Derived Categories of Modules and Coherent Sheaves

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Abstract

We present recent results on derived categories of modules and coherent sheaves, namely, tame–wild dichotomy and semi-continuity theorem for derived categories over finite dimensional algebras, as well as explicit calculations for derived categories of modules over nodal rings and of coherent sheaves over projective configurations of types A and $\tilde{A}$.

This paper is a survey of some recent results on the structure of derived categories obtained by the author in collaboration with Viktor Bekkert and Igor Burban [6, 11, 12]. The origin of this research was the study of Cohen–Macaulay modules and vector bundles by Gert-Martin Greuel and myself [27, 28, 29, 30] and some ideas from the work of Huisgen-Zimmermann and Saorín [42]. Namely, I understood that the technique of "matrix problems," briefly explained below in subsection 2.3, could be successfully applied to the calculations in derived categories, almost in the same way as it was used in the representation theory of finite-dimensional algebras, in study of Cohen–Macaulay modules, etc. The first step in this direction was the semi-continuity theorem for derived categories [26] presented in subsection 2.1. Then Bekkert and I proved the tame–wild dichotomy for derived categories over finite dimensional algebras (see subsection 2.2). At the same time, Burban and I described the indecomposable objects in the derived categories over nodal rings (see Section 3) and projective configurations of types A and $\tilde{A}$ (see Section 4). Note that it follows from [23, 29] that these are the only cases, where such a classification is possible; for all other pure noetherian rings (or projective curves) even the categories of modules (respectively, of vector bundles) are wild. In both cases the description reduces to a special class of matrix problems ("bunches of chains" or "clans"), which also arises in a wide range of questions from various areas of mathematics.

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I tried to explain the backgrounds, but, certainly, only sketched proofs, referring for the details to the original papers cited above.

1 Generalities

We first recall some definitions. Let $\mathbb{S}$ be a commutative ring. An $\mathbb{S}$-*category* is a category $\mathscr{A}$ such that all morphism sets $\mathscr{A}(A,B)$ are $\mathbb{S}$-modules and the multiplication of morphisms is $\mathbb{S}$-bilinear. We call $\mathscr{A}$

- *local* if every object $A \in \mathscr{A}$ decomposes into a finite direct sum of objects with local endomorphism rings;
- ω-*local* if every object $A \in \mathscr{A}$ decomposes into a finite or countable direct sum of objects with local endomorphism rings;
- *fully additive* if any idempotent morphism in $\mathscr{A}$ splits, that is defines a decomposition into a direct sum;
- *locally finite* (over $\mathbb{S}$) if all morphism spaces $\mathscr{A}(A,B)$ are finitely generated $\mathbb{S}$-modules. If $\mathbb{S}$ is a field, a locally finite category is often called *locally finite dimensional*. If, moreover, $\mathscr{A}$ has finitely many objects, we call it *finite* (over $\mathbb{S}$). Especially, if $\mathscr{A}$ is an $\mathbb{S}$-algebra (i.e. a $\mathbb{S}$-category with one object), we call it a *finite* $\mathbb{S}$-*algebra*.
- If $\mathscr{A}$ is fully additive and locally finite over $\mathbb{S}$, we shall call it a *falf* ($\mathbb{S}$-) category.

Mostly the ring $\mathbb{S}$ will be local and complete noetherian ring. Then, evidently, every falf $\mathbb{S}$-category is local; moreover, an endomorphism algebra $\mathscr{A}(A,A)$ in a falf category is a finite $\mathbb{S}$-algebra. It is known that any local (or ω-local) category is fully additive; moreover, a decomposition into a direct sum of objects with local endomorphism rings is always unique; in other words, any local (or ω-local) category is a *Krull–Schmidt* one, cf. [4, Theorem 3.6].

For a local category $\mathscr{A}$ we denote by $\operatorname{rad}\mathscr{A}$ its *radical*, that is the set of all morphisms $f : A \to B$, where $A, B \in \operatorname{Ob}\mathscr{A}$, such that no component of the matrix presentation of f with respect to some (hence any) decomposition of A and B into a direct sum of indecomposable objects is invertible. Note that if $f \notin \operatorname{rad}\mathscr{A}$, there is a morphism $g : B \to A$ such that $fgf = f$ and $gfg = g$. Hence both gf and fg are nonzero idempotents, which define decompositions $A \simeq A_1 \oplus A_2$ and $B \simeq B_1 \oplus B_2$ such that the matrix presentation of f with respect to these decompositions is diagonal: $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$, and f_1 is invertible. Obviously, if $\mathscr{A}$ is locally finite dimensional, then $\operatorname{rad}\mathscr{A}(A,B)$ coincide with the set of all morphisms $f : A \to B$ such that gf (or fg) is nilpotent for any morphism $g : B \to A$.

We denote by $\mathscr{C}(\mathscr{A})$ the category of complexes over $\mathscr{A}$, i.e. that of diagrams

$$(A_\bullet, d_\bullet): \qquad \dots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \dots,$$

where $A_n \in \operatorname{Ob}\mathscr{A}$, $d_n \in \mathscr{A}(A_n, A_{n-1})$, with relations $d_n d_{n+1} = 0$ for all n. Sometimes we omit $d_\bullet$ denoting this complex by $A_\bullet$. Morphisms between two such complexes, $(A_\bullet, d_\bullet)$ and $(A'_\bullet, d'_\bullet)$ are, by definition, commutative diagrams of the form

$$\begin{array}{cccccccccc}
 & \dots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \xrightarrow{d_{n-1}} & \dots \\
\phi_\bullet: & \dots & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \dots \\
 & \dots & \longrightarrow & A'_{n+1} & \xrightarrow{d'_{n+1}} & A'_n & \xrightarrow{d'_n} & A'_{n-1} & \xrightarrow{d'_{n-1}} & \dots
\end{array}$$

Note that we use "homological" notations (with down indices) instead of more usual "cohomological" ones (with upper indices). Two morphisms, $\phi_\bullet$ and $\psi_\bullet$, between $(A_\bullet, d_\bullet)$ and $(A'_\bullet, d'_\bullet)$ are called *homotopic* if there are morphisms $\sigma_n : A_n \to A'_{n+1}$ ($n \in \mathbb{N}$) such that $\phi_n - \psi_n = d'_{n+1}\sigma_n + \sigma_{n-1}d_n$ for all n. We denote it by $\phi \sim \psi$. We also often omit evident indices and write, for instance, $\phi - \psi = d'\sigma + \sigma d$. The *homotopy category* $\mathscr{H}(\mathscr{A})$ is, by definition, the factor category $\mathscr{C}(\mathscr{A})/\mathscr{C}_{\sim 0}$, where $\mathscr{C}_{\sim 0}$ is the ideal of morphisms homotopic to zero.

Suppose now that $\mathscr{A}$ is an abelian category. Then, for every complex $(A_\bullet, d_\bullet)$, its *homologies* $\mathrm{H}_\bullet = \mathrm{H}_\bullet(A_\bullet, d_\bullet)$ are defined, namely $\mathrm{H}_n(A_\bullet, d_\bullet) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$. Every morphism $\phi_\bullet$ as above induces morphisms of homologies $\mathrm{H}_n(\phi_\bullet) : \mathrm{H}_n(A_\bullet, d_\bullet) \to \mathrm{H}_n(A'_\bullet, d'_\bullet)$. It is convenient to consider $\mathrm{H}_\bullet(\mathscr{A}_\bullet, d_\bullet)$ as a complex with zero differential and we shall usually do so. Then $\mathrm{H}_\bullet$ becomes an endofunctor inside $\mathscr{C}(\mathscr{A})$. If $\phi_\bullet \sim \psi_\bullet$, then $\mathrm{H}_\bullet(\phi_\bullet) = \mathrm{H}_\bullet(\psi_\bullet)$, so $\mathrm{H}_\bullet$ can be considered as a functor $\mathscr{H}(\mathscr{A}) \to \mathscr{C}(\mathscr{A})$. We call $\phi_\bullet$ a *quasi-isomorphism* if $\mathrm{H}_\bullet(\phi_\bullet)$ is an isomorphism. Then we write $\phi_\bullet : (A_\bullet, d_\bullet) \approx (A'_\bullet, d'_\bullet)$ or sometimes $(A_\bullet, d_\bullet) \approx (A'_\bullet, d'_\bullet)$ if $\phi_\bullet$ is not essential. The *derived category* $\mathscr{D}(\mathscr{A})$ is defined as the category of fractions (in the sense of [34]) $\mathscr{H}(\mathscr{A})[\mathscr{Q}^{-1}]$, where $\mathscr{Q}$ is the set of quasi-isomorphisms. In particular, the functor of homologies $\mathrm{H}_\bullet$ becomes a functor $\mathscr{D}(\mathscr{A}) \to \mathscr{C}(\mathscr{A})$. Note that a morphism between two complexes with zero differential is homotopic to zero if and only if it is zero, and is a quasi-isomorphism if and only if it is an isomorphism. Moreover, any morphism between such complexes in the derived category is equal (in this category) to the image of a real morphism between these complexes in $\mathscr{C}(\mathscr{A})$. Thus we can consider the category $\mathscr{C}^0(\mathscr{A})$ of complexes with zero differential as a full subcategory of $\mathscr{H}(\mathscr{A})$ or of $\mathscr{D}(\mathscr{A})$. In particular, we can (and shall) identify every object $A \in \mathscr{A}$ with the complex $A_\bullet$ such that $A_0 = A$, $A_n = 0$ for $n \neq 0$. It gives a full embedding of $\mathscr{A}$ into $\mathscr{H}(\mathscr{A})$ or $\mathscr{D}(\mathscr{A})$.

We denote by $\mathscr{C}^-(\mathscr{A})$ (respectively, $\mathscr{C}^+(\mathscr{A})$, $\mathscr{C}^b(\mathscr{A})$) the categories of *right bonded* (respectively, *left bounded*, (*two-side*) *bounded*) complexes, i.e. such that $A_n = 0$ for $n \ll 0$ (respectively, $n \gg 0$ or both). Correspondingly, we consider the right (left, two-side) bounded homotopy categories $\mathscr{H}^-(\mathscr{A}), \mathscr{H}^+(\mathscr{A}), \mathscr{H}^b(\mathscr{A})$ and right (left, two-side) bounded derived categories $\mathscr{D}^-(\mathscr{A}), \mathscr{D}^+(\mathscr{A}), \mathscr{D}^b(\mathscr{A})$.

The categories $\mathscr{C}(\mathscr{A}), \mathscr{H}(\mathscr{A}), \mathscr{D}(\mathscr{A})$, as well as their bounded subcategories, are *triangulated categories* [38]. Namely, the *shift* maps a complex $A_\bullet$ to the complex $A_\bullet[1]$, where $A_n[1] = A_{n-1}$.[1] A *triangle* is a sequence isomorphic (as a diagram in the corresponding category) to a sequence of the form

$$A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} \mathsf{C}f_\bullet \xrightarrow{h_\bullet} A_\bullet[1],$$

where $f_\bullet$ is a morphism of complexes, $\mathsf{C}f_\bullet$ is the *cone* of this morphism, i.e. $\mathsf{C}f_n = A_{n-1} \oplus B_n$, the differential $\mathsf{C}f_n \to \mathsf{C}f_{n-1} = A_{n-2} \oplus B_{n-1}$ is given by the matrix $\begin{pmatrix} -d_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}$; $g(b) = (0, b)$ and $h(a, b) = a$.

If $\mathscr{A} = \mathscr{R}$-Mod, the category of modules over a pre-additive category $\mathscr{R}$ (for instance, over a ring), the definition of the right (left) bounded derived category can be modified. Namely, $\mathscr{D}^-(\mathscr{R}\text{-Mod})$ is equivalent to the homotopy category $\mathscr{H}^-(\mathscr{R}\text{-Proj})$, where $\mathscr{R}$-Proj is the category of projective $\mathscr{R}$-modules. Recall that a *module* over a pre-additive category $\mathscr{R}$ is a functor $M : \mathscr{R} \to \mathsf{Ab}$, the category of abelian groups. Such a module is projective (as an object of the category $\mathscr{R}$-Mod) if and only if it is isomorphic to a direct summand of a direct sum of *representable* modules $\mathscr{A}^A = \mathscr{A}(A, \_)$ ($A \in \operatorname{Ob} \mathscr{A}$). Just in the same way, the left bounded category $\mathscr{D}^+(\mathscr{R}\text{-Mod})$ is equivalent to the homotopy category $\mathscr{H}^+(\mathscr{R}\text{-Inj})$, where $\mathscr{R}$-Inj is the category of injective $\mathscr{R}$-modules. If the category $\mathscr{R}$ is *noetherian*, i.e. every submodule of every representable module is finitely generated, the right bounded derived category $\mathscr{D}^-(\mathscr{R}\text{-mod})$, where $\mathscr{R}$-mod denotes the category of finitely generated $\mathscr{R}$-modules, is equivalent to $\mathscr{H}^-(\mathscr{R}\text{-proj})$, where $\mathscr{R}$-proj is the category of finitely generated projective $\mathscr{R}$-modules.

In general, it is not true that $\mathscr{D}^b(\mathscr{R}\text{-Mod})$ is equivalent to $\mathscr{H}^b(\mathscr{R}\text{-Proj})$ (or to $\mathscr{H}^b(\mathscr{R}\text{-Inj})$). For instance, a projective resolution of a module M, which is isomorphic to M in $\mathscr{D}(\mathscr{R}\text{-Mod})$, can be left unbounded. Nevertheless, there is a good approximation of the two-side derived category by finite complexes of projective modules. Namely, consider the full subcategory $\mathscr{C}^{(N)} = \mathscr{C}^{(N)}(\mathscr{R}) \subseteq \mathscr{C}^b(\mathscr{R}\text{-proj})$ consisting of all bounded complexes $P_\bullet$ such that $P_n = 0$ for $n > N$ (note that we do not fix the right bound). We say that two morphisms, $\phi_\bullet, \psi_\bullet : P_\bullet \to P'_\bullet$, from $\mathscr{C}^{(N)}$ are *almost homotopic* and write $\phi \overset{N}{\sim} \psi$ if there are morphisms $\sigma_n : P_n \to P'_{n+1}$ such that

[1] Note again the *homological* (down) indices here.

$\phi_n - \psi_n = d'_{n+1}\sigma_n + \sigma_{n-1}d_n$ for all $n < N$ (not necessarily for $n = N$). We denote by $\mathscr{H}^{(N)} = \mathscr{H}^{(N)}(\mathscr{A})$ the factor category $\mathscr{C}^{(N)}/\mathscr{C}_{\overset{N}{\sim}0}$, where $\mathscr{C}_{\overset{N}{\sim}0}$ is the ideal consisting of all morphisms almost homotopic to zero. There are natural functors $\mathcal{I}_N : \mathscr{H}^{(N)} \to \mathscr{H}^{(N+1)}$. Namely, for a complex $P_\bullet \in \mathscr{H}^{(N)}$ find a homomorphism $d_{N+1} : P_{N+1} \to P_N$, where P_{N+1} is projective and $\operatorname{Im} d_{N+1} = \operatorname{Ker} d_N$. Then the complex

$$\mathcal{I}_N P_\bullet :\ P_{N+1} \xrightarrow{d_{N+1}} P_N \xrightarrow{d_N} P_{N-1} \to \dots$$

is uniquely defined up to isomorphism in $\mathscr{H}^{(N+1)}$. Moreover, any morphism $\phi_\bullet : P_\bullet \to P'_\bullet$ from $\mathscr{H}^{(N)}$ induces a morphism $\mathcal{I}_N\phi_\bullet : \mathcal{I}_N P_\bullet \to \mathcal{I}_N P'_\bullet$, which coincides with $\phi_\bullet$ for all places $n \le N$, and this morphism is also uniquely defined as a morphism from $\mathscr{H}^{(N+1)}$. It gives the functor $\mathcal{I}_N$. One can easily verify that actually all these functors are full embeddings and $\mathscr{D}^b(\mathscr{R}\text{-Mod}) \simeq \varinjlim_N \mathscr{H}^{(N)}(\mathscr{R})$. If $\mathscr{R}$ is noetherian, the same is true for the category $\mathscr{D}^b(\mathscr{R}\text{-mod})$ if we replace everywhere $\mathscr{R}$-Proj by $\mathscr{R}$-proj.

One can also consider the projection $\mathcal{E}_N : \mathscr{H}^{(N+1)} \to \mathscr{H}^{(N)}$, which just erases the term P_{N+1} in a complex $P_\bullet \in \mathscr{H}^{(N+1)}$, and show that $\mathscr{D}^-(\mathscr{R}\text{-Mod}) \simeq \varprojlim_N \mathscr{H}^{(N)}(\mathscr{R})$.

Suppose now that $\mathscr{A}$ is a falf category over a *complete local noetherian* ring $\mathbb{S}$. Then, evidently, the bounded categories $\mathscr{C}^b(\mathscr{A})$ and $\mathscr{H}^b(\mathscr{A})$ are also falf categories, hence Krull-Schmidt categories. In [11, Appendix A] it is proved that the same is true for *unbounded* categories $\mathscr{C}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A})$. The proof is based on the following analogue of the Hensel lemma (cf. [11, Corollary A.5]).

Lemma 1.1. *Let Λ be a finite algebra over a local noetherian ring $\mathbb{S}$ with maximal ideal $\mathfrak{m}$ and $a \in \Lambda$. For every $n \in \mathbb{N}$ there is a polynomial $g(x) \in \mathbb{S}[x]$ such that*

- $g(a)^2 \equiv g(a) \mod \mathfrak{m}^{n+1}$*;*
- $g(e) \equiv e \mod \mathfrak{m}^n$ *for every element e of an arbitrary finite $\mathbb{S}$-algebra such that $e^2 \equiv e \mod \mathfrak{m}^n$;*
- $g(a) \equiv 1 \mod \mathfrak{m}$ *if and only if a is invertible;*
- $g(a) \equiv 0 \mod \mathfrak{m}$ *if and only if a is nilpotent modulo $\mathfrak{m}$.*

Theorem 1.2. *Suppose that $\mathbb{S}$ is a complete local noetherian ring with maximal ideal $\mathfrak{m}$. If $\mathscr{A}$ is a falf category over $\mathbb{S}$, the categories $\mathscr{C}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A})$ are ω-local (in particular, Krull–Schmidt). Moreover, a morphism $f_\bullet : A_\bullet \to B_\bullet$ from one of these categories belongs to the radical if and only if all components $f_n g_n$ (or $g_n f_n$) are nilpotent modulo $\mathfrak{m}$ for any morphism $g_\bullet : B_\bullet \to A_\bullet$.*

Proof. Let $a_\bullet$ be an endomorphism of a complex $A_\bullet$ from $\mathscr{C}(\mathscr{A})$. Consider the sets $I_n \subset \mathbb{Z}$ defined as follows: $I_0 = \{0\}$, $I_{2k} = \{ l \in \mathbb{Z} \mid -k \le l \le k \}$ and $I_{2k-1} = \{ l \in \mathbb{Z} \mid -k < l \le k \}$. Obviously, $\bigcup_n I_n = \mathbb{Z}$, $I_n \subset I_{n+1}$ and $I_{n+1} \setminus I_n$ consists of a unique element l_n. Using corollary 1.1, we can construct a sequence of endomorphisms $a_\bullet^{(n)}$ such that, for each $i \in I_n$,

- $(a_i^{(n)})^2 \equiv a_i^{(n)} \mod \mathfrak{m}^n$;
- $a_i^{(n+1)} \equiv a_i^{(n)} \mod \mathfrak{m}^n$;
- $a_i^{(n)}$ is invertible or nilpotent modulo $\mathfrak{m}$ if and only if so is a_i.

Then one easily sees that setting $u_i = \lim_{n\to\infty} a_i^{(n)}$, we get an idempotent endomorphism $u_\bullet$ of $A_\bullet$, such that $u_i \equiv 0 \mod \mathfrak{m}$ ($u_i \equiv 1 \mod \mathfrak{m}$) if and only if a_i is nilpotent modulo $\mathfrak{m}$ (respectively a_i is invertible).

Especially, if either one of a_l is neither nilpotent nor invertible modulo $\mathfrak{m}$, or one of a_l is nilpotent modulo $\mathfrak{m}$ while another one is invertible, then $u_\bullet$ is neither zero nor identity. Hence the complex $A_\bullet$ decomposes. Thus $A_\bullet$ is indecomposable if and only if, for any endomorphism $a_\bullet$ of $A_\bullet$, either $a_\bullet$ is invertible or all components a_n are nilpotent modulo $\mathfrak{m}$. Since all algebras $\operatorname{End} A_n/\mathfrak{m}\operatorname{End} A_n$ are finite dimensional, neither product $\alpha\beta$, where $\alpha, \beta \in \operatorname{End} A_n$ and one of them is nilpotent modulo $\mathfrak{m}$, can be invertible. Therefore, the set of endomorphisms $a_\bullet$ of an indecomposable complex $A_\bullet$ such that all components a_n are nilpotent modulo $\mathfrak{m}$ form an ideal R of $\operatorname{End} A_\bullet$ and $\operatorname{End} A_\bullet/R$ is a skew field. Hence $R = \operatorname{rad}(\operatorname{End} A_\bullet)$ and $\operatorname{End} A_\bullet$ is local.

