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Cohen–Macaulay module type

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1. Introduction

In this paper we consider the problem of Cohen–Macaulay module type for orders or for Cohen–Macaulay algebras Λ which occur as local rings of reduced curve singularities. It is more or less known that with respect to the classification of (maximal) Cohen–Macaulay modules (or lattices) all orders Λ over a discrete valuation ring $\mathcal{O}$ split into three types:

- *finite*, when Λ has only a finite number of indecomposable Cohen–Macaulay modules:
- *tame*, when the indecomposable modules of any fixed rank form a finite number of 1-parameter families, together with, maybe, a finite set of "isolated" modules;
- *wild*, which can be defined in two ways: either as those algebras having n-parameter families of indecomposable modules of a fixed rank for arbitrary big n, or as those for which the classification of modules includes the classification of representations of all algebras.

In the geometrical case, when $\mathcal{O} = K[[t]]$ with algebraically closed field K, it was proved by the authors [DG] that any order of infinite type is either tame or wild (and, as a consequence, that both definitions of wildness coincide). Of course, one would like to have an effective criterion to check whether a given order is of finite, tame or wild type. In the commutative case, criteria of finiteness were given by Jacobinski [Ja] and Drozd–Roiter [DR]. Later on this was generalized to local orders by Drozd–Kirichenko [DK].

In the geometrical case the finiteness turned out to be closely connected with the behaviour under deformations. Namely, as Greuel and Knörrer observed [GK], the complete local ring of some point of a reduced algebraic curve (we call such rings curve singularities) is of finite type if and only if it dominates some of the so-called simple plane curve singularities A_n, D_n, E_6, E_7, E_8 (cf. [AVG] for their definition and characteristic properties).

A conjecture arose that the singularities of tame type should coincide with those dominating a (strictly) unimodal plane curve singularity (cf. [AVG] and [Wa] for the definition). Moreover, for the "parabolic" singularities T_{44} and T_{36} the tameness has just been proved in connection with integral representations of finite groups by Yakovlev [Ya], Dieterich [Di 1], [Di 2]. Moreover, Schappert [Sc] showed that all plane curve singularities which are not unimodal have 2-parameter families of ideals, so are wild in view of the cited result of [DG]. Nevertheless, finally this conjecture turned out to be wrong (by the way, it seems that the wildness of W_{12} was more or less known, although no proof of it was ever published).

Indeed, it happens that the "serial" singularities T_{pq} play quite another role than the "exceptional" unimodal ones. Namely, in this paper we prove the following criterion of tameness.

THEOREM 1. *Let Λ be a curve singularity of infinite Cohen–Macaulay type. Then it is of tame type if and only if it dominates one of the singularities T_{pq}.*

Recall that T_{pq} is the local ring at zero of the plane curve given by the equation $X^p + \lambda X^2Y^2 + Y^q$ ($\lambda \neq 0, 1$). Actually, except for $(p, q) = (3, 6)$ or $(4, 4)$, the parameter λ can be omitted as all its values (including $\lambda = 1$) lead to isomorphic rings, whilst for $(p, q) = (3, 6)$ or $(4, 4)$ both λ and the restriction $\lambda \neq 0$, 1 are indispensable.

We really use some geometry, namely, deformations, to prove the tameness of T_{pq}. It turns out to be much easier to show that some other series of uni-modal (though non-plane) singularities P_{pq} (cf. [Wa]) are tame. Since each T_{pq} (for $(pq) \neq (44)$ or (36)) is a deformation of P_{pq}, we are able to use a result of Knörrer [Kn] to obtain the tameness of T_{pq} and hence the sufficiency of the condition of Theorem 1.

To prove its necessity we introduce in Section 6 some other conditions in terms of over-rings of Λ resembling those used in [DR] to formulate a finiteness criterion. Rather standard, although sometimes cumbersome, matrix calculations show that whenever these conditions are not satisfied, Λ is of wild type. Finally, we check that these overring conditions imply that Λ dominates one of the singularities T_{pq}. That accomplishes the proof. The overring conditions are very useful for checking tameness of singularities which are not in normal form. We formulate them in Theorem 3 in Section 6.

At the same time we deal also with a more finer subdivision of tame type depending on the least possible number of irreducible 1-parameter families needed to obtain all indecomposable modules of given rank, except a finite set of them. Λ is said to be of *finite growth*, provided that this number is bounded by a constant (independent of the rank) and of infinite growth otherwise. In this context the main role is played by the parabolic singularities.

THEOREM 2. *Let Λ be a curve singularity of tame Cohen–Macaulay module type. Then Λ is of finite growth if and only if it dominates one of the singularities T_{44} or T_{36}.*

REMARK. Let Λ be a singularity T_{pq} with $(p, q) = (4, 4)$ or $(p, q) = (3, 6)$ and $\Lambda' = End\ \mathfrak{m}$ its unique minimal overring, $\mathfrak{m}$ the maximal ideal of Λ. Then Λ and Λ' are known to be non-domestic. Recall that a singularity Λ of tame *CMT* is called *domestic* if there exists a finite set $\mathscr{F}_1, \ldots, \mathscr{F}_m$ of strict 1-parameter families (i.e. without trivial subfamilies) such that for any rank vector r almost all indecomposable Cohen–Macaulay Λ-modules of rank r can be induced from $\{\mathscr{F}_1, \ldots, \mathscr{F}_m\}$, i.e. are isomorphic to $\mathscr{F}_i(L)$ for appropriate i and L (cf. Section 1 for notations). On the other hand, one can see that any proper overring of T'_{36} is domestic and all proper overrings of T'_{44} were proved to be domestic in [Di 2]. So, we obtain the following.

COROLLARY. *A singularity Λ of tame CMT is domestic if and only if it properly dominates one of the singularities T'_{36} or T'_{44}.*

Of course, just as for finite type, these results are of somewhat "zoological" nature as we see no a priori reason why these quite different classes of singularities should coincide. It is a really exciting problem to find such a reason and a proof of this fact, which does not go through the classification.

On the other hand, the proof of the tameness of T_{pq} is the only place where geometry is really used. That is why we hope that our criteria remain valid in the non-geometrical case too. But the lack of techniques compel us to restrict to curve singularities. Moreover, for the sake of simplicity, we suppose through the whole paper that *char* $K \neq 2$ (e.g. even the definition of T_{pq} has to be changed in characteristic 2).

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1. Preliminaries

Throughout the whole paper we identify a curve singularity with its complete local ring Λ, which we assume to be reduced. We suppose the ground field K to be algebraically closed and of characteristic not equal to 2. Then a K-algebra Λ is said to be a curve singularity if and only if it satisfies the following conditions:

(c1) Λ is complete, local and noetherian;

(c2) $\Lambda/\mathfrak{m} = K$ where $\mathfrak{m}$ denotes the unique maximal ideal of Λ;
(c3) $\dim \Lambda = 1$ where dim denotes the Krull dimension;
(c4) Λ contains no nilpotent elements.

Let F be the ring of fractions of Λ (with respect to all non-zero divisors). Then $F = \Pi_{i=1}^{s} F_i$ is a direct product of fields where s is the number of branches of the singularity. The normalization $\bar{\Lambda}$ of Λ in F is finitely generated as Λ-module and decomposes into a direct product of discrete valuation rings: $\bar{\Lambda} = \Pi_{1 \leqslant i \leqslant s} \Lambda_i$. Here Λ_i is the normalization of the projection of Λ onto F_i. As K is algebraically closed, $\Lambda_i \simeq K[[t_i]]$ where t_i is a uniformizing element of the ith branch.

Let $CM(\Lambda)$ denote the category of (maximal) Cohen–Macaulay Λ-modules (or Λ-lattices) which coincides in our case with the category of finitely generated torsion-free Λ-modules. If $M \in CM(\Lambda)$, the natural mapping $M \to \tilde{M} = M \otimes_\Lambda F$ is an injection and we shall always identify M with its image in $\tilde{M}$. Thus, if some singularity Γ is an overring of Λ (we also say that Γ *dominates* Λ), which means $\Lambda \subset \Gamma \subset \bar{\Lambda}$, then the Cohen–Macaulay Γ-module $M\Gamma \subset \tilde{M}$ is defined. One can easily check that $M\Gamma \simeq M \otimes_\Lambda \Gamma/T$ where T is the torsion part of $M \otimes_\Lambda \Gamma$. As $\tilde{M}$ is a finitely generated F-module, $\tilde{M} \simeq \bigoplus_{i=1}^{s} r_i F_i$ for some integers r_i. Call the vector $\mathbf{r}(M) = (r_1, \ldots, r_s)$ the *rank vector* of M and let $CM_{\mathbf{r}}(\Lambda)$ denote the set of isomorphism classes of Cohen–Macaulay Λ-modules having fixed rank vector $\mathbf{r}$.

Recall some definitions concerning the *Cohen–Macaulay module type* (CMT) of a curve singularity Λ and families of Λ-modules. Let B be any K-algebra. Denote by $CM(\Lambda, B)$ the category of $B-\Lambda$-bimodules $\mathscr{F}$ satisfying the following ("family-") conditions:

(F1) $\mathscr{F}$ is finitely generated as bimodule;
(F2) $\mathscr{F}_\Lambda$ is torsion-free;
(F3) ${}_B\mathscr{F}$ is flat;
(F4) for any B-module L, which is finite dimensional over K, the Λ-module $\mathscr{F}(L) = L \otimes_B \mathscr{F}$ belongs to $CM(\Lambda)$.

