LATTICES OVER NOETHERIAN ALGEBRAS

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An analog of the reduction theorem for modules over an integrally closed integral domain is proved for torsionfree modules over a semiprimary Noetherian algebra. The result is translated into concrete terms for pseudo-Bass and pseudohereditary algebras, and also used to study the structure of genera.

The purpose of this paper is to carry over some results from the theory of genera of integral representations (see, e.g., [1, 2]) to the case of modules over Noetherian algebras of Krull dimension greater than one (a ring A is a Noetherian algebra if its center C is Noetherian and A is a finitely generated C-module). This rather complicates the formulation of the results (for example, it is no longer true that all modules in one genus are either all decomposable or all indecomposable), but, as is well known, such complications are unavoidably associated with the specific features of many-dimensional rings. The carry-over process as a whole is based on the pattern of the "reduction theorem" for torsionfree modules over an integrally closed Noetherian ring [3], the one-dimensional case of which becomes a description of the modules over a Dedekind domain.

Since the only actual use of noethericity in the proofs involves one-dimensional localizations, the exposition will be phrased in rather more general terms, i.e., for pseudo-Noetherian algebras. In particular, this will imply an extension of the reduction theorem to arbitrary Krull rings.

The main theorem below will be used to describe the structure of modules over special Noetherian algebras: pseudo-Bass algebras and algebras in which all one-dimensional localizations of a torsionfree module are completely decomposable. In addition, in the final section we shall show how the results can be "globalized" to the case of quasiprojective varieties (incidentally, these "global" analogs are apparently new even for curves, i.e., one-dimensional varieties).

1. Recall that a commutative ring C is said to be pseudo-Noetherian if, for any $a \in C$ the set of prime ideals containing a contains only finitely many minimal ideals, and for any such minimal $\mathfrak{p}$ the ring $C_\mathfrak{p}$ is Noetherian [4]. A ring A is called a pseudo-Noetherian algebra if its center C = C(A) is a pseudo-Noetherian ring and for any prime ideal $\mathfrak{p} \in C$ such that $ht\mathfrak{p} \geqslant 1$, the ring $A_\mathfrak{p}$ is a finitely generated $C_\mathfrak{p}$-module.

Throughout, A will always denote a semiprimary pseudo-Noetherian algebera, C its center, P the set of prime ideals of height 1 in C. Let $\tilde{C}$ denote the full ring of quotients of C (the direct sum of the fields of quotients of the rings $C/\mathfrak{p}$, where $\mathfrak{p}$ ranges over the set of minimal prime ideals of C). Then $\tilde{A} = A \otimes_C \tilde{C}$ is the classical ring of quotients of A (in Ore's sense). An A-module M is called an A-lattice if the canonical homomorphism $\mu: M \to \tilde{M} = M \otimes_C \tilde{C}$ is a monomorphism, and moreover $\tilde{M}$ contains a finitely generated A-submodule N such that $\operatorname{Im}\mu \subset N$. In that case we shall identify M with $\operatorname{Im}\mu$, i.e., we shall assume that M is a submodule of $\tilde{M}$. If M is an A-lattice, then $\tilde{M}$ is a finitely generated module over the semisimple Artin ring $\tilde{A}$. Therefore, $\tilde{M} \simeq \bigotimes_{i=1}^{s} U_i^{m_i}$, where $U_1, \ldots, U_s$ are all pairwise nonisomorphic simple $\tilde{A}$-modules. Denote $r(M) = (m_1, \ldots, m_s)$. If $r(N) = (n_1, \ldots, n_s)$, we shall write $r(M) \gg r(N)$ if $m_i \geqslant n_i$, and $n_i \neq 0$ implies that $m_i > n_i$ for all $i = 1, \ldots, s$.