Now we want to show that any complex from $\mathscr{C}(\mathscr{A})$ has an indecomposable direct summand. Consider an arbitrary complex $A_\bullet$ and suppose that $A_0 \ne 0$. For any idempotent endomorphism $e_\bullet$ of $A_\bullet$ at least one of the complexes $e(A_\bullet)$ or $(1-e)(A_\bullet)$ has a non-zero component at the zero place. On the set of all endomorphisms of $A_\bullet$ we can introduce a partial ordering by writing $e_\bullet \ge e'_\bullet$ if and only if $e'_\bullet = e_\bullet e'_\bullet e_\bullet$ and both e_0 and e'_0 are non-zero. Let $e_\bullet \ge e'_\bullet \ge e''_\bullet \ge \dots$ be a chain of idempotent endomorphisms of $A_\bullet$. As all endomorphism algebras $\operatorname{End} A_l$ are finitely generated $\mathbf{S}$-modules, the sequences $e_l, e'_l, e''_l, \dots \in \operatorname{End} A_l$ stabilize for all l, so this chain has a lower bound (formed by the limit values of components). By Zorn's lemma, there is a minimal non-zero idempotent of $A_\bullet$, which defines an indecomposable direct summand.

Again, since all $\operatorname{End} A_l$ are finitely generated, for every n there is a decomposition $A_\bullet = B_\bullet^{(n)} \oplus \bigoplus_{i=1}^{r_n} B_{in\bullet}$ where all $B_{in\bullet}$ are indecomposable and $B_l^{(n)} = 0$ for $l \in I_n$. Moreover, one may suppose that $r_n \le r_m$ for $m > n$ and $B_{in\bullet} = B_{im\bullet}$ for $i \le r_n$. Evidently, it implies that $A_\bullet = \bigoplus_{i=1}^{r} B_{i\bullet}$ where $r = \sup_n r_n$ and $B_{i\bullet} = B_{in\bullet}$ for $i \le r_n$, which accomplishes the proof of the Theorem 1.2 for $\mathscr{C}(\mathscr{A})$.

Note now that the endomorphism ring of each complex $B_{i\bullet}$ in the category $\mathscr{H}(\mathscr{A})$ is a factor ring of its endomorphism ring in $\mathscr{C}(\mathscr{A})$. Hence it is either local or zero; in the latter case the image of $B_{i\bullet}$ in $\mathscr{H}(\mathscr{A})$ is a zero object. Therefore, the claim is also valid for $\mathscr{H}(\mathscr{A})$. $\square$

Since the derived category $\mathscr{D}^-(\mathscr{R}\text{-mod})$ is equivalent to $\mathscr{H}^-(\mathscr{R}\text{-proj})$, we get the following corollary.

Corollary 1.3. *Let* $\mathbb{S}$ *be* $\mathscr{R}$ *be a locally finite* $\mathbb{S}$*-category (e.g. a finite* $\mathbb{S}$*-algebra). Then the derived category* $\mathscr{D}^-(\mathscr{R}\text{-mod})$ *is* ω*-local, in particular, Krull–Schmidt.*

2 Finite Dimensional Algebras

2.1 Semi-Continuity

In this section we suppose that $\mathbb{S} = \Bbbk$ is an algebraically closed field and $\mathbf{A}$ is a finite dimensional $\Bbbk$-algebra with radical $\mathbf{J}$. In this case one can define, following the pattern of [28], the *number of parameters* for objects of the bounded derived category $\mathscr{D}^b(\mathbf{A}\text{-mod})$. First of all, every object M in the category $\mathbf{A}$-mod has a *projective cover*, i.e. an epimorphism $f: P \to M$, where P is a projective module, such that $\operatorname{Ker} f \subseteq \mathbf{J}P$. Moreover, this projective cover is unique up to an isomorphism. It implies that every right bounded complex of $\mathbf{A}$-modules is isomorphic in the homotopy category $\mathscr{H}^-(\mathbf{A}\text{-mod})$ to a *minimal complex*, i.e. such a complex of projective modules

$$P_\bullet : \quad \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots ,$$

that $\operatorname{Im} d_n \subseteq \mathbf{J}P_{n-1}$ for all n. Consider now the full subcategory $\mathscr{H}_0^{(N)} = \mathscr{H}_0^{(N)}(\mathbf{A})$ of $\mathscr{H}^{(N)}(\mathbf{A})$ consisting of minimal complexes. Then again $\mathscr{D}^b(\mathbf{A}\text{-mod}) \simeq \varinjlim \mathscr{H}_0^{(N)}$. Moreover, two complexes from $\mathscr{H}_0^{(N)}$ are isomorphic in $\mathscr{D}^b(\mathbf{A}\text{-mod})$ if and only if they are isomorphic as complexes. Using this approximation, we can prescribe a *vector rank* to every object $M_\bullet \in \mathscr{D}^b(\mathbf{A}\text{-mod})$. Namely, let $\{A_1, A_2, \dots, A_s\}$ be a set of representatives of isomorphism classes of indecomposable projective $\mathbf{A}$-modules. Every finitely generated projective $\mathbf{A}$-module P uniquely decomposes as $P \simeq \bigoplus_{i=1}^s r_i A_i$. We call the vector $\mathbf{r}(P) = (r_1, r_2, \dots, r_s)$, the *rank* of the projective module P and for every vector $\mathbf{r} = (r_1, r_2, \dots, r_s)$ set $\mathbf{r}\mathbf{A} = \bigoplus_{i=1}^s r_i A_i$. Given a finite complex $P_\bullet$ of projective modules, we define its *vector rank* as the function $\mathbf{rk}(P_\bullet): \mathbb{Z} \to \mathbb{N}^s$ mapping $n \in \mathbb{Z}$ to $\mathbf{r}(P_i)$. It is a function with finite support. Let Δ be the set of all functions $\mathbb{Z} \to \mathbb{N}^s$ with finite support. For every function $\mathbf{r}_\bullet \in \Delta$, let $\mathcal{C}(\mathbf{r}_\bullet) = \mathcal{C}(\mathbf{r}_\bullet, \mathbf{A})$ be the set of all minimal complexes $P_\bullet$

such that $P_n = \mathbf{r}_n\mathbf{A}$ (we write $\mathbf{r}_n$ for $\mathbf{r}_\bullet(n)$). This set can be considered as an affine algebraic variety over $\Bbbk$, namely, $\mathcal{C}(\mathbf{r}_\bullet)$ is isomorphic to the subvariety of the affine space $\mathcal{H} = \prod_n \mathrm{Hom}_{\mathbf{A}}(P_n, \mathbf{J}P_{n-1})$ consisting of all sequences (f_n) such that $f_n f_{n+1} = 0$ for all n. Set also $\mathbf{G}(\mathbf{r}_\bullet) = \prod_n \mathrm{Aut}\, P_n$. It is an affine algebraic group acting on $\mathcal{C}(\mathbf{r}_\bullet)$ and its orbits are just isomorphism classes of minimal complexes of vector rank $\mathbf{r}_\bullet$. It is convenient to replace affine varieties by projective ones, using the obvious fact that the sequences (f_n) and (λf_n), where $\lambda \in \Bbbk$ is a nonzero scalar, belong to the same orbit. So we write $\mathbb{H}(\mathbf{r}_\bullet)$ for the projective space $\mathbb{P}(\mathcal{H})$ and $\mathbb{D}(\mathbf{r}_\bullet)$ for the image in $\mathbb{H}(\mathbf{r}_\bullet)$ of $\mathcal{C}(\mathbf{r}_\bullet)$. Actually, we exclude the complexes with zero differential, but such a complex is uniquely defined by its vector rank, so they play a negligible role in classification problems.

We consider now *algebraic families* of $\mathbf{A}$-complexes, i.e. flat families over an algebraic variety X. Such a family is a complex $\mathcal{F}_\bullet = (\mathcal{F}_n, d_n)$ of flat coherent $\mathbf{A} \otimes \mathcal{O}_X$-modules. We always assume this complex bounded and *minimal*; the latter means that $\mathrm{Im}\, d_n \subseteq \mathbf{J}\mathcal{F}_{n-1}$ for all n. We also assume that X is connected; it implies that the vector rank $\mathbf{rk}(\mathcal{F}_\bullet(x))$ is constant, so we can call it the *vector rank of the family* $\mathcal{F}$ and denote it by $\mathbf{rk}(\mathcal{F}_\bullet)$ Here, as usually, $\mathcal{F}(x) = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$, where $\mathfrak{m}_x$ is the maximal ideal of the ring $\mathcal{O}_{X,x}$. We call a family $\mathcal{F}_\bullet$ *non-degenerate* if, for every $x \in X$, at least one of $d_n(x) : \mathcal{F}_n(x) \to \mathcal{F}_{n-1}(x)$ is non-zero. Having a family $\mathcal{F}_\bullet$ over X and a regular map $\phi : Y \to X$, one gets the inverse image $\phi^*(\mathcal{F})$, which is a family of $\mathbf{A}$-complexes over the variety Y such that $\phi^*(\mathcal{F})(y) \simeq \mathcal{F}(\phi(y))$. If $\mathcal{F}_\bullet$ is non-degenerate, so is $\phi^*(\mathcal{F})$. Given an ideal $\mathbf{I} \subseteq \mathbf{J}$, we call a family $\mathcal{F}_\bullet$ an $\mathbf{I}$-*family* if $\mathrm{Im}\, d_n \subseteq \mathbf{I}\mathcal{F}_{n-1}$ for all n. Then any inverse image $\phi^*(\mathcal{F})$ is an $\mathbf{I}$-family as well. Just as in [29], we construct some "almost versal" non-degenerate $\mathbf{I}$-families.

For each vector $\mathbf{r} = (r_1, r_2, \dots, r_s)$ denote $\mathbf{I}(\mathbf{r}, \mathbf{r}') = \mathrm{Hom}_{\mathbf{A}}(\mathbf{r}\mathbf{A}, \mathbf{I} \cdot \mathbf{r}'\mathbf{A})$, where $\mathbf{I}$ is an ideal contained in $\mathbf{J}$. Fix a vector rank of bounded complexes $\mathbf{r}_\bullet = (\mathbf{r}_k) \in \Delta$ and set $\mathcal{H}(\mathbf{r}_\bullet, \mathbf{I}) = \bigoplus_k \mathbf{I}(\mathbf{r}_k, \mathbf{r}_{k-1})$. Consider the projective space $\mathbb{P}(\mathbf{r}_\bullet, \mathbf{I}) = \mathbb{P}(\mathcal{H}(\mathbf{r}_\bullet, \mathbf{I}))$ and its closed subset $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I}) \subseteq \mathbb{P}$ consisting of all sequences (h_k) such that $h_{k+1}h_k = 0$ for all k. Because of the universal property of projective spaces [40, Theorem II.7.1], the embedding $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I}) \to \mathbb{P}(\mathbf{r}_\bullet, \mathbf{I})$ gives rise to a non-degenerate $\mathbf{I}$-family $\mathcal{V}_\bullet = \mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$:

$$\mathcal{V}_\bullet : \quad \mathcal{V}_n \xrightarrow{d_n} \mathcal{V}_{n-1} \xrightarrow{d_{n-1}} \dots \longrightarrow \mathcal{V}_m, \tag{1}$$

where $\mathcal{V}_k = \mathcal{O}_{\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})}(n-k) \otimes \mathbf{r}_k\mathbf{A}$ for all $m \le k \le n$. We call $\mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$ the *canonical* $\mathbf{I}$-*family of* $\mathbf{A}$-*complexes* over $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$. Moreover, regular maps $\phi : X \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$ correspond to non-degenerate $\mathbf{I}$-families $\mathcal{F}_\bullet$ with $\mathcal{F}_k = 0$ for $k > n$ or $k < m$ and $\mathcal{F}_k = \mathcal{L}^{\otimes(n-k)} \otimes \mathbf{r}_k\mathbf{A}$ for some invertible sheaf $\mathcal{L}$ over X. Namely, such a family can be obtained as $\phi^*(\mathcal{V}_\bullet)$ for a uniquely defined regular map ϕ. Moreover, the following result holds, which shows the "almost versality" of the families $\mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$.

Proposition 2.1. *For every non-degenerate family of* $\mathbf{I}$*-complexes* $\mathcal{F}_\bullet$ *of vector rank* $\mathbf{r}_\bullet$ *over an algebraic variety* X*, there is a finite open covering* $X = \bigcup_j U_j$ *such that the restriction of* $\mathcal{F}_\bullet$ *onto each* U_j *is isomorphic to* $\phi_j^*\mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$ *for a regular map* $\phi_j : U_j \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$*.*

Proof. For each $x \in X$ there is an open neighbourhood $U \ni x$ such that all restrictions $\mathcal{F}_k|_U$ are isomorphic to $\mathcal{O}_U \otimes \mathbf{r}_k\mathbf{A}$; so the restriction $\mathcal{F}_\bullet|_U$ is obtained from a regular map $U \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$. Evidently it implies the assertion. □

Note that the maps ϕ_j are not canonical, so we cannot glue them into a "global" map $X \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$.

The group $\mathbf{G} = \mathbf{G}(\mathbf{r}_\bullet) = \prod_k \mathrm{Aut}(\mathbf{r}_k\mathbf{A})$ acts on $\mathcal{H}(\mathbf{r}_\bullet, \mathbf{I})$: $(g_k) \cdot (h_k) = (g_{k-1}h_kg_k^{-1})$. It induces the action of $\mathbf{G}(\mathbf{r}_\bullet)$ on $\mathbb{P}(\mathbf{R}_\bullet, \mathbf{I})$ and on $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$. The definitions immediately imply that $\mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})(x) \simeq \mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})(x')$ $(x, x' \in \mathbb{D})$ if and only if x and x' belong to the same orbit of $\mathbf{G}$. Consider the sets

$$\mathbb{D}_i = \mathbb{D}_i(\mathbf{r}_\bullet, \mathbf{I}) = \{\, x \in \mathbb{D} \mid \dim \mathbf{G}x \le i \,\}\,.$$

It is known that they are closed (it follows from the theorem on dimensions of fibres, cf. [40, Exercise II.3.22] or [48, Ch. I, § 6, Theorem 7]). We set

$$\mathrm{par}(\mathbf{r}_\bullet, \mathbf{I}, \mathbf{A}) = \max_i \{\, \dim \mathbb{D}_i(\mathbf{r}_\bullet, \mathbf{I}) - i \,\}$$

and call this integer the *parameter number* of $\mathbf{I}$-complexes of vector rank $\mathbf{r}_\bullet$. Obviously, if $\mathbf{I} \subseteq \mathbf{I}'$, then $\mathrm{par}(\mathbf{r}_\bullet, \mathbf{I}, \mathbf{A}) \le \mathrm{par}(\mathbf{r}_\bullet, \mathbf{I}', \mathbf{A})$. Especially, the number $\mathrm{par}(\mathbf{r}_\bullet, \mathbf{A}) = \mathrm{par}(\mathbf{r}_\bullet, \mathbf{J}, \mathbf{A})$ is the biggest one.

Proposition 2.1, together with the theorem on the dimensions of fibres and the Chevalley theorem on the image of a regular map (cf. [40, Exercise II.3.19] or [48, Ch. I, § 5, Theorem 6]), implies the following result.

Corollary 2.2. *Let* $\mathcal{F}_\bullet$ *be an* $\mathbf{I}$*-family of vector rank* $\mathbf{r}_\bullet$ *over a variety* X*. For each* $x \in X$ *set* $X_x = \{\, x' \in X \mid \mathcal{F}_\bullet(x') \simeq \mathcal{F}_\bullet(x) \,\}$ *and denote*

$$X_i = \{\, x \in X \mid \dim X_x \le i \,\}\,,$$
$$\mathrm{par}(\mathcal{F}_\bullet) = \max_i \{\, \dim X_i - i \,\}\,.$$

Then all subsets X_x *and* X_i *are* constructible *(i.e. finite unions of locally closed sets) and* $\mathrm{par}(\mathcal{F}_\bullet) \le \mathrm{par}(\mathbf{r}_\bullet, \mathbf{I}, \mathbf{A})$*.*

Note that the bases $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$ of our almost versal families are *projective*, especially *complete* varieties. We shall exploit this property while studying the behaviour of parameter numbers in families of algebras. Since decompositions of algebras in families into direct sums of projective modules can differ, we restrict our considerations to the complexes of *free* modules. Namely, let $\mathbf{a} = \mathbf{r}(\mathbf{A})$. For every sequence $\mathbf{b} = (b_n, \dots, b_m)$ of integers we set $\mathbf{ba} = (b_n\mathbf{a}, \dots, b_m\mathbf{a})$ and write $\operatorname{par}(\mathbf{b}, \mathbf{I}, \mathbf{A})$ instead of $\operatorname{par}(\mathbf{ba}, \mathbf{I}, \mathbf{A})$.

A (flat) *family of algebras* over an algebraic variety X is a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras, which is coherent and flat (thus locally free) as a sheaf of $\mathcal{O}_X$-modules. For such a family and every sequence $\mathbf{b} = (b_m, b_{m+1}, \dots, b_n)$ one can define the function $\operatorname{par}(\mathbf{b}, \mathcal{A}, x) = \operatorname{par}(\mathbf{b}, \mathcal{A}(x))$. Our main result is the upper semi-continuity of these functions.

Theorem 2.3. *Let $\mathcal{A}$ be a flat family of finite dimensional algebras over an algebraic variety X. For every vector $\mathbf{b} = (b_m, b_{m+1}, \dots, b_n)$ the function* $\operatorname{par}(\mathbf{b}, \mathcal{A}, x)$ *is upper semi-continuous, i.e. all sets*

$$X_j = \{\, x \in X \mid \operatorname{par}(\mathbf{b}, \mathcal{A}, x) \geq j \,\}$$

are closed.

Proof. We may assume that X is irreducible. Let $\mathbf{K}$ be the field of rational functions on X. We consider it as a constant sheaf on X. Set $\mathbf{J} = \operatorname{rad}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathbf{K})$ and $\mathcal{J} = \mathbf{J} \cap \mathcal{A}$. It is a sheaf of nilpotent ideals. Moreover, if ξ is the generic point of X, the factor algebra $\mathcal{A}(\xi)/\mathcal{J}(\xi)$ is semisimple. Hence there is an open set $U \subseteq X$ such that $\mathcal{A}(x)/\mathcal{J}(x)$ is semisimple, thus $\mathcal{J}(x) = \operatorname{rad} \mathcal{A}(x)$ for every $x \in U$. Therefore, $\operatorname{par}(\mathbf{b}, \mathcal{A}, x) = \operatorname{par}(\mathbf{b}, \mathcal{J}(x), \mathcal{A}(x))$ for $x \in U$; so $X_j = X_j(\mathcal{J}) \cup X'_j$, where

$$X_j(\mathcal{J}) = \{\, x \in X \mid \operatorname{par}(\mathbf{b}, \mathcal{J}(x), \mathcal{A}(x)) \geq j \,\}$$

and $X' = X \setminus U$ is a closed subset in X. Using noetherian induction, we may suppose that X'_j is closed, so we only have to prove that $X_j(\mathcal{J})$ is closed too.

Consider the locally free sheaf $\mathcal{H} = \bigoplus_{k=m+1}^{n} \mathcal{H}om(b_k\mathcal{A}, b_{k-1}\mathcal{J})$ and the projective space bundle $\mathbb{P}(\mathcal{H})$ [40, Section II.7]. Every point $h \in \mathbb{P}(\mathcal{H})$ defines a set of homomorphisms $h_k : b_k\mathcal{A}(x) \to b_{k-1}\mathcal{J}(x)$ (up to a homothety), where x is the image of h in X, and the points h such that $h_k h_{k+1} = 0$ form a closed subset $\mathbb{D}(\mathbf{b}, \mathcal{A}) \subseteq \mathbb{P}(\mathcal{H})$. We denote by π the restriction onto $\mathbb{D}(\mathbf{b}, \mathcal{A})$ of the projection $\mathbb{P}(\mathcal{H}) \to X$; it is a projective, hence closed map. Moreover, for every point $x \in X$ the fibre $\pi^{-1}(x)$ is isomorphic to $\mathbb{D}(\mathbf{b}, \mathcal{A}(x), \mathcal{J}(x))$. Consider also the group variety $\mathcal{G}$ over X: $\mathcal{G} = \prod_{k=m}^{n} \mathrm{GL}_{b_k}(\mathcal{A})$. There is a natural action of $\mathcal{G}$ on $\mathbb{D}(\mathbf{b}, \mathcal{A})$ over X, and the sets $\mathbb{D}_i = \{\, z \in \mathbb{D}(\mathbf{b}, \mathcal{A}) \mid \dim \mathcal{G}z \leq i \,\}$ are closed in $\mathbb{D}(\mathbf{b}, \mathcal{A})$. Therefore, the sets $Z_i = \pi(\mathbb{D}_i)$ are closed in X, as well as $Z_{ij} = \{\, x \in Z_i \mid \dim \pi^{-1}(x) \geq i + j \,\}$. But $X_j(\mathcal{J}) = \bigcup_i Z_{ij}$, thus it is also a closed set. $\square$

2.2 Derived Tame and Wild Algebras

We are going to define derived tame and derived wild algebras. To do it, we consider families of complexes with non-commutative bases.