In this case call $\mathscr{F}$ a *family of* (*Cohen–Macaulay*) Λ-*modules* with base B. Of course, the condition (F4) has to be checked only for simple B-modules L. For instance, if B is an affine algebra (i.e. of finite type over K and commutative), then we only require $\mathscr{F}(x) := \mathscr{F}(B/\mathfrak{m}_x) \in CM(\Lambda)$ for any closed point $x \in Spec\, B$. If B is affine, $\dim B = n$ and if for any closed point $x \in Spec\, B$ the set $\{y \in Spec\, B \mid \mathscr{F}(y) \simeq \mathscr{F}(x)\}$ is finite, call $\mathscr{F}$ an *n-parameter family of* Λ-*modules*. $\mathscr{F}$ is called *irreducible* if *Spec B* is irreducible.

Call a family $\mathscr{F} \in CM(\Lambda, B)$ *strict* provided for all B-modules L and L' of finite K-dimension the following holds: $\mathscr{F}(L) \not\simeq \mathscr{F}(L')$ whenever $L \not\simeq L'$ and $\mathscr{F}(L)$ is indecomposable if L is.

Λ is said to be of:

- *finite CMT* if there are only finitely many indecomposable Cohen–Macaulay Λ-modules (up to isomorphism);
- *tame CMT* if it is not of finite CMT but for any fixed rank vector $\mathbf{r}$ there exists a finite set $\{\mathscr{F}_1, \ldots, \mathscr{F}_m\}$ of 1-parameter families of Λ-modules such that almost all (i.e. all but a finite number) indecomposable $M \in CM_{\mathbf{r}}(\Lambda)$ are isomorphic to $\mathscr{F}_i(x)$ for suitable i and x;
- *wild CMT* if for any finitely generated K-algebra B (not necessary commutative) there exists a strict family $\mathscr{F} \in CM(\Lambda, B)$.

It is well-known (cf. [DR]) that if Λ is not of finite CMT, then it possesses 1-parameter families of indecomposable modules for arbitrarily big rank. Indeed, such Λ possesses a strict family with base $K[t]$.

It was proved in [DG] that any curve singularity Λ of infinite CMT is either of tame or of wild CMT. Of course, in the last case Λ possesses n-parameter families of indecomposable modules for arbitrary n (thus Λ cannot be both of wild and of tame CMT). Moreover, it is well-known (cf. [GP]) that to prove the wildness one has only to construct a strict family wih base $K\langle x, y\rangle$ (free algebra with two generators) or $K[x, y]$ or even $K[[x, y]]$.

Suppose Λ to be of tame CMT. Denote by $f(\Lambda, \mathbf{r})$ the least number of irreducible 1-parameter families necessary to obtain almost all indecomposable Cohen–Macaulay Λ-modules of rank vector $\mathbf{r}$. Call Λ of:

- *finite growth* if there exists a constant c (depending on Λ but not on $\mathbf{r}$) such that $f(\Lambda, \mathbf{r}) \leqslant c$;
- *infinite growth* provided $f(\Lambda, \mathbf{r})$ is unbounded when $\mathbf{r}$ increases.

It is also known (cf. [Dr 1], [DG]) that if Λ is tame, then the 1-parameter families $\mathscr{F}_i$ used in the definition of tameness can always be chosen with base $B = K[t]$. Moreover, we can take the $\mathscr{F}_i$ even strict with some rational bases B_i, i.e. those of the form $B_i = K[t, g_i(t)^{-1}]$ for suitable polynomials $g_i(t)$.

2. Subspace categories

To classify the Cohen–Macaulay modules it is convenient to use the so-called subspace categories (cf. [Ri]). Let C be a vector space category, i.e. a subcategory (usually not full) of the category *Vect* of finite dimensional vector spaces over K. We shall always suppose that C is fully additive [Dr 1] which means here that any two objects of C possess a direct sum in C and the endomorphism algebra $C(X, X)$ is local for any indecomposable object X. In particular, for any finite dimensional K-vector space L and any $X \in obC$ we may consider $L \otimes X$ as an object of C, identifying it with $(\dim L)X$.

Define the *subspace category SubC*. Its objects are the pairs (X, V) with $X \in obC$ and V a subspace of X. A morphism $(X, V) \to (Y, W)$ is, by definition, a morphism $\varphi \in C(X, Y)$ such that $\varphi(V) \subset W$. For our purpose we need also the case when C is a category of A-modules, i.e. all its objects are modules over some K-algebra A and its morphisms are A-homomorphisms (again not necessarily a full subcategory). Then we consider the full subcategory $Sub^A C$ consisting of all pairs (X, V) such that $VA = X$. Call it the *generating subspace category*.

Let B be a K-algebra. Define a *family of subspaces* with base B as a pair $\mathscr{F} = (X, W)$ where $X \in obC$ and W is a finitely generated B-submodule in $B \otimes X$ such that the B-module $B \otimes X/W$ is flat over B (and hence projective as it is finitely generated, cf. [AC]). Of course, then W itself is flat (and projective). Note that if B is noetherian (e.g. affine), any submodule $W \subset B \otimes X$ is finitely generated. For any finite dimensional (over K) B-module L the tensor product $L \otimes_B W$ is a subspace in $L \otimes_B (B \otimes X) \simeq L \otimes X$. Hence, the object $\mathscr{F}(L) = (L \otimes X, L \otimes_B W)$ of $SubC$ is well-defined. If C is a category of A-modules we can impose the condition $WA = B \otimes X$ and speak about *families of generating subspaces*. Now we are able to define the *subspace type* (or *generating subspace type*) for vector space categories just as we have done for the Cohen–Macaulay module type in Section 1.

The following simple observation turn out to be of great use for the calculation of Cohen–Macaulay modules. Suppose Γ to be an overring of Λ such that $\Gamma\mathfrak{m} = \mathfrak{m}$, $\mathfrak{m}$ the maximal ideal of Λ. Put $A = \Gamma/\mathfrak{m}$ and consider the A-module category $C = C_\Lambda^\Gamma$ whose objects are of the form $N/N\mathfrak{m}$ where $N \in CM(\Gamma)$ and morphisms are just the mapping $N/N\mathfrak{m} \to N'/N'\mathfrak{m}$ induced by Γ-homomorphisms $N \to N'$. Define the functor $\varphi: CM(\Lambda) \to Sub^A C$ by putting $\varphi(M) = (M\Gamma/M\mathfrak{m}, M/M\mathfrak{m})$ (note that $M\Gamma\mathfrak{m} = M\mathfrak{m}$).

PROPOSITION 2.1. *The functor φ is full, dense, reflects isomorphisms and preserves indecomposability (i.e. $\varphi(M) \simeq \varphi(M')$ implies $M \simeq M'$ and if M is indecomposable, then so is $\varphi(M)$, too.)*

The proof is an evident consequence of the definitions.

COROLLARY 2.2. *For any Γ as above, the Cohen–Macaulay module type of Λ coincides with the generating subspace type of C_Λ^Γ.*

For the proof cf. [DG], where it is given in a slightly different but quite analogous situation.

Usually one takes $\Gamma = End\mathfrak{m} = \{\gamma \in F \mid \mathfrak{m}\gamma \subset \mathfrak{m}\}$. At least, we shall use this choice in the following considerations.

We need also the next result proved in [Ba].

PROPOSITION 2.3. *Suppose Λ to be Gorenstein and $\Lambda' = End\mathfrak{m}$. Then Λ' is the unique minimal overring of Λ and any indecomposable Cohen–Macaulay*

Λ-module is either a Λ'-module or isomorphic to Λ. Hence, the Cohen–Macaulay types of Λ and Λ' coincide.

3. The singularities P_{pq}

Let us apply the method of the last section to concrete calculations. Namely, consider the singularity $P_{p,q}$ which is, by definition [Wa], the ring $\Lambda = K[[x, y, z]]/(xy, x^p + y^q + z^2)$ with $p, q \geqslant 2$ and $(p, q) \neq (2, 2)$. Its normalization $\bar{\Lambda}$ and its location inside $\bar{\Lambda}$ depend on the parity of p and q. Namely:

- if p and q are both odd, then $\bar{\Lambda} = K[[t]]^2$ and Λ is the subalgebra of $\bar{\Lambda}$ generated by the elements $(t^2, 0)$, $(0, t^2)$, (t^p, t^q);
- if p is odd and q is even, then $\bar{\Lambda} = K[[t]]^3$ and Λ is the subalgebra generated by $(t, t, 0)$, $(0, 0, t^2)$, $(t^{q/2}, 0, t^p)$;
- if both p and q are even, then $\bar{\Lambda} = K[[t]]^4$ and Λ is generated by $(t, t, 0, 0)$, $(0, 0, t, t)$, $(t^{p/2}, 0, t^{q/2}, 0)$.

