp. normal) is equivalent to geometrically reduced (resp. geometrically normal).

A morphism $\varphi : A \to B$ of arbitrary rings is called *reduced* (resp. *normal*) if φ is flat and if all fibres $B \otimes_A \kappa(\mathfrak{p})$ are reduced (resp. normal), where $\mathfrak{p}$ runs through all $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathfrak{p}B \neq B$.

Theorem B.8.20. *Let $\varphi : A \to B$ be a local flat morphism of local Noetherian rings, which satisfies*

(i) the completion map $A \to \widehat{A}$ is reduced (resp. normal).
(ii) $B/\mathfrak{m}_A B$ is geometrically reduced (resp. geometrically normal).

Then B and φ are reduced (resp. normal).

Proof. [Nis, p. 157, (2.4)] yields that φ is reduced (resp. normal). That B is reduced (resp. normal) follows then from Theorem B.8.19. □

Note that analytic local K-algebras are "excellent" (see [ChL, Remark 3.2.1 and Footnote]), hence they satisfy condition (i) of Theorem B.8.20.

B.9 Auslander-Buchsbaum Formula

The relation between Tor and the Koszul homology has the following immediate consequence for regular local rings.

A free resolution of an A-module M over a local ring $(A, \mathfrak{m}, K)$

$$F_\bullet : \qquad \ldots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0$$

is called a *minimal free resolution* of M if $d_i(F_i) \subset \mathfrak{m}F_{i-1}$ for all $i \geq 1$, or equivalently, if $d_i \otimes_A A/\mathfrak{m} = 0$.

For instance, if $K(\boldsymbol{x}, M)_\bullet$ is the Koszul complex with $\boldsymbol{x} = (x_1, \ldots, x_n)$, $x_i \in \mathfrak{m}_A$, then the differentials satisfy $d_i(K(\boldsymbol{x}, M)_i) \subset \mathfrak{m}K(\boldsymbol{x}, M)_{i-1}$.

Using Nakayama's lemma, it is easy to see that a free resolution is minimal iff, for each i, a basis of F_i is mapped onto a minimal system of generators of $\mathrm{Ker}(d_{i-1})$. Hence, a minimal free resolution of a finitely generated A-module M has the minimal possible length of all free resolutions of M. The length of a minimal free resolution of M is called the *projective dimension* of M and denoted by $\mathrm{pd}_A(M)$. Note that $\mathrm{pd}_A(M) = 0$ iff M is free.

If we tensor a minimal resolution $F_\bullet$ of M with K over A, then all differentials become 0, hence we get by taking homology,

$$\mathrm{Tor}_i^A(M, K) = H_i(F_\bullet \otimes_A K) = F_i \otimes_A K\,.$$

It follows that $\mathrm{Tor}_p^A(M, K) \neq 0$ for $p = \mathrm{pd}_A(M)$, and $\mathrm{Tor}_i^A(M, K) = 0$ for $i > \mathrm{pd}_A(M)$.

If A is regular and $x_1, \ldots, x_d$ is a regular system of parameters, then $K(\boldsymbol{x})_\bullet$ is a minimal free resolution of $K = A/\mathfrak{m}$. Since $\mathrm{Tor}_i^A(M, K)$ can be also computed by taking a free resolution of K we get

$$\mathrm{Tor}_i^A(M, K) = H_i(M \otimes_A K(\boldsymbol{x})_\bullet)\,.$$

Hence, we have shown

Theorem B.9.1. *Let A be a regular local ring with residue field K and M a finitely generated A-module. Then*

(1) $\mathrm{pd}_A(M) \leq \dim(A)$,
(2) $\mathrm{pd}_A(K) = \dim(A)$.

Indeed, as shown by Auslander (cf. [Eis]), the assumption that M is finitely generated is not necessary in (1).

The converse of Theorem B.9.1 is also true. To formulate it, we define the *global dimension* of a ring A to be

$$\operatorname{gldim}(A) := \sup\{\operatorname{pd}_A(M) \mid M \text{ is an } A\text{-module}\}\,.$$

Theorem B.9.2 (Serre). *Let A be a Noetherian local ring with residue field K. Then the following are equivalent:*

(a) *A is regular.*
(b) $\operatorname{gldim}(A) < \infty$.
(c) $\operatorname{gldim}(A) = \dim(A)$.
(d) $\operatorname{pd}_A(K) < \infty$.

Proof. See [Mat2, Theorem 19.2 and Lemma 19.1]. □

For arbitrary local rings, the depth sensitivity of the Koszul complex provides an elegant proof of the Auslander-Buchsbaum formula.

Theorem B.9.3 (Auslander-Buchsbaum formula). *Let A be a local ring and M a finitely generated A-module with* $\operatorname{pd}_A(M) < \infty$. *Then*

$$\operatorname{pd}_A(M) + \operatorname{depth}(M) = \operatorname{depth}(A)\,.$$

Proof. We use induction on $\operatorname{pd}_A(M)$. If $\operatorname{pd}_A(M) = 0$ the formula is true since the depth of a free module is the same as the depth of the ring.

Let $\operatorname{pd}_A(M) > 0$ and choose a surjection $F \twoheadrightarrow M$ with F a free module of rank equal to the minimal number of generators of M. Let M' be the kernel and consider the exact sequence $0 \to M' \to F \to M \to 0$. Then $\operatorname{pd}_A(M') = \operatorname{pd}_A(M) - 1$ and, by induction, $\operatorname{pd}_A(M') + \operatorname{depth}(M') = \operatorname{depth}(A)$.

Hence, we have to show $\operatorname{depth}(M') = \operatorname{depth}(M) + 1$. Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a set of generators of $\mathfrak{m}_A$. Then, by Lemma B.6.1, we have an exact sequence

$$\dots \to H_p(\boldsymbol{x}, M') \to H_p(\boldsymbol{x}, F) \to H_p(\boldsymbol{x}, M) \to H_{p-1}(\boldsymbol{x}, M') \to \dots\,.$$

By Theorem B.7.3, $H_i(\boldsymbol{x}, M') = 0$ if $i > p - 1 := n - \operatorname{depth}(M')$, and it does not vanish for $i = p - 1$. Since $\operatorname{pd}_A(M') \geq 0$, we have $\operatorname{depth}(M') \leq \operatorname{depth}(A)$.

If $\operatorname{depth}(M') < \operatorname{depth}(A) = \operatorname{depth}(F)$ then we obtain $H_i(\boldsymbol{x}, F) = 0$ for $i > p - 1$, hence $H_p(\boldsymbol{x}, M) \neq 0$, $H_i(\boldsymbol{x}, M) = 0$ for $i > p$, and, therefore, $\operatorname{depth}(M) = n - p$ as desired.

If $\operatorname{depth}(M') = \operatorname{depth}(A)$ then $\operatorname{pd}_A(M') = 0$, M' is free, and we have to show that the map

$$M' \otimes_A H_{p-1}(\boldsymbol{x}, A) \cong H_{p-1}(\boldsymbol{x}, M) \to H_{p-1}(\boldsymbol{x}, F) \cong F \otimes_A H_{p-1}(\boldsymbol{x}, A)$$

is not injective. Since $F \to M$ is a minimal presentation, $M' \subset \mathfrak{m}_A F$. By Lemma B.6.1, $\mathfrak{m}_A H_\bullet(\boldsymbol{x}, A) = 0$. Hence, the above map is the zero map, in particular, not injective. □

Using Theorem B.9.1 and the fact that regular rings are Cohen-Macaulay (Corollary B.8.8), we get

Corollary B.9.4. *Let A be a regular local ring and M a finitely generated A-module. Then*

$$\mathrm{pd}_A(M) + \mathrm{depth}(M) = \dim(A)\,.$$

C

Formal Deformation Theory

In this appendix, we collect basic facts of formal deformation theory, obstruction theory and cotangent cohomology. The main references for Schlessinger's theory of deformations over Artin rings is Schlessinger's original article [Sch]; for obstruction theory we refer mainly to [FaM] and to the articles by Artin [Art2, Art3]. As reference for the cotangent cohomology, we recommend the articles by Palamodov [Pal, Pal1, Pal2], Flenner [Fle1, Fle] and Buchweitz [Buc]. A detailed account of Schlessinger's theory with applications to deformations of algebraic varieties can be found in the book by Sernesi [Ser]. For slightly different aspects of deformation theory, we refer to [Ste, Ste1] and [Lau].

C.1 Functors of Artin Rings

Let K be a fixed field. Throughout this appendix, we denote by $\mathcal{A}$ the category of Noetherian local K-algebras with residue field K. That is, for $A \in \mathcal{A}$ with maximal ideal $\mathfrak{m}_A = \mathfrak{m}$, the composition of the natural morphisms $K \to A \to A/\mathfrak{m}$ is the identity. Morphisms in $\mathcal{A}$ are local morphisms of K-algebras.

Moreover, $\widehat{\mathcal{A}}$, resp. $\mathcal{A}r$, denotes the full subcategory of complete, resp. Artinian, local K-algebras.