Definition 2.4. 1. Let $\mathbf{R}$ be a $\Bbbk$-algebra. A *family of* $\mathbf{A}$*-complexes based on* $\mathbf{R}$ is a complex of finitely generated projective $\mathbf{A} \otimes \mathbf{R}^{\mathrm{op}}$-modules $P_\bullet$. We denote by $\mathscr{C}^{(N)}(\mathbf{A}, \mathbf{R})$ the category of all bounded families with $P_n = 0$ for $n > N$ (again we do not prescribe the right bound). For such a family $P_\bullet$ and an $\mathbf{R}$-module L we denote by $P_\bullet(L)$ the complex $(P_n \otimes_{\mathbf{R}} L, d_n \otimes 1)$. If L is finite dimensional, $P_\bullet(L) \in \mathscr{C}^{(N)}(\mathbf{A}) = \mathscr{C}^{(N)}(\mathbf{A}, \Bbbk)$.

Obviously, if the algebra $\mathbf{R}$ is *affine*, i.e. commutative, finitely generated over $\Bbbk$ and without nilpotents, such families coincide in fact with families of complexes over the algebraic variety $\operatorname{Spec} \mathbf{R}$. Especially, if $\mathbf{R}$ is also connected (i.e. contains no nontrivial idempotents), the *vector rank* of such a family $\mathbf{rk}(P_\bullet)$ is defined as $\mathbf{rk}(P_\bullet \otimes_{\mathbf{R}} S)$, where S is a simple $\mathbf{R}$-module (no matter which one).

2. We call a family $P_\bullet$ *strict* if for every finite dimensional $\mathbf{R}$-modules L, L'

(a) $P_\bullet(L) \simeq P_\bullet(L')$ if and only if $L \simeq L'$;

(b) $P_\bullet(L)$ is indecomposable if and only if so is L.

3. We call $\mathbf{A}$ *derived wild* if it has a strict family of complexes over every finitely generated $\Bbbk$-algebra $\mathbf{R}$.

The following useful fact is well known.

Proposition 2.5. *An algebra* $\mathbf{A}$ *is derived wild if and only if it has a strict family over one of the following algebras:*

- *free algebra* $\Bbbk\langle x, y\rangle$ *in two variables;*
- *polynomial algebra* $\Bbbk[x, y]$ *in two variables;*
- *power series algebra* $\Bbbk[[x, y]]$ *in two variables.*

Definition 2.6. 1. A *rational algebra* is a $\Bbbk$-algebra $\Bbbk[t, f(t)^{-1}]$ for a non-zero polynomial $f(t)$. A *rational family* of $\mathbf{A}$-complexes is a family over a rational algebra $\mathbf{R}$. Equivalently, a rational family is a family over an open subvariety of the affine line.

2. An algebra $\mathbf{A}$ is called *derived tame* if there is a set of rational families of bounded $\mathbf{A}$-complexes $\mathfrak{P}$ such that:

(a) for each $\mathbf{r}_\bullet \in \Delta$, the set $\mathfrak{P}(\mathbf{r}_\bullet) = \{\, P_\bullet \in \mathfrak{P} \mid \mathbf{rk}(P_\bullet) = \mathbf{r}_\bullet \,\}$ is finite.

(b) for every $\mathbf{r}_\bullet$ all indecomposable complexes from $\mathcal{C}(\mathbf{r}_\bullet, \mathbf{A})$, except finitely many of them (up to isomorphism), are isomorphic to a complex $P_\bullet(L)$ for some $P_\bullet \in \mathfrak{P}$ and some finite dimensional L.

We call $\mathfrak{P}$ a *parameterizing set* of $\mathbf{A}$-complexes.

These definitions do not formally coincide with other definitions of derived tame and derived wild algebras, for instance, those proposed in [36, 37], but all of them are evidently equivalent. It is obvious (and easy to prove, like in [20]) that neither algebra can be both derived tame and derived wild. The following result ("tame–wild dichotomy for derived categories") has recently been proved by V. Bekkert and the author [6].

Theorem 2.7. *Every finite dimensional algebra over an algebraically closed field is either derived tame or derived wild.*

2.3 Sliced Boxes

The proof of Theorem 2.7 rests on the technique of representations of boxes ("matrix problems"). We recall now the main related notions. A *box* is a pair $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$, where $\mathscr{A}$ is a category and $\mathscr{V}$ is an $\mathscr{A}$-coalgebra, i.e. an $\mathscr{A}$-bimodule supplied with *comultiplication* $\mu : \mathscr{V} \to \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V}$ and *counit* $\iota : \mathscr{V} \to \mathscr{A}$, which are homomorphisms of $\mathscr{A}$-bimodules and satisfy the usual coalgebra conditions

$$(\mu \otimes 1)\mu = (1 \otimes \mu)\mu, \quad i_l(\iota \otimes 1)\mu = i_r(1 \otimes \iota)\mu = \mathrm{Id},$$

where $i_l : \mathscr{A} \otimes_{\mathscr{A}} \mathscr{V} \simeq \mathscr{V}$ and $i_r : \mathscr{V} \otimes_{\mathscr{A}} \mathscr{A} \simeq \mathscr{V}$ are the natural isomorphisms. The kernel $\overline{\mathscr{V}} = \operatorname{Ker} \iota$ is called the *kernel of the box*. A *representation* of such a box in a category $\mathscr{C}$ is a functor $M : \mathscr{A} \to \mathscr{C}$. Given another representation $N : \mathscr{A} \to \mathscr{C}$, a *morphism* $f : M \to N$ is defined as a homomorphism of $\mathscr{A}$-modules $\mathscr{V} \otimes_{\mathscr{A}} M \to N$, The composition gf of $f : M \to N$ and $g : N \to L$ is defined as the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu \otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{1 \otimes f} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{g} L,$$

while the identity morphism Id_M of M is the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\iota \otimes 1} \mathscr{A} \otimes_{\mathscr{A}} M \xrightarrow{i_l} M.$$

Thus we obtain the *category of representations* $\operatorname{Rep}(\mathfrak{A}, \mathscr{C})$. If $\mathscr{C} = \textsc{vec}$, the category of finite dimensional vector spaces, we just write $\operatorname{Rep}(\mathfrak{A})$. If f is a morphism and $\gamma \in \mathscr{V}(a, b)$, we denote by $f(\gamma)$ the morphism $f(b)(\gamma \otimes \_) : M(a) \to N(a)$. A box $\mathfrak{A}$ is called *normal* (or *group-like*) if there is a set of elements $\omega = \{\, \omega_a \in \mathscr{V}(a, a) \mid a \in \operatorname{Ob} \mathscr{A} \,\}$ such that $\iota(\omega_a) = 1_a$ and $\mu(\omega_a) = \omega_a \otimes \omega_a$ for every $a \in \operatorname{Ob} \mathscr{A}$. In this case, if f is an isomorphism, all morphisms $f(\omega_a)$ are isomorphisms $M(a) \simeq N(a)$. This set is called a *section* of $\mathfrak{A}$. For a normal box, one defines the *differentials* $\partial_0 : \mathscr{A} \to \overline{\mathscr{V}}$ and $\partial_1 : \overline{\mathscr{V}} \to \overline{\mathscr{V}} \otimes_{\mathscr{A}} \overline{\mathscr{V}}$ setting

$$\begin{aligned} \partial_0(\alpha) &= \alpha\omega_a - \omega_b\alpha \quad \text{ for } \alpha \in \mathscr{A}(a, b); \\ \partial_1(\gamma) &= \mu(\gamma) - \gamma \otimes \omega_a - \omega_b \otimes \gamma \quad \text{ for } \gamma \in \overline{\mathscr{V}}(a, b). \end{aligned}$$

Usually we omit indices, writing $\partial\alpha$ and $\partial\gamma$.

Recall that a *free category* $\Bbbk\Gamma$, where Γ is an oriented graph, has the vertices of Γ as its objects and the paths from a to b (a, b being two vertices) as a basis of the vector space $\Bbbk\Gamma(a, b)$.If Γ has no oriented cycles, such a category is locally finite dimensional. A *semi-free* category is a category of fractions $\Bbbk\Gamma[S^{-1}]$, where $S = \{\, g_\alpha(\alpha) \mid \alpha \in \mathfrak{L}\,\}$ and $\mathfrak{L}$ is a subset of the set of loops in Γ (called *marked loops*). The arrows of Γ are called the *free* (respectively, *semi-free*) generators of the free (semi-free) category. A *normal box* $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ is called *free* (*semi-free*) if such is the category $\mathscr{A}$, moreover, the kernel $\overline{\mathscr{V}} = \operatorname{Ker}\iota$ of the box is a free $\mathscr{A}$-bimodule and $\partial\alpha = 0$ for each marked loop α. A set of *free* (respectively, *semi-free*) generators of such a box is a union $\mathbf{S} = \mathbf{S}_0 \cup \mathbf{S}_1$, where $\mathbf{S}_0$ is a set of free (semi-free) generators of the category $\mathscr{A}$ and $\mathbf{S}_1$ is a set of free generators of the $\mathscr{A}$-bimodule $\overline{\mathscr{V}}$.

We call a category $\mathscr{A}$ *trivial* if it is a free category generated by a trivial graph (i.e. one with no arrows); thus $\mathscr{A}(a, b) = 0$ if $a \neq b$ and $\mathscr{A}(a, a) = \Bbbk$. We call $\mathscr{A}$ *minimal*, if it is a semi-free category with a set of semi-free generators consisting of loops only, at most one loop at each vertex. Thus $\mathscr{A}(a, b) = 0$ again if $a \neq b$, while $\mathscr{A}(a, a)$ is either $\Bbbk$ or a rational algebra. We call a normal box $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ *so-trivial* if $\mathscr{A}$ is trivial, and *so-minimal* if $\mathscr{A}$ is minimal and all its loops α are minimal too (i.e. with $\partial\alpha = 0$).

A *layered box* [15] is a semi-free box $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ with a section ω, a set of semi-free generators $\mathbf{S} = \mathbf{S}_0 \cup \mathbf{S}_1$ and a function $\rho : \mathbf{S}_0 \to \mathbb{N}$ satisfying the following conditions:

- A morphism ϕ from $\operatorname{Rep}(\mathfrak{A})$ is an isomorphism if all maps $\phi(\omega_a)$ ($a \in \operatorname{Ob}\mathscr{A}$)) are isomorphisms.
- There is at most one marked loops at each vertex.
- For each $\alpha \in \mathbf{S}_0$ the differential $\partial\alpha$ belongs to the $\mathscr{A}_\alpha$-sub-bimodule of $\overline{\mathscr{V}}$ generated by $\mathbf{S}_1$, where $\mathscr{A}_\alpha$ is the semi-free subcategory of $\mathscr{A}$ with the set of semi-free generators $\{\, \beta \in \mathbf{S}_0 \mid \rho(\beta) < \rho(\alpha)\,\}$.

Obviously, we may suppose, without loss of generality, that $\rho(\alpha) = 0$ for every marked loop α. The set $\{\, \omega, \mathbf{S}, \rho\,\}$ is called a *layer* of the box $\mathfrak{A}$.

In [21] (cf. also [15, 25]) the classification of representations of an arbitrary finite dimensional algebra was reduced to representations of a free layered box. To deal with derived categories we have to consider a wider class of boxes. First, a *factor-box* of a box $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ modulo an ideal $\mathscr{I} \subseteq \mathscr{A}$ is defined as the box $\mathfrak{A}/\mathscr{I} = (\mathscr{A}/\mathscr{I}, \mathscr{V}/(\mathscr{I}\mathscr{V} + \mathscr{V}\mathscr{I}))$ (with obvious comultiplication and counit). Note that if $\mathfrak{A}$ is normal, so is $\mathfrak{A}/\mathscr{I}$.

Definition 2.8. A *sliced box* is a factor-box $\mathfrak{A}/\mathscr{I}$, where $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ is a free layered box such that the set of its objects $\mathsf{V} = \operatorname{Ob}\mathscr{A}$ is a disjoint union $\mathsf{V} = \bigcup_{i \in \mathbb{Z}} \mathsf{V}_i$ so that the following conditions hold:

- $\mathscr{A}(a,a) = \Bbbk$ for each object $a \in \mathscr{A}$;
- $\mathscr{A}(a,b) = 0$ if $a \neq b$, $a \in \mathsf{V}_i$, $b \in \mathsf{V}_j$ with $j \geq i$;
- $\mathscr{V}(a,b) = 0$ if $a \in \mathsf{V}_i$, $b \in \mathsf{V}_j$ with $i \neq j$.

The partition $\mathsf{V} = \bigcup_i \mathsf{V}_i$ is called a *slicing*.

Certainly, in this definition we may assume that the elements of the ideal $\mathscr{I}$ are linear combinations of paths of length at least 2. Otherwise we can just eliminate one of the arrows from the underlying graph without changing the factor $\mathfrak{A}/\mathscr{I}$.

Note that for every representation $M \in \operatorname{Rep}(\mathfrak{A})$, where $\mathfrak{A}$ is a free (semi-free, sliced) box with the set of objects V, one can consider its *dimension* $\mathbf{dim}(M)$, which is a function $\mathsf{V} \to \mathbb{N}$, namely $\mathbf{dim}(M)(a) = \dim M(a)$. We call such a representation *finite dimensional* if its support $\operatorname{supp} M = \{\, a \in \mathsf{V} \mid M(a) \neq 0 \,\}$ is finite and denote by $\operatorname{rep}(\mathfrak{A})$ the category of finite dimensional representations. Having these notions, one can easily reproduce the definitions of families of representations, especially strict families, wild and tame boxes; see [21, 25] for details. The following procedure, mostly copying that of [21], allows to model derived categories by representations of sliced boxes.

Let $\mathbf{A}$ be a finite dimensional algebra, $\mathbf{J}$ be its radical. As far as we are interested in $\mathbf{A}$-modules and complexes, we can replace $\mathbf{A}$ by a Morita equivalent reduced algebra, thus suppose that $\mathbf{A}/\mathbf{J} \simeq \Bbbk^s$ [31]. Let $1 = \sum_{i=1}^s e_i$, where e_i are primitive orthogonal idempotents; set $\mathbf{A}_{ji} = e_j \mathbf{A} e_i$ and $\mathbf{J}_{ji} = e_j \mathbf{J} e_i$; note that $\mathbf{J}_{ji} = \mathbf{A}_{ji}$ if $i \neq j$. We denote by $\mathscr{S}$ the trivial category with the set of objects $\{\, (i,n) \mid n \in \mathbb{N},\ i = 1,2,\dots,s \,\}$ and consider the $\mathscr{S}$-bimodule $\mathscr{J}$ such that

$$\mathscr{J}\big((i,n),(j,m)\big) = \begin{cases} 0 & \text{if } m \neq n-1, \\ \mathbf{J}^*_{ji} & \text{if } m = n-1. \end{cases}$$

Let $\mathscr{B} = \mathscr{S}[\mathscr{J}]$ be the tensor category of this bimodule; equivalently, it is the free category having the same set of objects as $\mathscr{S}$ and the union of bases of all $\mathscr{J}\big((i,n),(j,m)\big)$ as a set of free generators. Denote by $\mathscr{U}$ the $\mathscr{S}$-bimodule such that

$$\mathscr{U}\big((i,n),(j,m)\big) = \begin{cases} 0 & \text{if } n \neq m, \\ \mathbf{A}^*_{ji} & \text{if } n = m \end{cases}$$

and set $\widetilde{\mathscr{W}} = \mathscr{B} \otimes_{\mathscr{S}} \mathscr{U} \otimes_{\mathscr{S}} \mathscr{B}$. Dualizing the multiplication $\mathbf{A}_{kj} \otimes \mathbf{A}_{ji} \to \mathbf{A}_{ki}$, we get homomorphisms

$$\lambda_r : \mathscr{B} \to \mathscr{B} \otimes_{\mathscr{S}} \widetilde{\mathscr{W}}, \quad \lambda_l : \mathscr{B} \to \widetilde{\mathscr{W}} \otimes_{\mathscr{S}} \mathscr{B}, \quad \tilde{\mu} : \widetilde{\mathscr{W}} \to \widetilde{\mathscr{W}} \otimes_{\mathscr{S}} \widetilde{\mathscr{W}}.$$

In particular, $\tilde{\mu}$ defines on $\widetilde{\mathscr{W}}$ a structure of $\mathscr{B}$-coalgebra. Moreover, the subbimodule $\mathscr{W}_0$ generated by $\operatorname{Im}(\lambda_r - \lambda_l)$ is a coideal in $\widetilde{\mathscr{W}}$, i.e. $\tilde{\mu}(\mathscr{W}_0) \subseteq \mathscr{W}_0 \otimes_{\mathscr{B}} \widetilde{\mathscr{W}} \oplus \widetilde{\mathscr{W}} \otimes_{\mathscr{B}} \mathscr{W}_0$. Therefore, $\mathscr{W} = \widetilde{\mathscr{W}}/\mathscr{W}_0$ is also a $\mathscr{B}$-coalgebra, so we get a box $\mathfrak{B} = (\mathscr{B}, \mathscr{W})$. One easily checks that it is free and triangular.

Dualizing multiplication also gives a map

$$\nu : \mathbf{J}_{ji}^* \to \bigoplus_{k=1}^{s} \mathbf{J}_{jk}^* \otimes \mathbf{J}_{ki}^*. \tag{2}$$

Namely, if we choose bases $\{\,\alpha\,\}$, $\{\,\beta\,\}$ $\{\,\gamma\,\}$ in the spaces, respectively, $\mathbf{J}_{ji}$, $\mathbf{J}_{jk}$, $\mathbf{J}_{ki}$, and dual bases $\{\,\alpha^*\,\}$, $\{\,\beta^*\,\}$, $\{\,\gamma^*\,\}$ in their duals, then $\beta^* \otimes \gamma^*$ occurs in $\nu(\alpha^*)$ with the same coefficient as α occurs in $\beta\gamma$. Note that the right-hand space in (2) coincide with each $\mathscr{B}\big((i,n),(j,n-2)\big)$. Let $\mathscr{I}$ be the ideal in $\mathscr{B}$ generated by the images of ν in all these spaces and $\mathfrak{D} = \mathfrak{B}/\mathscr{I} = (\mathscr{A}, \mathscr{V})$, where $\mathscr{A} = \mathscr{B}/\mathscr{I}$, $\mathscr{V} = \mathscr{W}/(\mathscr{I}\mathscr{W} + \mathscr{W}\mathscr{I})$. If necessary, we write $\mathfrak{D}(\mathbf{A})$ to emphasize that this box has been constructed from a given algebra $\mathbf{A}$. Certainly, $\mathfrak{D}$ is a sliced box, and the following result holds.

Theorem 2.9. *The category of finite dimensional representations* $\operatorname{rep}(\mathfrak{D}(\mathbf{A}))$ *is equivalent to the category* $\mathscr{C}^b_{\min}(\mathbf{A})$ *of bounded minimal projective* $\mathbf{A}$*-complexes.*

Proof. Let $A_i = \mathbf{A}e_i$; they form a complete list of non-isomorphic indecomposable projective $\mathbf{A}$-modules; set also $J_i = \operatorname{rad} A_i = \mathbf{J}e_i$. Then $\operatorname{Hom}_{\mathbf{A}}(A_i, J_j) \simeq \mathbf{J}_{ji}$. A representation $M \in \operatorname{rep}(\mathfrak{D})$ is given by vector spaces $M(i,n)$ and linear maps

$$M_{ji}(n) : \mathbf{J}_{ji}^* = \mathscr{A}\big((i,n),(j,n-1)\big) \to \operatorname{Hom}\big(M(i,n), M(j,n-1)\big)$$

subject to the relations

$$\sum_{k=1}^{s} \mathbf{m}\big(M_{jk}(n) \otimes M_{ki}(n+1)\big)\nu(\alpha) = 0 \tag{3}$$

for all i,j,k,n and all $\alpha \in \mathbf{J}_{ji}$, where $\mathbf{m}$ denotes the multiplication of maps

$$\operatorname{Hom}\big(M(k,n), M(j,n-1)\big) \otimes \operatorname{Hom}\big(M(i,n+1), M(k,n)\big) \\ \to \operatorname{Hom}\big(M(i,n+1), M(j,n-1)\big).$$

For such a representation, set $P_n = \bigoplus_{i=1}^{s} A_i \otimes M(i,n)$. Then

$$\operatorname{rad} P_n = \bigoplus_{i=1}^{n} J_i \otimes M(i,n)$$

and

$$\begin{aligned}\operatorname{Hom}_{\mathbf{A}}(P_n, \operatorname{rad} P_{n-1}) &\simeq \bigoplus_{i,j} \operatorname{Hom}_{\mathbf{A}}\big(A_i \otimes M(i,n), J_j \otimes M(j,n-1)\big)\\ &\simeq \bigoplus_{ij} \operatorname{Hom}\big(M(i,n), \operatorname{Hom}_{\mathbf{A}}\big(A_i, J_j \otimes M(j,n-1)\big)\big)\\ &\simeq \bigoplus_{ij} M(i,n)^* \otimes \mathbf{J}_{ji} \otimes M(j,n-1)\\ &\simeq \bigoplus_{ij} \operatorname{Hom}\big(\mathbf{J}_{ji}^*, \operatorname{Hom}\big(M(i,n), M(j,n-1)\big)\big).\end{aligned}$$

Thus the set $\{\, M_{ji}(n) \mid i,j = 1,2,\dots,s \,\}$ defines a homomorphism $d_n : P_n \to P_{n-1}$ and vice versa. Moreover, one easily verifies that the condition (3) is equivalent to the relation $d_n d_{n+1} = 0$. Since every projective $\mathbf{A}$-module can be given in the form $\bigoplus_{i=1}^s A_i \otimes V_i$ for some uniquely defined vector spaces V_i, we get a one-to-one correspondence between finite dimensional representations of $\mathfrak{D}$ and bounded minimal complexes of projective $\mathbf{A}$-modules. In the same way one also establishes one-to-one correspondence between morphisms of representations and of the corresponding complexes, compatible with their multiplication, which accomplishes the proof. $\square$

Corollary 2.10. *An algebra* $\mathbf{A}$ *is derived tame (derived wild) if and only if so is the box* $\mathfrak{D}(\mathbf{A})$.