As Λ is a complete intersection and hence a Gorenstein ring, we can use Proposition 2.3 and replace Λ by its unique minimal overring Λ'. In all three cases described above Λ' is generated by four generators: the first two generators of Λ and two components of the third one (e.g. $(t^2, 0)$, $(0, t^2)$, $(t^p, 0)$, $(0, t^q)$ in the first case).

From now on Λ will denote the minimal overring of some singularity $P_{p,q}$ and $\mathfrak{m}$ its maximal ideal. As the calculations are quite similar in all three cases, we shall only do them for the first one. To use Proposition 2.1, put $\Gamma = \mathit{End}\,\mathfrak{m}$, $A = \Gamma/\mathfrak{m}$. Then $\Gamma = \Gamma_1 \times_K \Gamma_2$ where $\Gamma_1 = K[[t^2, t^{p-2}]]$ and $\Gamma_2 = K[[t^2, q^{q-2}]]$. If N is a Γ-module, then $N = N_1 \oplus N_2$ with N_i a Γ_i-module. But Γ_i is a singularity of type A_n, thus the Γ_i-modules are well-known. Namely, the indecomposable ones are isomorphic to Γ_{ik} ($k = 1, \ldots, (p-1)/2$ for $i = 1$ and $k = 1, \ldots, (q-1)/2$ for $i = 2$) where $\Gamma_{ik} = K[[t^2, t^{2k-1}]]$ is considered as an overring of Γ_i. Hence,

$$N \simeq \bigoplus_{i,k} n_{ik}\Gamma_{ik} \quad \text{and} \quad N/N\mathfrak{m} \simeq \bigoplus_{i,k} n_{ik}X_{ik}$$

where $X_{ik} = \Gamma_{ik}/\Gamma_{ik}\mathfrak{m}$. Obviously, X_{ik} is the 2-dimensional space with basis $\{x_{ik}, y_{ik}\}$ where $x_{ik} = 1 + \Gamma_{ik}\mathfrak{m}$ and $y_{ik} = t^{2k-1} + \Gamma_{ik}\mathfrak{m}$. By the way, if $\mathfrak{m}_i$ denotes the maximal ideal of Γ_i, then $\Gamma_{ik}\mathfrak{m}_i = \Gamma_{ik}\mathfrak{m}$ whenever $\Gamma_{ik} \neq \Gamma_i$, i.e. $k \neq (p-1)/2$ for $i = 1$ resp. $k \neq (q-1)/2$ for $i = 2$. So, in these cases $\{x_{ik}, y_{ik}\}$ is the minimal system of generators of X_{ik} as an A-module. Of course, if $\Gamma_{ik} = \Gamma_i$, then just x_{ik} generates X_{ik}.

To determine the category $C = C_\Lambda^\Gamma$ we also have to find $C(X_{ik}, X_{jl}) \simeq \mathit{Hom}_\Gamma(\Gamma_{ik}, \Gamma_{jl})/\mathit{Hom}_\Gamma(\Gamma_{ik}, \Gamma_{jl}\mathfrak{m})$. Certainly, $\mathit{Hom}_\Gamma(\Gamma_{ik}, \Gamma_{jl}) = 0$ if $i \neq j$. Identifying $\varphi \in Hom_\Gamma(\Gamma_{ik}, \Gamma_{il})$ with the element $\varphi(1) \in \Gamma_{il}$, we can consider this space as a subspace of Γ_{il}, namely $\{\gamma \in \Gamma_{il} \mid \gamma\Gamma_{ik} \subset \Gamma_{il}\}$. But this is either Γ_{il} if $l \leqslant k$ or the ideal generated by $(t^{2(l-k)}, t^{2l-1})$ if $k < l$. It follows that $C(X_{ik}, X_{il})$ is a 2-dimensional vector space with basis $\{a_{ikl}, b_{ikl}\}$ where

$$a_{ikl}(x_{ik}) = \begin{cases} x_{il} & \text{if } l \leqslant k \\ 0 & \text{if } k < l \end{cases}$$

$$a_{ikl}(y_{ik}) = \begin{cases} 0 & \text{if } l < k \\ y_{il} & \text{if } k \leqslant l \end{cases}$$

(in particular, a_{ikk} is the identity map); $b_{ikl}(x_{ik}) = y_{il}$ and $b_{ikl}(y_{ik}) = 0$.

Let V be a subspace of $X = \bigoplus_{i,k} n_{ik} X_{ik}$ with basis $\{v_1, \ldots, v_n\}$. Take as a basis of X the set $\{x_{ik}\mathbf{e}_m, y_{ik}\mathbf{e}_m \mid m = 1, \ldots, n_{ik}\}$ (for all possible values of i, k). Here $\mathbf{e}_m$ denotes the standard basis vectors of $K^m : \mathbf{e}_m = (0, \ldots, 1, \ldots, 0)^T$, 1 at the mth place. Put

$$v_j = \sum_{ikm} \xi^j_{ikm} x_{ik} \mathbf{e}_m + \sum_{ikm} \eta^j_{ikm} y_{ik} \mathbf{e}_m.$$

Then V can be described by the set of matrices $\xi_{ik} = (\xi^j_{ikm})$ and $\eta_{ik} = (\eta^j_{ikm})$, both of size $n_{ik} \times m$. Of course, if we change the basis of V, the set $\{\xi_{ik}, \eta_{ik}\}$ is transformed to $\{\xi_{ik}\theta, \eta_{ik}\theta\}$ for some invertible $n \times n$ matrix θ. Moreover, the subspace V is generating (i.e. $VA = X$) if and only if the rows of each of two following matrices g_1 and g_2 are linearly independent, where

$$g_i = \begin{pmatrix} \xi_{il} \\ \vdots \\ \xi_{im} \\ \eta_{i1} \\ \vdots \\ \eta_{i,m-1} \end{pmatrix}, \qquad m = \begin{cases} \dfrac{p-1}{2} & \text{for } i = 1 \\[2mm] \dfrac{q-1}{2} & \text{for } i = 2. \end{cases}$$

Similarly, an endomorphism φ of X can be described by a set of matrices $\{\alpha_{ikl}, \beta_{ikl}\}$ – both of size $n_{il} \times n_{ik}$ for all possible values of i, k, l – formed by the coefficients of the components of $\varphi_{ikl}: n_{ik} X_{ik} \to n_{il} X_{il}$ with respect to the bases $\{a_{ikl}, b_{ikl}\}$ of $C(X_{ik}, X_{il})$ chosen above.

Moreover, φ is an automorphism of X if and only if all diagonal components φ_{ikk} i.e. all matrices α_{ikk}, are invertible. One can now easily calculate the set of matrices $\{\xi'_{ikl}, \eta'_{ikl}\}$ corresponding to the subspace of $\varphi(V)$, namely:

$$\xi'_{ik} = \sum_{k \leqslant l} \alpha_{ilk} \xi_{il},$$

$$\eta'_{ik} = \sum_{l \leqslant k} \alpha_{ilk} \eta_{il} + \sum_{l} \beta_{ilk} \xi_{il}.$$

Therefore, *Sub C* coincides with the category of "representations of two pairs of chains (E_1, F_1) and (E_2, F_2) with the relation $\sim$" in the sense of the work [Bo]. Here we put: $F_1 = \{Z_1\}$, $F_2 = \{Z_2\}$ (one point sets), $E_i = \{x_{im} < x_{i,m-1} < \cdots < x_{i1} < y_{i1} < y_{i2} < \cdots < y_{im}\}$ ($m = (p-1)/2$ for $i = 1$ and $(q-1)/2$ for $i = 2$) and the relation $\sim$ is given by $Z_1 \sim Z_2$, $x_{ik} \sim y_{ik}$ for all possible values of i and k. Hence, it follows from [Bo] that C is of tame subspace type and as well of tame generating subspace type. Moreover, using the list of indecomposable representations given in [Bo], one can see that it is of infinite growth. So we obtain by Corollary 2.2.

PROPOSITION 3.1. *All curve singularities P_{pq} are of tame CMT and of infinite growth.*

4. Using deformations

Recall a result of Knörrer [Kn] on the behaviour of modules in a family of curve singularities. As we need only affine families (even only with the affine line as a base space), we prefer an algebraic formulation. Thus, a family of (affine) curves over some base algebra B is a flat, finitely generated B-algebra $\mathscr{L}$ such that for any closed point $x \in Spec\, B$ the algebra $\mathscr{L}(x) = \mathscr{L} \otimes_B K(x)$ (where $K(x)$ is the residue field of the point x) is a reduced affine algebra of Krull dimension 1. In particular, there is only a finite number of singular points $y \in Spec\, \mathscr{L}(x)$.

Consider the completion Λ_y of the local ring $\mathscr{L}(x)_y$ and denote by $par(y, \mathbf{r})$ the greatest number n such that there exists an n-parameter family of Λ_y-modules of rank $\mathbf{r}$. Put $par(x, \mathbf{r}) = \Sigma_{y \in Spec\, \mathscr{L}(x)}\, par(y, \mathbf{r})$. As almost all $y \in Spec\, \mathscr{L}(x)$ are non-singular, this sum is well-defined.