Definition C.1.1. A *functor of Artin rings* is a covariant functor

$$F : \mathcal{A}r \longrightarrow \mathcal{S}ets\,.$$

For $A \in \mathcal{A}r$, the natural morphism to the residue field $A \to K$ induces a map $F(A) \to F(K)$. An element $\xi \in F(A)$ is called an *infinitesimal deformation* of its image $\xi_0 \in F(K)$; an infinitesimal deformation ξ is called a *deformation of first order* if $A = K[\varepsilon]$ where $K[\varepsilon]$ is the two-dimensional Artinian K-algebra with $\varepsilon^2 = 0$. The K-algebra A is called the *base* (or *base ring*) of $\xi \in F(A)$; we also say that ξ is a *deformation over A*. We refer to

$$t_F := F(K[\varepsilon])$$

as the *tangent space of the functor* F (it has a natural structure of a K-vector space if F satisfies further properties, see Lemma C.1.7 below).

Every functor F of Artin rings can be extended to a functor

$$\widehat{F} : \widehat{\mathcal{A}} \longrightarrow \mathcal{S}ets$$

in the following way: for $R \in \widehat{\mathcal{A}}$, let $R_n := R/\mathfrak{m}_R^{n+1}$, where $\mathfrak{m}_R$ is the maximal ideal of R and $\pi_n : R_n \to R_{n-1}$ the residue class map, $n \geq 0$. Then R_n is an object of $\mathcal{A}r$ and $\big(F(R_n), F(\pi_n)\big)$ is a projective system. We define the functor $\widehat{F}$ by taking the projective limit,

$$\widehat{F}(R) := \varprojlim_n F(R_n)\,.$$

Giving a morphism $\varphi : R \to S$ in $\widehat{\mathcal{A}}$ is equivalent to giving a compatible sequence of morphisms $\varphi_n : R_n \to S_n = S/\mathfrak{m}_s^{n+1}$ since φ is local. Then $\widehat{F}(\varphi)$ is induced by the maps $F(\varphi_n)$,

$$\widehat{F}(\varphi) = \varprojlim_n F(\varphi_n) : \widehat{F}(R) \longrightarrow \widehat{F}(S)\,.$$

An element $\widehat{\xi} \in \widehat{F}(R)$ is thus given by a sequence $(\xi_n)_{n\geq 0}$, $\xi_n \in F(R_n)$, such that $F(\pi_n)(\xi_n) = \xi_{n-1}$ for all $n \geq 1$. We call $\widehat{\xi}$ a *formal element* of F.

Schlessinger's theory applies to functors of Artin rings F such that $F(K)$ contains only one element ξ_0. Such functors arise naturally as follows.

Let $R \in \widehat{\mathcal{A}}$ be a fixed object and define the functor $h_R : \mathcal{A}r \to \mathcal{S}ets$ by setting, for $A \in \mathcal{A}r$,

$$h_R(A) := \operatorname{Mor}_{\widehat{\mathcal{A}}}(R, A)\,.$$

For $\varphi \in \operatorname{Mor}_{\widehat{\mathcal{A}}}(A, B)$, the map

$$h_R(\varphi) : h_R(A) \longrightarrow h_R(B)$$

is defined by mapping a morphism $R \to A$ to $R \to A \xrightarrow{\varphi} B$.

Of course, h_R extends to a functor $\widetilde{h}_R : \widehat{\mathcal{A}} \to \mathcal{S}ets$ by the same formulas.

Definition C.1.2. A functor of Artin rings $F : \mathcal{A}r \to \mathcal{S}ets$ is called *prorepresentable* if F is isomorphic to h_R for some $R \in \widehat{\mathcal{A}}$. If F extends to a functor $\widetilde{F} : \widehat{\mathcal{A}} \to \mathcal{S}ets$ such that $\widetilde{F}$ is isomorphic to $\widetilde{h}_R$ for some $R \in \widehat{\mathcal{A}}$, then F (or $\widetilde{F}$) is called *representable.*

If F is representable and isomorphic to $\widetilde{h}_R$ then there exists an element $\xi_R \in F(R)$ corresponding to $\mathrm{id}_R \in \operatorname{Mor}_{\widehat{\mathcal{A}}}(R, R)$ satisfying the following property: for each $\xi \in F(A)$, there exists a unique morphism $\varphi : R \to A$ such that $\xi = F(\varphi)(\xi_R)$. We then also say that ξ_R is a *universal deformation* of $\xi_0 = F(K)$ with base R. It has the property that every deformation $\xi \in F(A)$ of ξ_0 with base $A \in \widehat{\mathcal{A}}$ is induced from ξ_R by a unique morphism $R \to A$.

Remark C.1.2.1. Let F be a functor of Artin rings, $R \in \widehat{\mathcal{A}}$, and assume that there is a morphism of functors $h_R \to F$. Via this morphism id_R induces an element $\widehat{\xi}_R = (\xi_{R,n})_{n\geq 0} \in \widehat{F}(R)$ by setting

$$\xi_{R,n} = \text{image of } \varepsilon_n \in h_R(R_n) \text{ in } F(R_n)$$

where $\varepsilon_n : R \to R_n$ is the canonical projection.

Conversely, an element $\widehat{\xi} = (\xi_n)_{n\geq 0} \in \widehat{F}(R)$ induces a morphism of functors $h_R \to F$ in the following way: if A is Artinian, then $A = A_n$ for sufficiently large n and, hence, a morphism $\varphi : R \to A$ is equivalent to a sequence of compatible morphisms $\varphi_n : R_n \to A$ for sufficiently large n. The desired morphism $h_R \to F$ is given by

$$h_R(A) \ni \varphi \longmapsto F(\varphi_n)(\xi_n) \in F(A) \tag{C.1.1}$$

for sufficiently large n and $A \in \mathcal{A}r$. Under this morphism, we have $\widehat{\xi} = \widehat{\xi}_R$.

Using the above notations, the following lemma follows now immediately.

Lemma C.1.3. *A functor of Artin rings F is prorepresentable iff there exists a complete local K-algebra $R \in \widehat{\mathcal{A}}$ and a formal element $\widehat{\xi}_R = (\xi_{R,n}) \in \widehat{F}(R)$ such that, for each $\xi \in F(A)$, $A \in \mathcal{A}r$, there exists a unique morphism $\varphi = (\varphi_n) : R \to A$ satisfying $\xi = F(\varphi_n)(\xi_{R,n})$ for sufficiently large n.*

With the above notations, $(R, \widehat{\xi}_R)$ is called a *formal universal couple* and $\widehat{\xi}_R$ is called a *formal universal element* for F. Using the terminology of deformations, we also say that $\widehat{\xi}_R$ is a *formal universal deformation* of $\xi_0 \in F(K)$ with base R. It has the property that every infinitesimal deformation $\xi \in F(A)$, $A \in \mathcal{A}r$, of ξ_0 is induced from $\widehat{\xi}_R$ by a unique morphism $\varphi : R \to A$.

Note that $\widehat{\xi}_R$ is just a projective system $(\xi_n \in F(R_n))_{n\geq 0}$ but in general not an element of $F(R)$ (even in the case that F happens to extend to $\widehat{\mathcal{A}}$).

Representable, resp. prorepresentable functors are important. However, many functors of interest (for example, deformation functors of isolated singularities) are not representable, resp. prorepresentable. We introduce therefore some weaker notions.

Let $F \to G$ be a morphism of functors of Artin rings. Then, for any morphism $B \to A$ in $\mathcal{A}r$, we have a commutative diagram

$$\begin{array}{ccc} F(B) & \xrightarrow{\widetilde{\beta}} & F(A) \\ {\scriptstyle\widetilde{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ G(B) & \xrightarrow[\beta]{} & G(A) \end{array}$$

which induces a morphism

$$F(B) \longrightarrow F(A) \times_{G(A)} G(B) = \big\{(a,b) \in F(A) \times G(B) \bigm| \alpha(a) = \beta(b)\big\}$$

via $c \mapsto \big(\widetilde{\beta}(c), \widetilde{\alpha}(c)\big)$.

Definition C.1.4. A morphism $F \to G$ of functors of Artin rings is called *smooth* if, for every surjection $A' \to A$ in $\mathcal{A}r$, the induced map

$$F(A') \longrightarrow F(A) \times_{G(A)} G(A') \tag{C.1.2}$$

is surjective.

The functor F is called *smooth* if the morphism from F to the constant functor $A \mapsto \{*\}$, where $\{*\}$ is the set consisting of one element, is smooth. This means that $F(\varphi) : F(A') \to F(A)$ is surjective for every surjection $\varphi : A' \to A$ in $\mathcal{A}r$.

A surjection $A' \to A$ is called a *small extension* if its kernel J is one-dimensional over K; then $J^2 = 0$ by Nakayama's lemma. Since every surjection in $\mathcal{A}r$ factors as a finite sequence of small extensions it is sufficient to require the surjectivity of (C.1.2) for small extensions.

Note that a smooth morphism $F \to G$ is surjective, that is, $F(A) \to G(A)$ is surjective for each $A \in \mathcal{A}r$; it follows that the induced morphism $\widehat{F} \to \widehat{G}$ is surjective, too.

Definition C.1.5. Let F be a functor of Artin rings. Let $R \in \widehat{\mathcal{A}}$ and $\widehat{\xi}_R \in \widehat{F}(R)$, and consider the morphism $h_R \to F$ defined by $\widehat{\xi}_R$ as in Remark C.1.2.1. Then

(1) $\widehat{\xi}_R$ is called *formally complete* if $h_R \to F$ is surjective;
(2) $\widehat{\xi}_R$ is called *formally versal* if $h_R \to F$ is smooth;
(3) $\widehat{\xi}_R$ is called *formally semiuniversal* if it is formally versal and if the map of tangent spaces

$$t_{h_R} \longrightarrow t_F$$

is a bijection. In this case, $\widehat{\xi}_R$ is called a *(prorepresentable) hull* of F.

