CUBIC RINGS AND THEIR IDEALS

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Abstract. We give an explicit description of cubic rings over a discrete valuation ring, as well as a description of all ideals of such rings.

Introduction

Ideals of local rings have been studied by a lot of authors from quite different viewpoints. One of the questions that arise with this respect is on the *number of parameters* $\mathrm{par}(\mathbf{C})$ defining the ideals of such a ring $\mathbf{C}$ up to isomorphism, especially when it is reduced and of Krull dimension 1. Certainly, it makes sense if the residue field $\mathbf{k}$ is infinite. In [8] it was shown that $\mathrm{par}(\mathbf{C}) = 0$, i.e. $\mathbf{C}$ has a finite number of ideals (up to isomorphism), if and only if $\mathbf{C}$ is *Cohen–Macaulay finite*, i.e. has a finite number of indecomposable non-isomorphic Cohen–Macaulay modules (in the 1-dimensional reduced case they coincide with torsion free modules). Then Schappert [12] proved that a plane curve singularity has at most 1-parameter families of ideals if and only if it dominates one of the *strictly unimodal* plane curve singularities in the sense of [14], or, the same, *unimodal* and *bimodal* plane curve singularities in the sense of [1]. In [7] this result was generalized to all curve singularities. Note that this time $\mathrm{par}(\mathbf{C}) = 1$ does not imply that $\mathbf{C}$ is *Cohen–Macaulay tame*, i.e. has at most 1-dimensional families of indecomposable Cohen–Macaulay modules. Tameness means that $\mathbf{C}$ dominates a singularity of type T_{pq} [5]. The case $\mathrm{par}(\mathbf{C}) > 1$ had not been studied before the second author described the one branch singularities of type W such that $\mathrm{par}(\mathbf{C}) \leq 2$ [13].

In this paper we study the *cubic rings*. We describe all such rings, their ideals and, in particular, establish the value $\mathrm{par}(\mathbf{C})$ for any cubic ring $\mathbf{C}$. As a consequence, we show that a cubic ring is Gorenstein if and only if it is a plane curve singularity (i.e. its embedding dimension equals 2).

1. Generalities

We denote by $\mathbf{D}$ a discrete valuation ring with the ring of fractions $\mathbf{K}$, the maximal ideal $\mathfrak{m} = t\mathbf{D}$ and the residue field $\mathbf{k} = \mathbf{D}/t\mathbf{D}$. A *cubic ring* over $\mathbf{D}$ is, by definition, a $\mathbf{D}$-subalgebra $\mathbf{C}$ in a 3-dimensional

2000 *Mathematics Subject Classification.* Primary 13C05, Secondary 16G60.

semisimple $\mathbf{K}$-algebra $\mathbf{L}$, which is a free $\mathbf{D}$-module of rank 3. We also denote $\mathbf{A}$ the integral closure of $\mathbf{D}$ in $\mathbf{L}$ and always suppose that $\mathbf{A}$ is finitely generated as $\mathbf{C}$-module. Equivalent condition (see, for instance, [3]): the $\mathfrak{m}$-adic completion $\hat{\mathbf{C}}$ of the ring $\mathbf{C}$ has no nilpotent elements. It is always the case if the algebra $\mathbf{L}$ is *separable*, for instance, if $\operatorname{char}\mathbf{K} = 0$. We also set $\mathbf{A}_m = t^m\mathbf{A} + \mathbf{D}$ and $\mathbf{J}_m = t\mathbf{A}_{m-1} = \operatorname{rad}\mathbf{A}_m$ $(m > 0)$.

In what follows, an *ideal* means a *fractional* $\mathbf{C}$*-ideal* in $\mathbf{K}$, i.e. a finitely generated $\mathbf{C}$-submodule $M \subseteq \mathbf{K}$ such that $\mathbf{K}M = \mathbf{K}$. Then M is a free $\mathbf{D}$-module of rank 3. We are going to describe all ideals of cubic rings up to isomorphism. It is known (see, for instance, [9]) that there is a one-to-one correspondence between $\mathbf{C}$-ideals and $\hat{\mathbf{C}}$-ideals, mapping M to its $\mathfrak{m}$-adic completion. This correspondence *reflects isomorphisms*, i.e. maps non-isomorphic ideals to non-isomorphic. So, in what follows we may (and will) suppose that $\mathbf{D}$ is *complete* with respect to the $\mathfrak{m}$-adic topology.