PROPOSITION 4.1 (cf. [Kn]). *The function $par(x, \mathbf{r})$ is upper semi-continuous on Spec B, i.e. for any fixed m the set $\{x \in Spec\, B \mid par(x, \mathbf{r}) \geqslant m\}$ is closed in Spec B.*[1]

[1] Knörrer proves the theorem only for the case $r_1 = \cdots = r_s$, but the general case can be proved in the same way.

As a module of rank $\mathbf{r} = (r_1, \dots, r_s)$ can split into at most $r_1 + \cdots + r_s$ indecomposable ones, Λ_y is of tame *CMT* if and only if $par(y, \mathbf{r}) \leqslant r_1 + \cdots + r_s$ where s is the number of branches, which is, of course, bounded for all y. On the other hand it follows from [DG] that, if Λ_y is of wild *CMT*, $par(y, \mathbf{r})$ grows quadratically with $\mathbf{r}$, just as the number of parameters of families of non-conjugate pairs of matrices. This gives us the following

COROLLARY 4.2. *The set* $W(\mathscr{L}) = \{x \in Spec\, B \mid \Lambda_y$ *is of wild CMT for some* $y \in Spec\, \mathscr{L}(x)\}$ *is the union of a countable number of closed subvarieties of* $Spec\, B$.

In particular, suppose the family $\mathscr{L}$ over B has a section σ such that all singularities $\Lambda_{\sigma(x)}$ except possibly $\Lambda_{\sigma(0)}$ are isomorphic. Then $\Lambda_y := \Lambda_{\sigma(x), x \neq 0}$ is called a deformation of $\Lambda_0 := \Lambda_{\sigma(0)}$.

COROLLARY 4.3. *If* Λ_y *is a deformation of* Λ_0 *and* Λ_0 *is of tame CMT, then* Λ_y *is also of tame CMT.*

Now we are able to prove the sufficiency of the condition stated in Theorem 1.

COROLLARY 4.4. *The singularities* $T_{p,q}$, *and hence all their overrings, are of tame CMT.*

Proof. The tameness of T_{44} was proved in [Di 2] (cf. also [Ya]) and that of T_{36} in [Bo] and [Di 1]. So we may suppose that $(p, q) \neq (4, 4)$ and $(p, q) \neq (3, 6)$. Then all values of the parameter λ in the equation of $T_{p,q}$ give isomorphic rings.

Consider the algebra $\mathscr{L}$ over the polynomial ring $B = K[\lambda]$ (i.e. over the affine line): $\mathscr{L} = B[x, y]/(xy - \lambda z,\ x^p + y^q + z^2)$. We identify the points of $Spec\, B$ with the corresponding values of λ. Evidently, the only singular point of each $\mathscr{L}(\lambda)$ is $(0, 0)$. Moreover, if $\Lambda(\lambda)$ denotes the corresponding singularity, then $\Lambda(\lambda) \simeq T_{p,q}$ for all $\lambda \neq 0$ whilst $\Lambda(0) \simeq P_{p,q}$. Therefore, $T_{p,q}$ is a deformation of $P_{p,q}$ and hence is tame by Corollary 4.3 and Proposition 3.1.

REMARK 4.5. It is a very exciting observation that if we would have tried to calculate the $T_{p,q}$-modules using the method of Section 2, this would have been much more complicated than the calculations for $P_{p,q}$ made in Section 3. Indeed, as $T_{p,q}$ is Gorenstein, we can consider instead of $T_{p,q}$ itself its unique minimal overring Λ. But then, if we put $\Gamma = End\, \mathfrak{m}$, the ring Γ turns out to be isomorphic to the unique minimal overring of $P_{p-2,q-2}$. So we have first to do the whole procedure for the P's and only afterwards we should be able to start with the T's.

Unfortunately, we cannot do the same for the "parabolic" cases (3.6) or (4.4) since then $T_{p,q}$ really depends on the parameter λ. Thus Corollary 4.2 only implies that $T_{p,q}(\lambda)$ is tame for "almost all" values of λ. Note, however, that if the conjectured extension of Knörrer's theorem in the sense that the function $par(x, \mathbf{r})$ is constant for families of plane curve singularities with constant Milnor number, the tameness of $T_{p,q}(\lambda)$ for all λ would already follow from the tameness of P_{pq}.

5. The singularities $\tilde{P}_{33}$ and $\tilde{P}_{34}$

We also need to consider two special singularities $\tilde{P}_{33}$ and $\tilde{P}_{34}$. Namely:

- $\tilde{P}_{33}$ is the subalgebra of $K[[t]]^2$ generated by the elements (t, t^2), $(t^2, 0)$ and $(0, t^5)$;
- $\tilde{P}_{34}$ is the subalgebra of $K[[t]]^3$ generated by (t, t, t), $(0, 0, t^2)$ and $(0, t^3, 0)$.

They are no longer Gorenstein, hence not complete intersections. Of course, $\tilde{P}_{33}$ dominates T_{37} and $\tilde{P}_{34}$ dominates T_{38}, so they are of tame *CMT* by Corollary 4.4. But we have yet to prove

PROPOSITION 5.1. *$\tilde{P}_{33}$ and $\tilde{P}_{34}$ are both of infinite growth.*

REMARK 5.2. One can easily see that $\tilde{P}_{33}$ deforms to P_{33} and $\tilde{P}_{34}$ deforms to P_{34}. If we had an analogue of Knörrer's theorem for the number of families $f(\Lambda, r)$, then Proposition 5.1 would become a corollary of Proposition 3.1. But unfortunately, although we are quite sure that such analogue is true, we are not able to prove it ad hoc (it can be deduced, of course, from Theorem 2).

Proof. As the calculations for both cases are very similar, we shall do them only for $\Lambda = \tilde{P}_{33}$, which is a little bit more complicated. Again we use the procedure of Proposition 2.1 and Corollary 2.2. Put $\Gamma = End\,\mathfrak{m} = \{\gamma \in F \mid \gamma\mathfrak{m} \subset \mathfrak{m}\}$. Now Γ is the subring of $\bar{\Lambda} = K[[t]]^2$ generated by $(t, 0)$, $(0, t^2)$ and $(0, t^3)$. Then $A = \Gamma/\mathfrak{m}$ is the 3-dimensional K-algebra with basis $\{1, a_1, a_2\}$ where $a_1 = (0, t^2) + \mathfrak{m}$ and $a_2 = (0, t^3) + \mathfrak{m}$. In particular, $a_i a_j = 0$ for any i, j. Consider the overring Δ of Γ generated by $(t, 0)$ and $(0, t)$. Then $A' = \Delta/\Delta\mathfrak{m}$ is also 3-dimensional with basis $\{1', b_1, b_2\}$ where $b_1 = (0, t) + \Delta\mathfrak{m}$ and $b_2 = (0, t^2) + \Delta\mathfrak{m}$. (We write $1'$ here to distinguish the units of A and A'). Now $b_2 = b_1^2$ and $b_1 b_2 = 0$ in A'. Note also that $\{1', b_1\}$ is a minimal set of generators of A' as A-module.

Consider the full subcategory D of C_Λ^Γ consisting of all direct sums of copies of A and A'. It is enough to show that D is of infinite growth (with respect to families of generating subspaces). Then C_Λ^Γ is of infinite growth and hence Λ by Corollary 2.2. Of course, $D(A, A) = A$ and $D(A', A') = A'$ as algebras. As

$Hom_\Gamma(\Gamma, \Delta) \simeq \Delta$, all morphisms of $D(A, A')$ are generated by multiplication of elements of Γ by those of Δ. So there are three linear independent morphisms:

$$\begin{aligned}
&c_0\colon 1 \mapsto 1', \quad a_1 \mapsto b_2, \quad a_2 \mapsto 0;\\
&c_1\colon 1 \mapsto b_1, \quad a_1 \mapsto 0, \quad a_2 \mapsto 0;\\
&c_2\colon 1 \mapsto b_2, \quad a_1 \mapsto 0, \quad a_2 \mapsto 0.
\end{aligned}$$

Morphisms of $D(A', A)$ are also generated by multiplication of elements of Δ by those of Γ, but the last should be taken from the set $\{\gamma \in \Gamma \mid \gamma\Delta \subset \Delta\}$ which coincides in this case with the maximal ideal of Γ. Hence, we have again three linear independent morphisms:

$$\begin{aligned}
&d_0\colon 1' \mapsto a_1, \quad b_1 \mapsto a_2, \quad b_2 \mapsto 0;\\
&d_1\colon 1' \mapsto 0, \quad b_1 \mapsto a_2, \quad b_2 \mapsto 0;\\
&d_2\colon 1' \mapsto a_2, \quad b_1 \mapsto 0, \quad b_2 \mapsto 0.
\end{aligned}$$

Let V be a subspace of $X = \mathfrak{m}A \oplus \mathfrak{m}'A'$ with basis $\{v_1, \dots, v_n\}$. It can be described by the set of matrices $\alpha_k = (\alpha_{kij})$ and $\beta_k = (\beta_{kij})V$, $k = 0, 1, 2$ where $v_j = \Sigma_i(\alpha_{0ij}1 + \alpha_{1ij}a_1 + \alpha_{2ij}a_2)\mathbf{e}^i + \Sigma_i(\beta_{0ij}1' + \beta_{1ij}b_1 + \beta_{2ij}b_2)\mathbf{e}^i$. Moreover, the subspace V is generating if and only if the rows of the matrix

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \beta_1 \end{pmatrix}$$

are linear independent. Hence, changing the basis of V, we may suppose that

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}.$$

Moreover, consider only the case when

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \beta_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Using the morphism b_1 (more precisely, the multiplication by b_1 in A'), we can transform the whole set of matrices into the form:

$$\begin{pmatrix}\alpha_0\\ \beta_0\\ \beta_1\\ \alpha_2\\ \alpha_1\\ \beta_2\end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\\ 0 & 0 & \gamma_1 & \gamma_0\\ 0 & 0 & 0 & \gamma_2\end{pmatrix}$$

for appropriate matrices γ_0, γ_1, γ_2.