If $\mathfrak{p}$ is a prime ideal in C, we write $M(\mathfrak{p}) = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$, and let $x(\mathfrak{p})$ denote the image of an element $x \in M$ under the canonical map $M \to M(\mathfrak{p})$, $\varphi_\mathfrak{p}: M_\mathfrak{p} \to N_\mathfrak{p}$ and $\varphi(\mathfrak{p}): M(\mathfrak{p}) \to N(\mathfrak{p})$ the homomorphisms induced by a homomorphism $\varphi: M \to N$.

Let X be a finite subset of P. Denote $\overline{X} = C \setminus \bigcup_{\mathfrak{p} \in P} \mathfrak{p}$ and $M_X = \overline{X}^{-1}M$. If $X \supset Y$, we have a well-defined canonical homomorphism $M_X \to M_Y$, so that we can construct the module $\overline{M} = \varprojlim M_X$,

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where X ranges over all finite subsets of P. The module $\overline{M}$ thus defined is naturally identified with a certain submodule of $\tilde{M}$ containing M. Note that if C is an integral domain then $\overline{M}=\bigcap_{\mathfrak{p}\in P} M_{\mathfrak{p}}$, and moreover $M_{\mathfrak{p}}=\overline{M}_{\mathfrak{p}}$ for all $\mathfrak{p}\in P$. If $M = \overline{M}$, the lattice M is said to be closed. It is easy to see that if $\overline{A}$ is a lattice, then for any lattice M the module $\overline{M}$ is also a lattice.

The main theorem of this paper is

THEOREM 1. Let M and N be A-lattices such that $r(N) \ll r(M)$ and for any $\mathfrak{p}\in P$ there exists a split monomorphism $N_{\mathfrak{p}}\to M_{\mathfrak{p}}$. Then there exists a monomorphism $\varphi: N\to M$, such that for any $\mathfrak{p}\in P$ the monomorphism $\varphi_{\mathfrak{p}}$ is split. If the lattice N is closed, the factor module $M/\operatorname{Im}\varphi$ is also a lattice.

We first prove a few lemmas.

LEMMA 1. Let C be a semilocal ring of dimension 1. Then the semigroup of A-lattices is a semigroup with cancellation, i.e., if M, N and N' are A-lattices such that $M\otimes N\simeq M\otimes N'$, then $N \simeq N'$.

Proof. Follows from [5].

LEMMA 2. Let C be a local Noetherian ring, M and N finitely generated A-modules and φ $N \to M$ a split monomorphism. Then if $\psi: N \to M$ is a homomorphism such that $\varphi(\mathfrak{p})=\psi(\mathfrak{p})$, then ψ is also a split monomorphism.

Proof. We identify M with $N\oplus L$, and φ with the homomorphism $N\to N\oplus L$, defined by the matrix $\begin{pmatrix}1\\0\end{pmatrix}$. Let $\psi: N\to N\oplus L$ be defined by the matrix $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Then $\alpha(\mathfrak{p})=1$, so that $\alpha \equiv 1\ (\operatorname{mod}\operatorname{rad}\operatorname{End}_A M)$ and therefore α is invertible. Consequently, if $\eta\in\operatorname{End}_A M$ is defined by the matrix

$$\begin{pmatrix}\alpha^{-1} & 0\\ -\alpha^{-1}\beta & 1\end{pmatrix},$$

then $\eta\psi=\varphi$. Hence ψ is a split monomorphism.

LEMMA 3. Let C be a local Noetherian ring, $\mathfrak{m}$ a maximal ideal in it, M a finitely generated A-module, $\varphi: M\to M$. Then $\{a\in C\mid \varphi-a\cdot 1$ is invertible$\}$ consists of only finitely many cosets modulo