Recall also that the *embedding dimension* $\operatorname{edim}\mathbf{C}$ of a local noetherian ring $\mathbf{C}$ with the maximal ideal $\mathbf{J}$ and the residue filed $\mathbf{k}$ is defined as $\dim_{\mathbf{k}}\mathbf{J}/\mathbf{J}^2$. If $\mathbf{C}$ is of Krull dimension 1 and $\operatorname{edim}\mathbf{C} = 2$, $\mathbf{C}$ is called a *plane curve singularity*. In the geometric case, when $\mathbf{C}$ contains a subfield of representatives of $\mathbf{k}$, it actually means that there is a plane curve C such that $\mathbf{C}$ is the completion of the local ring of a singular point $x \in C$.

From the general theory of ramification in finite extensions we see that the following cases can happen:

One branch, ramified case: $\mathbf{L}$ is a field, the maximal ideal of $\mathbf{A}$ equals $\tau\mathbf{A}$, $\mathbf{A}/\tau\mathbf{A} \simeq \mathbf{k}$ and $t\mathbf{A} = \tau^3\mathbf{A}$.

One branch, non-ramified case: $\mathbf{L}$ is a field, the maximal ideal of $\mathbf{A}$ equals $t\mathbf{A}$ and $\mathbf{A}/t\mathbf{A} = \mathbf{k}[\bar{\theta}]$ is a cubic extension of the field $\mathbf{k}$, where $\bar{\theta}$ is a root of an irreducible cubic polynomial $f(x) \in \mathbf{k}[x]$.

Two branches, ramified case: $\mathbf{L} = \mathbf{K}_1 \times \mathbf{K}$, where $\mathbf{K}_1$ is a quadratic extension of $\mathbf{K}$, $\mathbf{A} = \mathbf{D}_1 \times \mathbf{D}$, the maximal ideal of $\mathbf{D}_1$ is $\tau\mathbf{D}_1$, $\mathbf{D}_1/\tau\mathbf{D}_1 \simeq \mathbf{k}$ and $t\mathbf{D}_1 = \tau^2\mathbf{D}_1$.

Two branches, non-ramified case: $\mathbf{L} = \mathbf{K}_1 \times \mathbf{K}$, where $\mathbf{K}_1$ is a quadratic extension of $\mathbf{K}$, $\mathbf{A} = \mathbf{D}_1 \times \mathbf{D}$, the maximal ideal of $\mathbf{D}_1$ is $t\mathbf{D}_1$ and $\mathbf{D}_1/\tau\mathbf{D}_1 = \mathbf{k}[\bar{\theta}]$ is a quadratic extension of the field $\mathbf{k}$, where $\bar{\theta}$ is a root of an irreducible quadratic polynomial $f(x) \in \mathbf{k}[x]$.

Three branches case: $\mathbf{L} = \mathbf{K}^3$, $\mathbf{A} = \mathbf{D}^3$.

We recall [10, 2] that, for any cubic ring $\mathbf{C}$, every ideal of $\mathbf{C}$ is isomorphic either to an *over-ring* of $\mathbf{C}$, i.e. a cubic ring $\mathbf{B}$ such that $\mathbf{C} \subseteq \mathbf{B} \subset \mathbf{L}$, or to the *dual ideal* $\mathbf{B}^* = \operatorname{Hom}_{\mathbf{D}}(\mathbf{B}, \mathbf{D})$ of such an over-ring. Hence, to describe all ideals of $\mathbf{C}$, we only need to describe over-rings of $\mathbf{C}$. Obviously, any cubic ring in $\mathbf{L}$ contains some $\mathbf{A}_m$. Therefore, to describe all cubic rings (so their ideals as well), we have to describe the over-rings of $\mathbf{A}_m$. If $\mathbf{B}$ is an over-ring of $\mathbf{C}$, they also say that $\mathbf{B}$ *dominates* $\mathbf{C}$.