One can check that the use of all other morphisms of D leads to the following permissible transformations of the triple $(\gamma_0, \gamma_1, \gamma_2)$:

$$\begin{aligned}&\gamma_0 \mapsto \sigma\gamma_0\sigma^{-1},\\ &\gamma_1 \mapsto \sigma\gamma_1\tau^{-1} + \gamma_0\eta_1,\\ &\gamma_2 \mapsto \tau\gamma_2\sigma^{-1} + \eta_2\gamma_0\end{aligned}$$

for some invertible matrices σ, τ and arbitrary η_1, η_2.

Consider the case when

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & I\\ 0 & 0 & 0\end{pmatrix}$$

(i.e. a direct sum of 1×1 and 2×2 Jordan cells with eigenvalue 0). Then γ_1 and γ_2 can be transformed to:

$$\gamma_1 \mapsto \begin{pmatrix}\gamma_{11}\\ 0\\ \gamma_{12}\end{pmatrix}, \quad \gamma_2 \mapsto (\gamma_{21}, \gamma_{22}, 0).$$

The permissible transformations for γ_{ij} will be:

$$\begin{aligned}&\gamma_{11} \mapsto \sigma_1\gamma_{11}\tau^{-1} + \xi_1\gamma_{12};\\ &\gamma_{12} \mapsto \sigma_2\gamma_{12}\tau^{-1};\\ &\gamma_{21} \mapsto \tau\gamma_{21}\sigma_1^{-1} + \gamma_{22}\xi_2;\\ &\gamma_{22} \mapsto \tau\gamma_{22}\sigma_2^{-1}.\end{aligned}$$

Therefore, the quadruple $\{\gamma_{ij}\}$ is indeed a "representation of two pairs of chains (E_1, F_1) and (E_2, F_2) with the relation $\sim$" in the sense of [Bo] where $E_1 = \{x_2 < x_1\}$, $F_1 = \{z_1\}$, $E_2 = \{z_2\}$, $F_2 = \{y_2 < y_1\}$ with $x_i \sim y_i$ $(i = 1, 2)$ and $z_1 \sim z_2$. Again, it follows from the list of representations given in [Bo] that D and hence Λ is of infinite growth.

6. The overring condition

To prove the necessity of the condition of Theorem 1, we shall introduce a necessary and sufficient condition for a singularity Λ to be of tame type. These conditions apply to some overrings of Λ close to its normalization $\bar{\Lambda}$ and are quite useful for checking tameness.

We introduce some notations which will be used through the remainder of the paper. Let Γ be an overring of Λ and $\Gamma/\Gamma\mathfrak{m} = A_1 \times \cdots \times A_m$ with local algebras A_i of dimensions $d_1, \ldots, d_m$. Denote $\mathbf{d}(\Gamma) = [d_1, \ldots, d_m]$, the *multiplicity vector* of Λ with respect to Γ, and $d(\Gamma) = d_1 + \cdots + d_m$, the (*total*) *multiplicity*. We always arrange d_i such that $d_1 \leqslant \cdots \leqslant d_m$. Certainly, $d(\Gamma)$ is the minimal number of generators of Γ considered as Λ-module and $\mathbf{d}(\bar{\Lambda})$ is the usual multiplicity of the singularity Λ.

Suppose that $\bar{\Lambda} = \Pi_{i=1}^{s} \Delta_i$ with $\Delta_i = K[[t_i]]$ and that e_i is the idempotent of $\bar{\Lambda}$ for which $\Delta_i = e_i\Lambda$. Let $t = (t_1, \ldots, t_s)$ and $\theta \in \bar{\Lambda}$ such element that $\bar{\Lambda}\mathfrak{m} = \theta\bar{\Lambda}$. Of course, we may (and we shall) suppose that $\theta \in \mathfrak{m}$. Put $\Lambda' = t\bar{\Lambda} + \Lambda$ (the *weak normalization* of Λ or its biggest local overring), $\Lambda'' = \theta t\bar{\Lambda} + \Lambda$ and $\Lambda'_i = \Lambda' + Ke_i$. Call an indempotent e_i *admissible* provided $e_i\mathfrak{m} \subset \mathfrak{m} + \theta t\bar{\Lambda}$. Note that $\theta\bar{\Lambda} = (t_1^{d_1}, \ldots, t_s^{d_s})\bar{\Lambda}$ where $\mathbf{d}(\bar{\Lambda}) = [d_1, \ldots, d_s]$.

Geometrically, Λ' is local and is the transverse union of s smooth branches (i.e. glued together at one point with independent tangent directions), Λ'_i is semilocal, obtained from Λ' by separating the ith branch. Λ'' is a kind of biggest local overring of Λ such that Λ and Λ'' have the same embedding dimension and multiplicity vector with respect to $\bar{\Lambda}$.

Now we are able to state our overring conditions.

THEOREM 3. *Let Λ be a curve singularity of infinite CMT. The following condition are necessary and sufficient for Λ to be of tame CMT*:

(O1) $d(\bar{\Lambda}) \leqslant 4$ *and* $\mathbf{d}(\bar{\Lambda}) \notin \{[4], [1, 3], [3]\}$,
(O2) $d(\Lambda') \leqslant 3$ *and* $\mathbf{d}(\Lambda'_i) \neq [1, 3]$ *for any admissible indempotent* e_i,
(O3) *if* $d(\bar{\Lambda}) = 3$, *then* $d(\Lambda'') \leqslant 2$.

REMARK 6.1. As we shall see, the condition $\mathbf{d}(\Lambda'_i) \neq [1, 3]$ is satisfied for any idempotent e_i provided Λ is of tame *CMT*. The point is that one has to check it only for admissible idempotents. Of course, in this case it means that if $d(\Lambda') = 3$, then $\Delta_i\mathfrak{m} \neq t_i\Delta_i$.

Recall first the criterion for Λ to be of finite *CMT* as given in [DR] (cf. also [GK]):

PROPOSITION 6.2. *Λ is of finite CMT if and only if $d(\bar{\Lambda}) \leqslant 3$ and $d(\theta\bar{\Lambda} + \Lambda) \leqslant 2$.*

We shall use also the next simple fact proved in [Dr 2]:

PROPOSITION 6.3. *$d(\bar{\Lambda})$ is an upper bound for the number of generators of all Λ-ideals. In particular, $d(\Gamma) \leqslant d(\bar{\Lambda})$ for any overring Γ.*

From Theorem 3 we obtain:

COROLLARY 6.4. *If Λ is an integral domain (i.e. has only one branch), then it is either of finite or of wild CMT.*

To prove both Theorems 1 and 3, we shall show that the tameness implies conditions (O1–O3) and then check that any ring satisfying these conditions dominates some $T_{p,q}$.

In view of Corollary 4.4, that will do.

First of all, we prove the following main lemma.

LEMMA 6.5. *Suppose that Γ is an overring of Λ with either $d(\Gamma) \geqslant 5$ or $\mathbf{d}(\Gamma) = [4]$ or $\mathbf{d}(\Lambda) = [1, 3]$. Then Λ is of wild CMT.*

Proof. Note that $\mathbf{d}(\Gamma)$ does not change if we replace Λ by $\Gamma\mathfrak{m} + \Lambda$. Therefore, we may later on suppose that $\Gamma\mathfrak{m} = \mathfrak{m}$. Denote $A = \Gamma/\mathfrak{m}$.

If $d(\Gamma) \geqslant 5$, an easy count of parameters shows that $par(\Lambda, r)$ grows quadratically with r (cf. [Dr 1]), so Λ is of wild *CMT*. Let $\mathbf{d}(\Gamma)q = [4]$, i.e. A is a local 4-dimensional algebra. We shall construct a strict family $\mathscr{F}$ of generating subspaces in C_Λ^Γ with base $B = K\langle x, y\rangle$. There are the following possibilities for A:

(i) $A = K[a]$, $a^4 = 0$:

(ii) $A = K[a, b]$, $ab = 0$, $a^2 = b^2$;

(iii) $A = K[a, b]$, $ab = b^2 = 0$, $a^3 = 0$;

(iv) $A = K[a_1, a_2, a_3]$, $a_ia_j = 0$ for any i, j.