2.4 Proof of Dichotomy

Now we are able to prove Theorem 2.7. Namely, according to Corollary 2.10, it follows from the analogous result for sliced boxes.

Theorem 2.11. *Every sliced box is either tame or wild.*

Actually, just as in [21] (see also [15, 25]), we shall prove this theorem in the following form.

Theorem 2.11a. *Suppose that a sliced box* $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ *is not wild. For every dimension* $\mathbf{d}$ *of its representations there is a functor* $F_{\mathbf{d}} : \mathscr{A} \to \mathscr{M}$*, where* $\mathscr{M}$ *is a minimal category, such that every representation* $M : \mathscr{A} \to \text{VEC}$ *of* $\mathfrak{A}$ *of dimension* $\mathbf{dim}(M) \le \mathbf{d}$ *is isomorphic to the inverse image* $F^*N = N \circ F$ *for some functor* $N : \mathscr{M} \to \text{VEC}$*. Moreover,* F *can be chosen* strict*, which means that* $F^*N \simeq F^*N'$ *implies* $N \simeq N'$ *and* F^*N *is indecomposable if so is* N*.*

Remark. We can consider the induced box $\mathfrak{A}^F = (\mathscr{M}, \mathscr{M} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{M})$. It is a so-minimal box, and F^* defines a full and faithful functor $\operatorname{rep}(\mathfrak{A}^F) \to \operatorname{rep}(\mathfrak{A})$. Its image consists of all representations $M : \mathscr{A} \to \text{VEC}$ that factorize through F.

Proof. As we fix the dimension $\mathbf{d}$, we may assume that the set of objects is finite (namely, $\operatorname{supp}\mathbf{d}$). Hence the slicing $\mathsf{V} = \bigcup_i \mathsf{V}_i$ (see Definition 2.8) is finite too: $\mathsf{V} = \bigcup_{i=1}^m \mathsf{V}_i$ and we use induction by m. If $m = 1$, $\mathfrak{A}$ is free, and our claim follows from [21, 15]. So we may suppose that the theorem is true for smaller values of m, especially, it is true for the restriction $\mathfrak{A}' = (\mathscr{A}', \mathscr{V}')$ of the box $\mathfrak{A}$ onto the subset $\mathsf{V}' = \bigcup_{i=2}^m \mathsf{V}_i$. Thus there is a strict functor $F' : \mathscr{A}' \to \mathscr{M}$, where $\mathscr{M}$ is a minimal category, such that every representation of $\mathfrak{A}'$ of dimension smaller than $\mathbf{d}$ is of the form F'^*N for $N : \mathscr{M} \to \text{VEC}$. Consider now the amalgamation $\mathscr{B} = \mathscr{A} \bigsqcup^{\mathscr{A}'} \mathscr{M}$ and the box $\mathfrak{B} = (\mathscr{B}, \mathscr{W})$, where $\mathscr{W} = \mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B}$. The functor F' extends to a functor $F : \mathscr{A} \to \mathscr{B}$ and induces a homomorphism of $\mathbf{A}$-bimodules $\mathscr{V} \to \mathscr{W}$; so it defines a functor $F^* : \operatorname{rep}(\mathfrak{B}) \to \operatorname{rep}(\mathfrak{A})$, which is full and faithful. Moreover, every representation of $\mathfrak{A}$ of dimension smaller than $\mathbf{d}$ is isomorphic to F^*N for some N, and all possible dimensions of such N are restricted by some vector $\mathbf{b}$. Therefore, it is enough to prove the claim of the theorem for the box $\mathfrak{B}$.

Note that the category $\mathscr{B}$ is generated by the loops from $\mathscr{M}$ and the images of arrows from $\mathscr{A}(a, b)$ with $b \in \mathsf{V}_1$ (we call them *new arrows*). It implies that all possible relations between these morphisms are of the form $\sum_\beta \beta g_\beta(\alpha) = 0$, where $\alpha \in \mathscr{B}(a, a)$ is a loop (necessarily minimal, i.e. with $\partial\alpha = 0$), g_β are some polynomials, and β runs through the set of new arrows from a to b for some $b \in \mathsf{V}_1$. Consider all of these relations for a fixed b; let them be $\sum_\beta \beta g_{\beta,k}(\alpha) = 0$ $(k = 1, \dots, r)$. Their coefficients form a matrix $\big(g_{\beta,k}(\alpha)\big)$. Using linear transformations of the set $\{\,\beta\,\}$ and of the set of relations, we can make this matrix diagonal, i.e. make all relations being $\beta f_\beta(\alpha) = 0$ for some polynomials f_β. If one of f_β is zero, the box $\mathfrak{B}$ has a sub-box

$$\begin{tikzcd} a \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & b \end{tikzcd},$$

with $\partial\alpha = \partial\beta = 0$, which is wild; hence $\mathfrak{B}$ and $\mathfrak{A}$ are also wild. Otherwise, let $f(\alpha) \neq 0$ be a common multiple of all $f_\beta(\alpha)$, $\Lambda = \{\,\lambda_1, \lambda_2, \dots, \lambda_r\,\}$ be the set of roots of $f(\alpha)$. If $N \in \operatorname{rep}(\mathfrak{B})$ is such that $N(\alpha)$ has no eigenvalues from Λ, then $f(N(\alpha))$ is invertible; thus $N(\beta) = 0$ for all $\beta : a \to b$. So we can apply the *reduction of the loop* α with respect to the set Λ and the dimension $d = \mathbf{b}(a)$, as in [21, Propositions 3,4] or [25, Theorem 6.4]. It gives a new box that has the same number of loops as $\mathfrak{B}$, but the loop corresponding to α is "isolated," i.e. there are no more arrows starting or ending at the same vertex. In the same way we are able to isolate all loops, obtaining a semi-free layered box $\mathfrak{C}$ and a morphism $G : \mathfrak{B} \to \mathfrak{C}$ such that G^* is full and faithful and all representations of $\mathfrak{B}$ of dimensions smaller than $\mathbf{b}$ are of the form G^*L. As the theorem is true for semi-free boxes, it accomplishes the proof. $\square$

Remark. Applying reduction functors, like in the proof above, we can also extend to sliced boxes (thus to derived categories) other results obtained before for free boxes. For instance, we mention the following theorem, quite analogous to that of Crawley-Boevey [17].

Theorem 2.12. *If an algebra* $\mathbf{A}$ *is derived tame, then, for any vector rank* $\mathbf{r}_\bullet \in \Delta = (\mathbf{r}_n \,|\, n \in \mathbb{Z})$*, there is at most finite set of* generic $\mathbf{A}$-complexes *of endolength* $\mathbf{r}_\bullet$*, i.e. such indecomposable minimal bounded complexes* $P_\bullet$ *of projective* $\mathbf{A}$*-modules, not all of which are finitely generated, that* $\mathrm{length}_{\mathbf{E}}(P_n) = \mathbf{r}_n$ *for all* n*, where* $\mathbf{E} = \mathrm{End}_{\mathbf{A}}(P_\bullet)$.

Its proof reproduces again that of [17], with obvious changes necessary to include sliced boxes into consideration.

2.5 Deformations of Derived Tame Algebras

Combining the semi-continuity properties with tame–wild dichotomy, we can prove the results on deformations of derived tame algebras, analogous to those of [28, 35]. Note first the following easy observation.

Proposition 2.13. *Let* $\mathbf{A}$ *be a finite dimensional algebra. For every vector* $\mathbf{r} = (r_1, r_2, \dots, r_s)$ *set* $|\mathbf{r}| = \sum_{i=1}^{s} r_i$*. For every vector rank* $\mathbf{r}_\bullet \in \Delta(\mathbf{A})$ *set* $|\mathbf{r}_\bullet| = \sum_n \mathbf{r}_n$.

1. $\mathbf{A}$ *is derived tame if and only if* $\mathrm{par}(\mathbf{r}_\bullet, \mathbf{A}) \le |\mathbf{r}_\bullet|$ *for every* $\mathbf{r}_\bullet \in \Delta$.
2. $\mathbf{A}$ *is derived wild if and only if there is a vector rank* $\mathbf{r}_\bullet$ *such that* $\mathrm{par}(k\mathbf{r}_\bullet, \mathbf{A}) \ge k^2$ *for every* $k \in \mathbb{N}$.

Proof. The necessity of these conditions follows from the definitions of derived tameness and wildness. Certainly, they exclude each other. Since every algebra is either derived tame or derived wild, the sufficiency follows. □

This proposition together with Theorem 2.3 immediately implies the following result.

Corollary 2.14. *For a family of algebras* $\mathcal{A}$ *over* X *denote*

$$X_{\mathrm{tame}} = \{\, x \in X \,|\, \mathcal{A}(x) \text{ is } \textit{derived tame} \,\},$$
$$X_{\mathrm{wild}} = \{\, x \in X \,|\, \mathcal{A}(x) \text{ is } \textit{derived wild} \,\}.$$

Then X_{tame} *is a countable intersection of open subsets and* X_{wild} *is a countable union of closed subsets.*

Proof. By Theorem 2.3 the set $Z(\mathbf{r}_\bullet) = \{\, x \in X \mid \mathrm{par}(\mathbf{r}_\bullet, \mathbf{A}) \le |\mathbf{r}_\bullet| \,\}$ is open. But $X_{\mathrm{tame}} = \bigcap_{\mathbf{r}} Z(\mathbf{r})$ and hence $X_{\mathrm{wild}} = \bigcup_{\mathbf{r}} (X \setminus Z(\mathbf{r}))$. □

The following conjecture seems very plausible, though even its analogue for usual tame algebras has not yet been proved. (Only for *representation finite* algebras the corresponding result was proved in [33].)

Conjecture 2.15. For any (flat) family of algebras over an algebraic variety X the set X_{tame} is open.

Recall that an algebra $\mathbf{A}$ is said to be a (flat) *degeneration* of an algebra $\mathbf{B}$, and $\mathbf{B}$ is said to be a (flat) *deformation* of $\mathbf{A}$, if there is a (flat) family of algebras $\mathcal{A}$ over an algebraic variety X and a point $p \in X$ such that $\mathcal{A}(x) \simeq \mathbf{B}$ for all $x \neq p$, while $\mathcal{A}(p) \simeq \mathbf{A}$. One easily verifies that we can always assume X to be a *non-singular curve*. Corollary 2.14 obviously implies

Corollary 2.16. *Suppose that an algebra* $\mathbf{A}$ *is a (flat) degeneration of an algebra* $\mathbf{B}$*. If* $\mathbf{B}$ *is derived wild, so is* $\mathbf{A}$*. If* $\mathbf{A}$ *is derived tame, so is* $\mathbf{B}$*.*

If we consider non-flat families, the situation can completely change. The reason is that the dimension is no more constant in these families. That is why it can happen that such a "degeneration" of a derived wild algebra may become derived tame, as the following example due to Brüstle [10] shows.

Example 2.17. There is a (non-flat) family of algebras $\mathcal{A}$ over an affine line $\mathbb{A}^1$ such that all of them except $\mathcal{A}(0)$ are isomorphic to the derived wild algebra $\mathbf{B}$ given by the quiver with relations

$$\begin{tikzcd} & & \bullet & \\ \bullet \arrow[r, "\alpha"] & \bullet \arrow[r, "\beta_1"] & \bullet \arrow[u, "\gamma_1"'] \arrow[d, "\gamma_2"] & \bullet \arrow[l, "\beta_2"'] \\ & & \bullet & \end{tikzcd} \qquad \beta_1\alpha = 0,$$

while $\mathcal{A}(0)$ is isomorphic to the derived tame algebra $\mathbf{A}$ given by the quiver with relations

$$\begin{tikzcd} & & \bullet & \\ \bullet \arrow[r, "\alpha"] & \bullet \arrow[r, "\beta_1"] \arrow[ur, "\xi_1"] & \bullet \arrow[u, "\gamma_1"'] \arrow[d, "\gamma_2"'] & \bullet \arrow[l, "\beta_2"'] \arrow[dl, "\xi_2"] \\ & & \bullet & \end{tikzcd} \qquad \beta_1\alpha = \gamma_1\beta_1 = \gamma_2\beta_2 = 0. \tag{4}$$

Namely, one has to define $\mathcal{A}(\lambda)$ as the factor algebra of the path algebra of the quiver as in (4), but with the relations $\beta_1\alpha = 0,\ \gamma_1\beta_1 = \lambda\xi_1,\ \gamma_2\beta_2 = \lambda\xi_2$. Note that $\dim \mathbf{A} = 16$ and $\dim \mathbf{B} = 15$, which shows that this family is not flat.

Actually, in such a situation the following result always holds.

Proposition 2.18. *Let $\mathcal{A}$ be a family (not necessarily flat) of algebras over a non-singular curve X such that $\mathcal{A}(x) \simeq \mathbf{B}$ for all $x \neq p$, where p is a fixed point, while $\mathcal{A}(p) \simeq \mathbf{A}$. Then there is a flat family $\mathcal{B}$ over X such that $\mathcal{B}(x) \simeq \mathbf{B}$ for all $x \neq p$ and $\mathcal{B}(p) \simeq \mathbf{A}/I$ for some ideal I.*

Proof. Note that the restriction of $\mathcal{A}$ onto $U = X \setminus \{ p \}$ is flat, since $\dim \mathcal{A}(x)$ is constant there. Let $n = \dim \mathbf{B}$, Γ be the quiver of the algebra $\mathbf{B}$ and $\mathbf{G} = \Bbbk\Gamma$ be the path algebra of Γ. Consider the Grassmannian $\mathbf{Gr}(n, \mathbf{G})$, i.e. the variety of subspaces of codimension n of $\mathbf{G}$. The ideals form a closed subset $\mathrm{Alg} = \mathrm{Alg}(n, \mathbf{G}) \subset \mathbf{Gr}(n, \mathbf{G})$. The restriction of the canonical vector bundle $\mathcal{V}$ over the Grassmannian onto Alg is a sheaf of ideals in $\mathcal{G} = \mathbf{G} \otimes \mathcal{O}_{\mathrm{Alg}}$, and the factor $\mathcal{F} = \mathcal{G}/\mathcal{V}$ is a universal family of factor algebras of $\mathbf{G}$ of dimension n. Therefore, there is a morphism $\phi : U \to \mathrm{Alg}$ such that the restriction of $\mathcal{A}$ onto U is isomorphic to $\phi^*(\mathcal{F})$. Since Alg is projective and X is non-singular, ϕ can be continued to a morphism $\psi : X \to \mathrm{Alg}$. Let $\mathcal{B} = \psi^*(\mathcal{F})$; it is a flat family of algebras over X. Moreover, $\mathcal{B}$ coincides with $\mathcal{A}$ outside p. Since both of them are coherent sheaves on a non-singular curve and $\mathcal{B}$ is locally free, it means that $\mathcal{B} \simeq \mathcal{A}/\mathcal{T}$, where $\mathcal{T}$ is the torsion part of $\mathcal{A}$, and $\mathcal{B}(p) \simeq \mathcal{A}(p)/\mathcal{T}(p)$. □

Corollary 2.19. *If a degeneration of a derived wild algebra is derived tame, the latter has a derived wild factor algebra.*

In Brüstle's example 2.17, to obtain a derived wild factor algebra of $\mathbf{A}$, one has to add the relation $\xi_1\alpha = 0$, which obviously holds in $\mathbf{B}$.

By the way, as a factor algebra of a tame algebra is obviously tame (which is no more true for derived tame algebras!), we get the following corollary (cf. also [18, 29]).

Corollary 2.20. *Any deformation (not necessarily flat) of a tame algebra is tame. Any degeneration of a wild algebra is wild.*

3 Nodal Rings

3.1 Backström Rings

We consider a class of rings, which generalizes in a certain way local rings of *ordinary multiple points* of algebraic curves. Following the terminology used in the representations theory of orders, we call them *Backström rings*. In this section we suppose all rings being *noetherian* and *semi-perfect* in the sense of [3]; the latter means that all idempotents can be lifted modulo radical, or, equivalently, that every finitely generated module M has a projective cover, i.e. such an epimorphism $f : P \to M$, where P is projective and $\operatorname{Ker} f \subseteq$

rad P. Hence, just as for finite dimensional algebras, the derived category $\mathscr{D}^-(\mathbf{A}\text{-mod})$ is equivalent to the homotopy category of right bounded *minimal complexes*, i.e. such complexes of finitely generated projective modules

$$\dots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \dots$$

that $\operatorname{Im} d_n \subseteq \operatorname{rad} P_{n-1}$ for all n.

Definition 3.1. A ring $\mathbf{A}$ (noetherian and semi-perfect) is called a *Backström ring* if there is a hereditary ring $\mathbf{H} \supseteq \mathbf{A}$ (also semi-perfect and noetherian) and a (two-sided, proper) $\mathbf{H}$-ideal $\mathbf{J}_{\mathbf{A}}$ such that both $\mathbf{R} = \mathbf{H}/\mathbf{J}$ and $\mathbf{S} = \mathbf{A}/\mathbf{J}$ are semi-simple.

For Backström rings there is a convenient way to the calculations in derived categories. Recall that for a hereditary ring $\mathbf{H}$ every object $C_\bullet$ from $\mathscr{D}^-(\mathbf{H}\text{-mod})$ is isomorphic to the direct sum of its homologies. Especially, any indecomposable object from $\mathscr{D}^-(\mathbf{H}\text{-mod})$ is isomorphic to a shift $N[n]$ for some $\mathbf{H}$-module N, or, the same, to a "short" complex $0 \to P' \xrightarrow{\alpha} P \to 0$, where P and P' are projective modules and α is a monomorphism with $\operatorname{Im} \alpha \subseteq \operatorname{rad} P$ (maybe $P' = 0$). Thus it is natural to study the category $\mathscr{D}^-(\mathbf{A}\text{-mod})$ using this information about $\mathscr{D}^-(\mathbf{H}\text{-mod})$ and the functor $T : \mathscr{D}^-(\mathbf{A}\text{-mod}) \to \mathscr{D}^-(\mathbf{H}\text{-mod})$ mapping $C_\bullet$ to $\mathbf{H} \otimes_{\mathbf{A}} C_\bullet$. (Of course, we mean here the left derived functor of $\otimes$, but when we consider complexes of projective modules, it restricts indeed to the usual tensor product.)

Consider a new category $\mathscr{T} = \mathscr{T}(\mathbf{A})$ (the *category of triples*) defined as follows:

- Objects of $\mathscr{T}$ are triples $(A_\bullet, B_\bullet, \iota)$, where
  - $A_\bullet \in \mathscr{D}^-(\mathbf{H}\text{-mod})$;
  - $B_\bullet \in \mathscr{D}^-(\mathbf{S}\text{-mod})$;
  - ι is a morphism $B_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ from $\mathscr{D}^-(\mathbf{S}\text{-mod})$ such that the induced morphism $\iota^R : \mathbf{R} \otimes_{\mathbf{S}} B_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ is an isomorphism in $\mathscr{D}^-(\mathbf{R}\text{-mod})$.
- A morphism from a triple $(A_\bullet, B_\bullet, \iota)$ to a triple $(A'_\bullet, B'_\bullet, \iota')$ is a pair (Φ, ϕ), where
  - $\Phi : A_\bullet \to A'_\bullet$ is a morphism from $\mathscr{D}^-(\mathbf{H}\text{-mod})$;
  - $\phi : B_\bullet \to B'_\bullet$ is a morphism from $\mathscr{D}^-(\mathbf{S}\text{-mod})$;
  - the diagram

$$\begin{array}{ccc} B_\bullet & \xrightarrow{\iota} & \mathbf{R} \otimes_{\mathbf{H}} A_\bullet \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle 1\otimes\Phi} \\ B'_\bullet & \xrightarrow{\iota'} & \mathbf{R} \otimes_{\mathbf{H}} A'_\bullet \end{array} \tag{5}$$

    commutes in $\mathscr{D}^-(\mathbf{S}\text{-mod})$.