Proof. Let $\varphi\in B=\operatorname{End}_A M$. The module B is finitely generated as a C-module. Consider $\overline{B}=B/\mathfrak{m}B$, $\overline{B}$ is a finite algebra over $k=C/\mathfrak{m}$. Then $\varphi(\mathfrak{m})\in\overline{B}$, and since $\mathfrak{m}B\subset\operatorname{rad}B$, it follows that φ is invertible if and only if $\varphi(\mathfrak{m})$ is invertible. Now consider $f\in\operatorname{End}_k\overline{B}$, where f is the multiplication operator on $\varphi(\mathfrak{m})$. The difference $\varphi(\mathfrak{m})-a(\mathfrak{M})$ is noninvertible if and only if $f-a(\mathfrak{m})$ is noninvertible. But this means that $a(\mathfrak{m})$ is a characteristic value of f. Thus $\varphi(\mathfrak{m})-a(\mathfrak{m})$ is noninvertible for only finitely many values of $a(\mathfrak{m})$, and hence there are only finitely many cosets $\operatorname{mod}\mathfrak{M}$, for which $\varphi-a\cdot 1$ is noninvertible.

LEMMA 4. Let $\mathfrak{p}$ and I be ideals of C such that $I\not\subset\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal and the field of residues $k(\mathfrak{p})$ is infinite. Then $|I+\mathfrak{p}/\mathfrak{p}|=\infty$.

Proof. Obvious.

Proof of Theorem 1. We first observe that the last assertion follows from the previous ones. Indeed, since $\operatorname{Im}\varphi\simeq N$, it follows that $\operatorname{Im}\varphi$ is a closed lattice. Therefore, if $x\in M$, but $x\notin\operatorname{Im}\varphi$, then there exists $\mathfrak{p}\in P$ such that $x\notin\operatorname{Im}\varphi_{\mathfrak{p}}$. Then if $a\in C$ is a nondivisor of zero, it is also true that $ax\notin\operatorname{Im}\varphi_{\mathfrak{p}}$, since $\operatorname{Im}\varphi_{\mathfrak{p}}$ is a direct summands in $M_{\mathfrak{p}}$, and so $ax\notin\operatorname{Im}\varphi$. Consequently, $M/\operatorname{Im}\varphi$ is a torsionfree C-module and is therefore an A-lattice.

Let $\tilde{N}=\oplus_i U_i^{n_i}$, $\tilde{M}=\oplus_i U_i^{m_i}$. In each U_i pick a lattice L_i and define $N'=\oplus_i L_i^{n_i}$, $M'=\oplus_i L_i^{m_i}$. Then N' and M' are lattices in $\tilde{N}$ and $\tilde{M}$, respectively, and we may assume, multiplying N and M by nondivisors of zero if necessary, that $N'\supset N\supset cN'$ and $M'\supset M\supset cM'$ for some nondivisor of zero $c\in C$. Denote $X=\{\mathfrak{p}\in P\mid c\notin\mathfrak{p}\}$. This set is finite, and the ring C_X is semilocal. It follows from the assumptions of the theorem that there exists a monomorphism $\psi: N \to M$ such that ψ_X is split, i.e., $\psi_{\mathfrak{p}}$ is split for all $\mathfrak{p}\in X$.

Define $Y=\{\mathfrak{p}\in P\mid\psi_{\mathfrak{p}}$ is not split$\}$. This set is also finite. For every $\mathfrak{p}\notin X$ and $i = 1,\dots,s$, let $r_{i\mathfrak{p}}(\psi)$ denote the largest number d such that $\operatorname{Im}\psi_{\mathfrak{p}}$ contains a direct summand of $M_{\mathfrak{p}}$, isomorphic to $L_{i\mathfrak{p}}^{d}$. Obviously, $r_{i\mathfrak{p}}(\psi)=n$ for $\mathfrak{p}\notin Y$. But if $r_{i\mathfrak{p}}(\psi)=n_i$ for all $\mathfrak{p}$ and i, then $\mathrm{Im}\,\psi_{\mathfrak{p}}$ contains a direct summand isomorphic to $\Psi_{\mathfrak{p}}$ whence it follows that $N_{\mathfrak{p}}$, is split. Since the set of pairs $(\mathfrak{p}, i)$, for which $r_{i\mathfrak{p}}(\psi)<n_i$, is finite, the proof is now reduced to the following

LEMMA 5. Let $\psi:N \to M$ be a monomorphism and $Y \subset P$ a finite subset satisfying the following conditions:

1) $Y \cap X = \emptyset$;

2) $\psi_{\mathfrak{p}}$ is split for $\mathfrak{p}\notin Y$;

3) there exist $\mathfrak{q}\in Y$ and an index j such that $r_{i\mathfrak{p}}(\psi)<n_j$.