$\mathscr{F}$ will always be of the form (nA, W) for some n and some $W \subset n(B \otimes A)$. In all cases W contains the standard basis vectors $\mathbf{e}_i$ $(i = 1, \ldots, n)$, so we shall write down only the matrix E whose columns are the extra generators of W as B-module. Here is the list of E's for all cases:

$$n=5,\quad E=\begin{pmatrix} a & 0 & a^2 & 0 \\ 0 & a & 0 & a^2 \\ a^2 & 0 & 0 & a^3 \\ 0 & a^2 & a^3x & a^3y \\ a^3 & 0 & 0 & 0 \end{pmatrix}; \tag{i}$$

$$n=3,\quad E=\begin{pmatrix} a & b \\ bx & a+by \\ a^2 & 0 \end{pmatrix}; \qquad \text{(ii) and (iii)}$$

$$n=1,\quad E=(a_1xa_2ya_3). \tag{iv}$$

The proof of strictness is quite similar in all cases and involves only some routine but tedious calculations. So we shall include it only for the case (ii) (middle as for complexity).

If L is an m-dimensional B-module, then $L\otimes_B W$ is the subspace in $L\otimes nA \simeq nmA$ generated by all possible e_i ($i=1,\dots,mn$) and the columns of the matrix $E(L)$ obtained from E by replacing x and y by the matrices X and Y defining the multiplication by, x and y in L. Of course, all units have to be replaced by unit matrices. So, in the case (ii) $E(L)$ will be:

$$E(L)=\begin{pmatrix} aI & bI \\ bX & aI+bY \\ a^2I & 0 \end{pmatrix}.$$

An isomorphism of $\mathscr{F}(L)$ onto $\mathscr{F}(L')$ is an automorphism φ of nmA such that $\varphi(L\otimes_B W)=L'\otimes_B W$. Consider φ as $nm\times nm$ matrix with coefficients in A. As both $L\otimes_B W$ and $L'\otimes_B W$ contain all e_i, the columns of this matrix belong to $L'\otimes_B W$. Moreover, as all elements of $E(L)$ lie in $rad\, A$, the columns of $E(L)$ are linear independent with e_i. Hence, $\varphi(L\otimes_B W)=L'\otimes_B W$ means that $\varphi E(L)=E(L')\sigma$ for some invertible matrix σ (with coefficients in K). As $a^3=ba^2=0$, we may omit in φ all parts containing a^2 as a multiple. Write φ as $n\times n$ block matrix: $\varphi=(\alpha_{ij}+\beta_{ij}a+\gamma_{ij}b)$ where $i,j=1,\dots,n$ and α_{ij}, β_{ij}, γ_{ij} are $m\times m$ matrices with entries from K. Write also σ as $k\times k$ block matrix (σ_{ij}) where k is the number of columns in E and σ_{ij} are $m\times m$ matrices.

Consider, for the case (ii), the place (23) in the matrices $\varphi E(L)=E(L')\sigma$. Then we have:

$$\alpha_{31}b + \alpha_{32}a + \alpha_{32}Yb = \sigma_{12}a^2,$$

whence $\sigma_{12} = \alpha_{32} = \alpha_{31} = 0$. The place (13) gives then $\alpha_{33}a^2 = \sigma_{11}a^2$, i.e. $\alpha_{33} = \sigma_{11}$. From now on we need only equalities modulo a^2. The place (12) gives now:

$$\alpha_{11}b + \alpha_{12}a = b\sigma_{22}, \quad \text{i.e. } \alpha_{11} = \sigma_{22}, \alpha_{12} = 0.$$

Then the place (11) gives:

$$\alpha_{11}a = \sigma_{11}a + \sigma_{21}b, \quad \text{i.e. } \alpha_{11} = \sigma_{11}, \sigma_{21} = 0.$$

Now the place (21) gives:

$$\alpha_{21}a + \alpha_{22}Xb = X'\sigma_{11}b, \quad \text{i.e. } \alpha_{22}X = X'\sigma_{11}, \alpha_{21} = 0.$$

At last the place (22) gives:

$$\alpha_{22}a + \alpha_{22}Yb = \sigma_{22}a + Y'\sigma_{22}b, \quad \text{i.e. } \alpha_{22}Y = Y'\sigma_{22} \text{ and } \alpha_{22} = \sigma_{22}.$$

Therefore, $\alpha_{11} = \alpha_{22} = \alpha_{33} = \sigma_{11} = \sigma_{22} = \sigma$ is an invertible matrix and $X' = \sigma X\sigma^{-1}$, $Y' = \sigma Y\sigma^{-1}$, i.e. $L \simeq L'$ as B-modules. The same calculation applied to an endomorphism ψ of $\mathscr{F}(L)$ shows that modulo radical

$$\psi = \begin{pmatrix} \varepsilon & 0 & \xi \\ 0 & \varepsilon & \eta \\ 0 & 0 & \varepsilon \end{pmatrix}$$

where ε is an endomorphism of L. If $\psi^2 = \psi$, then $\varepsilon^2 = \varepsilon$. If L is indecomposable, we have either $\varepsilon = 0$ or $\varepsilon = 1$, when either $\psi = 0$ or $\psi = 1$, i.e. $\mathscr{F}(L)$ is also indecomposable. Hence $\mathscr{F}$ is strict.

Quite analogous calculations prove the case $\mathbf{d}(A) = [1,3]$, i.e. $A = A_1 \times A_2$ with $A_1 = K$ and A_2 a 3-dimensional local algebra. There are two possibilities:

(v) $A_2 = K[a]$, $a^2 = 0$;
(vi) $A_2 = K[a_1, a_2]$, $a_ia_j = 0$ for any i, j.

Here are strict families $\mathscr{F} = (X, W)$ of generating subspaces over $B = K\langle x, y\rangle$ given by X and a matrix E whose columns generate W as B-submodule in $B \otimes X$:

(v) $X = 6A_1 \oplus 5A_2$,

$$E = \left(\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & a^2 & a & 0 \\
0 & 1 & 0 & 0 & 0 & a^2x & a^2y & 0 & a \\
0 & 0 & 1 & 0 & 0 & a^2 & 0 & a^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & a^2 & 0 & a^2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & a^2 & 0
\end{array}\right)$$

(vi) $X = 3A_1 \oplus 4A_2$,

$$E = \left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & a_1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & a_1 \\
0 & 0 & 1 & 0 & 0 & 0 & a_2 \\
0 & 0 & 0 & 1 & a_2 & a_2x & a_2y
\end{array}\right)$$

(the upper block corresponds to the A_1-part and the lower one to the A_2-part of generators for W). We omit the check-up of strictness.

Certainly, Lemma 6.4 implies conditions (O1) and (O2) of Theorem 3 except $\mathbf{d}(\bar{\Lambda}) \neq [3]$. Note that Λ' is local, so $d(\Lambda') = 4$ implies $\mathbf{d}(\Lambda') = [4]$. But the remaining case is, probably, the most cumbersome.

LEMMA 6.6. *If* $\mathbf{d}(\bar{\Lambda}) = [3]$, *then* Λ *is either of finite or of wild CMT.*

Proof. In this case Λ is a subring of $\bar{\Lambda} = K[[t]]$ containing t^3 but neither t nor t^2.

If Λ contains r^4 or t^5, then $d(\mathfrak{m}\bar{\Lambda} + \Lambda) = 2$ and Λ is of finite CMT (these are the singularities of type E_6 and E_8). So we have only to prove that the ring $\Lambda + t^6\bar{\Lambda}$ is of wild CMT. Thus, we may suppose that $\Lambda = t^6\bar{\Lambda} + K1 + Kt^3$. Then $\{\gamma \in F \mid \gamma\mathfrak{m} \subset \mathfrak{m}\} = \Gamma = t^3\bar{\Lambda} + K1$. Consider the Λ-submodule $U \subset 2\bar{\Lambda}$ generated by the elements

$$u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

We have to determine $\Delta = End\ U$. As $\bar{U} = U\bar{\Lambda} = 2\bar{\Lambda}$, Δ is the subring in $Mat_2(\bar{\Lambda})$ consisting of all matrices β such that $\beta u_i \in U$ $(i = 1, 2, 3)$. It is easy to check that $U \supset t^3\bar{U}$, so $\Delta \supset t^3\, Mat_2(\bar{\Lambda})$. If $\beta u_i \in U (i = 1, 2)$, then the columns of β lie in U, i.e. β modulo t^3 is of the shape:

$$\begin{pmatrix} \xi_2 + \xi_3 t & \eta_2 + \eta_3 t \\ \xi_1 + \xi_3 t^2 & \eta_1 + \eta_3 t^2 \end{pmatrix}$$

for some ξ_1, $\eta_j \in K$. But $\beta u_3 \in U$ gives then:

$$\xi_1 = 0, \quad \xi_2 = \eta_1 \quad \text{and} \quad \xi_3 = -\eta_2.$$

Hence, β is of the shape:

$$\begin{pmatrix} \xi - \eta t & \eta + \zeta t \\ -\eta t^2 & \xi + \zeta t^2 \end{pmatrix}.$$

In particular, $\beta \equiv \xi I (mod\ t)$, whence Δ is local and U is indecomposable.