One can define a functor $\mathbf{F} : \mathscr{D}^-(\mathbf{A}\text{-mod}) \to \mathscr{T}(\mathbf{A})$ setting

$$\mathbf{F}(C_\bullet) = (\mathbf{H} \otimes_{\mathbf{A}} C_\bullet, \mathbf{S} \otimes_{\mathbf{A}} C_\bullet, \iota)\,,$$

where $\iota : \mathbf{S} \otimes_{\mathbf{A}} C_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} (\mathbf{H} \otimes_{\mathbf{A}} C_\bullet) \simeq \mathbf{R} \otimes_{\mathbf{A}} C_\bullet$ is induced by the embedding $\mathbf{S} \to \mathbf{R}$. The values of $\mathbf{F}$ on morphisms are defined in an obvious way.

Theorem 3.2. *The functor* $\mathbf{F}$ *is a* full representation equivalence, *i.e. it is*

- dense, *i.e. every object from* $\mathscr{T}$ *is isomorphic to an object of the form* $\mathbf{F}(C_\bullet)$*;*
- full, *i.e. each morphism* $\mathbf{F}(C_\bullet) \to \mathbf{F}(C'_\bullet)$ *is of the form* $\mathbf{F}(\gamma)$ *for some* $\gamma : C_\bullet \to C'_\bullet$*;*
- conservative, *i.e.* $\mathbf{F}(\gamma)$ *is an isomorphism if and only if so is* γ*;*

As a consequence, $\mathbf{F}$ *maps non-isomorphic objects to non-isomorphic and indecomposable to indecomposable.*

Note that in general $\mathbf{F}$ is not *faithful*: it is possible that $\mathbf{F}(\gamma) = 0$ though $\gamma \neq 0$ (cf. Example 3.10.3 below).

Proof (sketched). Consider any triple $T = (A_\bullet, B_\bullet, \iota)$. We may suppose that $A_\bullet$ is a minimal complex from $\mathscr{C}^-(\mathbf{A}\text{-proj})$, while $B_\bullet$ is a complex with zero differential (since $\mathbf{S}$ is semi-simple), and the morphism ι is a usual morphism of complexes. Note that $\mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ is also a complex with zero differential. We have an exact sequence of complexes:

$$0 \longrightarrow \mathbf{J}A_\bullet \longrightarrow A_\bullet \longrightarrow \mathbf{R} \otimes_{\mathbf{H}} A_\bullet \longrightarrow 0.$$

Together with the morphism $\iota : B_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ it gives rise to a commutative diagram in the category of complexes $\mathscr{C}^-(\mathbf{A}\text{-mod})$

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{J}A_\bullet \arrow[r] \arrow[d, equal] & C_\bullet \arrow[r] \arrow[d, "\alpha"'] & B_\bullet \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \mathbf{J}A_\bullet \arrow[r] & A_\bullet \arrow[r] & \mathbf{R} \otimes_{\mathbf{H}} A_\bullet \arrow[r] & 0,
\end{tikzcd}$$

where $C_\bullet$ is the preimage in $A_\bullet$ of $\operatorname{Im} \iota$. The lower row is also an exact sequence of complexes and α is an embedding. Moreover, since ι^R is an isomorphism, $\mathbf{J}A_\bullet = \mathbf{J}C_\bullet$. It implies that $C_\bullet$ consists of projective $\mathbf{A}$-modules and $\mathbf{H} \otimes_{\mathbf{A}} C_\bullet \simeq A_\bullet$, wherefrom $T \simeq \mathbf{F}C_\bullet$.

Let now $(\Phi, \phi) : \mathbf{F}C_\bullet \to \mathbf{F}C'_\bullet$. We suppose again that both $C_\bullet$ and $C'_\bullet$ are minimal, while $\Phi : \mathbf{H} \otimes_{\mathbf{A}} C_\bullet \to \mathbf{H} \otimes_{\mathbf{A}} C'_\bullet$ and $\phi : \mathbf{S} \otimes_{\mathbf{A}} C_\bullet \to \mathbf{S} \otimes_{\mathbf{A}} C'_\bullet$ are morphisms of complexes. Then the diagram (5) is commutative in the category of complexes, so $\Phi(C_\bullet) \subseteq C'_\bullet$ and Φ induces a morphism $\gamma : C_\bullet \to C'_\bullet$. It is evident from the construction that $\mathbf{F}(\gamma) = (\Phi, \phi)$. Moreover, if (Φ, ϕ) is an isomorphism, so are Φ and ϕ (since our complexes are minimal). Therefore, $\Phi(C_\bullet) = C'_\bullet$, i.e. $\operatorname{Im}\gamma = C'_\bullet$. But $\ker\gamma = \ker\Phi \cap C_\bullet = 0$, thus γ is an isomorphism too. $\square$

Evident examples of Backström rings are completions of local rings of ordinary multiple points of algebraic curves. If $\mathbf{A}$ is such a ring, $\mathbf{H}$ is its *normalization* (i.e. integral closure in the full ring of fractions) and $\mathbf{J}$ is the radical of $\mathbf{A}$ (or, the same, of $\mathbf{H}$). If the field $\Bbbk$ is algebraically closed, $\mathbf{A}$ is actually isomorphic to a *bouquet* of power series rings $\Bbbk[[t]]$, i.e. to the subring in $\Bbbk[[t]]^m$, where m is the multiplicity of the singularity, consisting of all sequences $(f_1, f_2, \dots, f_m)$ such that all $f_i(t)$ have the same constant term. Backström rings also include important classes of finite dimensional algebras, such as *gentle, skew-gentle* and others (cf. [13]). Certainly, most of Backström rings are actually wild (hence derived wild). Nevertheless, some of them are derived tame and their derived categories behave very well. An important class of such rings, called *nodal rings*, will be considered in the next subsection.

3.2 Nodal Rings: Strings and Bands

Definition 3.3. A Backström ring $\mathbf{A}$ is called a *nodal ring* if it is *pure noetherian*, i.e. has no minimal ideals, while the hereditary ring $\mathbf{H}$ and the ideal $\mathbf{J}$ from Definition 3.1 satisfy the following conditions:

1. $\mathbf{J} = \operatorname{rad}\mathbf{A} = \operatorname{rad}\mathbf{H}$.
2. $\operatorname{length}_{\mathbf{A}}(\mathbf{H} \otimes_{\mathbf{A}} U) \le 2$ for every simple left $\mathbf{A}$-module U and $\operatorname{length}_{\mathbf{A}}(V \otimes_{\mathbf{A}} \mathbf{H}) \le 2$ for every simple right $\mathbf{A}$-module V.

Note that condition 2 must be imposed both on left and on right modules.

In this situation the hereditary ring $\mathbf{H}$ is also pure noetherian. It is known (cf. e.g. [9]) that such a hereditary ring is Morita equivalent to a direct product of rings $\mathbf{H}(\mathbf{D}, n)$, where $\mathbf{D}$ is a discrete valuation ring (maybe non-commutative) and $\mathbf{H}(\mathbf{D}, n)$ is the subring of $\operatorname{Mat}(n, \mathbf{D})$ consisting of all matrices (a_{ij}) with non-invertible entries a_{ij} for $i < j$. Especially, $\mathbf{H}$ and $\mathbf{A}$ are *semi-prime* (i.e. without nilpotent ideals). For the sake of simplicity we shall only consider the *split case*, when the factor $\mathbf{H}/\mathbf{J}$ is a finite dimensional algebra over a field $\Bbbk$ and $\mathbf{A}/\mathbf{J}$ is its subalgebra.

Remark. In [23] the author showed that if $\mathbf{A}$ is pure noetherian, but not a nodal ring, then the category of $\mathbf{A}$-modules of finite length is wild. All the more so are the categories $\mathbf{A}$-mod and $\mathscr{D}^b(\mathbf{A}\text{-mod})$.

Example 3.4. 1. The first example of a nodal ring is the completion of the local ring of a *simple node* (or a simple double point) of an algebraic curve over a field $\Bbbk$. It is isomorphic to $\mathbf{A} = \Bbbk[[x,y]]/(xy)$ and can be embedded into $\mathbf{H} = \Bbbk[[x_1]] \times \Bbbk[[x_2]]$ as the subring of pairs (f,g) such that $f(0) = g(0)$: x maps to $(x_1, 0)$ and y to $(0, x_2)$. Evidently this embedding satisfies conditions of Definition 3.3.

2. The *dihedral algebra* $\mathbf{A} = \Bbbk\langle\langle\, x, y\,\rangle\rangle/(x^2, y^2)$ is another example of a nodal ring. In this case $\mathbf{H} = \mathbf{H}(\Bbbk[[t]], 2)$ and the embedding $\mathbf{A} \to \mathbf{H}$ is given by the rule

$$x \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

3. The "Gelfand problem," arising from the study of Harish-Chandra modules over the Lie group $\mathrm{SL}\,(2,\mathbb{R})$, is that of classification of diagrams with relations

$$\begin{tikzcd} 2 \arrow[r, bend left, "x_+"] & 1 \arrow[l, bend left, "x_-"] \arrow[r, bend right, "y_-"'] & 3 \arrow[l, bend right, "y_+"'] \end{tikzcd} \qquad x_+x_- = y_+y_-.$$

If we consider the case when x_+x_- is nilpotent (the nontrivial part of the problem), such diagrams are just modules over the ring $\mathbf{A}$, which is the subring of $\mathrm{Mat}(3, \Bbbk[[t]])$ consisting of all matrices (a_{ij}) with $a_{12}(0) = a_{13}(0) = a_{23}(0) = a_{32}(0) = 0$. The arrows of the diagram correspond to the following matrices:

$$x_+ \mapsto te_{12}, \quad x_- \mapsto e_{21}, \quad y_+ \mapsto te_{13}, \quad y_- \mapsto e_{31},$$

where e_{ij} are the matrix units. It is also a nodal ring with $\mathbf{H}$ being the subring of $\mathrm{Mat}(3, \Bbbk[[t]])$ consisting of all matrices (a_{ij}) with $a_{12}(0) = a_{13}(0) = 0$ (it is Morita equivalent to $\mathbf{H}(\Bbbk[[t]], 2)$). More general cases, arising in representation theory of Lie groups $\mathrm{SO}\,(1,n)$, were considered in [41] (cf. also [11, Section 7], where the corresponding diagrams are treated as nodal rings).

4. The classification of *quadratic functors*, which play an important role in algebraic topology (cf. [5]), reduces to the study of modules over the ring $\mathbf{A}$, which is the subring of $\mathbb{Z}_2^2 \times \mathrm{Mat}(2, \mathbb{Z}_2)$ consisting of all triples

$$\left(a, b, \begin{pmatrix} c_1 & 2c_2 \\ c_3 & c_4 \end{pmatrix}\right) \quad \text{with} \quad a \equiv c_1 \mod 2 \quad \text{and} \quad b \equiv c_4 \mod 2,$$

where $\mathbb{Z}_2$ is the ring of 2-adic integers [24]. It is again a (split) nodal ring: one can take for $\mathbf{H}$ the ring of all triples as above, but without congruence conditions; then $\mathbf{H} = \mathbb{Z}_2^2 \times \mathbf{H}(\mathbb{Z}_2, 2)$.

Certainly, we shall apply Theorem 3.2 to study the derived categories of modules over nodal rings. Moreover, in this case the resulting problem belongs to a well-known type, considered in [7, 8, 16] (for its generalization to the non-split case, see [19]). We denote by $U_1, U_2, \dots, U_s$ indecomposable non-isomorphic projective (left) modules over $\mathbf{A}$, by $V_1, V_2, \dots, V_r$ those over $\mathbf{H}$ and consider the decompositions of $\mathbf{H} \otimes_{\mathbf{A}} U_i$ into direct sums of V_j. Condition 2 from Definition 3.3 implies that there are three possibilities:

1. $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j$ for some j and V_j does not occur as a direct summand in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \neq i$;
2. $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j \oplus V_{j'}$ $(j \neq j')$ and neither V_j nor $V_{j'}$ occur in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \neq i$;
3. There are exactly two indices $i \neq i'$ such that $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq \mathbf{H} \otimes_{\mathbf{A}} U_{i'} \simeq V_j$ and V_j does not occur in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \notin \{ i, i' \}$.

We denote by H_j the indecomposable projective $\mathbf{H}$-module such that $H_j / \mathbf{J} H_j \simeq V_j$. Since $\mathbf{H}$ is a semi-perfect *hereditary order*, any indecomposable complex from $\mathscr{D}^-(\mathbf{H}\text{-mod})$ is isomorphic either to $0 \to H_k \xrightarrow{\phi} H_j \to 0$ or to $0 \to H_j \to 0$ (it follows, for instance, from [22]). Moreover, the former complex is completely defined by either j or k and the length $l = \operatorname{length}_{\mathbf{H}}(\operatorname{Cok} \phi)$. We shall denote it both by $C(j, -l, n)$ and by $C(k, l, n+1)$, while the latter complex will be denoted by $C(j, \infty, n)$, where n is the number of the place of H_j in the complex (so the number of the place of H_k is $n+1$). We denote by $\widetilde{\mathbb{Z}}$ the set $(\mathbb{Z} \setminus \{ 0 \}) \cup \{ \infty \}$ and consider the ordering $\leq$ on $\widetilde{\mathbb{Z}}$, which coincides with the usual ordering separately on positive integers and on negative integers, but $l < \infty < -l$ for any positive l. Note that for each j the submodules of H_j form a chain with respect to inclusion. It immediately implies the following result.

Lemma 3.5. *There is a homomorphism $C(j, l, n) \to C(j, l', n)$, which is an isomorphism on the n-th components, if and only if $l \leq l'$ in $\widetilde{\mathbb{Z}}$. Otherwise the n-th component of any homomorphism $C(j, l, n) \to C(j, l', n)$ is zero modulo* $\mathbf{J}$.

We transfer the ordering from $\widetilde{\mathbb{Z}}$ to the set $\mathfrak{E}_{j,n} = \left\{ C(j, l, n) \,\middle|\, l \in \widetilde{\mathbb{Z}} \right\}$, so the latter becomes a chain with respect to this ordering. We also consider one element sets $\mathfrak{F}_{j,n} = \{ (j, n) \}$ and denote

$$\mathfrak{F}^*_{j,n} = \{ (i, j, n) \,|\, V_j \text{ is a direct summand of } \mathbf{H} \otimes_{\mathbf{A}} U_i \} \; .$$

If j is fixed, there can be at most two such values of i. It happens when case 3 from page 103 occurs: $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq \mathbf{H} \otimes_{\mathbf{A}} U_{i'} \simeq V_j$. Then we write $(j, n) \sim (j, n)$. We also write $C(j, -l, n) \sim C(k, l, n+1)$ if these symbols denote the same complex $0 \to H_k \xrightarrow{\phi} H_j \to 0$, and $(j, n) \sim (j', n)$ $(j \neq j')$ if case 2 from page 103 occurs: $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j \oplus V_{j'}$ (if j is fixed, there can be only one j' with this property). Thus a triple $(A_\bullet, B_\bullet, \iota)$ from the category $\mathcal{T}(\mathbf{A})$ is given by homomorphisms $\phi^{ijn}_{jln} : d_{i,j,n}U_i \to r_{j,l,n}V_j$, where $C(j,l,n) \in \mathfrak{E}_{j,n}$ and $(i,j,n) \in \mathfrak{F}^*_{j,n}$. Here the left U_i comes from B_n and the right V_j comes from the direct summands $r_{j,l,n}C(j,l,n)$ of $A_\bullet$ after tensoring by $\mathbf{R}$. Note that if $C(j,-l,n) \sim C(k,l,n+1)$, we have $r_{j,-l,n} = r_{k,l,n+1}$, and if $(j,n) \simeq (j',n)$, we have $d_{i,j,n} = d_{i,j',n}$ for the unique possible value of i. We present ϕ^{ijn}_{jln} by its matrix $M^{ijn}_{jln} \in \mathrm{Mat}(r_{j,l,n} \times d_{i,j,n}, \Bbbk)$. Then Lemma 3.5 implies the following

Proposition 3.6. *Two sets of matrices $\{ M^{ijn}_{jln} \}$ and $\{ N^{ijn}_{jln} \}$ describe isomorphic triples if and only if one of them can be transformed to the other one by a sequence of the following "elementary transformations":*

1. *For any given values of i,n, simultaneously $M^{ijn}_{jln} \mapsto M^{ijn}_{jln}S$ for all j,l such that $(i,j,n) \in \mathfrak{F}^*_{j,n}$, where S is an invertible matrix of appropriate size.*
2. *For any given values of j,l,n, simultaneously $M^{ijn}_{jln} \mapsto S'M^{ijn}_{jln}$ for all $(i,j,n) \in \mathfrak{F}^*_{j,n}$ and $M^{i,k,n-\mathrm{sgn}\, l}_{k,-l,n-\mathrm{sgn}\, l} \mapsto S'M^{i,k,n-\mathrm{sgn}\, l}_{k,-l,n-\mathrm{sgn}\, l}$ for all $(i,k,n-\mathrm{sgn}\, l) \in \mathfrak{F}^*_{k,n-\mathrm{sgn}\, l}$, where S' is an invertible matrix of appropriate size and $C(j,l,n) \sim C(k,-l,n-\mathrm{sgn}\, l)$. If $l = \infty$, it just means $M^{ijn}_{j\infty n} \mapsto S'M^{ijn}_{j\infty n}$.*
3. *For any given values of j, $l' < l$, n, simultaneously $M^{ijn}_{jln} \mapsto M^{ijn}_{jln} + RM^{ijn}_{jl'n}$ for all $(i,j,n) \in \mathfrak{F}^*_{j,n}$, where R is an arbitrary matrix of appropriate size.* (Note that, unlike the preceding transformation, this one does not touch the matrices $M^{i,k,n-\mathrm{sgn}\, l}_{k,-l,n-\mathrm{sgn}\, l}$ such that $C(j,l,n) \sim C(k,-l,n-\mathrm{sgn}\, l)$.)

This sequence can be infinite, but must contain finitely many transformations for every fixed values of j and n.

Therefore, we obtain *representations of the bunch of chains* $\{ \mathfrak{E}_{j,n}, \mathfrak{F}_{j,n} \}$ considered in [7, 8],[2] so we can deduce from these papers a description of indecomposables in $\mathscr{D}^-(\mathbf{A}\text{-mod})$ (for infinite words, which correspond to infinite strings, see [12]). We arrange it in terms of *strings and bands* often used in representation theory.

Definition 3.7. 1. We define the *alphabet* $\mathfrak{X}$ as the set $\bigcup_{j,n}(\mathfrak{E}_{j,n} \cup \mathfrak{F}_{j,n})$. We define symmetric relations $\sim$ and $-$ on $\mathfrak{X}$ by the following exhaustive rules:

(a) $C(j,l,n) - (j,n)$ for all $l \in \mathbb{Z}$;

(b) $C(j,-l,n) \sim C(k,l,n+1)$ if these both symbols correspond to the same complex $0 \to H_k \xrightarrow{\phi} H_j \to 0$;

[2] Note that in [7, 8] they are called "bunches of semichained sets," but we prefer to say "bunches of chains," as in [29, 11].

(c) $(j,n) \sim (j',n)$ $(j' \neq j)$ if $V_j \oplus V_{j'} \simeq \mathbf{H} \otimes_{\mathbf{A}} U_i$ for some i;
(d) $(j,n) \sim (j,n)$ if $V_j \simeq \mathbf{H} \otimes_{\mathbf{A}} U_i \simeq \mathbf{H} \otimes_{\mathbf{A}} U_{i'}$ for some $i' \neq i$.

2. We define an $\mathfrak{X}$-*word* as a sequence $w = x_1 r_1 x_2 r_2 x_3 \dots r_{m-1} x_m$, where $x_k \in \mathfrak{X}$, $r_k \in \{-, \sim\}$ such that

(a) $x_k r_k x_{k+1}$ in $\mathfrak{X}$ for $1 \leq k < m$;
(b) $r_k \neq r_{k+1}$ for $1 \leq k < m-1$.

We call x_1 and x_m the *ends of the word* w.

3. We call an $\mathfrak{X}$-word w *full* if

(a) $r_1 = r_{m-1} = -$
(b) $x_1 \not\sim y$ for each $y \neq x_1$;
(c) $x_m \not\sim z$ for each $z \neq x_m$.

Condition (a) reflects the fact that ι^R must be an isomorphism, while conditions (b,c) come from generalities on bunches of chains [8, 11].

4. A word w is called *symmetric* if $w = w^*$, where $w^* = x_m r_{m-1} x_{m-1} \dots r_1 x_1$ (the *inverse word*), and *quasi-symmetric* if there is a shorter word v such that $w = v \sim v^* \sim \dots \sim v^* \sim v$.

5. We call the end x_1 (x_m) of a word w *special* if $x_1 \sim x_1$ and $r_1 = -$ (respectively, $x_m \sim x_m$ and $r_{m-1} = -$). We call a word w

(a) *usual* if it has no special ends;
(b) *special* if it has exactly one special end;
(c) *bispecial* if it has two special ends.