Note that we have the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and that any of the conditions (1) – (3) implies that $F(K)$ contains exactly one element ξ_0.

We also say that $\widehat{\xi}_R$ is a *complete*, resp. *versal*, resp. *semiuniversal*, *formal deformation* of ξ_0 if (1), resp. (2), resp. (3), holds.

Remark C.1.5.1. (1) If $\widehat{\xi}_R$ and $\widehat{\xi}'_{R'}$ are two semiuniversal formal deformations for F, then there exists an isomorphism

$$\varphi : R \xrightarrow{\cong} R'$$

such that $\widehat{F}(\varphi)(\widehat{\xi}_R) = \widehat{\xi}'_{R'}$.

The isomorphism φ is in general not unique (it is unique if F is prorepresentable). The proof of $R \cong R'$ uses that $h_R(K[\varepsilon])$ is a vector space over K and isomorphic to the *Zariski tangent space*

$$T_R := \operatorname{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$$

of R, which is easy to see. Then $\mathfrak{m}_R/\mathfrak{m}_R^2 \cong \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2$ and, hence, $R \cong R'$ by the lifting Lemma 1.14.

(2) In the same manner, one proves the following: let $\widehat{\xi}_R$ be a semiuniversal and $\widehat{\eta}_S$ a versal deformation of ξ_0, then there is a (non-unique) isomorphism

$$\varphi : R[[x_1, \dots, x_n]] \xrightarrow{\cong} S$$

for some $n \geq 0$ such that $\widehat{F}(\varphi \circ j)(\widehat{\xi}_R) = \widehat{\eta}_S$ where $j : R \to R[[x_1, \dots, x_n]]$ is the inclusion.

For detailed proofs see Proposition II.1.14 or [Ser], [Fle].

Schlessinger's main theorem in [Sch] gives a criterion for deciding whether a functor of Artin rings has a semiuniversal deformation. Now, we explain this criterion, which is usually easy to verify in practice.

Let $\varphi' : A' \to A$ and $\varphi'' : A'' \to A$ be morphisms in $\mathcal{A}r$. Then we have a Cartesian square in $\mathcal{A}r$,

$$\begin{tikzcd} A' \times_A A'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A' \arrow[d, "\varphi'"] \\ A'' \arrow[r, "\varphi''"'] & A \end{tikzcd}$$

where $A' \times_A A''$ is the pull-back $\{(a', a'' \in A' \times A'' \mid \varphi'(a') = \varphi''(a'')\}$ with componentwise addition and multiplication.

Applying the functor F to the above diagram, we get a commutative diagram in $\mathcal{S}ets$ and, hence, a map

$$\alpha : F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A''),$$

where objects of the pull-back on the right-hand side may be viewed as pairs of objects of $F(A')$ and $F(A'')$ glued along their image in $F(A)$.

Consider the following (hull) conditions (*Schlessinger's conditions*):

(H_0) $F(K)$ consists of one element (denoted by ξ_0);
(H_1) α is a surjection for $A' \to A$ a small extension;
(H_2) α is a bijection for $A = K$, $A' = K[\varepsilon]$;
(H_3) $F(K[\varepsilon])$ is a finite dimensional K-vector space;
(H_4) α is a bijection for $A' \to A$ a small extension.

Theorem C.1.6 (Schlessinger). *Let F be a functor of Artin rings satisfying (H_0).*

(1) F has a semiuniversal formal deformation iff the properties (H_1), (H_2), (H_3) hold.
(2) F is prorepresentable iff properties (H_1), (H_2), (H_3), (H_4) hold.

Let us describe the idea of Schlessinger's construction; for the proof we refer to [Sch] or [Ser].

Let F satisfy $(\mathrm{H}_0)-(\mathrm{H}_3)$. The semiuniversal formal deformation $\widehat{\xi}_R$ for F is constructed inductively by constructing a projective system

$$\big(R_n, \pi_{n+1} : R_{n+1} \to R_n\big)_{n\geq 0}$$

of Artinian K-algebras and a sequence $\big(\xi_n \in F(R_n)\big)_{n\geq 0}$ such that

$$F(\pi_n)(\xi_n) = \xi_{n-1}, \quad n \geq 1 .$$

Start with $R_0 = K$ and ξ_0 the unique element of $F(K)$ (by (H_0)). By (H_3), $t_F = F(K[\varepsilon])$ has finite dimension, say r, and let $t_1, \dots, t_r$ be a basis of t_F.

Let $K[\boldsymbol{T}] = K[T_1, \dots, T_r]$ and define the R_n as successive quotients of $K[\boldsymbol{T}]$. Start with $R_0 = K$ and ξ_0 and continue with

$$\begin{aligned} R_1 &:= K[\boldsymbol{T}]/\langle \boldsymbol{T}\rangle^2 \cong K[\varepsilon] \times_K \dots \times_K K[\varepsilon] , \\ \xi_1 &:= (t_1, \dots, t_r) \in F(R_1) = t_F \times \dots \times t_F \quad (r \text{ factors}) . \end{aligned}$$

If $(R_0, \xi_0), \dots, (R_{n-1}, \xi_{n-1})$ have already been constructed, $R_q = K[\boldsymbol{T}]/J_q$ and $\xi_q \in F(R_q)$ for $q = 0, \dots, n-1$, we look for an ideal $J_n \subset K[\boldsymbol{T}]$ minimal among the ideals $J \subset K[\boldsymbol{T}]$ satisfying

(a) $J_{n-1} \supset J \supset \langle \boldsymbol{T}\rangle J_{n-1}$,
(b) ξ_{n-1} lifts to $K[\boldsymbol{T}]/J$.

Using (H_1), one shows that the intersection of two such ideals satisfies again (a) and (b) and, therefore, $J_n \subset J_{n-1}$ is the intersection of all ideals satisfying (a) and (b). We set $R_n = K[\boldsymbol{T}]/J_n$ and choose any lift ξ_n of ξ_{n-1} to R_n.

Thus, if I is the intersection of all J_n, $n \geq 0$, then $R = K[\boldsymbol{T}]/I$ is the projective limit of $(R_n)_{n\geq 0}$ and $\widehat{\xi}_R = \big(\xi_n \in F(R_n)\big)_{n\geq 0}$.

Then one shows that $\widehat{\xi}_R$ is indeed a semiuniversal formal deformation for F. Note that, by construction, we have an isomorphism

$$t_F \cong T_R ,$$

where T_R is the Zariski tangent space of R.

The following lemma states that t_F is always a vector space (not necessarily finite dimensional) if (H_0) and (H_2) are satisfied:

Lemma C.1.7. *Let F be a functor of Artin rings satifying (H_0) and (H_2). Then $t_F = F(K[\varepsilon])$ carries a natural structure of a K-vector space.*

Proof. Using (H_2), we define the addition on $F(K[\varepsilon])$ by

$$+ : F(K[\varepsilon]) \times F(K[\varepsilon]) \xrightarrow{\alpha^{-1}} F(K[\varepsilon] \times_K K[\varepsilon]) \xrightarrow{F(+)} F(K[\varepsilon]) ,$$

where $F(+)$ is the morphism obtained by applying F to

$$+ : K[\varepsilon] \times_K K[\varepsilon] \to K[\varepsilon]\,, \quad (a + b\varepsilon, a + b'\varepsilon) \mapsto a + (b + b')\varepsilon\,.$$

The zero element is the image of $\xi_0 \in F(K)$ under $F(K) \to F(K[\varepsilon])$. For $c \in K$, the morphism

$$c : K[\varepsilon] \to K[\varepsilon]\,, \quad a + b\varepsilon \mapsto a + cb\varepsilon\,,$$

induces a map $F(c) : F(K[\varepsilon]) \to F(K[\varepsilon])$. The scalar multiplication on $F(K[\varepsilon])$ is given by

$$c \cdot F(a + b\varepsilon) = F(c)(a + b\varepsilon)\,.$$

It is easy to check that $\big(F(K[\varepsilon]), +, \cdot\big)$ is a K-vector space. □

Remark C.1.7.1. Let $p : A' \to A$ be a small extension with $\ker(p) = \langle t\rangle$, and let F satisfy (H_0) and (H_2). Generalizing the construction in the proof of Lemma C.1.7, we define an action of t_F on $F(A')$ by

$$\tau : F(K[\varepsilon]) \times F(A') \xrightarrow{\alpha^{-1}} F(K[\varepsilon] \times_K A') \xrightarrow{F(\beta)} F(A')\,,$$

where $\beta : K[\varepsilon] \times_K A' \to A'$ is the morphism $(a + b\varepsilon, a') \mapsto a' + bt$.