Since the unique (up to isomorphism) $\mathbf{A}$-ideal is $\mathbf{A}$ itself, we proceed by induction: supposing that all over-rings of $\mathbf{A}_m$ are known, we find all over-rings of $\mathbf{A}_{m+1}$. If $\mathbf{C}$ is an over-ring of $\mathbf{A}_{m+1}$, then $\mathbf{B} = \mathbf{C}\mathbf{A}_m$ is an over-ring of $\mathbf{A}_m$, $t\mathbf{B} \subset \mathbf{C}$ and $\mathbf{C}/t\mathbf{B}$ is a $\mathbf{k}$-subalgebra in $\mathbf{B}/t\mathbf{B}$. If $\mathbf{B} \supseteq \mathbf{A}_{m-1}$, then $t\mathbf{B} \supseteq \mathbf{J}_m$, hence, $\mathbf{C} \supseteq \mathbf{J}_m + \mathbf{D} = \mathbf{A}_m$. Therefore, the following procedure gives all over-rings of $\mathbf{A}_{m+1}$ which are not over-rings of $\mathbf{A}_m$:

Procedure.
- For every over-ring $\mathbf{B}$ of $\mathbf{A}_m$, which is not an over-ring of $\mathbf{A}_{m-1}$, calculate $\bar{\mathbf{B}} = \mathbf{B}/t\mathbf{B}$. Set $\bar{\mathbf{A}} = (\mathbf{A}_m + t\mathbf{B})/t\mathbf{B} \subseteq \bar{\mathbf{B}}$.
- Find all proper subalgebras $\mathbf{S} \subset \bar{\mathbf{B}}$ such that $\bar{\mathbf{A}}\mathbf{S} = \bar{\mathbf{B}}$. Equivalently, the natural map $\mathbf{S} \to \mathbf{B}/\mathbf{B}\mathbf{J}_m$ must be surjective.
- For each such $\mathbf{S}$ take its preimage in $\mathbf{B}$.

2. Calculations

2.1. **One branch, ramified case.** We set

$$\mathbf{C}_{2r}(\alpha) = \mathbf{D} + t^r\alpha\mathbf{D} + t^{2r}\mathbf{A}, \text{ where } v(\alpha) = 1,$$
$$\mathbf{C}_{2r+1}(\alpha) = \mathbf{D} + t^r\alpha\mathbf{D} + t^{2r+1}\mathbf{A}, \text{ where } v(\alpha) = 2,$$

where v is the discrete valuation related to the ring $\mathbf{A}$, i.e. $v(\alpha) = k$ means that $\alpha \in \tau^k\mathbf{D} \setminus \tau^{k+1}\mathbf{D}$. Note that $\mathbf{C}_0(\alpha) = \mathbf{A}$. Obviously, α can be uniquely chosen as $\tau + a\tau^2$ for $\mathbf{C}_{2r}$ and as $\tau^2 + at\tau$ for $\mathbf{C}_{2r+1}$, where $a \in \mathbf{D}$ is defined modulo t^r.

Theorem 2.1. *Every over-ring of* $\mathbf{A}_m$ *coincides with* $t^k\mathbf{C}_r(\alpha) + \mathbf{D}$ *for some* k, r *such that* $r + k \le m$ *and some* α. *The rings* $\mathbf{C}_r(\alpha)$ *are just all plane curve singularities in this case.*

Proof. For $m = 1$ it is easy and known [8, 11]. So, we use the Procedure for $m > 1$, setting $\mathbf{B} = t^k\mathbf{C}_r(\alpha) + \mathbf{D}$, where $k + r = m$. Then the basis of $\bar{\mathbf{B}}$ consists of the classes of the elements $\{1, t^h\alpha, t^m\tau^s\}$, where $h = k + [r/2]$, $s \in \{1, 2\}$ and $s \equiv r \pmod 2$. Since $t^h\alpha \notin \mathbf{J}_m$, the subalgebra $\mathbf{S}$ necessarily contains the class of $t^h\alpha + ct^m\tau^s$ for some $c \in \mathbf{D}$. If $k = 0$, then $m = r$ and $v(t^m\tau^s) = 2v(t^h\alpha)$. Therefore, $\bar{\mathbf{B}}$ has no proper subalgebra containing the class of $t^h\alpha + ct^m\tau^s$. If $k > 0$, the preimage of $\mathbf{S}$ is $\mathbf{D} + (t^h\alpha + ct^m\tau^s)\mathbf{D} + t^{m+1}\mathbf{A}$. It coincides with $t^{k-1}\mathbf{C}_{r+2}(\alpha') + \mathbf{D}$ where $\alpha' = \alpha + ct^{m-h}\tau^s$.