Note that a special word is never symmetric, a quasi-symmetric word is always bispecial, and a bispecial word is always full.

6. We define a *cycle* as a word w such that $r_1 = r_{m-1} = \sim$ and $x_m - x_1$. Such a cycle is called *non-periodic* if it cannot be presented in the form $v - v - \dots - v$ for a shorter cycle v. For a cycle w we set $r_m = -$, $x_{qm+k} = x_k$ and $r_{qm+k} = r_k$ for any $q, k \in \mathbb{Z}$.

7. A k-th *shift* of a cycle w, where k is an even integer, is the cycle $w^{[k]} = x_{k+1} r_{k+1} x_{k+2} \dots r_{k-1} x_k$. A cycle w is called *symmetric* if $w^{[k]} = w^*$ for some k.

8. We also consider *infinite words* of the sorts $w = x_1 r_1 x_2 r_2 \dots$ (with one end) and $w = \dots x_0 r_0 x_1 r_1 x_2 r_2 \dots$ (with no ends) with the following restrictions:

(a) every pair (j, n) occurs in this sequence only finitely many times;

(b) there is an n_0 such that no pair (j, n) with $n < n_0$ occurs.

We extend to such infinite words all above notions in the obvious manner.

Definition 3.8 (String and band data). 1. *String data* are defined as follows:

(a) a *usual string datum* is a full usual non-symmetric $\mathfrak{X}$-word w;

(b) a *special string datum* is a pair (w, δ), where w is a full special word and $\delta \in \{0, 1\}$;

(c) a *bispecial string datum* is a quadruple $(w, m, \delta_1, \delta_2)$, where w is a bispecial word that is neither symmetric nor quasi-symmetric, $m \in \mathbb{N}$ and $\delta_1, \delta_2 \in \{0, 1\}$.

2. A *band datum* is a triple (w, m, λ), where w is a non-periodic cycle, $m \in \mathbb{N}$ and $\lambda \in \Bbbk^*$; if w is symmetric, we also suppose that $\lambda \neq 1$.

The results of [7, 8] (and [11] for infinite words) imply

Theorem 3.9. *Every string or band datum* $\mathbf{d}$ *defines an indecomposable object* $C_\bullet(\mathbf{d})$ *from* $\mathscr{D}^-(\mathbf{A}\text{-mod})$*, so that*

1. *Every indecomposable object from* $\mathscr{D}^-(\mathbf{A}\text{-mod})$ *is isomorphic to* $C_\bullet(\mathbf{d})$ *for some* $\mathbf{d}$.

2. *The only isomorphisms between these complexes are the following:*

   (a) $C(w) \simeq C(w^*)$ and $C(w, \delta) \simeq C(w^*, \delta)$;

   (b) $C(w, m, \delta_1, \delta_2) \simeq C(w^*, m, \delta_2, \delta_1)$;

   (c) $C(w, m, \lambda) \simeq C(w^{[k]}, m, \lambda) \simeq C(w^{*[k]}, m, 1/\lambda)$ if $k \equiv 0 \mod 4$;

   (d) $C(w^*, m, \lambda) \simeq C(w^{[k]}, m, 1/\lambda) \simeq C(w^{*[k]}, m, \lambda)$ if $k \equiv 2 \mod 4$.

3. *Every object from* $\mathscr{D}^-(\mathbf{A}\text{-mod})$ *uniquely decomposes into a direct sum of indecomposable objects.*

The construction of complexes $C_\bullet(\mathbf{d})$ is rather complicated, especially in the case, when there are pairs (j, n) with $(j, n) \sim (j, n)$ (e.g. special ends are involved). So we only show several examples arising from simple node, dihedral algebra and Gelfand problem.

3.3 Examples

3.3.1 Simple Node

In this case there is only one indecomposable projective **A**-module (**A** itself) and two indecomposable projective **H**-modules H_1, H_2 corresponding to the first and the second direct factors of the ring **H**. We have $\mathbf{H} \otimes_{\mathbf{A}} \mathbf{A} \simeq \mathbf{H} \simeq H_1 \oplus H_2$. So the $\sim$-relation is given by:

1. $(1, n) \sim (2, n)$;
2. $C(j, l, n) \sim C(j, -l, n - \operatorname{sgn} l)$ for any $l \in \mathbb{Z} \setminus \{ 0 \}$.

Therefore, there are no special ends at all. Moreover, any end of a full string must be of the form $C(j, \infty, n)$. Note that the homomorphism in the complex corresponding to $C(j, -l, n)$ and $C(j, l, n+1)$ ($l \in \mathbb{N}$) is just multiplication by x_j^l. Consider several examples of strings and bands.

Example 3.10. 1. Let w be the cycle

$$\begin{aligned} C(2,1,1) &\sim C(2,-1,0) - (2,0) \sim (1,0) - C(1,-2,0) \sim C(1,2,1) - (1,1) \\ &\sim (2,1) - C(2,4,1) \sim C(2,-4,0) - (2,0) \sim (1,0) - C(1,-1,0) \\ &\sim C(1,1,1) - (1,1) \sim (2,1) - C(2,-3,1) \sim C(2,3,2) - (2,2) \\ &\sim (1,2) - C(1,2,2) \sim C(1,-2,1) - (1,1) \sim (2,1) \end{aligned}$$

Then the band complex $C_\bullet(w, 1, \lambda)$ is obtained from the complex of **H**-modules

$$\begin{tikzcd}
 & H_2 \arrow[r, "x_2"] \arrow[ddddd, dashed, dash, bend right, "\lambda"'] & H_2 \arrow[d, dashed, dash] \\
 & H_1 \arrow[r, "x_1^2"] \arrow[d, dashed, dash] & H_1 \\
 & H_2 \arrow[r, "x_2^4"] & H_2 \arrow[d, dashed, dash] \\
 & H_1 \arrow[r, "x_1"] \arrow[d, dashed, dash] & H_1 \\
H_2 \arrow[r, "x_2^3"] \arrow[d, dashed, dash] & H_2 & \\
H_1 \arrow[r, "x_1^2"] & H_1 &
\end{tikzcd}$$

by gluing along the dashed lines (they present the $\sim$ relations $(1, n) \sim (2, n)$). All gluings are trivial, except the last one marked with 'λ'; the latter must be twisted by λ. It gives the **A**-complex

$$\begin{tikzcd}
 & \mathbf{A} \arrow[r, "y"] & \mathbf{A} \\
\mathbf{A} \arrow[ur, "\lambda x^2"] \arrow[dr, "y^3"'] & \mathbf{A} \arrow[ur, "x^2"] \arrow[r, "y^4"] & \mathbf{A} \\
 & \mathbf{A} \arrow[ur, "x"'] &
\end{tikzcd} \tag{6}$$

Here each column presents direct summands of a non-zero component C_n (in our case $n = 2, 1, 0$) and the arrows show the non-zero components of the differential. According to the embedding $\mathbf{A} \to \mathbf{H}$, we have to replace x_1 by x and x_2 by y. Gathering all data, we can rewrite this complex as

$$\mathbf{A} \xrightarrow{\begin{pmatrix} \lambda x^2 \\ 0 \\ y^3 \end{pmatrix}} \mathbf{A} \oplus \mathbf{A} \oplus \mathbf{A} \xrightarrow{\begin{pmatrix} y & 0 \\ x^2 & y^4 \\ 0 & x \end{pmatrix}} \mathbf{A} \oplus \mathbf{A}\,,$$

though the form (6) seems more expressive, so we use it further. If $m > 1$, one only has to replace $\mathbf{A}$ by $m\mathbf{A}$, each element $a \in \mathbf{A}$ by aE, where E is the identity matrix, and λa by $aJ_m(\lambda)$, where $J_m(\lambda)$ is the Jordan $m \times m$ cell with eigenvalue λ. So we obtain the complex

$$\begin{tikzcd}
 & m\mathbf{A} \arrow[r, "yE"] & m\mathbf{A} \\
m\mathbf{A} \arrow[ur, "x^2 J_m(\lambda)"] \arrow[dr, "y^3 E"'] & m\mathbf{A} \arrow[ur, "x^2E"] \arrow[r, "y^4E"] & m\mathbf{A} \\
 & m\mathbf{A} \arrow[ur, "xE"'] &
\end{tikzcd}$$

or, the same,

$$m\mathbf{A} \xrightarrow{\begin{pmatrix} x^2 J_m(\lambda) \\ 0 \\ y^3 E \end{pmatrix}} m\mathbf{A} \oplus m\mathbf{A} \oplus m\mathbf{A} \xrightarrow{\begin{pmatrix} yE & 0 \\ x^2E & y^4E \\ 0 & xE \end{pmatrix}} m\mathbf{A} \oplus m\mathbf{A}\,.$$

2. Let w be the word

$$\begin{aligned}
C(1,\infty,1) &- (1,1) \sim (2,1) - C(2,2,1) \sim C(2,-2,0) - (2,0) \\
&\sim (1,0) - C(1,-3,0) \sim C(1,3,1) - (1,1) \sim (2,1) - C(2,-1,1) \\
&\sim C(2,1,2) - (2,2) \sim (1,2) - C(1,1,2) \sim C(1,-1,1) - (1,1) \\
&\sim (2,1) - C(2,2,1) \sim C(2,-2,0) - (2,0) \sim (1,0) - C(1,\infty,0)\,.
\end{aligned}$$

Then the string complex $C_\bullet(w)$ is

$$\begin{tikzcd}
 & \mathbf{A} \arrow[r, "y^2"] & \mathbf{A} \\
\mathbf{A} \arrow[r, "y"] \arrow[dr, "x"'] & \mathbf{A} \arrow[ur, "x^3"'] & \\
 & \mathbf{A} \arrow[r, "y^2"] & \mathbf{A}
\end{tikzcd}$$

Note that for string complexes (which are always usual in this case) there are no multiplicities m and all gluings are trivial.

3. Set $a = x + y$. Then the factor $\mathbf{A}/a\mathbf{A}$ is represented by the complex $\mathbf{A} \xrightarrow{a} \mathbf{A}$, which is the band complex $C_\bullet(w, 1, 1)$, where

$$\begin{aligned} w = C(1,1,1) &\sim C(1,-1,0) - (1,0) \sim (2,0) - C(2,-1,0) \\ &\sim C(2,1,1) - (2,1) \sim (1,1). \end{aligned}$$

Consider the morphism of this complex to $\mathbf{A}[1]$ given on the 1-component by multiplication $\mathbf{A} \xrightarrow{x} \mathbf{A}$. It is non-zero in $\mathscr{D}^-(\mathbf{A}\text{-mod})$ (presenting a non-zero element from $\operatorname{Ext}^1(\mathbf{A}/a\mathbf{A}, \mathbf{A})$), but the corresponding morphism of triples is $(\Phi, 0)$, where Φ arises from the morphism of the complex $\mathbf{H} \xrightarrow{a} \mathbf{H}$ to $\mathbf{H}[1]$ given by multiplication with x_1. But Φ is homotopic to 0: $x_1 = e_1 a$, where $e_1 = (1,0) \in \mathbf{H}$, thus $(\Phi, 0) = 0$ in the category of triples. So the functor $\mathbf{F}$ from Theorem 3.2 is not faithful in this case.

4. The string complex $C_\bullet(\mathbf{l}, 0)$, where w is the word

$$\begin{aligned} &C(1,\infty,0) - (1,0) \sim (2,0) - C(2,-1,0) \sim C(2,1,1) - (2,1) \\ &\quad \sim (1,1) - C(1,-2,1) \sim C(1,1,2) - (1,2) \sim (2,2) - C(2,-1,2) \\ &\quad \sim C(2,1,3) - (2,3) \sim (1,3) - C(1,-2,3) \sim C(1,2,4) - \dots, \end{aligned}$$

is

$$\dots\ \mathbf{A} \xrightarrow{x^2} \mathbf{A} \xrightarrow{y} \mathbf{A} \xrightarrow{x^2} \mathbf{A} \xrightarrow{y} \mathbf{A} \longrightarrow 0.$$

Its homologies are not left bounded, so it does not belong to $D^b(\mathbf{A}\text{-mod})$.

3.3.2 Dihedral Algebra

This case is very similar to the preceding one. Again there is only one indecomposable projective $\mathbf{A}$-module ($\mathbf{A}$ itself) and two indecomposable projective $\mathbf{H}$-modules H_1, H_2, corresponding to the first and the second columns of matrices from the ring $\mathbf{H}$, and we have $\mathbf{H} \otimes_{\mathbf{A}} \mathbf{A} \simeq \mathbf{H} \simeq H_1 \oplus H_2$. The main difference is that now the unique maximal submodule of H_j is isomorphic to H_k, where $k \neq j$. So the $\sim$-relation is given by:

1. $(1,n) \sim (2,n)$;
2. $C(j,l,n) \sim C(j,-l,n - \operatorname{sgn} l)$ if $l \in \mathbb{Z} \setminus \{0\}$ is even, and $C(j,l,n) \sim C(j',-l,n-\operatorname{sgn} l)$, where $j' \neq j$, if $l \in \mathbb{Z} \setminus \{0\}$ is odd.

Again there are no special ends. The embeddings $H_k \to H_j$ are given by right multiplications with the following elements from $\mathbf{H}$:

$$\begin{aligned} H_1 \to H_1 &\ - \text{ by } t^r e_{11} \quad (\text{colength } 2r), \\ H_1 \to H_2 &\ - \text{ by } t^r e_{12} \quad (\text{colength } 2r-1), \\ H_2 \to H_1 &\ - \text{ by } t^r e_{21} \quad (\text{colength } 2r+1), \\ H_2 \to H_2 &\ - \text{ by } t^r e_{22} \quad (\text{colength } 2r). \end{aligned}$$

When gluing **H**-complexes into **A**-complexes we have to replace them respectively

$$\begin{array}{llll} t^r e_{11} \ - \ \text{by} \ (xy)^r, & & t^r e_{22} \ - \ \text{by} \ (yx)^r, \\ t^r e_{12} \ - \ \text{by} \ (xy)^{r-1}x, & & t^r e_{21} \ - \ \text{by} \ (yx)^r y. \end{array}$$

The gluings are quite analogous to those for simple node, so we only present the results, without further comments.

Example 3.11. 1. Consider the band datum $(w, 1, \lambda)$, where

$$\begin{aligned} w = C(1,-2,0) \sim C(1,2,1) - (1,1) \sim (2,1) - C(2,-5,1) \\ \sim \ C(1,5,2) - (1,2) \sim (2,2) - C(2,4,2) \sim C(2,-4,1) - (2,1) \\ \sim \ (1,1) - C(1,3,1) \sim C(2,-3,0) - (2,0) \sim (1,0). \end{aligned}$$

The corresponding complex $C_\bullet(w, m, \lambda)$ is

$$\begin{tikzcd} & m\mathbf{A} \arrow[r, "xyE"] & m\mathbf{A} \\ m\mathbf{A} \arrow[ur, "(xy)^2xE"] \arrow[r, "(yx)^2E"'] & m\mathbf{A} \arrow[ur, "xyxJ_m(\lambda)"'] & \end{tikzcd}$$

2. Let w be the word

$$\begin{aligned} C(2,\infty,0) - (2,0) \sim (1,0) - C(1,-1,0) \sim C(2,1,1) - (2,1) \\ \sim \ (1,1) - C(1,3,1) \sim C(2,-3,0) - (2,0) \sim (1,0) - C(1,-3,0) \\ \sim \ C(2,3,1) - (2,1) \sim (1,1) - C(1,\infty,1). \end{aligned}$$

Then the string complex $C_\bullet(w)$ is

$$\begin{tikzcd} \mathbf{A} \arrow[r, "e_{21}"] \arrow[dr, "t^2e_{12}"] & \mathbf{A} \\ \mathbf{A} \arrow[r, "te_{21}"'] & \mathbf{A} \end{tikzcd}$$

3. The factor $\mathbf{A}/\mathbf{J}$ is described by the infinite string complex $C_\bullet(w)$:

$$\begin{tikzcd} \cdots \arrow[r, "e_{21}"] & \mathbf{A} \arrow[r, "te_{12}"] & \mathbf{A} \arrow[r, "e_{21}"] & \mathbf{A}. \\ \cdots \arrow[r, "te_{12}"] & \mathbf{A} \arrow[r, "e_{21}"] & \mathbf{A} \arrow[ur, "te_{12}"'] & \end{tikzcd}$$

The corresponding word w is

$$\begin{aligned} \cdots - C(2,1,2) \sim C(1,-1,1) - (1,1) \sim (2,1) - C(2,1,1) \\ \sim \ C(1,-1,0) - (1,0) \sim (2,0) - C(2,-1,0) \sim C(1,1,1) - (1,1) \\ \sim \ (2,1) - C(2,-1,1) \sim C(1,1,2) - \ldots \end{aligned}$$

3.3.3 Gelfand Problem

In this case there are 2 indecomposable projective $\mathbf{H}$-modules H_1 (the first column) and H_2 (both the second and the third columns). There are 3 indecomposable $\mathbf{A}$-projectives A_i $(i = 1, 2, 3)$; A_i correspond to the i-th column of $\mathbf{A}$. We have $\mathbf{H} \otimes_{\mathbf{A}} A_1 \simeq H_1$ and $\mathbf{H} \otimes_{\mathbf{A}} A_2 \simeq \mathbf{H} \otimes_{\mathbf{A}} A_3 \simeq H_2$. So the relation $\sim$ is given by:

1. $(2, n) \sim (2, n)$;
2. $C(j, l, n) \sim C(j, -l, n - \operatorname{sgn} l)$ if l is even;
3. $C(j, l, n) \sim C(j', -l, n - \operatorname{sgn} l)$ $(j' \neq j)$ if l is odd.

Hence a special end is always $(2, n)$.

Example 3.12. 1. Consider the special word w:

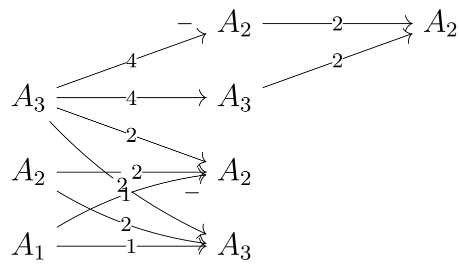
$$\begin{aligned}(2,0) - C(2,-2,0) &\sim C(2,2,1) - (2,1) \sim (2,1) - C(2,-4,1) \\ &\sim \ C(2,4,2) - (2,2) \sim (2,2) - C(2,2,2) \sim C(2,-2,1) - (2,1) \\ &\sim \ (2,1) - C(2,-1,1) \sim C(1,1,2) - (1,2)\,.\end{aligned}$$

The complex $C_\bullet(w, 0)$ is obtained by gluing from the complex of $\mathbf{H}$-modules

$$\begin{tikzcd}
 & H_2 \arrow[r, "2"] \arrow[d, dashed] & H_2 \\
H_2 \arrow[r, "4"] & H_2 & \\
H_2 \arrow[r, "2"] \arrow[u, dashed] & H_2 \arrow[d, dashed] & \\
H_1 \arrow[r, "1"] & H_2 &
\end{tikzcd}$$

Here the numbers inside arrows show the colengths of the corresponding images. We mark dashed lines defining gluings with arrows going from the bigger complex (with respect to the ordering in $\mathfrak{C}_{j,n}$) to the smaller one. When we construct the corresponding complex of $\mathbf{A}$-modules, we replace each H_2 by A_2 and A_3 starting with A_2 (since $\delta = 0$; if $\delta = 1$ we start from A_3). Each next choice is arbitrary with the only requirement that every dashed line must touch both A_2 and A_3. (Different choices lead to isomorphic complexes: one can see it from the pictures below.) All horizontal mappings must be duplicated by slanting ones, carried along the dashed arrow from the starting point or opposite the dashed arrow with the opposite sign from the ending point (the latter procedure will be marked by '$-$' near the duplicated arrow).

So we get the **A**-complex

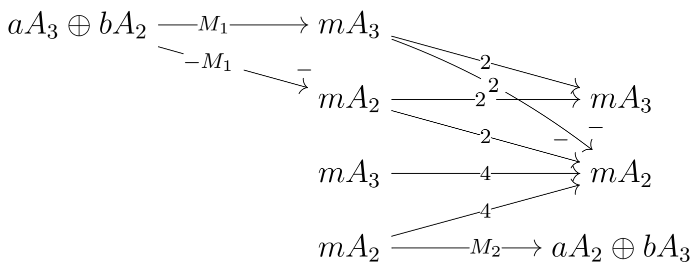


All mappings are uniquely defined by the colengths in the **H**-complex, so we just mark them with 'l.'