The action t_F induces an action on the fibres of $F(p) : F(A') \to F(A)$. Indeed, the isomorphism

$$\beta \times \mathrm{pr}_2 : K[\varepsilon] \times_K A' \xrightarrow{\cong} A' \times_A A'$$

induces a map $\tau \times \mathrm{id} : F(K[\varepsilon]) \times F(A') \to F(A') \times_{F(A)} F(A')$ making the following diagram commute:

$$\begin{tikzcd}
F(K[\varepsilon]) \times F(A') \arrow[r, "\tau\times\mathrm{id}"] \arrow[d, "\alpha^{-1}"'] & F(A') \times_{F(A)} F(A') \\
F(K[\varepsilon] \times_K A') \arrow[r, "F(\beta\times\mathrm{pr}_2)"'] & F(A' \times_A A') \arrow[u, "\alpha"']
\end{tikzcd}$$

In particular, $\xi' \in F(A')$ and $\tau(v, \xi')$ have the same image in $F(A)$ for all $v \in F(K[\varepsilon])$.

It follows that the action of t_F on the nonempty fibres of $F(A') \to F(A)$ is transitive if F satisfies (H_1), resp. transitive and free if F satisfies (H_4).

C.2 Obstructions

Let F be a functor of Artin rings, $A \in \mathcal{A}r$, and $\xi \in F(A)$. Obstruction theory formalizes the problem of lifting ξ to $\xi' \in F(A')$ if $A' \twoheadrightarrow A$ is a small extension of A.

Usually, such a lifting does not exist; there is an obstruction against lifting ξ to ξ'. Indeed, in Schlessinger's construction of a semiuniversal formal object for F the problem was to find an appropriate extension

$$R_n = K[\boldsymbol{T}]/J_n \twoheadrightarrow K[\boldsymbol{T}]/J_{n-1} = R_{n-1}$$

such that a lifting of $\xi_{n-1} \in F(R_{n-1})$ to $\xi_n \in F(R_n)$ exists. There are two extreme situations:

Either $J_n = J_{n-1}$ for all $n \geq 1$, which implies that $J_n = \langle \boldsymbol{T} \rangle$. If this happens, the base R of the semiuniversal deformation satisfies $R = K$ and ξ_0 does not allow any lifting at all. In this case, F is called *rigid*.

For the other extreme, let $r > 0$ and $J_n = \langle \boldsymbol{T} \rangle J_{n-1}$ for all n. This means that there is no obstruction to lift ξ_{n-1} from $R_{n-1} = K[\boldsymbol{T}]/\langle \boldsymbol{T} \rangle^n$ to $R_n = K[\boldsymbol{T}]/\langle \boldsymbol{T} \rangle^{n+1}$. In this case, F is called *unobstructed* and $R = K[[\boldsymbol{T}]]$.

Under some mild assumptions, which are fulfilled in our applications, a functor of Artin rings admits an obstruction theory. We give a short overview of the basic notations and properties and refer for details to [FaM] and [Ser] (see also [Art2], [Fle]).

Definition C.2.1. Let $B \to A$ be a ring homomorphism. A *B-extension* of A by I is an exact sequence

$$(A', \varphi): \quad 0 \to I \to A' \xrightarrow{\varphi} A \to 0\,,$$

where A' is a B-algebra, φ a morphism of B-algebras and $I = \ker(\varphi)$ is an ideal of A' satisfying $I^2 = 0$. Then I is an A-module (via $a \cdot i = a'i$ where $\varphi(a') = a$).

Two B-extensions (A', φ) and (A'', ψ) of A by I are *isomorphic* if there exists a B-isomorphism $A' \xrightarrow{\cong} A''$ inducing the identity on A and I.

The *trivial B-extension* of A by I is $A[I]$, where $A[I] = A \oplus I$ as B-modules with multiplication

$$(a, i) \cdot (b, j) = (ab, aj + bi)\,,$$

and $\varphi : A[I] \to A$ is the projection.

We denote by $\mathrm{Ex}_B(A, I)$ the set of isomorphism classes of B-extensions of A by I and by $[A', \varphi]$ the class of extensions (A', φ).

The set $\mathrm{Ex}_B(A, I)$ carries a natural A-module structure, defined as follows: for $a \in A$ and $[A', \varphi] \in \mathrm{Ex}_B(A, I)$ define

$$a \cdot [A', \varphi] := [a_*(A', \varphi)]\,,$$

where $a_*(A', \varphi)$ is the push-forward of (A', φ) by $a : I \to I$. Recall that the *push-forward* (or *pushout*) of (A', φ) by a morphism $\alpha : I \to J$ of A-modules is the extension of A by J,

$$0 \to J \xrightarrow{\beta} A'' := \frac{A'[J]}{\{(-i, \alpha(i)) \mid i \in I\}} \xrightarrow{\psi} A \to 0\,,$$

with $\beta(j) = (0, j)$ and $\psi(a', j) = \varphi(a')$.

The addition in $\mathrm{Ex}_B(A, I)$ is defined by

$$[A', \varphi] + [A'', \psi] := [\delta_*(A' \times_A A'', \phi)] \,,$$

where $(A' \times_A A'', \phi)$ is the B-extension of $A' \times_A A''$ by $I \oplus I$,

$$0 \to I \oplus I \to A' \times_A A'' \xrightarrow{\phi} A \to 0 \,,$$

$\phi(a', a'') = \varphi(a') = \psi(a'')$, where $\delta : I \oplus I \to I$ is the sum $(i, j) \mapsto i + j$ and where $\delta_*(A' \times_A A'', \phi)$ is the push-forward of $(A' \times_A A'', \phi)$ by δ.

If $f : C \to A$ is a morphism of B-algebras then we define the *pull-back* of (A', φ) by f to be the B-extension of C by I,

$$f^*(A', \varphi) : \quad 0 \to I \to A' \times_A C \to C \to 0 \,,$$

with $I \to A' \times_A C$ defined by $i \mapsto (i, 0)$ and $A' \times_A C \to C$ by $(a', c) \mapsto c$.

The pull-back f^* preserves isomorphism classes and, hence, induces a map

$$f^* : \mathrm{Ex}_B(A, I) \longrightarrow \mathrm{Ex}_B(C, I)$$

which is a homomorphism of C-modules.

Definition C.2.2. The A-module $\mathrm{Ex}_B(A, I)$ is called the *first cotangent module of A over B with values in I*. It is denoted by

$$T^1_{A/B}(I) := \mathrm{Ex}_B(A, I) \,.$$

Moreover, we set

$$T^1_{A/B} := T^1_{A/B}(A) \ \text{ and } \ T^1_A := T^1_{A/K} \,.$$

$T^1_{A/B}$ is called the *first cotangent module of A over B* and T^1_A is called the *first cotangent module of A*.

Obstruction theory in our context is the systematic investigation of obstructions against smoothness. For example, if $f : B \to A$ is a morphism in $\mathcal{A}$,

$$\mathrm{ob}(A/B) := T^1_{A/B}(K)$$

is called the *obstruction space of A/B* and A/B is called *unobstructed* if $\mathrm{ob}(A/B) = 0$.

For $B \in \widehat{\mathcal{A}}$ it is shown in [Ser, Chapter 2.1] that $\mathrm{ob}(A/B) \cong \mathrm{ob}(\widehat{A}/B)$, where $\widehat{A}$ is the $\mathfrak{m}_A$-adic completion of A and that

$$\mathrm{ob}(A/B) = 0 \iff \widehat{A} \cong B[[T_1, \dots, T_d]] \,,$$

where $d = \dim_K \mathfrak{m}_A/(\mathfrak{m}_B A + \mathfrak{m}_A^2)$. That is, A/B is unobstructed iff $\widehat{A}$ is a formal power series ring over B. (A proof for analytic algebras is given in Lemma II.1.30, p. 263).

In particular, for $B = K$, we get $\widehat{A} \cong K[[T_1, \dots, T_d]]$ iff

$$\mathrm{ob}(A) := T^1_A(K) = 0\,.$$

Moreover, we have

$$\dim_K T_A \geq \dim A \geq \dim_K T_A - \dim_K \mathrm{ob}(A)\,,$$

where T_A is the Zariski tangent space of A.

We turn now to obstructions for functors F of Artin rings satisfying Schlessinger's conditions (H_0), (H_1) and (H_2). For details, we refer to [FaM].

Definition C.2.3. An *obstruction theory* for F consists of a K-vector space V_F (the *obstruction space*) and, for each $A \in \mathcal{A}r$ and for each extension $e : 0 \to I \to A' \to A \to 0$ of A by I, of a map (the *obstruction map*)

$$v_e : F(A) \longrightarrow V_F \otimes_K I$$

such that the following holds:

(a) If $e \to \widetilde{e}$ is a morphism of extensions,

$$\begin{array}{rcccccccccc} e: & 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\ & & & {\scriptstyle\lambda}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\pi} & & \\ \widetilde{e}: & 0 & \longrightarrow & \widetilde{I} & \longrightarrow & \widetilde{A}' & \longrightarrow & \widetilde{A} & \longrightarrow & 0 \end{array}$$

then the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{v_e} & V_F \otimes I \\ {\scriptstyle F(\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\lambda} \\ F(\widetilde{A}) & \xrightarrow[v_{\widetilde{e}}]{} & V_F \otimes \widetilde{I} \end{array}$$

commutes (*base change property*);

(b) an element $\xi \in F(A)$ lifts to $\xi' \in F(A')$ iff $v_e(\xi) = 0$.

For $\xi \in F(A)$, $v_e(\xi)$ is called the *obstruction of* ξ *against lifting* to A'.