Now one easily checks that $\operatorname{edim} \mathbf{C}_r(\alpha) = 2$, while $\operatorname{edim} \mathbf{C} = 3$ for all other rings. $\square$

2.2. **One branch, non-ramified case.** We set $\mathbf{C}_r(\alpha) = \mathbf{D} + t^r\alpha\mathbf{D} + t^{2r}\mathbf{A}_0$, where $\alpha \in \mathbf{A}^\times \setminus \mathbf{D}$. Again $\mathbf{C}_0(\alpha) = \mathbf{A}_0$. Note that α can be uniquely chosen as $\theta + a\theta^2$, where θ is a fixed preimage of $\bar{\theta}$ in $\mathbf{D}_1$ and $a \in \mathbf{D}$ is defined modulo t^r.

Theorem 2.2. *Every over-ring of $\mathbf{A}_m$ coincides with $t^k\mathbf{C}_r(\alpha) + \mathbf{D}$ for some k, r and α with $2r + k \le m$. The rings $\mathbf{C}_r(\alpha)$ are just all plane curve singularities in this case.*

Proof. For $m = 1$ it is obvious. So, using the Procedure for $m > 1$, we set $\mathbf{B} = t^k\mathbf{C}_r(\alpha) + \mathbf{D}$ with $2r + k = m$. Then a basis of $\bar{\mathbf{B}}$ consists of the classes of elements $\{1, t^{r+k}\alpha, t^m\alpha^2\}$ for some $\alpha^2 \in \mathbf{A}^\times \setminus (\mathbf{D} + \alpha\mathbf{D})$. Since $t^{r+k}\alpha \notin \mathbf{J}_m$, $\mathbf{S}$ must contain the class of an element $t^{r+k}\alpha' = t^{r+k}\alpha + ct^m\alpha^2$ for some $c \in \mathbf{D}$. As above, it is impossible if $k = 0$. If $k > 0$, then the preimage of $\mathbf{S}$ is $\mathbf{D} + t^{r+k}\alpha' + t^{m+1}\mathbf{A} = t^{k-1}\mathbf{C}_{r+1}(\alpha') + \mathbf{D}$.

Now one easily checks that $\operatorname{edim} \mathbf{C}_r(\alpha) = 2$, while $\operatorname{edim} \mathbf{C} = 3$ for all other rings. □

2.3. **Two branches, ramified case.** We denote by v the valuation defined by the ring $\mathbf{D}_1$, by e the idempotent in $\mathbf{A}$ such that $e\mathbf{A} = \mathbf{D}_1$ and set

$$\mathbf{C}_{l,q}(\alpha) = \mathbf{D} + t^l(e + t^q\alpha)\mathbf{D} + t^r\mathbf{A}, \text{ where } r = 2l + q,$$
$$\mathbf{C}_r(\alpha) = \mathbf{D} + t^r\alpha\mathbf{D} + t^{2r+1}\mathbf{A}.$$

In both cases $\alpha \in \mathbf{D}_1$ and $v(\alpha) = 1$, where v is the valuation defined by the ring $\mathbf{D}_1$. Obviously, α can be uniquely chosen as $a\tau$, where $a \in \mathbf{D}$ is defined modulo r. Note that $\mathbf{C}_{0,q}(\alpha) = \mathbf{D} + e_1\mathbf{D} + t^q\mathbf{A}$ are just all decomposable rings in this case and $\mathbf{C}_{0,0}(\alpha) = \mathbf{A}$.