2. Let w be the bispecial word

$$
\begin{aligned}
&(2,2)-C(2,2,2)\sim C(2,-2,1)-(2,1)\sim (2,1)-C(2,2,1)\\
&\qquad\sim\ C(2,-2,0)-(2,0)\sim (2,0)-C(2,-4,0)\sim C(2,4,1)-(2,1)\\
&\qquad\sim\ (2,1)-C(2,6,1)\sim C(2,-6,0)-(2,0)
\end{aligned}
$$

The complex $C_\bullet(w,m,1,0)$ is the following one:



where $a=[(m+1)/2]$, $b=[m/2]$, so $a+b=m$. (The change of δ_1,δ_2 transpose A_2 and A_3 at the ends.) All arrows are just $\alpha_l E$, where α_l is defined by the colength l, except of the "end" matrices M_i. To calculate the latter, write $\alpha_l E$ for one of them (say, M_1) and $\alpha_l J$ for another one (say, M_2), where J is the Jordan $m\times m$ cell with eigenvalue 1, then put the odd rows or columns into the first part of M_i and the even ones to its second part. In our example we get

$$
M_1=\alpha_2\left(\begin{array}{ccc|cc}
1&0&0&0&0\\
0&0&0&1&0\\
0&1&0&0&0\\
0&0&0&0&1\\
0&0&1&0&0
\end{array}\right),\qquad
M_2=\alpha_6\left(\begin{array}{ccccc}
1&1&0&0&0\\
0&0&1&1&0\\
0&0&0&0&1\\
\hline
0&1&1&0&0\\
0&0&0&1&1
\end{array}\right).
$$

(We use columns for M_1 and rows for M_2 since the left end is the source and the right end is the sink of the corresponding mapping.)

3. The band complex $C_\bullet(w, 1, \lambda)$, where w is the cycle

$$\begin{aligned}(2,1) &\sim (2,1) - C(2,-2,1) \sim C(2,2,2) - (2,2) \sim (2,2) - C(2,4,2)\\ &\sim C(2,-4,1) - (2,1) \sim (2,1) - C(2,6,1) \sim C(2,-6,0) - (2,0)\\ &\sim (2,0) - C(2,-4,0) \sim C(2,4,1)\end{aligned}$$

is

$$\begin{tikzcd}
mA_2 \arrow[r, "2" description] \arrow[dddr, "2" description, "-"' near end] & mA_2 \arrow[ddr, bend left, "4^\lambda" description, "-" near end] \arrow[dddr, bend left, "4^\lambda" description] & \\
mA_3 \arrow[ur, "2" description] \arrow[r, "4" description] \arrow[dr, "4" description, "-"' near end] \arrow[ddr, bend right, "2" description, "-"' near end] & mA_2 \arrow[dr, "6" description] & \\
 & mA_3 \arrow[r, "6" description] & mA_2 \\
 & mA_3 \arrow[ur, "4^\lambda" description, "-"' near end] \arrow[r, "4^\lambda" description] & mA_3
\end{tikzcd}$$

Superscript 'λ' denotes that the corresponding mapping must be twisted by $J_m(\lambda)$.

4. The projective resolution of the simple $\mathbf{A}$-module U_1 is

$$\begin{tikzcd}
 & A_2 \arrow[r, "1" description] & A_1 \\
A_1 \arrow[ur, "1" description, "-" near end] \arrow[r, "1" description] & A_3 \arrow[ur, "1" description] &
\end{tikzcd}$$

It coincides with the usual string complex $C_\bullet(w)$, where w is

$$\begin{aligned}(1,0) - C(1,-1,0) &\sim C(2,1,1) - (2,1) \sim (2,1) - C(2,-1,1)\\ &\sim C(1,1,2) - (1,2).\end{aligned}$$

The projective resolution of U_2 (U_3) is $A_1 \to A_2$ (respectively $A_1 \to A_3$), which is the special string complex $C_\bullet(w, 0)$ (respectively $C_\bullet(w, 1)$), where

$$w = (2,0) - C(2,-1,0) \sim C(1,1,1) - (1,1).$$

Note that $\operatorname{gl.dim} \mathbf{A} = 2$. It is due to the fact that the case 1 from page 103 occur: $\mathbf{H} \otimes_{\mathbf{A}} A_1 \simeq H_1$. One can prove the following consequence of the above calculations.

Corollary 3.13. *Let* $\mathbf{A}$ *be a nodal ring. Suppose that there is no simple* $\mathbf{A}$-*module* U *such that* $\mathbf{H} \otimes_{\mathbf{A}} U$ *is a simple* $\mathbf{H}$-*module. Then* $\operatorname{gl.dim} \mathbf{A} = \infty$*; moreover, the* finitistic dimension *(in the sense of* [3]*) of* $\mathbf{A}$ *equals* 1*, i.e. for every* $\mathbf{A}$-*module* M *either* $\operatorname{proj.dim} M \leq 1$ *or* $\operatorname{proj.dim} M = \infty$.

4 Projective Curves

In this section we consider "global" analogues of the results of the preceding one, namely, the derived categories of the categories $\operatorname{Coh} X$ of coherent sheaves over some projective curves X. Again we first consider a general framework ("projective configurations," which are an analogue of Backström rings), when the calculations in $\operatorname{Coh} X$ can be reduced to some matrix problems. Then we apply this technique to those classes of projective configurations, where the resulting matrix problem is tame. Throughout this section we suppose that the field $\Bbbk$ is algebraically closed. Analogous results can also be deduced for non-closed fields using the technique of [19], though the picture becomes more complicated.

4.1 Projective Configurations

Definition 4.1. Let X be a projective curve over $\Bbbk$, which we suppose reduced but possibly reducible. We denote by $\pi : \tilde{X} \to X$ its *normalization*; then $\tilde{X}$ is a disjoint union of smooth curves. We call X a *projective configuration* if all components of $\tilde{X}$ are rational curves (i.e. of genus 0) and all singular points p of X are *ordinary*, i.e. the dimension of the tangent cone at p or, the same, the number of linear independent tangent directions at this point equals its multiplicity. Algebraically it means that, if $\pi^{-1}(p) = \{ y_1, y_2, \dots, y_m \}$, the image of $\mathcal{O}_{X,p}$ in $\prod_{i=1}^m \mathcal{O}_{\tilde{X},y_i}$ contains $\prod_{i=1}^m \mathfrak{m}_i$, where $\mathfrak{m}_i$ is the maximal ideal of $\mathcal{O}_{\tilde{X},y_i}$.

We denote by S the set of singular points of X, by $\tilde{S} = \pi^{-1}(S)$ its preimage in $\tilde{X}$ and consider S ($\tilde{S}$) as a closed subvariety of X (resp. $\tilde{X}$). Let $\varepsilon : S \to X$ and $\tilde{\varepsilon} : \tilde{S} \to \tilde{X}$ be their embeddings, and $\overline{\pi} : \tilde{S} \to S$ be the restriction of π onto $\tilde{S}$. We also put $\mathcal{O} = \mathcal{O}_X$, $\widetilde{\mathcal{O}} = \mathcal{O}_{\tilde{X}}$, $\mathcal{S} = \mathcal{O}_S$, $\mathcal{R} = \mathcal{O}_{\tilde{S}}$, and denote by $\mathcal{J}$ the conductor of $\widetilde{\mathcal{O}}$ in $\mathcal{O}$, i.e. the maximal sheaf of $\pi_*\widetilde{\mathcal{O}}$-ideals contained in $\mathcal{O}$. Note that $\mathcal{S}_p \simeq \mathcal{O}_p/\mathcal{J}_p$ and $\mathcal{R}_y \simeq \widetilde{\mathcal{O}}_y/(\pi_*\mathcal{J})_y$. Since S and $\tilde{S}$ are 0-dimensional, hence affine, the categories $\operatorname{Coh} S$ and $\operatorname{Coh} \tilde{S}$ can be identified with the categories of modules, respectively, $\mathbf{S}$-mod and $\mathbf{R}$-mod, where $\mathbf{S} = \prod_{p \in S} \mathcal{S}_p$ and $\mathbf{R} = \prod_{y \in \tilde{S}} \mathcal{R}_y$. If X is a projective configuration, these algebras are semisimple, namely $\mathcal{S}_p \simeq \Bbbk(p)$ and $\mathcal{R}_y \simeq \Bbbk(y)$. Moreover, one easily sees that $\mathcal{J} \simeq \pi_*\widetilde{\mathcal{O}}(-\tilde{S})$, where $\widetilde{\mathcal{O}}(-\tilde{S}) = \widetilde{\mathcal{O}}(-\sum_{y \in \tilde{S}} y)$.

Since X is a projective variety, Serre's theorem [40, Theorem III.5.17] shows that for every coherent sheaf $\mathcal{F} \in \operatorname{Coh} X$ there is an integer n_0 such that all sheaves $\mathcal{F}(n)$ for $n \geq n_0$ are generated by their global sections, or, the same, there are epimorphisms $m\mathcal{O} \to \mathcal{F}(n)$. It easily implies that the derived category $\mathscr{D}^-(\operatorname{Coh} X)$ can be identified with the category of fractions $\mathscr{H}^-(\mathscr{VB}\, X)[\mathcal{Q}^{-1}]$, where $\mathscr{VB}\, X$ is the category of locally free coherent sheaves (equivalently, the category of vector bundles [40, Exercise II.5.18]) over X and $\mathcal{Q}$ is the set of quasi-isomorphisms in $\mathscr{H}^-(\mathscr{VB}\,X)$. So we always present objects from $\mathscr{D}^-(\mathrm{Coh}\,X)$ and from $\mathscr{D}^-(\mathrm{Coh}\,\tilde{X})$ as complexes of vector bundles. We denote by $T : \mathscr{D}^-(\mathrm{Coh}\,X) \to \mathscr{D}^-(\mathrm{Coh}\,\tilde{X})$ the left derived functor $L\pi^*$. Again if $\mathcal{C}_\bullet$ is a complex of vector bundles, $T\mathcal{C}_\bullet$ coincides with $\pi^*\mathcal{C}_\bullet$.

Just as in Subsection 3.1, we define the *category of triples* $\mathcal{T} = \mathcal{T}(X)$: Objects of $\mathcal{T}$ are triples $(\mathcal{A}_\bullet, \mathcal{B}_\bullet, \iota)$, where

- $\mathcal{A}_\bullet \in \mathscr{D}^-(\mathrm{Coh}\,\tilde{X})$ (we always present it as a complex of vector bundles);
- $\mathcal{B}_\bullet \in \mathscr{D}^-(\mathrm{Coh}\,\mathcal{S})$ (we always present it as a complex with zero differential);
- ι is a morphism $\mathcal{B}_\bullet \to \overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}_\bullet$ from $\mathscr{D}^-(\mathrm{Coh}\,\mathcal{S})$ such that the induced morphism $\iota^R : \overline{\pi}^*\mathcal{B}_\bullet \to \tilde{\varepsilon}^*\mathcal{A}_\bullet$ is an isomorphism in $\mathscr{D}^-(\mathrm{Coh}\,\mathcal{R})$.

A morphism from a triple $(\mathcal{A}_\bullet, \mathcal{B}_\bullet, \iota)$ to a triple $(\mathcal{A}'_\bullet, \mathcal{B}'_\bullet, \iota')$ is a pair (Φ, ϕ), where

- $\Phi : \mathcal{A}_\bullet \to \mathcal{A}'_\bullet$ is a morphism from $\mathscr{D}^-(\mathrm{Coh}\,\tilde{X})$;
- $\phi : \mathcal{B}_\bullet \to \mathcal{B}'_\bullet$ is a morphism from $\mathscr{D}^-(\mathrm{Coh}\,\mathcal{S})$;
- the diagram

$$\begin{tikzcd} \mathcal{B}_\bullet \arrow[r, "\iota"] \arrow[d, "\phi"'] & \overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}_\bullet \arrow[d, "\overline{\pi}_*\tilde{\varepsilon}^*\Phi"] \\ \mathcal{B}'_\bullet \arrow[r, "\iota'"] & \overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}'_\bullet \end{tikzcd} \tag{7}$$

  commutes in $\mathscr{D}^-(\mathrm{Coh}\,\mathcal{S})$.

We define a functor $\mathbf{F} : \mathscr{D}^-(\mathrm{Coh}\,X) \to \mathcal{T}(X)$ setting $\mathbf{F}(\mathcal{C}_\bullet) = (\pi^*\mathcal{C}_\bullet, \varepsilon^*\mathcal{C}_\bullet, \iota)$, where $\iota : \varepsilon^*\mathcal{C}_\bullet \to \overline{\pi}_*\tilde{\varepsilon}^*(\pi^*\mathcal{C}_\bullet)$ is induced by the natural isomorphism $\overline{\pi}^*\varepsilon^*\mathcal{F}_\bullet \simeq \tilde{\varepsilon}^*\pi^*\mathcal{F}_\bullet$.. Just as in Section 1, the following theorem holds (with almost the same proof, see [12]).

Theorem 4.2. *The functor* $\mathbf{F}$ *is a representation equivalence, i.e. it is dense and conservative.*

Remark. We do not now whether it is *full*, though it seems very plausible.

Just as for Backström rings, most projective configurations are *vector bundle wild*. Namely, in [29] it was shown that the only projective curves, which are not vector bundle wild, are the following:

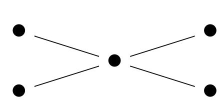
- Projective line $\mathbb{P}^1$.
- Elliptic curves, i.e. smooth projective curves of genus 1, or, the same, smooth plane cubics.

- Projective configurations of types A and $\tilde{A}$ (see the next subsection for their definitions).

Actually, projective line and projective configurations of type A are *vector bundle finite*, i.e. have only finitely many indecomposable vector bundles (up to isomorphism and natural twists), while elliptic curves and projective configurations of type $\tilde{A}$ are vector bundle tame. Since the derived category $\mathscr{D}^-(\operatorname{Coh} X)$ (even $\mathscr{D}^b(\operatorname{Coh} X)$) contains $\operatorname{Coh} X$ as a full subcategory, it can never be representation finite. We always have one-parameter family of skyscrapers, such as $\Bbbk(x)$ ($x \in X$). If the curve X is smooth, the category $\operatorname{Coh} X$ is hereditary, thus its indecomposable objects are just shifts of sheaves. Moreover, every coherent sheaf is a direct sum of a vector bundle and several *skyscrapers*, i.e. sheaves supported in one point. The latter are just $\mathcal{O}/\mathfrak{m}_x^k$ for some $x \in X$ and some integer k, so they form one-parameter families. Hence, if a smooth curve is vector bundle tame, it is derived tame as well. It happens, just as in the case of pure noetherian rings, that *all* vector bundle tame projective curves are also derived tame, though for projective configurations of types A and $\tilde{A}$ the structure of skyscrapers is more complicated (it involve modules over local rings, which are nodal) and, moreover, there are "mixed" sheaves, which are neither vector bundles (even not torsion free) nor skyscrapers.

4.2 Configurations of Types A and $\tilde{A}$

Now we suppose that X is a projective configurations and all singular points of X are *nodes* (or double points). To such a curve one associates a graph $\Delta(X)$ called its *intersection graph* or *dual graph*. The vertices of $\Delta(X)$ are the irreducible components of X and the edges of $\Delta(X)$ are the singular points of X. The ends of an edge p are the components containing this point. In particular, if p only belongs to one component, it is a loop in $\Delta(X)$. Note that the graph $\Delta(X)$ does not completely define X. For instance, consider the case, when $\Delta(X)$ is the graph of type $\tilde{D}_4$, i.e.



The component corresponding to the central point contains 4 singular points. Therefore, their *harmonic ratio* is invariant under isomorphisms of $\mathbb{P}^1$ and can be an arbitrary scalar $\lambda \in \Bbbk \setminus \{0,1\}$ (these points can always be chosen as $0,1,\lambda,\infty$). Thus the configurations with this dual graph but different values of λ are not isomorphic.

We say that a projective configuration X is

- *of type* A if its intersection graph is a chain:

$$1 \;\text{———}\; 2 \;\text{———}\; \cdots \;\text{———}\; s$$

- *of type* $\tilde{A}$ if its intersection graph is a cycle:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[rrr, dash, bend right] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & s \end{tikzcd}$$

(If $s = 1$, the projective configuration of type A is just a projective line, while the projective configuration of type $\tilde{A}$ is a nodal cubic.)

In other words, in the A-case irreducible components $X_1, X_2, \dots, X_s$ and singular points $p_1, p_2, \dots, p_{s-1}$ can be arranged so that $p_i \in X_i \cap X_{i+1}$, while in the $\tilde{A}$-case the components $X_1, X_2, \dots, X_s$ and the singular points $p_1, p_2, \dots, p_s$ can be so arranged that $p_i \in X_i \cap X_{i+1}$ for $i < s$ and $p_s \in X_s \cap X_1$. Note that in the A-case $s > 1$, while in the $\tilde{A}$-case $s = 1$ is possible: then there is one component with one ordinary double point (a nodal plane cubic). These projective configurations are global analogues of nodal rings, and the calculations according Theorem 4.2 are quite similar to those of Section 3. We present here the calculations for the $\tilde{A}$-case and add remarks explaining what changes should be made for the A-case.

If $s > 1$, the normalization of X is just a disjoint union $\bigsqcup_{i=1}^{s} X_i$; for uniformity, we write $X_1 = \tilde{X}$ if $s = 1$. We also denote $X_{qs+i} = X_i$. Certainly, $X_i \simeq \mathbb{P}_1$ for all i. Every singular point p_i has two preimages p_i', p_i'' in $\tilde{X}$; we suppose that $p_i' \in X_i$ corresponds to the point $\infty \in \mathbb{P}^1$ and $p_i'' \in X_{i+1}$ corresponds to the point $0 \in \mathbb{P}^1$. Recall that any indecomposable vector bundle over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d)$ for some $d \in \mathbb{Z}$. So every indecomposable complex from $\mathscr{D}^-(\operatorname{Coh} \tilde{X})$ is isomorphic either to $0 \to \mathcal{O}_i(d) \to 0$ or to $0 \to \mathcal{O}_i(-lx) \to \mathcal{O}_i \to 0$, where $\mathcal{O}_i = \mathcal{O}_{X_i}$, $d \in \mathbb{Z}$, $l \in \mathbb{N}$ and $x \in X_i$. The latter complex corresponds to the indecomposable sky-scraper sheaf of length l and support $\{\, x \,\}$. (It is isomorphic in the derived category to any complex $0 \to \mathcal{O}_i((k-l)x) \to \mathcal{O}_i(kx) \to 0$ with arbitrary $k \in \mathbb{Z}$.) We denote this complex by $C(x, -l, n)$ and by $C(x, l, n+1)$. The complex $0 \to \mathcal{O}_i(d) \to$ is denoted by $C(p_i', d\omega, n)$ and by $C(p_{i-1}'', d\omega, n)$. As before, n is the unique place, where the complex has non-zero homologies. We define the symmetric relation $\sim$ for these symbols setting $C(x, -l, n) \sim C(x, l, n+1)$ and $C(p_i', d\omega, n) \sim C(p_{i-1}'', d\omega, n)$.

Let $\mathbb{Z}^\omega = (\mathbb{Z} \oplus \{\, 0 \,\}) \cup \mathbb{Z}\omega$, where $\mathbb{Z}\omega = \{\, d\omega \mid d \in \mathbb{Z} \,\}$. We introduce an ordering on $\mathbb{Z}^\omega$, which is natural on $\mathbb{N}$, on $-\mathbb{N}$ and on $\mathbb{Z}\omega$, but $l < d\omega < -l$ for each $l \in \mathbb{N}$, $d \in \mathbb{Z}$. Recall that $\operatorname{Hom}(\mathcal{O}_i(d), \mathcal{O}_i(d'))$ can be considered as the space of homogeneous polynomial of degree $d' - d$ in homogeneous coordinates on $\mathbb{P}^1$ if $d' \geq d$; otherwise it is zero. Note also that $\mathcal{C}_n(x) \simeq \Bbbk$ if $\mathcal{C} = C(x, l, n)$ for some $l \in \mathbb{Z}^\omega$. It easily implies the following analogue of Lemma 3.5.

Lemma 4.3. *There is a morphism of complexes $\mathcal{C}_\bullet = C(x,z,n) \to \mathcal{C}'_\bullet = C(x,z',n)$ such that its* nth *component induces a non-zero mapping (actually an isomorphism) $\mathcal{C}_n(x) \to \mathcal{C}'_n(x)$ if and only if $z \le z'$ in $\mathbb{Z}^\omega$. Moreover, if $z = d\omega$, $z' = d'\omega$, $d' > d$ and $x \in S$, hence also $\mathcal{C}_\bullet = C(x',z,n)$ and $\mathcal{C}'_\bullet = C(x',z',n)$ for another singular point x', there is a morphism $\phi : \mathcal{C}_\bullet \to \mathcal{C}'_\bullet$ such that $\phi(x) \ne 0$, but $\phi(x') = 0$.*

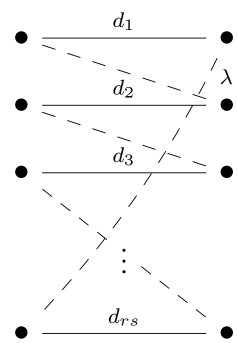
We introduce the ordered sets $\mathfrak{E}_{x,n} = \{\, C(x,z,n) \mid z \in \mathbb{Z}^\omega \,\}$ with the ordering inherited from $\mathbb{Z}^\omega$, We also put $\mathfrak{F}_{x,n} = \{\,(x,n)\,\}$ and $(p'_i,n) \sim (p''_{i-1},n)$ for all i,n. Lemma 4.3 shows that the category of triples $\mathcal{T}(X)$ can be again described in terms of the bunch of chains $\{\, \mathfrak{E}_{x,n},\, \mathfrak{F}_{x,n} \,\}$. Thus we can describe indecomposable objects in terms of strings and bands just as for nodal rings. We leave the corresponding definitions to the reader; they are quite analogous to those from Section 3. If we consider a configuration of type A, we have to exclude the points p'_s, p''_s and the corresponding symbols $C(p'_s,z,n)$, $C(p''_s,z,n)$, (p'_s,n), (p''_s,n). Thus in this case $C(p''_{s-1},d\omega,n)$ and $C(p'_1,d\omega,n)$ are not in $\sim$ relation with any symbol. It makes possible finite or one-side infinite full strings, while in the $\tilde{A}$-case only two-side infinite strings are full. Note that an infinite word must contain a finite set of symbols (x,n) with any fixed n; moreover there must be n_0 such that $n \ge n_0$ for all entries (x,n) that occur in this word.