Remark C.2.3.1. (1) In [FaM], an obstruction map is a map

$$F(A) \times I^* \to V_F\,, \quad I^* = \mathrm{Hom}_K(I, K)\,,$$

with certain properties. It is called linear if the induced map $(\xi, \_\_) : I^* \to V_F$ is linear.

For a linear obstruction theory, giving a map $F(A) \times I^* \to V_F$ is equivalent to giving a map $F(A) \to V_F \otimes_K I$. Our condition (b) then corresponds to completeness in [FaM]. Hence, our notion of an obstruction theory is equivalent to the notion of a complete linear obstruction theory in [FaM].

(2) If an obstruction theory for F exists, then there exists also a *universal obstruction theory* (O_F, ob_e) having the property that for every obstruction theory (V_F, v_e) there exists a unique morphism $(O_F, \mathrm{ob}_e) \to (V_F, v_e)$. A *morphism* $(W_F, w_e) \to (V_F, v_e)$ *of obstruction theories* is a K-linear map $\alpha : W_F \to V_F$ such that $v_e = (\alpha \otimes \mathrm{id}) \circ w_e$.

The universality implies functoriality in the following sense: if $\nu : F \to G$ is a morphism of functors of Artin rings, then there exists a natural K-linear map $O(\nu) : O_F \to O_G$ between the universal obstruction spaces. The functor F is called *unobstructed* if it has 0 as obstruction space.

(3) The base change property implies that $v_e = v_{\widetilde{e}}$ if e and $\widetilde{e}$ are isomorphic extensions of A by I. Hence, v_e induces for each $\xi \in F(A)$ a map

$$\mathrm{Ex}(A, I) \to O_F \otimes_K I\,, \quad e \mapsto v_e(\xi)$$

which is K-linear. The kernel of this map consists exactly of the isomorphism classes of liftable extensions. This is the point of view taken in [Ser].

The existence of an obstruction theory is guaranteed be the following theorem (see [FaM, Prop. 4.3 and Thm. 6.11]):

Theorem C.2.4 (Fantechi, Manetti). *A functor F of Artin rings satisfying (H_0), (H_1) and (H_2) has an obstruction theory iff it satisfies the following (linearity) condition:*

(L) Let $0 \to I \to A' \to A \to 0$ be an extension of A, let

$$C = A' \times_K A' / \{(i, i) \mid i \in I\}$$

and let p, q be the natural maps

$$F(C) \xrightarrow{p} F(A \times_K A) \xrightarrow{q} F(A) \times F(A)\,.$$

Then every element in $F(A \times_K A)$ which maps to the diagonal under q is in the image of p.

It is easy to see that (L) is satisfied if q is bijective.

The following criterion for smoothness is very useful:

Proposition C.2.5. *Let $\nu : F \to G$ be a morphism of functors of Artin rings satisfying (H_0), (H_1), (H_2) and (L). Let t_F and O_F, resp. t_G and O_G, be the tangent spaces and the universal obstruction spaces of F, resp. G. Then ν is smooth iff the induced sequence of vector spaces*

$$t_F \to t_G \to 0 \to O_F \to O_G$$

is exact. In particular, F is smooth iff it is unobstructed.

For a proof, see [FaM, Lemma 6.1].

We close this section by citing another useful result:

Proposition C.2.6. *Let F be a functor of Artin rings having a semiuniversal formal deformation $\widehat{\xi}_R$. If F has an obstruction theory with finite dimensional obstruction space V_F then*

$$\mathrm{edim}(R) \geq \dim(R) \geq \mathrm{edim}(R) - \dim_K(V_F)$$

See [Ser, Cor. 2.2.11] for a proof.

C.3 The Cotangent Complex

We recall the definition of the cotangent complex of a morphism. Main references for this part are [Pal2], [Fle] and [Buc] where detailed proofs and further references are given.

The theory of cotangent complexes for a morphism of algebras was developed by André [And] and Quillen [Qui] and later generalized to morphisms of schemes by Illusie [Ill]. The first steps were made before by Lichtenbaum and Schlessinger in [LiS]. The complex analytic counterpart for complex spaces was first developed by Palamodov [Pal, Pal1, Pal2] and later generalized to morphisms by Flenner [Fle, Fle1].

Below, we sketch only the principle of the construction for analytic algebras and state the main properties following [Fle]. We assume in this section that K is a real valued field of characteristic 0.

Definition C.3.1. A *graded anticommutative analytic K-algebra* is an associative ring with 1, together with a grading $R = \bigoplus_{i\leq 0} R^i$ such that

(a) for each $x \in R^i$, $y \in R^j$ we have $xy = (-1)^{ij} yx \in R^{i+j}$;
(b) R^0 is an analytic K-algebra in the usual sense;
(c) R^i is a finitely generated R^0-module for $i \leq 0$.

The module R^i is called the *i-th homogeneous component* of R.

A *morphism* of such algebras is a morphism of rings which is compatible with the gradings and which is a morphism of analytic K-algebras on the 0-th homogeneous component.

Note that a usual analytic K-algebra A is graded anticommutative by setting $R^0 = A$ and $R^i = 0$ for $i < 0$.

A *graded R-module* is a two-sided R-module M together with a grading $M = \bigoplus_{i\in\mathbb{Z}} M^i$ such that $xm = (-1)^{ij} mx \in M^{i+j}$ for all $x \in R^i$, $m \in M^j$.

A *morphism* $\varphi : M \to N$ *of degree* d between two graded R-modules is a morphism of right modules such that $\varphi(M^i) \subset N^{i+d}$ for all $i \in \mathbb{Z}$.

For a graded R-module M we define the *analytic symmetric algebra* $\widehat{S}(M)$ of M over R.

Let $S_R(M) = \bigoplus_{p\geq 0} S_R^p(M)$ be the symmetric algebra of M, that is, the quotient of the tensor algebra $T_R(M) = \bigoplus_{p\geq 0} M^{\otimes p}$, where $S_R^p(M)$ is the quotient of the p-fold tensor product $M^{\otimes p}$ modulo the relations generated by all elements of the form

$$m_1 \otimes \ldots \otimes m_i \otimes m_{i+1} \otimes \ldots \otimes m_p - (-1)^{d_i d_{i+1}} m_1 \otimes \ldots \otimes m_{i+1} \otimes m_i \otimes \ldots \otimes m_p \,,$$

where $d_i = \deg(m_i)$. Then $S_R(M)$ is a graded anticommutative R-algebra. However, the degree 0 component is in general not an analytic algebra. To make it analytic, we define a modification $\widehat{S}(M)$ of $S_R(M)$ as follows:

Definition C.3.2. Let M be a graded R-module with $M^i = 0$ for $i > 0$ and with M^i being finitely generated as R^0-module for all $i < 0$. Let $\widehat{R}^0$ be the analytic R^0-algebra associated to the localization of $S_{R^0}(M^0)$ with respect to the homogeneous maximal ideal $S_{R^0}(M^0)_+$. Define

$$\widehat{S}(M) := \widehat{S}_R(M) := S_R(M) \otimes_{S_{R^0}(M^0)} \widehat{R}^0 \,.$$

For the degree 0 part we get $\widehat{S}(M)^0 = \widehat{R}^0$.

For example, if M^0 is a free R^0-module with basis $x_1, \ldots, x_n$, then

$$S_{R^0}(M^0) \cong R^0[x_1, \ldots, x_n] \,, \quad \widehat{R}^0 = R^0\langle x_1, \ldots, x_n\rangle$$

and similarly for a quotient module of a free module.

If E is a set and $\deg : E \to \mathbb{Z}_{\leq 0}$ a (degree) map such that for each $i \in \mathbb{Z}_{\leq 0}$ the set $E^i := \{e \in E \mid \deg(e) = i\}$ is finite, then we denote by R^E the free anticommutative graded R-module with basis E.

We set $R[E] := \widehat{S}(R^E)$ and call it the *free anticommutative graded analytic K-algebra over E.*

If A is a usual analytic K-algebra, $E = E^0 \cup E^{<0}$, $E^0 = \{x_1, \ldots, x_n\}$, then

$$(A^E)^0 = A^{E^0} \cong A^n \,, \quad S_A(A^{E^0}) = A[x_1, \ldots, x_n]$$

and, hence,

$$A[E] = \widehat{S}(A^E) = (A\langle x_1, \ldots, x_n\rangle)[E^{<0}] \,.$$

Definition C.3.3. Let $B \to R$ be a morphism of graded anticommutative analytic K-algebras and M a right R-module. A *B-derivation of R* with values in M is a B-linear map $s : R \to M$ which satifies the *graded Leibniz rule*

$$s(xy) = s(x)y + (-1)^{ij} s(y)x$$

for all homogeneous elements $x \in R^i$, $y \in R^j$.

A *DG-algebra* (or *differentially graded analytic K-algebra*) is a graded anticommutative analytic K-algebra R together with an R^0-derivation $s : R \to R$ of degree 1 with $s^2 = 0$.

A *morphism of DG-algebras* is a morphism $\varphi : B \to A$ of graded anticommutative analytic K-algebras which is compatible with the derivations.

A *DG-module* over a DG-algebra R is a graded R-module M together with a differential $s : M \to M$ of degree 1 such that

$$s(xm) = s(x)m + (-1)^{ij} s(m)x$$

for homogeneous elements $x \in R^i$, $m \in M^j$.