Theorem 2.3. *Every over-ring of $\mathbf{A}_m$ coincides with either $t^k\mathbf{C}_{l,r}(\alpha) + \mathbf{D}$ or $t^k\mathbf{C}_r(\alpha) + \mathbf{D}$, where $k + r \le m$. The rings $\mathbf{C}_{l,q}(\alpha)$ and $\mathbf{C}_r(\alpha)$ are just all plane curve singularities in this case.*

Proof. The case $m = 1$ is obvious. So, using the Procedure, we suppose that $m > 1$ and $k + r = m$. If $\mathbf{B} = t^k\mathbf{C}_{l,q}(\alpha) + \mathbf{D}$, a basis of $\bar{\mathbf{B}}$ consists of the classes of $\{1, t^{k+l}(e + t^q\alpha), t^m\tau\}$. Since $t^{k+l}(e + t^q\alpha) \notin \mathbf{J}_m$, the subalgebra $\mathbf{S}$ must contain the classe of $t^{k+l}(e + t^q\alpha')$ for some $\alpha' \in \mathbf{D}_1$ with $v(\alpha') = 1$. Again the case $k = 0$ is impossible. If $k > 0$, the preimage of $\mathbf{S}$ coincides with $t^{k-1}\mathbf{C}_{l+1,q} + \mathbf{D}$. If $\mathbf{B} = t^k\mathbf{C}_r(\alpha) + \mathbf{D}$, the calculations are quite similar.

Now one easily checks that $\operatorname{edim} \mathbf{C}_{l,q}(\alpha) = \operatorname{edim} \mathbf{C}_r(\alpha) = 2$, while $\operatorname{edim} \mathbf{C} = 3$ for all other rings. □

2.4. **Two branches, non-ramified case.** We set

$$\mathbf{C}_{l,q}(\alpha) = \mathbf{D} + t^l(e_1 + t^q\alpha)\mathbf{D} + t^r\mathbf{A}, \text{ where } r = 2l + q$$
$$\text{and } \alpha \in \mathbf{D}_1 \setminus (e_1\mathbf{D} + t\mathbf{D}).$$

Then α can be chosen as $a\theta$, where θ is a fixed preimage of $\bar{\theta}$ in $\mathbf{D}_1$ and $a \in \mathbf{D}$ is uniquely defined modulo t^l. Again $\mathbf{C}_{0,q}(\alpha) = \mathbf{D} + e_1\mathbf{D} + t^q\mathbf{A}$ are just all decomposable rings in this case. Especially, $\mathbf{C}_{0,0}(\alpha) = \mathbf{A}$.

Theorem 2.4. *Every over-ring of* $\mathbf{A}_m$ *coincides with one of the rings* $t^k\mathbf{C}_{l,q}(\alpha) + \mathbf{D}$, *where* $k + r \leq m$. *The rings* $\mathbf{C}_{l,q}(\alpha)$ *are just all plane curve singularities in this case.*

We omit the proof in this case, since it practically repeats the calculations in the other cases.

2.5. **Three branches case.** We set

$$\mathbf{C}_{l,q}(\alpha) = \mathbf{D} + t^l\alpha\mathbf{D} + t^r\mathbf{A},$$

where $\alpha = e + t^q ae'$, $e \neq e'$ are primitive idempotent in $\mathbf{A}$, $r = 2l + q$, $a \in \mathbf{D}^\times$ and $a \not\equiv 1 (\mathrm{mod}\ t)$ if $q = 0$. Obviously, a is unique modulo t^l. Again $\mathbf{C}_{0,q}(\alpha) = \mathbf{D} + e\mathbf{D} + t^q\mathbf{A}$ are just all decomposable rings in this case and $\mathbf{C}_{0,0} = \mathbf{A}$. Note also that if $\mathbf{C} = \mathbf{D} + t^l\alpha\mathbf{D} + t^r\mathbf{A}$, where $\alpha = e + ae'$ as above with $a \equiv 1 (\mathrm{mod}\ t)$, then, for $a \equiv 1 (\mathrm{mod}\ t^l)$, $\mathbf{C} = t^l\mathbf{C}_{0,q}(1 - e - e') + \mathbf{D}$, and for $a \equiv 1 (\mathrm{mod}\ t^q)$ with $0 < q < l$, $\mathbf{C} = \mathbf{C}_{l,q}(\alpha')$ for some α'.