Then there exists a monomorphism $\psi':N\to M$ such that $\psi'_{\mathfrak{p}}$ is split for all $\mathfrak{p}\notin Y$, $r_{i\mathfrak{p}}(\psi')\geqslant r_{i\mathfrak{p}}(\psi)$ for all $\mathfrak{p}\in Y$, and $r_{j\mathfrak{q}}(\psi')>r_{j\mathfrak{q}}(\psi)$.

Proof. Denote $L=L_j$, $r=r_{j\mathfrak{q}}(\psi)$. Since $n_j < m_j$, there is a split monomorphism $\alpha:L \to M'$ such that $\mathrm{Im}\,\alpha\cap\mathrm{Im}\,\psi=0$. Define $\theta = c\alpha$ and consider θ as a homomorphism $L \to M$. Since $M_Y = M'_Y$ and c is invertible in C_Y, it follows that θ_Y is split. Consider the homomorphism $\eta=(\psi,\theta):N\oplus L\to M$. Then η is a monomorphism. Let $Z=X\cup\{\mathfrak{p}\in P\mid\mathfrak{p}\notin Y$ and $\eta_{\mathfrak{p}}$ is not split. Let us look for ψ' in the form $\psi + \theta\xi$, where $\xi:N \to L$ and $\xi(\mathfrak{p})=0$ for $\mathfrak{p}\in Z\cup Y\setminus\{\mathfrak{q}\}$. Then $\psi'(\mathfrak{p})=\psi(\mathfrak{p})$ for $\mathfrak{p}\in Z$, and so $\psi'_{\mathfrak{p}}$ is split by Lemma 2. If $\mathfrak{p}\notin Y\cup Z$, then even $\eta_{\mathfrak{p}}$ is split, and so, a fortiori, is $\psi'_{\mathfrak{p}}$. We now observe that the inequality $r_{i_{\mathfrak{p}}}(\psi)\geqslant d$ implies the existence of decompositions $N_{\mathfrak{p}}\simeq L_{\mathfrak{p}}^{d}\oplus N_1$ and $M_{\mathfrak{p}}\simeq L_{\mathfrak{p}}^{d}\oplus M_1$, relative to which ψ is defined by a matrix $\begin{pmatrix}\psi_0 & 0\\ 0 & \psi_1\end{pmatrix}$, where ψ_0 is an automorphism of L^d, or, what is the same, $\psi_0(\mathfrak{p})$ is invertible. Therefore, if $\mathfrak{p}\in Y\setminus\{\mathfrak{q}\}$ or $i \neq j$, we have $r_{i\mathfrak{p}}(\psi')\geqslant r_{i\mathfrak{p}}(\psi)$.