Following the method of Section 2, consider the vector space category $C = C_\Lambda^\Gamma$ and even its full subcategory consisting of all direct sums $nA \otimes \mathfrak{m}H$ for $A = \Gamma/\mathfrak{m}$ and $H = U/U\mathfrak{m}$. Here A is a 3-dimensional algebra with basis $\{1, a_1, a_2\}$ where $a_1 = t^4 + \mathfrak{m}$, $a_2 = t^5 + \mathfrak{m}$ and H is a 6-dimensional A-module with basis $\{h_i \mid i = 1, \ldots, 6\}$ where $h_1 = u_1 + U\mathfrak{m}$, $h_2 = U_2 + U\mathfrak{m}$, $h_3 = u_3 + U\mathfrak{m}$, $h_4 = t^4 u_1 + U\mathfrak{m}$, $h_5 = t^5 u_1 + U\mathfrak{m} = -t^4 u_2 + U\mathfrak{m}$, and $h_6 = t^5 u_2 + U\mathfrak{m}$. Moreover, $a_i a_j = 0$ in A for any i, j and in H we have:

$$h_1 a_1 = h_4, \quad h_1 a_2 = h_5, \quad h_2 a_1 = -h_5, \quad h_2 a_2 = h_6, \quad h_3 a_1 = h_6, \quad h_3 a_2 = 0$$

and

$$h_i a_j = 0 \quad \text{for } i \geqslant 4 \text{ and any } j.$$

Of course, we may identify $C(A, A)$ with A and $C(A, H)$ with H. Besides, $C(H, H) = \Delta/I$ where $I = \{\beta \in \Delta \mid \beta(H) \subset H\mathfrak{m}\}$. Therefore, it consists of elements b_1, b_2 such that

$$b_1\colon h_1 \mapsto h_2,\ h_2 \mapsto -h_3,\ h_4 \mapsto h_5,\ h_5 \mapsto h_6,\ h_6 \mapsto 0;$$
$$b_2\colon h_1 \mapsto h_3,\ h_2 \mapsto 0,\ h_3 \mapsto h_4,\ h_5 \mapsto 0,\ h_6 \mapsto 0$$

and d_{ij} $(i = 1, 2, 3; j = 4, 5, 6)$ such that $d_{ij}(h_i) = h_j$, $d_{ij}(h_k) = 0$ if $k \neq j$ plus the identity.

Now we have to calculate $C(H, A) = Hom_\Gamma(U, \Gamma)/Hom_\Gamma(U, \mathfrak{m})$. A homomorphism $\varphi: U \to \Gamma$ is given by a pair (φ_1, φ_2) of elements of F. Moreover, $\varphi(h_1) = \varphi_2$ and $\varphi(h_2) = \varphi_1$ have to belong to Γ whence they are of the shape $\varphi = (\xi_1 + \xi_2 t^3 + \xi_3 t^4 + \cdots, \eta_1 + \eta_2 t^3 + \eta_3 t^4 + \cdots)$. The condition $\varphi(h_3) \in \Gamma$ gives also $\xi_1 = \eta_1 = 0$. Thus $C(H, A)$ has a basis consisting of elements $c_{ij}(i = 1, 2, 3; j = 1, 2)$ such that $c_{ij}(h_i) = a_j$ and $c_{ij}(h_k) = 0$ if $k \neq j$.

Knowing all morphisms, we are able to construct a strict family of generating subspaces in $X = 4A \oplus 2H$ over the base $B = K\langle x, y\rangle$. The B-submodule $W \subset B \otimes X$ defining this family is generated by the standard basis vector $\mathbf{e}_1, \ldots, \mathbf{e}_4$ of $4A$, all $h_i \mathbf{e}_j$ for $i = 1, 2, 3$, $j = 1, 2$ in $2H$ and the columns of the following matrix

$$\left(\begin{array}{ccccccccc} 0 & 0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 \\ \hline h_4 & 0 & 0 & h_5 + xh_6 & 0 & yh_6 & 0 & 0 & h_6 \\ 0 & h_4 & 0 & h_6 & 0 & h_5 & 0 & 0 & 0 \end{array}\right)$$

Here the upper part corresponds to the direct summand $4A$ whilst the lower one corresponds to $2H$.

Again we omit the checking of wildness as it is nothing more than routine calculation, since we know all morphisms of C.

The last step in proving the necessity of conditions (O1–O3) is the easiest one. In view of Proposition 6.1, it remains to prove:

LEMMA 6.7. *If* $d(\Lambda') = d(\Lambda'') = 3$, *then* Λ *is of wild CMT.*

Proof. First of all note that $rad\,\Lambda' = t\Lambda$ and $\Lambda' = t\bar{\Lambda} + K1$ is local. $\Lambda'\mathfrak{m} = t\theta\bar{\Lambda} + \mathfrak{m}$ and $\dim(t\bar{\Lambda}/t\theta\bar{\Lambda}) = \dim(\bar{\Lambda}/\theta\bar{\Lambda}) = d(\bar{\Lambda}) = 3$ (cf. Proposition 6.2). Hence, $\dim(t\theta\bar{\Lambda} + \mathfrak{m}/t\theta\bar{\Lambda}) = 1$, i.e. $\Lambda'\mathfrak{m} = \theta t\bar{\Lambda} + K\theta$. Then $\Lambda'' = \theta t\bar{\Lambda} + K\theta + K1$, $rad\,\Lambda'' = \theta t\bar{\Lambda} + K\theta = \Lambda'\mathfrak{m}$ and $\Lambda''\mathfrak{m} = \theta^2 t\bar{\lambda} + \mathfrak{m}$. Again $\dim(\theta t\bar{\Lambda}/\theta^2 t\bar{\Lambda}) = 3$ implies that $\Lambda''\mathfrak{m} = \theta^2 t\bar{\Lambda} + K\theta + K\theta^2$, whence $(rad\,\Lambda'')^2 \subset \Lambda''\mathfrak{m}$.

Of course, without changing $d(\Lambda')$ and $d(\Lambda'')$, we can replace Λ by $\Lambda''\mathfrak{m} + \Lambda$, thus we shall suppose that $\Lambda''\mathfrak{m} = \mathfrak{m}$. Now we are able to apply Proposition 2.1 to $\Gamma = \Lambda''$. Then $A = \Gamma/\mathfrak{m}$ is a 3-dimensional local algebra with $(rad\,A)^2 = 0$, so it has a basis $\{1, a_1, a_2\}$ with $a_i a_j = 0$ for any i, j. Put $A' = \Lambda'/\Lambda'\mathfrak{m}$ which is also local 3-dimensional. Moreover, for any homomorphism $\varphi: \Gamma \to \Lambda'$ we have $\varphi(rad\,\Gamma) \subset \Lambda'\ rad\,\Gamma = \Lambda'\mathfrak{m}$, so the induced mapping $A \to A'$ maps a_1 and a_2 to 0. On the other hand, put $I = \{\psi \in Hom_\Gamma(\Lambda', \Gamma) \mid \psi(rad\,\Lambda') \subset \mathfrak{m}\Gamma\}$. Then $I \supset \theta t\Lambda$. Certainly, $Hom_\Gamma(\Lambda', \Gamma) = \Lambda'\mathfrak{m}$, thus $\dim Hom_\Gamma(\Lambda', \Gamma)/I \leqslant 1$. As $rad\,A'$ is 2-dimensional, we can choose its basis $\{1', b_1, b_2\}$ in such way that $\beta(b_2) = 0$ for any $\beta \in C(A', A)$. Now a strict family of generating submodules in $X = A' \oplus 2A$ with base $B = K\langle x, y\rangle$ can be defined by the B-submodule $W \subset B \otimes X$ generated by the standard basic vectors $\mathbf{e}_1$, $\mathbf{e}_2$ from $2A$, elements $1'$ and b_1 from A' and the columns of the matrix:

$$\left(\begin{array}{ccc} b_2 & 0 & 0 \\ \hline 0 & a_1 & a_2 \\ a_2 & a_2 x & a_1 + a_2 y \end{array}\right)$$

Here the checking of strictness is quite easy and, of course, we again omit it.

7. End of the proof

To accomplish the proof of Theorems 1 and 2 we have to show, for a singularity Λ of infinite CMT, that:

- if Λ satisfies conditions (O1–O3) of Theorem 3, then it dominates some $T_{p,q}$;
- if, moreover, Λ dominates neither $T_{4,4}$ nor $T_{3,6}$, then it is of infinite growth.

Recall the parameterization of $T_{p,q}$. As for $P_{p,q}$, it depends on the parity of p and q. Namely:

- if p and q are both odd, then $T_{p,q}$ is isomorphic to the subalgebra of $K[[t]]^2$ generated by the elements (t^2, t^{p-2}) and (t^{q-2}, t^2);
- if p is odd and q is even, then $T_{p,q}$ is isomorphic to the subalgebra of $K[[t]]^3$ generated by (t, t, t^{p-2}) and $(0, t^{q/2-1}, t^2)$ if $(p, q) \neq (3, 6)$, and by (t, t, t) and $(0, t^2, \lambda t^2)$ with $\lambda \notin \{0, 1\}$ if $(p, q) = (3, 6)$;
- if both p and q are even, $T_{p,q}$ is isomorphic to the subalgebra of $K[[t]]^4$ generated by $(t, t, t^{p/2-1}, 0)$ and $(t^{q/2-1}, 0, t, t)$ if $(p, q) \neq (4, 4)$, and by $(t, 0, t, t)$ and $(0, t, t, \lambda t)$ with $\lambda \notin \{0, 1\}$ if $(p, q) = (4, 4)$.