Theorem 2.5. *Every over-ring of* $\mathbf{A}_m$ *coincides with* $t^k\mathbf{C}_{l,q}(\alpha) + \mathbf{D}$ *for some* α *and some* l, q *with* $k + r \leq m$. *The rings* $\mathbf{C}_{l,q}(\alpha)$ *are just all plane curve singularities in this case.*

We also omit the proof in this case, since it practically repeats the calculations in the other cases.

2.6. **Table of plane curve cubic singularities.** We present in Table 1 below all plane curve cubic singularities. In this table s is the number of branches, * marks the unramified cases (related to the residue field extensions, hence impossible if $\mathbf{k}$ is algebraically closed); x, y are generators of the maximal ideal, $v(a)$ denotes the *multivaluation* of an element $a \in \mathbf{A}$, i.e. the vector of valuations of its components with respect to the decomposition of $\mathbf{A}$ into the product of discrete valuation rings. The column "type" shows the correspondence with the Arnold's classification [1, § 15]. If $\mathrm{char}\,\mathbf{k} = 0$ and $\mathbf{A}$ is ramified, it actually shows the place of the rings in this classification. If $\mathrm{char}\,\mathbf{k} = 0$ and $\mathbf{C}$ is non-ramified, it shows the place of the ring in this classification after the natural extension of the field $\mathbf{k}$. The validation of this column is given in [7, Section 2.3]. Note that, following [7], we denote by $E_{l,q}$ the singularities $J_{l,q}$ in the sense of [1]. Such notations seem more uniform. Note also that the singularities of types E_1 and E_2 are actually not cubic, but quadratic, and coincide with those of types A_1 and A_2 of [1]. Finally, the last column, "par" shows the number of parameters p from the residue field $\mathbf{k}$ which define a unique ring of this type. We will consider this value in the last section. It does not coincide with the *modality* in the sense of [1]; the latter equals $p - 1$.

Table 1.

s	name	$v(x)$	$v(y)$	type	par
1	$\mathbf{C}_{2r}(\alpha)$ $\mathbf{C}_{2r+1}(\alpha)$	(3) (3)	$(3r+1)$ $(3r+2)$	E_{6r} E_{6r+2}	r r
1^*	$\mathbf{C}_r(\alpha)$	(1)	(r)	$E^*_{r,0}$	r
2	$\mathbf{C}_r(\alpha)$ $\mathbf{C}_{l,q}(\alpha)$	$(2,1)$ $(2,1)$	$(2r+1,\infty)$ $(2l,\infty)$	E_{6r+1} $E_{l,2q+1}$	r l
2^*	$\mathbf{C}_{l,q}(\alpha)$	$(1,1)$	(l,∞)	$E^*_{l,2q}$	l
3	$\mathbf{C}_{l,q}(\alpha)$	$(1,1,1)$	$(l,l+q,\infty)$	$E_{l,2q}$	l

Remark. The *tame* cubic plane curve singularities $T_{3,q}$ $(q \geq 6)$ [4, 5] coincide with those of types $E_{2,q-6}$.

3. Ideals

As we have mentioned above, every ideal of a cubic ring $\mathbf{C}$ is isomorphic either to an over-ring $\mathbf{B} \supseteq \mathbf{C}$ or to its dual $\mathbf{B}^* = \mathrm{Hom}_{\mathbf{D}}(\mathbf{B},\mathbf{D})$. If $\mathbf{C}$ is *Gorenstein* (for instance, if it is a plane cubic singularity), then $\mathbf{C}^* \simeq \mathbf{C}$, thus $\mathbf{B}^* \simeq \mathrm{Hom}_{\mathbf{C}}(\mathbf{B},\mathbf{C})$. Therefore, to calculate $\mathbf{B}^*$, one has to choose a Gorenstein subring $\mathbf{C} \subseteq \mathbf{B}$ and to calculate

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{B},\mathbf{C}) \simeq \{\lambda \in \mathbf{L} \mid \lambda\mathbf{B} \subseteq \mathbf{C}\} = \{\lambda \in \mathbf{C} \mid \lambda\mathbf{B} \subseteq \mathbf{C}\}$$

(the latter equality holds since $1 \in \mathbf{B}$). This remark easily leads to the following result.