Now let $\mathfrak{p}=\mathfrak{q}$, $d=r$. Then by Lemma 1 $N_1\simeq L_{\mathfrak{p}}^{n_j-r}\oplus(\oplus_{i\neq j}L_{i\mathfrak{p}}^{n_i})$ and $M_1\simeq M_{\mathfrak{p}}^{m_j-r}\oplus(\oplus_{i\neq j}L_{i\mathfrak{p}}^{m_i})$. Since $\theta_{\mathfrak{p}}$ is split and $\mathrm{Im}\,\theta_{\mathfrak{q}}\cap\mathrm{Im}\,\psi_{\mathfrak{q}}=0$, $\theta_{\mathfrak{q}}$ is defined by a column (θ_k) in which the element θ_k is invertible for some $k > r$. Let $\xi:N \to L$ be the restriction to N of the homomorphism $N'' \to L$ defined by the row $(0,\ldots,0, \beta, 0,\ldots,0)$, with $\beta:L \to L$ in the k-th place. Then the matrix $\psi'_{\mathfrak{q}}=\psi_{\mathfrak{q}}+\theta_{\mathfrak{q}}\xi_{\mathfrak{q}}$ contains in the (kk) place the element $\psi_{kk} + \theta_k\beta$, and all its columns except the k-th coincide with the corresponding columns of $\psi_{\mathfrak{q}}$. If $k(\mathfrak{q})$ is infinite, it follows from Lemmas 3 and 4 that there exists $\beta\in C$ for which $\psi_{kk} + \theta_k\beta$ is invertible, and moreover $\beta\in cI$, where I is the intersection of all $\mathfrak{p}\in Y\cup Z\setminus\{\mathfrak{q}\}$. But if $k(\mathfrak{q})$ is finite, then $\mathfrak{q}$ is a maximal ideal and by the Chinese Remainder Theorem there exists $\beta\in\mathrm{End}\,L$ such that $\beta \equiv 0\ (\mathrm{mod}\ cI)$ and $\beta\equiv\theta_k^{-1}(1-\psi_{kk})(\mathrm{mod}\,\mathfrak{q})$, i.e., $\psi_{kk} + \theta_k\beta$ is again invertible. Thus $r_{j\mathfrak{q}}(\psi')>r$, as required.

Remark 1. If $\mathrm{Kr{\cdot}dim}\,C = 1$, then the above exact sequence is split and Theorem 1 becomes a well-known result from the theory of integral representations [1]. In the general case, however, it is generally impossible to construct a split exact sequence of the above kind even in the simplest case, when A = C is a regular Noetherian ring and M and N projective modules: there are well-known examples of indecomposable projective modules M such that $\tilde{M} = A^n$, where $n > 1$. A similar remark applies, naturally, to the results of the next sections.

Remark 2. In actual fact, a split monomorphism $N_{\mathfrak{p}}\to M_{\mathfrak{p}}$ surely exists for all $\mathfrak{p}\in P$, except possibly a finite number (in any case, for all $\mathfrak{p}\notin X$, where X is defined as in the proof of Theorem 1).

2. We now consider some applications of Theorem 1.

The genus g(M) of a lattice M is defined as the set of all A-lattices N such that $M_{\mathfrak{p}}\simeq N_{\mathfrak{p}}$ for all $\mathfrak{p}\in P$. The sum of genera g(M) and g(N) is the genus $g(M\oplus N)$. Accordingly, a genus is said to be indecomposable if all its lattices are indecomposable. Henceforth we shall assume for simplicity's sake that $\tilde{A}$ is a lattice (this is so, for example, in the important particular case in which the integral closure of the ring C in $\tilde{C}$ is a finitely generated C-module). Then every genus will contain a closed lattice. Fix one such lattice and denote it by M(g). Obviously, if lattices M and M' lie in the same lattice, then r(M) = r(M'). Denote this common value by r(g). A genus g is said to be minimal if there exists no decomposition $g = g_1 + g_2$ such that $r(g) \gg r(g_2)$, i.e., in any such decomposition there exists i such that the i-th coordinate of r(g) does not vanish but that of $r(g_1)$ does.

Let G denote the set of all indecomposable genera of A-lattices.

THEOREM 2. For any A-lattice M there exists an exact sequence of lattices $0 \to N \to M \to L \to 0$ whose localization with respect to any prime ideal $\mathfrak{p}\in P$ is split, and moreover $N=\otimes_i N_i$, where each lattice N_i is isomorphic to $M(g_i)$ for some $g_i \in G$, and $g(L)$ is a minimal genus.