(In the cases of $T_{3,6}$ and $T_{4,4}$ different values of λ really lead to non-isomorphic rings.)

We keep the notation of the preceding paragraph. From now on we suppose that Λ is of infinite CMT and satisfies (O1–O3). Then $2 \leqslant s \leqslant 4$ as $s \leqslant d(\bar{\Lambda})$ and the case $s = 1$ is excluded by (O1) together with Propositions 6.1 and 6.2.

Consider first the case $s = 4$. Then $\mathbf{d}(\Lambda) = [1, 1, 1, 1]$, hence $\theta = t$. As $d(\Lambda') = \dim(t\bar{\Lambda} + \Lambda/\theta t\bar{\Lambda} + \mathfrak{m}) \leqslant 3$ and $\dim(\Lambda'/\theta t\bar{\Lambda}) = 5$ in this case, $\mathfrak{m}$ has to contain at least two elements linear independent modulo $t^2\Lambda$. Of course, if there are four of them, then $\Lambda = t\bar{\Lambda} + K1$ dominates all $T_{4,4}$. Let there be three such elements. Changing t_i, if necessary, and the numbering of branches, we may suppose these three elements to be $(t_1, 0, 0, at_4)$, $(0, t_2, 0, bt_4)$, $(0, 0, t_3, t_4)$ for some a, $b \in \Gamma_4$. Note that we must have uniformizing elements at all positions as $\bar\Lambda\mathfrak{m} = t\bar\Lambda$. If either $a(0) \neq 0$ or $b(0) \neq 0$, one can easily check that Λ dominates some $T_{4,4}$. If both $a(0) = 0$ and $b(0) = 0$, then Λ contains elements $(t_1, t_2, 0, t_4^{m-1})$ and $(0, 0, t_3, t_4)$ for some $m \geqslant 4$ and hence $(0, t_2^{n-1}, t_3, t_4)$ since $\Lambda \supset t^{n-1}\bar\Lambda$ for sufficiently large n. Thus, it dominates $T_{2m,2n}$. Moreover, in the last case $\Lambda'' = \Lambda + t^2\Lambda$ is the minimal overring P'_{24} of the singularity P_{24}. By Propositions 2.3 and 3.1, P'_{24} and hence Λ are of infinite growth.

If $\mathfrak{m}$ contains only two elements, linear independent modulo $t^2\bar\Lambda$, we may suppose them to be $(t_1, 0, at_3, t_3)$ and $(0, t_2, bt_3, ct_4)$ for some $a, b \in \Gamma_3$ and $c \in \Gamma_4$ such that either $a(0) \neq 0$ or $b(0) \neq 0$. Moreover, we claim that also either $b(0) \neq 0$ or $c(0) \neq 0$: otherwise $\mathbf{d}(\Lambda'_2) = [3, 1]$ in contradiction with (O2). If $c(0) = 0$, the second element can be taken as $(0, t_2, t_3, 0)$ and permuting the branches into the order 1, 4, 2, 3, we see that Λ dominates some $T_{p,q}$ with $(p, q) \neq (4, 4)$. Note that in this case it is again a subring of P'_{24}, and therefore of infinite growth. The cases $b(0) = 0$ (hence $a(0) \neq 0$ and $c(0) \neq 0$ or $a(0)c(0) = b(0)$ are the same. Finally, if $a(0)c(0) \neq b(0)$ and all of them are non-zero, one can easily check that Λ dominates one of the singularities $T_{4,4}$.

Let now $s = 3$. There are two possibilities for $\mathbf{d}(\bar\Lambda)$, namely, [1, 1, 1] and [1, 1, 2]. If $\mathbf{d}(\Lambda) = [1, 1, 1]$, then again $\theta = t$. By Propositions 6.1 and 6.2, $d(\Lambda') = \dim(\Lambda'/t^2\bar\Lambda + \mathfrak{m}) = 3$ whence $\dim(t\bar\Lambda/t^2\bar\Lambda + \mathfrak{m}) = 2$, i.e. $\mathfrak{m}$ contains only one element of $t\bar\Lambda/t^2\bar\Lambda$. Of course, we may suppose that it is t. On the other hand, as $d(\bar\Lambda) = 3$, (O3) implies that $d(\Lambda'') \leqslant 2$, whence $\mathfrak{m}$ contains at least two elements of $t^2\bar\Lambda$ linear independent modulo $t^2\bar\Lambda$. If there are three of them, then $\Lambda \supset t^2\bar\Lambda$ and dominates all $T_{3,6}$. So we may suppose that there are exactly two of them: t^2 and $(0, at_2^2, bt_3^2)$ for some $a \in \Gamma_2$, $b \in \Gamma_3$ and, say, $b(0) \neq 0$. If also $a(0) \neq 0$ and $a(0) \neq b(0)$, then Λ dominates one of the singularities $T_{3,6}$. Let $a(0) = 0$ ($a(0) = b(0)$ can be obviously reduced to this one). Then Λ contains an element $(0, t^{m-1}, t^2)$ for some $m \geqslant 4$ and hence dominates $T_{3,2m}$. Moreover, in this case its overring $\Lambda + t^3\bar\Lambda$ coincides with the singularity $\bar P_{34}$ considered in Section 5. But $\bar P_{34}$ is of infinite growth by Proposition 5.2, hence so is also Λ.

If $\mathbf{d}(\bar\Lambda) = [1, 1, 2]$, then $\theta = (t_1, t_2, t_3^2)$. Again (O2) implies that $\dim(t\bar\Lambda/\theta t\bar\Lambda + \mathfrak{m}) \leqslant 2$ i.e. $\mathfrak{m}$ contains at least two elements linear independent modulo $\theta t\bar\Lambda$. If there are three of them, then it is easy to see that Λ contains the elements (t, t, t^{p-2}) for some odd $p \geqslant 5$ and $(0, t^{m-1}, t^2)$ for some $m \geqslant 3$, hence, dominates $T_{p,qm}$. Moreover, again Λ is a subring of $P'_{2,3}$, and therefore of infinite growth.

Suppose now that $\mathfrak{m}$ contains exactly two elements linear independent modulo $\theta t\bar\Lambda$. They can be chosen as $(at_1, 0, t_3^2)$ and (bt_1, t_2, ct_3^{p-2}) for some odd $p \geqslant 5$ and $c(0) \neq 0$ (again we use the inclusion $\Lambda \supset t^n\Lambda$). If $b(0) \neq (0)$, then Λ dominates $T_{p,q}$ for some even q and again is of infinite growth as a subring of P'_{23}. Otherwise, one can check that $\mathbf{d}(\Lambda'_2) = [1,3]$ in contradiction with (O2).

Consider finally the case $s = 2$. As the calculations are quite similar to those for $s = 3$, we only sketch them here. Again there are two possibilities for $\mathbf{d}(\bar{\Lambda})$, namely, $[1, 2]$ and $[2, 2]$. Let first $\mathbf{d}(\bar{\Lambda}) = [1, 2]$. Then $\theta = (t_1, t_2^2)$ and $d(\Lambda') = 3$ whence $\dim(t\bar{\Lambda}/\theta t\bar{\Lambda} + \mathfrak{m}) = 2$, i.e. θ is the only element lying in $\mathfrak{m}\backslash\theta t\bar{\Lambda}$. On the other hand, condition (O3) implies that $\mathfrak{m}$ contains at least two elements of $\theta t\bar{\Lambda}$, linear independent modulo $\theta^2 t\bar{\Lambda}$. If there are three of them, then $\Lambda \supset \theta t\bar{\Lambda}$ and Λ dominates the ring Δ generated by (t_1, t_2^2) and (t_1^2, t_2^3). But Δ is of finite CMT by Proposition 5.1 (indeed it is the simple plane curve singularity of type E_7) which is impossible. Thus Λ contains exactly two such elements, namely θ and (at^2, bt^3) with $a(0) \neq 0$ or $b(0)$ $\bar{\Lambda}$. But $b(0) \neq 0$ again implies that Λ is of finite CMT (of type E_7). Therefore, $b(0) = 0$, $a(0) \neq 0$ and hence Λ dominates $T_{3,q}$ for some $q \geqslant 7$. Moreover, Λ has an overring $\Lambda + \theta^2\bar{\Lambda}$ which coincides with the singularity $\tilde{P}_{33}$. As the latter is of infinite growth by Proposition 5.1, Λ is also of infinite growth.

Now let $\mathbf{d}\bar{\Lambda} = [2,\ 2]$. Then $\theta = (t_1^2,\ t_2^2)$. Condition (O2) implies that $\mathfrak{m}$ contains two linear independent elements modulo $\theta t\bar{\Lambda}$. But $\dim(\theta\bar{\Lambda}/\theta t\bar{\Lambda}) = 2$, hence Λ contains both (t^2, t^{p-2}) and (t^{q-2}, t^2) for some odd $p, q \geqslant 5$. Therefore, Λ dominates $T_{p,q}$ and is a subring of P'_{33}, this is of tame CMT and of infinite growth.

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