Note that a DG-algebra (R, s) is a complex with R^0-linear differential $s : R^i \to R^{i+1}$, and we can consider the (finitely generated) cohomology modules

$$H^i(R) = Z^i_R / B^i_R\,,$$

where $Z^i_R = \ker(R^i \to R^{i+1})$ and $B^i_R = \operatorname{Im}(R^{i-1} \to R^i)$.

Any analytic K-algebra A in the usual sense is a DG-algebra R via $R^0 := A$, $R^i = 0$ for $i \neq 0$ and $s = 0$.

Definition C.3.4. Let $\varphi : B \to A$ be a morphism of DG-algebras. A *(Tjurina) resolvent* of A/B is a DG-B-algebra $R_{A/B}$ which is a free B-algebra, together with a surjective morphism $R_{A/B} \to A$ which is a *quasi-isomorphism* of complexes, that is, inducing an isomorphism $H^i(R_{A/B}) \xrightarrow{\cong} H^i(A)$ for all i.

It is proved in [Fle, Satz 1.4] (see also [Pal, Prop. 1.1] and [Pal1, Ch. 2, 1.2]) that for any morphism $B \to A$ of DG-algebras a resolvent $R_{A/B}$ exists. If $B = K$, we write R_A instead of $R_{A/K}$.

If $B \to A$ is a morphism of usual analytic K-algebras, then a Tjurina resolvent $R_{A/B}$ of A/B consists, hence, of

(i) a B-algebra epimorphism $p : R^0_{A/B} \cong B\langle x_1, \dots, x_n\rangle \to A$ of analytic K-algebras,
(ii) a free graded anticommutative $R^0_{A/B}$-algebra $R_{A/B} = R^0_{A/B}[E]$ with E a set of elements of negative degree, the number of elements in each degree being finite, and
(iii) an $R^0_{A/B}$-derivation $s : R_{A/B} \to R_{A/B}$ of degree 1 such that the sequence

$$\ldots \to R^{-2}_{A/B} \xrightarrow{s} R^{-1}_{A/B} \xrightarrow{s} R^0_{A/B} \xrightarrow{p} A \to 0$$

is exact.

The construction of $R := R_{A/B}$ is by induction and proceeds as follows. Choose a minimal set of generators $\{x_1, \dots, x_{n_0}\}$ of $\mathfrak{m}_A$ and let

$$p : R^0 = R^0_{A/B} = B\langle X_1, \dots, X_{n_0}\rangle \to A$$

be the surjection of B-algebras sending X_i to x_i. Note that for $B = \mathbb{C}$ this corresponds to an embedding $(X_0, \mathbf{0}) \subset (\mathbb{C}^{n_0}, \mathbf{0})$ if $A = \mathcal{O}_{X_0,\mathbf{0}}$ is the local ring of a complex germ $(X_0, \mathbf{0})$.

Let $I = \ker(p) = \langle f_1, \dots, f_{n_1} \rangle \subset R^0$ (for $B = \mathbb{C}$ this corresponds to the ideal of $(X_0, \mathbf{0}) \subset (\mathbb{C}^{n_0}, \mathbf{0})$). Now, consider the Koszul complex $K(\boldsymbol{f})$ (see B.6) of the sequence $(f_1, \dots, f_{n_1})$ in R^0 and denote by R_1 the complex $K(\boldsymbol{f})^\bullet$ with $R_1^{-i} := K(\boldsymbol{f})^{-i} = K(\boldsymbol{f})_i$, $i \geq 0$. Then

$$R_1 : \quad \dots \to K(\boldsymbol{f})^{-2} \xrightarrow{s} K(\boldsymbol{f})^{-1} \xrightarrow{s} K(\boldsymbol{f})^{-0} = R^0$$

is a free anticommutative graded R^0-algebra, generated by $e_1, \dots, e_{n_1}$ in degree -1 with $s(e_i) = f_i$.

If A is a *complete intersection over* B, that is, $f_1, \dots, f_{n_1}$ can be chosen as a regular sequence in R^0, then R_1 is a resolution of A (by Theorem B.6.3). In particular, for a complete intersection $A = K\langle x_1, \dots, x_n \rangle / \langle f_1, \dots, f_{n_1} \rangle$ the Koszul complex $K(\boldsymbol{f})^\bullet$ can be chosen as a Tjurina resolvent of A.

If A is not a complete intersection over B, then $H^{-1}(R_1) \neq 0$ by Theorem B.6.3 and we have to modify the construction. The R_0-module $H^{-1}(R_1)$ can be generated by finitely many elements $f'_1, \dots, f'_{n_2}$. Hence, there is a surjection

$$\left(\bigoplus_{i=1}^{n_2} R^0 e'_i \right) \oplus R_1^{-2} \xrightarrow{s} \ker\bigl(R_1^{-1} \xrightarrow{s} R^0\bigr)$$

with s defined on R_1^{-2} as Koszul derivation and $s(e'_i) = f'_i$ for $i = 1, \dots, n_2$. Setting $\deg(e'_i) = -2$, we define R_2 as the free anticommutative graded R^0-algebra generated by the generators of degree -2 (that is, the e'_i and the elements of a homogeneous basis of R_1^{-2}) and of degree -1 (the e_i). The morphism s extends uniquely to R_2. By construction, $H^{-1}(R_2) = 0$. If $H^{-2}(R_2) \neq 0$, we add new generators in degree -3 as above and obtain R_3. Continuing in this manner, we choose free generators of degree -4, -5, etc. Then R^{-i} is the free R^0-module generated by the elements of degree $-i$. With the induced differential this defines $R_{A/B}$. Note that $R_{A/B}$ is not unique.

To define the cotangent complex we need, besides a Tjurina resolvent, differential modules. As for morphisms of analytic algebras, one can define, for each morphism $B \to R$ of graded anticommutative analytic K-algebras, the (universal finite) *differential module* $\Omega_{R/B}$.

If $R = B[E]$ is a free B-algebra, then $\Omega_{R/B}$ is just the free R-module with basis E, where we denote the basis elements in $\Omega_{R/B}$ by de, $e \in E$ with $\deg(e) = \deg(de)$. Then the map $e \mapsto de$ induces a unique B-derivation $d_{B[E]/B} : B[E] \to \Omega_{B[E]/B}$.

If $R = B[E]/I$, then $\Omega_{R/B}$ is the quotient of $\Omega_{B[E]/B} \otimes_{B[E]} R$ modulo the submodule generated by the elements $d_{B[E]/B}(x)$, $x \in I$.

The B-derivation $d_{B[E]/B}$ induces a B-derivation

$$d = d_{R/B} : R \to \Omega_{R/B}$$

of degree 0 such that the following universal property holds for the pair $(\Omega_{R/B}, d)$:

(i) $\Omega_{R/B}$ is a graded R-module such that the homogeneous components $\Omega^i_{R/B}$, $i \in \mathbb{Z}$, are finitely generated R^0-modules.
(ii) Let M be a graded R-module which is separated as R^0-module and $d' : R \to M$ a B-derivation. Then there is a unique R-linear map $h : \Omega_{R/B} \to M$ such that $d' = h \circ d$.

If $B \to R$ is a morphism of DG-algebras, then $\Omega_{R/B}$ has a natural structure as DG-module with differential defined by

$$s\bigl(yd(x)\bigr) = s(y)d(x) + (-1)^i yd\bigl(s(x)\bigr)$$

for homogeneous $x, y \in R$, $y \in R^i$. Thus, $(\Omega_{R/B}, s)$ is a complex of B-modules.

Definition C.3.5. For a morphism $B \to A$ of analytic K-algebras choose a Tjurina resolvent $R = R_{A/B} \to A$ which is a morphism of B-algebras, and let $\Omega_{R/B}$ be the module of differentials. The complex of A-modules

$$L^{\bullet}_{A/B} := \Omega_{R/B} \otimes_R A$$

is called the *cotangent complex* of A over B.

This complex is unique in the derived category[1] $D^-\bigl(\mathrm{Mod}(A)\bigr)$, that is, independent of the choice of the Tjurina resolvent. Moreover, $L^{\bullet}_{A/B}$ is functorial in A and B. For proofs we refer to [Fle].

The construction of the cotangent complex was generalized in [Fle] to the cotangent complex $L^{\bullet}_{B\to A/C}$ for a morphism of C-algebras $\varphi : B \to A$, that is, for a commutative diagram of analytic K-algebras

$$\begin{array}{ccc} B & \xrightarrow{\ \varphi\ } & A \\ & \nwarrow \quad \nearrow & \\ & C & \end{array}$$

and even more generally for a sequence $C \to A_1 \to A_2 \to \ldots \to A_n$ of morphisms of analytic algebras over C. Most important for applications is the case that C is the ground field K. Even more generally, there exists a cotangent complex for any finite diagram of morphisms (see [Ill] and [Buc]).

[1] Let $\mathrm{Mod}(A)$ denote the category of A-modules and $K^-\bigl(\mathrm{Mod}(A)\bigr)$ the category of right bounded complexes of A-modules with morphisms the homotopy classes of morphisms of complexes of degree 0. Then the derived category $D^-\bigl(\mathrm{Mod}(A)\bigr)$ is the localization of $K^-\bigl(\mathrm{Mod}(A)\bigr)$ with respect to quasiisomorphisms (see [GeM] or [Eis]).