Proof. Obviously, the genus $g(M)$ may always be expressed as $g' + g''$, where g'' is minimal and $f(M) \gg r(g')$. Let $g' = \sum_i g_i$, where $g_i \in G$. Define $N = \oplus_i M(g_i)$. Then Theorem 1 can be applied to the lattices M and N, i.e., there exists an exact sequence $0 \to N \to M \to L \to 0$ which is split for every $\mathfrak{p} \in P$, in which L is also a lattice. Since $M_\mathfrak{p} \simeq (N \oplus L)$ for all $\mathfrak{p} \in P$, it follows that $g(M) = g(N) + g(L)$, whence $g(L) = g''$, as claimed.

Let us define an overring of an algebra A to be a ring B such that $A \subset B \subset \tilde{A}$, and such that B is also an A-lattice. Call an algebra A bounded if it has a maximal overring (necessary and sufficient conditions for an algebra to be bounded were established in [4]).

COROLLARY. Let A be a bounded algebra such that for any $\mathfrak{p} \in P$ there exist finitely many isomorphism classes of $\mathfrak{X}$-lattices. Then there exists a finite set of closed A-lattices such that all genera $g(X)$ for $X \in \mathfrak{X}$ are indecomposable and distinct and for any A-lattice M there exists an exact sequence of lattices $0 \to N \to M \to L \to 0$ whose localization with respect to any $\mathfrak{p} \in P$ is split, and moreover $N = \oplus_i N_i$, $N_i \in \mathfrak{X}$, the genus $g(L)$ is minimal, and $g(L) = \sum_i g(L_i)$, where L_i are distinct lattices in $\mathfrak{X}$.

Proof. By analogy with the proof of Lemma (81.22) in [6], it can be shown that under these conditions A has finitely many indecomposable genera, so that we need only put $\mathfrak{X} = \{M(g) \mid g \in G\}$.

3. A ring A is called a pseudo-Bass ring if, for any prime ideal $\mathfrak{p} \in P$, $A_\mathfrak{p}$ is a Bass ring [7], i.e., $A_\mathfrak{p}$ and all its overrings are Gorenstein rings.

THEOREM 3. Let A be a pseudo-Bass ring and M a torsionfree A-module. Then there exist ideals $I_1, \ldots, I_{k+1}$ and an exact sequence $0 \to \bigoplus_{i=1}^{k} I_i \to M \to L \to 0$, which is split for any prime ideal $\mathfrak{p} \in P$, such that $g(L) = g(I_{k+1})$.

Proof. If A is pseudo-Bass, then for any prime ideal $\mathfrak{p} \in P$ the ring $A_\mathfrak{p}$ is Bass. Therefore, if $M_\mathfrak{p}$ is a torsionfree $A_\mathfrak{p}$-module, it will follow from Proposition 7.2 of [8] that $M_\mathfrak{p} = M_{1\mathfrak{p}} \oplus M_{2\mathfrak{p}}$, where $M_{1\mathfrak{p}}$ is an ideal, $r(M_{1\mathfrak{p}})$ is independent of $\mathfrak{p}$ and $r(M_\mathfrak{p}) \gg r(M_{2\mathfrak{p}})$. There exist lattices L_i such that $\tilde{M} = \tilde{L}_1 \oplus \tilde{L}_2$, where $r(L_1) = r(M_{1\mathfrak{p}})$. Denote $M' = L_1 \oplus L_2$, $S = \{\mathfrak{p} \in P \mid M_\mathfrak{p}$ $M'_\mathfrak{p}\}$. Then there exist lattices $N_i \subset \tilde{L}_i$ such that $N_{i\mathfrak{p}} = M_{i\mathfrak{p}}$ if $\mathfrak{p} \in S$ and $N_{i\mathfrak{p}} = L_{i\mathfrak{p}}$ if $\mathfrak{p} \notin S$. Hence $g(M) = g(N_1) + g(N_2)$, where $g(M) \gg g(N_2)$. We have thus shown that any minimal (in particular, indecomposable) genus is the genus of some ideal, and it remains only to apply Theorem 2.