Theorem 3.1. *The duals to the cubic rings are as follows:*

One branch ramified case: *If* $\mathbf{B} = \mathbf{D} + t^k\mathbf{C}_r(\alpha)$, *then* $\mathbf{B}^* \simeq \mathbf{D} + t^{[r/2]}\alpha\mathbf{D} + t^{k+r}\mathbf{A}$.

One branch non-ramified case: *If* $\mathbf{B} = \mathbf{D} + t^k\mathbf{C}_r(\alpha)$, *then* $\mathbf{B}^* \simeq \mathbf{D} + t^r\alpha\mathbf{D} + t^{k+2r}\mathbf{A}$.

Two branches ramified case: (1) *If* $\mathbf{B} = \mathbf{D} + t^k\mathbf{C}_{l,q}(\alpha)$, *then* $\mathbf{B}^* \simeq \mathbf{D} + t^l(e + t^q\alpha)\mathbf{D} + t^{k+2l+q}\mathbf{A}$.

(2) *If* $\mathbf{B} = \mathbf{D} + t^k\mathbf{C}_r(\alpha)$, *then* $\mathbf{B}^* \simeq \mathbf{D} + t^r\alpha\mathbf{D} + t^{k+2r+1}\mathbf{A}$.

Two branches non-ramified case: *If* $\mathbf{B} = \mathbf{D} + t^k\mathbf{C}_{l,q}(\alpha)$, *then* $\mathbf{B}^* \simeq \mathbf{D} + t^l(e + t^q\alpha)\mathbf{D} + t^{k+2l+q}\mathbf{A}$.

Three branches case: *If* $\mathbf{B} = \mathbf{D} + t^k\mathbf{C}_{l,q}(\alpha)$, *then* $\mathbf{B}^* \simeq \mathbf{D} + t^l\alpha\mathbf{D} + t^{k+2l+q}\mathbf{A}$.

Proof. The proof is immediate if we choose for a Gorenstein subring $\mathbf{C} \subseteq \mathbf{B}$ the plane curve singularity $\mathbf{C} = \mathbf{C}_{k+r}(\alpha)$ or $\mathbf{C}_{k+l,q}(\alpha)$ depending on the shape of $\mathbf{B}$. For instance, in two branches ramified case, if $\mathbf{B} = \mathbf{D} + t^k\mathbf{C}_{l,q}(\alpha)$ and $\mathbf{C} = \mathbf{D} + \mathbf{C}_{l+k,q}(\alpha)$, then

$$\begin{aligned}\mathbf{B}^* &\simeq \{\lambda \in \mathbf{C} \mid \lambda\mathbf{B} \subseteq \mathbf{C}\} = t^k\mathbf{D} + t^{k+l}(e + t^q\alpha)\mathbf{D} + t^{2k+2l+q}\mathbf{A} \\ &\simeq \mathbf{D} + t^l(e + t^q\alpha)\mathbf{D} + t^{k+2l+q}\mathbf{A}.\end{aligned}$$

□

Corollary 3.2. *If a cubic ring is Gorenstein, it is a plane curve singularity.*

Note that it is no more the case for the extensions of bigger degrees. For instance, the rings P_{pq} from [5], which are quartic, are Gorenstein (they are complete intersections) but of embedding dimension 3.