If $x \notin \tilde{S}$ (thus $z \notin \mathbb{Z}\omega$), the complex $C(x,z,n)$ vanishes under $\tilde{\varepsilon}^*$, so gives no essential input into the category of triples. It gives rise to the n-th shift of a sky-scraper sheaf with support at the regular point $\pi(x)$. In the language of bunches of chains it follows from the fact that $(x,n) \not\sim (x',n)$ for any $x' \ne x$, hence the only full words containing (x,n) are $(x,n) - C(x,l,n)$ for some $l \in \mathbb{Z} \setminus \{\,0\,\}$. Therefore, in the following examples we only consider complexes $C(x,z,n)$ with $x \in \tilde{S}$. Moreover, we confine most examples to the case $s = 1$ (so X is a nodal cubic). If $s > 1$, one must distribute vector bundles in the pictures below among the components of $\tilde{X}$.

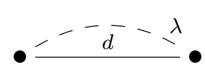
Example 4.4. 1. First of all, even a classification of vector bundles is non-trivial in $\tilde{A}$ case. They correspond to the bands concentrated at 0 place, i.e. such that the underlying cycle w is of the form

$$\begin{aligned}(p'_s,0) &\sim (p''_s,0) - C(p''_s,d_1\omega,0) \sim C(p'_1,d_1\omega,0) - (p'_1,0)\\ &\sim (p''_1,0) - C(p''_1,d_2\omega,0) \sim C(p'_2,d_2\omega,0) - (p'_2,0)\\ &\sim (p''_2,0) - C(p''_2,d_3\omega,0) \sim \cdots \sim C(p'_s,d_{rs}\omega,0)\end{aligned}$$

(obviously, its length must be a multiple of s, and we can start from any place p'_k,p''_k). Then $\mathcal{C}_\bullet(w,m,\lambda)$ is actually a vector bundle, which can be schematically described as the following gluing of vector bundles over $\tilde{X}$.



Here horizontal lines symbolize line bundles over X_i of the superscripted degrees, their left (right) ends are basic elements of these bundles at the point ∞ (respectively 0), and the dashed lines show which of them must be glued. One must take m copies of each vector bundle from this picture and make all gluings trivial, except one going from the uppermost right point to the lowermost left one (marked by 'λ'), where the gluing must be performed using the Jordan $m \times m$ cell with eigenvalue λ. In other words, if $e_1, e_2, \dots, e_m$ and $f_1, f_2, \dots, f_m$ are bases of the corresponding spaces, one has to identify f_1 with λe_1 and f_k with $\lambda e_k + e_{k-1}$ for $k > 1$. We denote this vector bundle over X by $\mathcal{V}(\mathbf{d}, m, \lambda)$, where $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$; it is of rank mr and of degree $m \sum_{i=1}^{r} d_i$. If $r = s = 1$, this picture becomes



If $r = m = 1$, we obtain all line bundles: they are $\mathcal{V}((d_1, d_2, \dots, d_s), 1, \lambda)$ (of degree $\sum_{i=1}^{s} d_i$). Thus the Picard group is $\mathbb{Z}^s \times \Bbbk^*$.

In the A-case, there are no bands concentrated at 0 place, but there are finite strings of this sort:

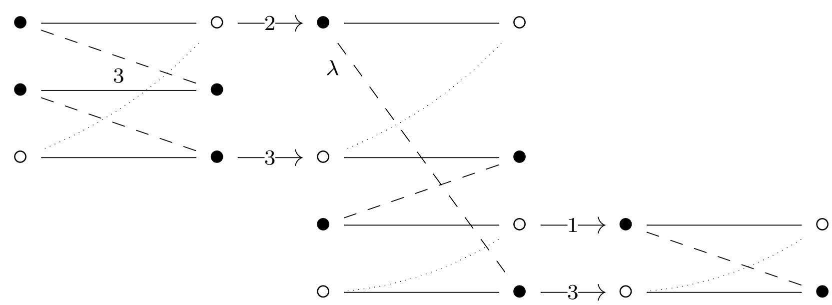
$$\begin{aligned} &C(p_1'', d_1\omega, 0) - (p_1', 0) \sim (p_1'', 0) - C(p_1'', d_2\omega, 0) \sim \\ &\qquad \sim C(p_2', d_2, 0) - (p_2', 0) \sim (p_2'', 0) - C(p_2'', d_3, 0) \sim \\ &\qquad \cdots \sim C(p_{s-1}', d_{s-1}\omega, 0) - (p_{s-1}', 0) \sim (p_{s-1}'', 0) - C(p_{s-1}'', d_s\omega, 0) \end{aligned}$$

So vector bundles over such configurations are in one-to-one correspondence with integral vectors $(d_1, d_2, \dots, d_s)$; in particular, all of them are line bundles and the Picard group is $\mathbb{Z}^s$. In the picture above one has to set $r = 1$ and to omit the last gluing (marked with 'λ').

2. From now on $s=1$, so we write p instead of p_1. Let w be the cycle

$$\begin{aligned}(p'',1) &\sim (p',1) - C(p',-2,1) \sim C(p',2,2) - (p',2) \sim (p'',2)-\\ &- C(p'',3\omega,2) \sim C(p',3\omega,2) - (p',2) \sim (p'',2) - C(p'',3,2) \sim\\ &\sim C(p'',-3,1) - (p'',1) \sim (p',1) - C(p',1,1) \sim C(p',-1,0)-\\ &\qquad - (p',0) \sim (p'',0) - C(p'',-2,0) \sim C(p'',2,1).\end{aligned}$$

Then the band complex $\mathcal{C}_\bullet(w,m,\lambda)$ can be pictured as follows:

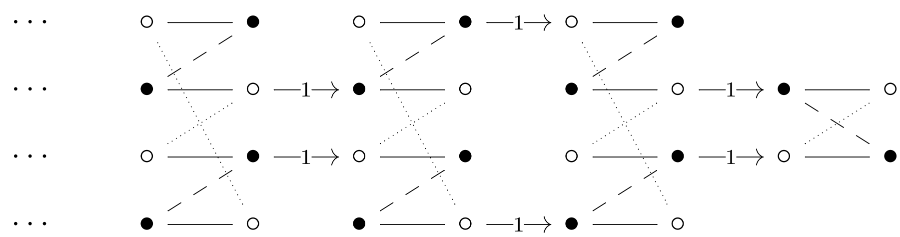


Again horizontal lines describe vector bundles over $\tilde{X}$. Bullets and circles correspond to the points ∞ and 0; circles show those points, where the corresponding complex gives no input into $\overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}_\bullet$. Horizontal arrows show morphisms in $\mathcal{A}_\bullet$; the numbers l inside give the lengths of factors. For instance, the first row in this picture describes the complex $C(p',-2,1)$, the second one is $C(p',3\omega,2)$ (or, the same, $C(p'',3\omega,2)$) and the last one is $C(p'',-3,0)$. Dashed and dotted lines describe gluings. Dashed lines (between bullets) correspond to mandatory gluings arising from relations $(p',n)\sim(p'',n)$ in the word w, while dotted lines (between circles) can be drawn arbitrarily; the only conditions are that each circle must be an end of a dotted line and the dotted lines between circles sitting at the same level must be parallel (in our picture they are between the 1st and 3rd levels and between the 4th and 5th levels). The degrees of line bundles in complexes $C(x,z,n)$ with $z\in\mathbb{N}\cup(-\mathbb{N})$ (they are described by the levels containing 2 lines) can be chosen as $d-l$ and d with arbitrary d, otherwise (in the second row) they are superscripted over the line. We set $d=1$ in the last row and $d=0$ elsewhere. Thus the resulting complex is

$$\mathcal{V}((-2,3,-3),m,1) \longrightarrow \mathcal{V}((0,0,-1,-2),m,\lambda) \longrightarrow \mathcal{V}((0,1),m,1)$$

(we do not precise mappings, but they can be easily restored). Note that our choice of d's enables to consider the components of this complex as the "standard" vector bundles $\mathcal{V}(\mathbf{d},m,\lambda)$ from the preceding example.

3. If $s = 1$, the sky-scraper sheaf $\Bbbk(p)$ is described by the complex



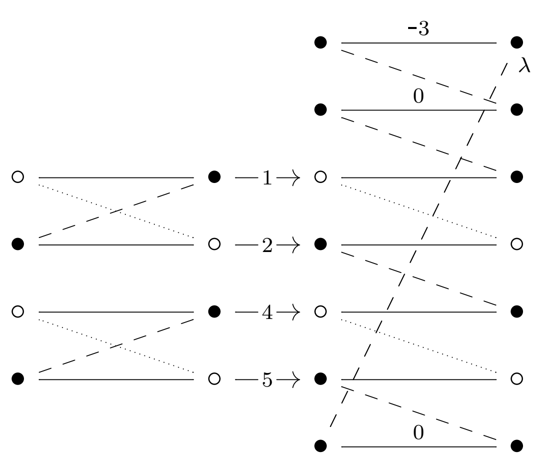
which is the string complex corresponding to the word

$$\begin{aligned}\dots C(p',-1,2) - (p',2) \sim (p'',2) - C(p'',1,2) \sim C(p'',-1,1)-\\ - (p'',1) \sim (p',1) - C(p',1,1) \sim C(p',-1,0) - (p',0) \sim\\ \sim (p'',0) - C(p'',-1,0) \sim C(p'',1,1) - (p'',1) \sim (p',1)-\\ - C(p',-1,1) \sim C(p',1,2) - (p',2) \sim (p'',2) - C(p'',-1,2) \dots\end{aligned}$$

4. The band complex $\mathcal{C}(w,m,\lambda)$, where w is the cycle

$$\begin{aligned}(p',0) \sim (p'',0) - C(p'',-3\omega,0) \sim C(p',-3\omega,0)-\\ - (p',0) \sim (p'',0) - C(p'',0\omega,0) \sim C(p',0\omega,0) - (p',0) \sim\\ \sim (p'',0) - C(p'',-1,0) \sim C(p'',1,1) - (p'',1) \sim (p',1)-\\ - C(p'',2,1) \sim C(p',-2,0) - (p',0) \sim (p'',0) - C(p'',-4,0) \sim\\ \sim C(p'',4,1) - (p'',1) \sim (p',1) - C(p',5,1) \sim C(p',-5,0)-\\ - (p',0) \sim (p'',0) - C(p'',0\omega,0) \sim C(p',0\omega,0)\end{aligned}$$

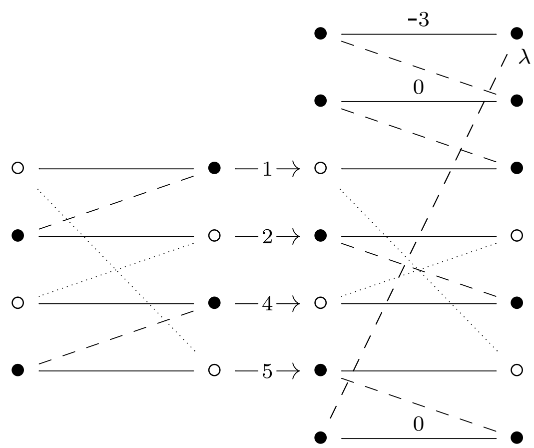
describes the complex



or

$$\mathcal{V}((0,0),m,1) \oplus \mathcal{V}((0,0),m,1) \longrightarrow \mathcal{V}((-3,0,1,2,4,5,0),m,\lambda).$$

Its homologies are zero except the place 0, so it corresponds to a coherent sheaf. One can see that this sheaf is a “mixed” one (neither torsion free nor sky-scraper). Note that this time we could trace dotted lines another way, joining the first free end with the last one and the second with the third:



It gives an isomorphic object in $\mathscr{D}(\operatorname{Coh} X)$:

$$\mathcal{V}((0,0,0,0),m,1) \longrightarrow \mathcal{V}((\text{-}3,0,1,5,0),m,\lambda) \oplus \mathcal{V}((2,4),m,1).$$

Remark 4.5. In [12] we used another encoding of strings and bands for projective configurations, which is equivalent but uses more specifics of the situation. In this paper we prefer to use a uniform encoding, which is the same both for nodal rings and for projective configurations.

4.3 Application to Cohen–Macaulay Modules

The description of vector bundles has an important application in the theory of Cohen–Macaulay modules over *surface singularities*.

Definition 4.6. 1. By a *normal surface singularity* over the field $\Bbbk$, which we suppose algebraically closed, we mean a complete noetherian $\Bbbk$-algebra $\mathbf{A}$ such that:

- $\operatorname{Kr.dim} \mathbf{A} = 2$;
- $\mathbf{A}/\mathfrak{m} \simeq \Bbbk$, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{A}$;
- $\mathbf{A}$ has no zero divisors and is *normal*, i.e. integrally closed in its field of fractions;
- $\mathbf{A}$ is not regular, i.e. $\operatorname{gl.dim} \mathbf{A} = \infty$.

We denote by X the scheme $\operatorname{Spec} \mathbf{A}$, by $p \in X$ the point corresponding to the maximal ideal $\mathfrak{m}$ (the unique closed point of X) and by $\check{X}$ the open subscheme $X \setminus \{p\}$.

2. A *resolution* of such a singularity is a morphism of $\Bbbk$-schemes $\pi : \tilde{X} \to X$ such that:

- $\tilde{X}$ is smooth;
- π is projective (hence closed) and birational;
- the restriction of π onto $\tilde{X} \setminus E$, where $E = \pi^{-1}(p)_{\mathrm{red}}$, is an isomorphism $\tilde{X} \setminus E \to \breve{X}$; we shall identify $\tilde{X} \setminus E$ with $\breve{X}$ using this isomorphism.

We call E the *exceptional curve* of the resolution π (it is indeed a projective curve) and denote by $E_1, E_2, \dots, E_s$ its irreducible components.

3. A resolution $\pi : \tilde{X} \to X$ is called *minimal*, if it cannot be decomposed as $\tilde{X} \to X' \to X$, where X' is also smooth.

Recall that such a resolution, as well as a minimal resolution, always exists (cf. e.g. [47]).

In [43] Kahn established a one-to-one correspondence between Cohen–Macaulay modules over a normal surface singularity $\mathbf{A}$ and a class of vector bundles over a *reduction cycle* $Z \subseteq \tilde{X}$, which is given by a specially chosen effective divisor $\sum_{i=1}^{s} m_i E_i$ $(m_i > 0)$. His result becomes especially convenient if this singularity is *minimally elliptic* in the sense of [46]. It means that $\mathbf{A}$ is Gorenstein (i.e. $\operatorname{inj.dim} \mathbf{A} = 2$) and $\dim \mathrm{H}^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of a minimally elliptic singularity, Z be its *fundamental cycle*, i.e. the smallest effective cycle such that all intersection numbers $(Z.E_i)$ are non-positive. Then Z is a reduction cycle in the sense of Kahn, and the following result holds.

Theorem 4.7 (Kahn). *There is one-to-one correspondence between Cohen–Macaulay modules over* $\mathbf{A}$ *and vector bundles* $\mathcal{F}$ *over* Z *such that* $\mathcal{F} \simeq \mathcal{G} \oplus n\mathcal{O}_Z$*, where*

(i) $\mathcal{G}$ *is* generically spanned*, i.e. global sections from* $\Gamma(E, \mathcal{G})$ *generate* $\mathcal{G}$ *everywhere, except maybe finitely many closed points;*

(ii) $\mathrm{H}^1(E, \mathcal{G}) = 0$*;*

(iii) $n \geq \dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{G}(Z))$*.*

Especially, indecomposable Cohen–Macaulay $\mathbf{A}$*-modules correspond to vector bundles* $\mathcal{F} \simeq \mathcal{G} \oplus n\mathcal{O}_Z$*, where either* $\mathcal{G} = 0$*,* $n = 1$ *or* $\mathcal{G}$ *is indecomposable, satisfies the above conditions* (i,ii) *and* $n = \dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{G}(Z))$*. (The vector bundle* $\mathcal{O}_Z$ *corresponds to the regular* $\mathbf{A}$*-module, i.e.* $\mathbf{A}$ *itself.)*

Kahn himself deduced from this theorem and the results of Atiyah [1] a description of Cohen–Macaulay modules over *simple elliptic* singularities, i.e. such that E is an elliptic curve (smooth curve of genus 1). Using the results of subsection 4.2, one can obtain an analogous description for *cusp singularities*, i.e. such that E is a projective configuration of type $\tilde{A}$. Briefly, one gets the following theorem (for more details see [30]).

Theorem 4.8. *There is a one-to-one correspondence between indecomposable Cohen–Macaulay modules over a cusp singularity* $\mathbf{A}$*, except the regular module* $\mathbf{A}$*, and vector bundles* $\mathcal{V}(\mathbf{d}, m, \lambda)$*, where* $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$ *satisfies the following conditions:*

- $\mathbf{d} > \mathbf{0}$*, i.e.* $d_i \geq 0$ *for all* i *and* $\mathbf{d} \neq (0, 0, \dots, 0)$*;*
- *no shift of* $\mathbf{d}$*, i.e. a sequence* $(d_{k+1}, \dots, d_{rs}, d_1, \dots, d_k)$*, contains a subsequence* $(0, 1, 1, \dots, 1, 0)$*, in particular* $(0, 0)$*;*
- *no shift of* $\mathbf{d}$ *is of the form* $(0, 1, 1, \dots, 1)$*.*

Moreover, from Theorem 4.7 and the results of [29] one gets the following corollary [30]:

Theorem 4.9. *If a minimally elliptic singularity* $\mathbf{A}$ *is neither simple elliptic nor cusp, it is Cohen–Macaulay wild, i.e. the classification of Cohen–Macaulay* $\mathbf{A}$*-modules includes the classification of representations of all finitely generated* $\mathbb{k}$*-algebras.*

An important example of Cohen–Macaulay tame minimally elliptic singularities are the surface singularities of type T_{pqr}, i.e. factor rings

$$\mathbb{k}[[x, y, z]]/(x^p + y^q + z^r + \lambda xyz) \quad (1/p + 1/q + 1/r \leq 1).$$

They are simple elliptic if $1/p + 1/q + 1/r = 1$ and cusp otherwise [49].

As a consequence of Theorem 4.8 and the Knörrer periodicity theorem [44, 50], one also obtains a description of Cohen–Macaulay modules over *hypersurface singularities* of type T_{pqr}, i.e. factor rings

$$\mathbb{k}[[x_1, x_2, \dots, x_n]]/(x_1^p + x_2^q + x_3^r + \lambda x_1 x_2 x_3 + Q) \quad (1/p + 1/q + 1/r \leq 1),$$

where Q is a non-degenerate quadratic form of $x_4, \dots, x_n$, and over *curve singularities* of type T_{pq}, i.e. factor rings

$$\mathbb{k}[[x, y]]/(x^p + y^q + \lambda x^2 y^2) \quad (1/p + 1/q \leq 1/2).$$

The latter fills up a flaw in the result of [27], where one has only proved that the curve singularities of type T_{pq} are Cohen–Macaulay tame, but got no explicit description of modules.

Suppose that $\operatorname{char} \mathbb{k} = 0$. Then it is known [2, 32] that a normal surface singularity $\mathbf{A}$ is Cohen–Macaulay finite, i.e. has only a finite number of non-isomorphic indecomposable Cohen–Macaulay modules, if and only if it is a *quotient singularity*, i.e. $\mathbf{A} \simeq \Bbbk[[x, y]]^G$, where G is a finite group of automorphisms. (I do not know a criterion of finiteness if char $\Bbbk > 0$). Just in the same way one can show that all singularities of the form $\mathbf{A} = \mathbf{B}^G$, where $\mathbf{B}$ is either simple elliptic or cusp, are Cohen–Macaulay tame, and obtain a description of Cohen–Macaulay modules in this case. Actually such singularities coincide with the so called *log-canonical* singularities [45]. There is an evidence that all other singularities are Cohen–Macaulay wild, so Table 1 completely describes Cohen–Macaulay types of isolated singularities (for the curve case see [27]; we mark by '?' the places, where the result is still a conjecture).

Table 1.

Cohen–Macaulay types of singularities

CM type	curves	surfaces	hypersurfaces
finite	dominate A-D-E	quotient	simple (A-D-E)
tame	dominate T_{pq}	log-canonical (only ?)	T_{pqr} (only ?)
wild	all other	all other ?	all other ?

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