A torsionfree A-module M is said to be irreducible if M is a simple $\tilde{A}$-module [9]; a torsionfree A-module M is said to be completely decomposable if it can be expressed as the direct sum of irreducible torsionfree A-modules [10]. The proof of the following theorem is analogous to that of Theorem 3.

THEOREM 4. Let A be a ring such that for any prime ideal $\mathfrak{p} \in P$ all torsionfree $A_\mathfrak{p}$-modules are completely decomposable. Then for any torsionfree A-module M there exists an exact sequence $0 \to N \to M \to L \to 0$, where N is a completely decomposable and L an irreducible A-module, which is split for any $\mathfrak{p} \in P$

A ring A is said to be pseudohereditary if, for any prime ideal $\mathfrak{p}$ of height 1, the ring $A_\mathfrak{p}$ is hereditary. Obviously, in that case all torsionfree $A_\mathfrak{p}$-modules are completely decomposable for any prime ideal $\mathfrak{p}$ of height 1. Consequently, Theorem 4 is true for pseudohereditary rings. We cite one more result generalizing the "reduction theorem" for lattices over integrally closed Noetherian rings [3].

THEOREM 5. Assume that the ring A is closed and for any $\mathfrak{p} \in P$ the ring $A_\mathfrak{p}$ is local [e.g., A = C). Then for any A-lattice M such that $M_\mathfrak{p}$ is free for all $\mathfrak{p} \in P$ (e.g., for any projective A-module) there exists an exact sequence $0 \to N \to M \to L \to 0$ which is split for all $\mathfrak{p} \in P$, such that M is a free A-module and L an ideal of A.

Proof. Follows directly from Theorem 1 by putting $N = A^{n-1}$, where $\tilde{M} = A^n$.

4. Theorem 1 also carries over to sheaves of lattices on schemes with an ample sheaf [11]. Let us call a separable scheme X pseudo-Noetherian if it has a finite open cover $X = \bigcup_i U_i$,

in which each U_i is isomorphic to the spectrum of some pseudo-Noetherian commutative ring. Assume that X is a reduced pseudo-Noetherian scheme, with a fixed ample sheaf $\mathcal{L}$, and let $\mathcal{A}$. be a given quasicoherent sheaf of semiprimary $\mathcal{O}_X$ -algebras such that for every point $x \in X$ of height 1 the algebra $\mathcal{A}_x$ is a finitely generated torsionfree $\mathcal{O}_{x,X}$ -module (the height of a point is the codimension of its closure). Sheaves of $\mathcal{A}$-lattices and so on are defined in the obvious way. As usual, we write $\mathcal{M}(n)$ for $\mathcal{M} \otimes \mathcal{L}^{\otimes n}$. Since $\mathrm{Hom}_{\mathcal{A}}(\mathcal{N}, \mathcal{M})(n) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{N}, \mathcal{M}(n))$ and for sufficiently large n this sheaf is generated by its global sections, one can prove the following result, repeating the proof of Theorem 1 almost word for word.

THEOREM 6. Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of $\mathcal{A}$-lattices such that $r(\mathcal{M}) \gg r(\mathcal{N})$ and for any point $x \in X$ of height i there exists a split monomorphism $\mathcal{N}_x \to \mathcal{M}_x$. Then for some n there exists a monomorphism $\varphi: \mathcal{N} \to \mathcal{M}$ such that φ_x is split for all points $x \in X$ of height 1.

COROLLARY. Assume that for every point $x \in X$ of height 1 the ring $\mathcal{A}_x$ is local, and let be a locally free sheaf of $\mathcal{A}$-modules of rank m. Then for some n there exists a monomorphism $\varphi: \mathcal{A}^{m-1} \to \mathcal{M}(n)$ such that φ_x is split for all points $x \in X$ of height 1.

Remark. Of course, unlike the affine case, the monomorphism φ here need not be split even when X is a one-dimensional Noetherian scheme.

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