4. Geometric case. Number of parameters

In this section we suppose that our rings are of *geometric nature*, i.e. $\mathbf{D} = \mathbf{k}[[t]]$, where $\mathbf{k}$ is algebraically closed. Then one can consider the *number of parameters* $\mathrm{par}(\mathbf{C})$ defining $\mathbf{C}$-ideals (see [4, Section 2.2] or [6, Section 3], where it is denoted by $\mathrm{par}(1; \mathbf{C}, \mathbf{A})$). Actually, it coincides with the minimal possible number p for which there is a finite set of *families of ideals* $\mathcal{I}_k$ $(1 \le k \le m)$ of dimensions at most p such that every $\mathbf{C}$-ideal is isomorphic to one belonging to some family $\mathcal{I}_k$. Equivalently, it is the maximal possible p such that is a p-dimensional family of ideals $\mathcal{I}$ where every isomorphism class of ideals only occurs finitely many times. In [7] a criterion was established in order that $\mathrm{par}(\mathbf{C}) \le 1$. For cubic rings it means that $\mathbf{C}$ dominates a singularity of type E_m $(18 \le m \le 20)$ or $E_{3,i}$. The following results give the exact value of $\mathrm{par}(\mathbf{C})$ for all cubic rings of geometric nature. (Note that no unramified case can occur for such rings.)

Theorem 4.1. *If* $\mathbf{C}$ *is a cubic ring of geometric nature,* $\mathrm{par}(\mathbf{C}) \le n$ *if and only if* $\mathbf{C}$ *dominates one of the singularities of type* E_{12n+i} $(6 \le i \le 8)$ *or* $E_{2n+1,q}$ $(q \ge 0)$.

Proof. Certainly, we have to prove that

(1) every ring of one of the listed types have at most n-parameter families of ideals;
(2) if $\mathbf{C}$ dominates no ring of the listed types, it has $(n+1)$-parameter families of ideals.

Since the calculations in all cases are similar, we only consider the one branch ramified case. Note first that the rings $\mathbf{C}_{2r}(\alpha)$ as well as

$\mathbf{C}_{2r+1}(\alpha)$ form a r-parametric family. Indeed, we can choose in the first case $\alpha = \tau + a\tau^2$, and in the second one $\alpha = \tau^2 + a\tau^4$, where $a \in \mathbf{D}$ is defined modulo t^r, and such a presentation is unique. The same is true also for $t^k\mathbf{C}_{2r}(\alpha) + \mathbf{D}$ and $t^k\mathbf{C}_{2r+1}(\alpha) + \mathbf{D}$ for any k. Since $\mathbf{C}_{2r}(\alpha) \supseteq \mathbf{A}_{2r}$ for all α, we get $\mathrm{par}(\mathbf{A}_{2r}) \geq r$.

Let $\mathbf{C}$ dominate neither a ring of type E_{12n+6}, i.e. $\mathbf{C}_{4n+2}(\alpha)$, nor a ring of type E_{12n+8}, i.e. $\mathbf{C}_{4n+3}(\alpha)$. Then it contains no element of valuation smaller than $6n+6$, so $\mathbf{C} \subseteq \mathbf{A}_{2n+2}$. Hence, $\mathrm{par}(\mathbf{C}) \geq n+1$.

On the other hand, consider the ring $\mathbf{C}_{2r+q}(\alpha)$, where $q \in \{0,1\}$. Its over-rings are of the kind $\mathbf{D} + t^k\mathbf{C}_{2m+q}(\beta)$, where $k+m \leq r$ and $k+2m \leq 2r$. Moreover, let $\alpha = \tau^{q+1} + a\tau^{2q+2}$ and $\beta = \tau^{q+1} + b\tau^{2q+2}$. Then b is defined modulo t^m and $b \equiv a \pmod{t^{r-m-k}}$. Therefore, the over-rings with the fixed m, k form a p-parameter family, where $p = \min(m, r-m-k)$. Hence, $2p \leq r$ and $p \leq [r/2]$. If we set $r = 2n+1$, we get that $\mathrm{par}(\mathbf{C}_{4n+2}(\alpha)) \leq n$ and $\mathrm{par}(\mathbf{C}_{4n+3}(\alpha)) \leq n$ for all possible α. It accomplishes the proof. □

Obvious considerations give the number of parameters for special rings.

Corollary 4.2.

$$\begin{aligned} \mathrm{par}(\mathbf{C}_r(\alpha)) &= [r/2], \\ \mathrm{par}(\mathbf{C}_{l,q}(\alpha)) &= [l/2], \\ \mathrm{par}(\mathbf{A}_m) &= [m/2]. \end{aligned}$$

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