C.4 Cotangent Cohomology

Having introduced the cotangent complex, we can define the cotangent cohomology functors T^i.

Let $B \to A$ be a morphism of analytic K-algebras and M an A-module.

Definition C.4.1. We define

$$T^i_{A/B}(M) := \operatorname{Ext}^i_A(L^\bullet_{A/B}, M) := H^i\big(\operatorname{Hom}_A(L^\bullet_{A/B}, M)\big)\,,$$

the *i-th cotangent cohomology* of A/B with values in M. We write

$$T^i_{A/B} := T^i_{A/B}(A)\,, \quad T^i_A(M) := T^i_{A/K}(M)\,, \quad T^i_A := T^i_A(A)\,.$$

Note that $T^i_{A/B}(M)$ is an A-module which is zero for $i < 0$.

We collect now the most important properties of the cotangent cohomology. Proofs can be found in [Fle], [Buc] and [Rim]. The T^i are functorial in M and A/C and give rise to exact sequences.

Lemma C.4.2. *For each short exact sequence $0 \to M' \to M \to M'' \to 0$ of A-modules there is a long exact cohomology sequence*

$$0 \to T^0_{A/B}(M') \to T^0_{A/B}(M) \to T^0_{A/B}(M'') \to T^1_{A/B}(M') \to \dots\,.$$

Lemma C.4.3. *Let $C \to B \to A$ be morphisms of analytic algebras and M an A-module. Then there is a long exact sequence (*Zariski-Jacobi sequence*)*

$$0 \to T^0_{A/B}(M) \to T^0_{A/C}(M) \to T^0_{B/C}(M) \to T^1_{A/B}(M) \to \dots\,.$$

Lemma C.4.4. *Let*

$$\begin{tikzcd} A' & A \arrow[l] \\ B' \arrow[u] \arrow[ur, phantom, "\square"] & B \arrow[l, "\psi"] \arrow[u, "\varphi"'] \end{tikzcd}$$

be a Cartesian diagram of analytic algebras with φ or ψ being flat. Then, for every A'-module M, the natural map

$$T^i_{A/B}(M) \xrightarrow{\cong} T^i_{A'/B'}(M)\,, \quad i \geq 0$$

is an isomorphism.

The proofs of the above lemmas use standard techniques from homological algebra.

The lemma below shows that T^0 and T^1 are closely related to deformations (we shall see below that T^2 is related to obstructions).

Lemma C.4.5. *With the above notations,*

$$T^0_{A/B}(M) = \mathrm{Der}_B(A, M)\,,$$
$$T^1_{A/B}(M) = \mathrm{Ex}_B(A, M)\,.$$

The proof of this lemma and of Proposition C.4.6 below is a rather straightforward consequence of the explicit construction of the Tjurina resolvent in low degrees.

We collect now some properties of T^i which are in particular useful for computations:

Proposition C.4.6. *(1) If $B \to A$ is smooth, then $T^i_{A/B}(M) = 0$ for every A-module M and all $i \geq 1$.*
(2) If $T^1_{A/B}(K) = 0$, then $B \to A$ is smooth.
(3) If $A = B\langle x_1, \ldots, x_n\rangle/\langle f_1, \ldots, f_k\rangle$ with $f_1, \ldots, f_k$ a regular sequence in $B\langle x_1, \ldots, x_n\rangle$, then $T^i_{A/B}(M) = 0$ for every A-module M and all $i \geq 2$.
(4) If $T^2_{A/B}(K) = 0$, then $A \cong B\langle x_1, \ldots, x_n\rangle/\langle f_1, \ldots, f_k\rangle$ for some $n \geq 0$ and $f_1, \ldots, f_k$ a regular sequence in $B\langle x_1, \ldots, x_n\rangle$.
(5) If $0 \to I \to B \to A \to 0$ is exact, then

$$T^1_{A/B}(M) \cong \mathrm{Hom}_A(I/I^2, M)\,.$$

(6) Let $P = B\langle x_1, \ldots, x_n\rangle$ and $0 \to I \to P \to A \to 0$ exact. Then there is an exact sequence

$$0 \to T^0_{A/B}(M) \to \mathrm{Der}_B(P, M) \to \mathrm{Hom}_A(I/I^2, M) \to T^1_{A/B}(M) \to 0\,.$$

The statements (1) – (4) are an immediate consequence of the construction of the Tjurina resolvent $R_{A/B}$ and of the properties of the Koszul complex. Statements (5) and (6) follow similarly (see also Proposition II.1.25 with $A = \mathcal{O}_{X,\mathbf{0}}$, $B = \mathbb{C}$ and $M = A$).

The construction of the cotangent complex and its cohomology can be globalized. Because of the non-uniqueness of the Tjurina resolvent, the construction is rather complicated. For morphisms of schemes this was done by Illusie [Ill], for morphisms of complex spaces by Palamodov and Flenner (see [Pal2, Fle]).

Let $f : X \to Y$ be a morphism of complex spaces. Then there exists a cotangent complex $L^\bullet_{X/Y}$ which is a complex of $\mathcal{O}_X$-modules and which is unique in the derived category $D^-\big(\mathrm{Mod}(\mathcal{O}_X)\big)$ of right bounded complexes of $\mathcal{O}_X$-modules. The complex $L^\bullet_{X/Y}$ has the property that for $x \in X$ the stalks satisfy

$$L^\bullet_{X/Y,x} = L^\bullet_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}\,,$$

that is, $L^\bullet_{X/Y,x}$ is a cotangent complex for $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ as defined in C.3.

Definition C.4.7. For an $\mathcal{O}_X$-module $\mathcal{M}$ define

$$T^i_{X/Y}(\mathcal{M}) := \operatorname{Ext}^i_{\mathcal{O}_X}(L^\bullet_{X/Y}, \mathcal{M}),$$
$$\mathscr{T}^i_{X/Y}(\mathcal{M}) := \mathcal{E}xt^i_{\mathcal{O}_X}(L^\bullet_{X/Y}, \mathcal{M}),$$

where Ext denotes Ext-groups and $\mathcal{E}xt$ the Ext-sheaves.

The following theorem is proved in [Fle1]:

Theorem C.4.8. *Let $f : X \to Y$ be a morphism of complex spaces and $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ coherent $\mathcal{O}_X$-modules.*

(1) If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is exact, then there is a long exact sequence

$$0 \to \mathscr{T}^0_{X/Y}(\mathcal{M}') \to \mathscr{T}^0_{X/Y}(\mathcal{M}) \to \mathscr{T}^0_{X/Y}(\mathcal{M}'') \to \mathscr{T}^1_{X/Y}(\mathcal{M}') \to \dots$$

(2) $T^0_{X/Y}(\mathcal{M}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{M})$, $\mathscr{T}^0_{X/Y}(\mathcal{M}) \cong \mathscr{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{M})$.

(3) There is a (local to global) spectral sequence

$$E_2^{pq} = H^q\big(X, \mathscr{T}^p_{X/Y}(\mathcal{M})\big) \Longrightarrow T^{p+q}_{X/Y}(\mathcal{M}).$$

(4) $\mathscr{T}^i_{X/Y}(\mathcal{M})$ is a coherent $\mathcal{O}_X$-module with $\mathscr{T}^i_{X/Y}(\mathcal{M}) = 0$ for $i < 0$.

(5) $\mathscr{T}^i_{X/Y}(\mathcal{M})_x \cong T^i_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}(\mathcal{M}_x)$ for all $x \in X$ and all i;
$\mathscr{T}^i_{X/Y}(\mathcal{M}) = 0$ for $i \neq 0$ if $X \to Y$ is smooth.

(6) If $X \subset Y$ is a closed embedding and $\mathcal{O}_X = (\mathcal{O}_Y/\mathcal{I})|_X$ then

$$\mathscr{T}^i_{X/Y} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{M}),$$
$$T^i_{X/Y} \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{M}).$$

It follows that the sheaves $\mathscr{T}^i_{X/Y}(\mathcal{M})$, $i \neq 0$, are concentrated on $\operatorname{Sing}(f)$, the set of non-smooth points of f. For $i \geq 2$, $\mathscr{T}^i_{X/Y}(\mathcal{M})$is concentrated on the points $x \in X$ where $\mathcal{O}_{X,x}$ is not a relative complete intersection over $\mathcal{O}_{Y,f(x)}$.

C.5 Relation to Deformation Theory

Formal deformation and obstruction theory has its main applications in deformation theory of geometric objects. For example, deformations of morphisms of schemes (see [Ill, Buc]), of complex spaces or singularities, or deformations of modules and sheaves (see [Fle1, Pal2]). In this context, deformation theory has been used, for instance, to study local properties of moduli spaces (see [Ser]). It has been used by the authors to study the geometry of families of varieties with prescribed singularities (see the survey article [GLS] and the references given there or our forthcoming book [GLS1]).

Here, we restrict to deformations of (morphisms of) complex spaces or singularities (that is, complex space germs).

Because of the anti-equivalence between the categories (see I.1.4, p. 57),

$$\begin{array}{ccc} (\text{complex space germs}) & \longrightarrow & (\text{analytic } \mathbb{C}\text{-algebras}) \\ (X,x) & \longmapsto & \mathcal{O}_{X,x} \\ f:(X,x)\to(Y,y) & \longmapsto & f^\sharp:\mathcal{O}_{Y,y}\to\mathcal{O}_{X,x} \end{array}$$

the results of the previous sections apply immediately to deformations of (morphisms of) singularities.

In particular, the deformation functor $\underline{\mathcal{D}ef}_{(X,x)}$ from the category of complex space germs to the category of sets (see Definition II.1.4, p. 227) satisfies Schlessingers conditions (H_0), (H_1) and (H_2) (see [Sch], [Sch2]). It follows that the deformation functor for a sequence of morphisms or, more generally, for diagrams of complex space germs satisfies (H_0), (H_1) and (H_2) (see [Buc, Ch. II]). The cotangent cohomology provides an obstruction theory with obstruction space T^2 (see [Tju], [Sch1] or [LiS])

All this has been generalized by work of Illusie [Ill], Palamodov [Pal, Pal1, Pal2], Bingener [Bin, Bin1] and Flenner [Fle, Fle1] to deformations of morphisms (more generally, to deformations of diagrams of complex spaces). Since many definitions related to deformations of morphisms hold for complex spaces as well as for germs, we unify the notations.

Let $f:X\to Y$ denote a morphism of complex spaces or of complex space germs. A *deformation* of f over a pointed complex space T or a germ T is a diagram of morphisms of complex spaces or of germs

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d] \\ Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{Y} \arrow[d] \\ \{\mathrm{pt}\} \arrow[r, hook] & T \end{tikzcd} \qquad \text{(C.5.3)}$$

such that $\mathscr{Y}\to T$ and the composed map $\mathscr{X}\to T$ are flat.

A *morphism* of two such diagrams is the usual commutative diagram (see Definition II.1.21, p. 248) with identity maps on $X\to Y\to\{\mathrm{pt}\}$. A *morphism of diagrams over* T is a morphism of diagrams being the identity on T.

Associated to $f:X\to Y$, there are six naturally defined deformations functors which we introduce now:

$$\underline{\mathcal{D}ef}_{X\to Y}(T) = \left\{ \text{isomorphism classes of diagrams (C.5.3) over } T \right\},$$

$$\underline{\mathcal{D}ef}_{X/Y}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of diagrams (C.5.3) over } T \\ \text{with } \mathscr{Y} = Y\times T \text{ and with morphisms being} \\ \text{the identity on } Y\times T \end{array} \right\},$$

$$\underline{\mathcal{D}ef}_{X\backslash Y}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of diagrams (C.5.3) over } T \\ \text{with } \mathscr{X} = X \times T \text{ and with morphisms being} \\ \text{the identity on } X \times T \end{array} \right\},$$

$$\underline{\mathcal{D}ef}_{X\backslash X\to Y/Y}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of diagrams (C.5.3) over } T \\ \text{with } \mathscr{X} = X \times T,\ \mathscr{Y} = Y \times T \text{ and with mor-} \\ \text{phisms being the identity on } X \times T \text{ and } Y \times T \end{array} \right\},$$

$$\underline{\mathcal{D}ef}_{X}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of diagrams (C.5.3) over } T \\ \text{with } Y \hookrightarrow \mathscr{Y} \text{ omitted} \end{array} \right\},$$

$$\underline{\mathcal{D}ef}_{Y}(T) = \left\{ \begin{array}{l} \text{isomorphism classes of diagrams (C.5.3) over } T \\ \text{with } X \hookrightarrow \mathscr{X} \text{ omitted} \end{array} \right\}.$$

These six deformation functors (for germs as well as for spaces) satisfy Schlessinger's conditions (H_0), (H_1) and (H_2) (see [Sch3], [Fle] and [Buc]). Hence, they have a formal semiuniversal deformation iff they satisfy additionally condition (H_3).

The following commutative diagram of natural transformations between the functors is obvious. It is obtained by inclusion of functors or by forgetting parts of the diagram (C.5.3):

$$\begin{tikzcd}[column sep=small] & & \underline{\mathcal{D}ef}_{X\backslash X\to Y/Y} \arrow[dl] \arrow[dr] & & \\ \underline{\mathcal{D}ef}_{Y} & \underline{\mathcal{D}ef}_{X\backslash Y} \arrow[l] \arrow[dr] & & \underline{\mathcal{D}ef}_{X/Y} \arrow[r] \arrow[dl] & \underline{\mathcal{D}ef}_{X}\,. \\ & & \underline{\mathcal{D}ef}_{X\to Y} & & \end{tikzcd}$$

For any of these functors there is a cotangent complex in a derived category of right bounded complexes giving rise to the K-vector spaces $T^i_{X\to Y}$, $T^i_{X/Y}$, $T^i_{X\backslash Y}$, $T^i_{X\backslash X\to Y/Y}$, T^i_X, and T^i_Y. Moreover, there is an isomorphism ([Buc, 2.4.2.2])

$$T^i_{X\backslash X\to Y/Y} \cong T^{i-1}_Y(f_*\mathcal{O}_X)\,.$$

The following theorem of Illusie relates the cotangent cohomology to deformations (see [Ill, III.2] and [Buc, 2.4.4]).

To simplify notations, we write $\underline{\mathcal{D}ef}_{XY}$ for any of the above six functors and T^i_{XY} for the corresponding cotangent cohomology groups. Recall that T_ε denotes the fat point with $\mathcal{O}_{T_\varepsilon} = K[\varepsilon]$.

Theorem C.5.1 (Illusie). *With the above notations, we have:*

(1) $\underline{\mathcal{D}ef}_{XY}(T_\varepsilon) \cong T^1_{XY}$.
(2) $\underline{\mathcal{D}ef}_{XY}$ *has an obstruction theory with obstruction space* T^2_{XY}.

Moreover, using the above relations between the deformation functors, we get the following corollary (see [Buc, 2.4.5.2]):

Corollary C.5.2. *The vanishing of the groups on the left implies the smoothness of the morphism of functors on the right:*

$T^2_{X/Y}$: $\underline{\mathcal{D}ef}_{X\to Y} \to \underline{\mathcal{D}ef}_{Y}$

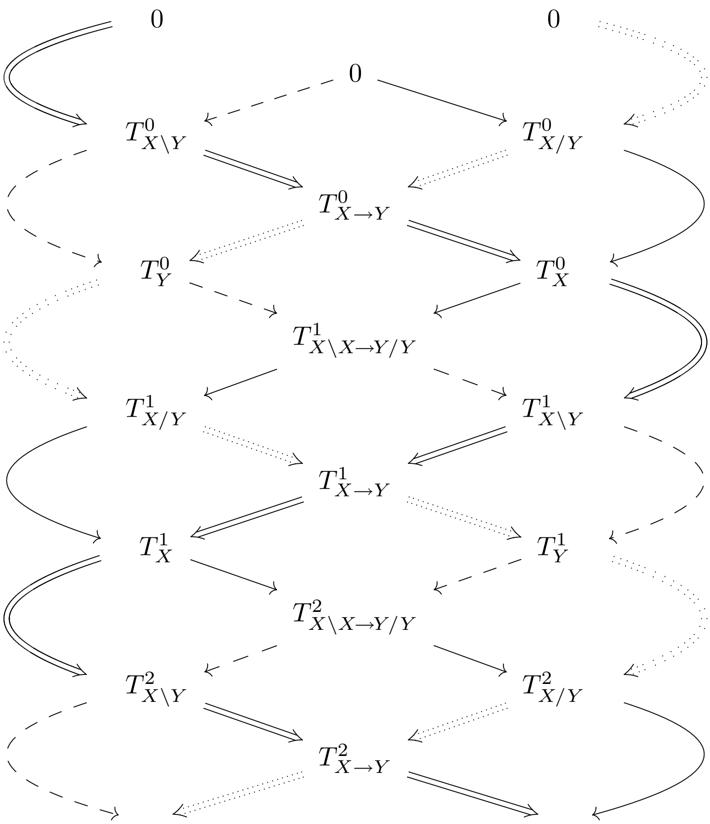
$T^2_{X\backslash Y}$: $\underline{\mathcal{D}ef}_{X\to Y} \to \underline{\mathcal{D}ef}_{X}$

T^1_{X}: $\underline{\mathcal{D}ef}_{X\backslash X\to Y/Y} \to \underline{\mathcal{D}ef}_{X/Y}$ *and* $\underline{\mathcal{D}ef}_{X\backslash Y} \to \underline{\mathcal{D}ef}_{X\to Y}$

T^1_{Y}: $\underline{\mathcal{D}ef}_{X\backslash X\to Y/Y} \to \underline{\mathcal{D}ef}_{X\backslash Y}$ *and* $\underline{\mathcal{D}ef}_{X/Y} \to \underline{\mathcal{D}ef}_{X\to Y}$

$T^1_{X\backslash X\to Y/Y}$: $\underline{\mathcal{D}ef}_{X/Y} \to \underline{\mathcal{D}ef}_{X}$ *and* $\underline{\mathcal{D}ef}_{X\backslash Y} \to \underline{\mathcal{D}ef}_{Y}$

The interdependence of the cotangent cohomology of the six functors can be expressed beautifully in terms of a braid of four long exact sequences, due to Buchweitz [Buc]:



The cotangent braid of a morphism $X \to Y$.

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Index

A

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