TILTING ON NON-COMMUTATIVE RATIONAL PROJECTIVE CURVES

IGOR BURBAN AND YURIY DROZD

Dedicated to Helmut Lenzing on the occasion of his 70th birthday

Abstract. In this article we introduce a new class of non-commutative projective curves and show that in certain cases the derived category of coherent sheaves on them has a tilting complex. In particular, we prove that the right bounded derived category of coherent sheaves on a reduced rational projective curve with only nodes and cusps as singularities, can be fully faithfully embedded into the right bounded derived category of the finite dimensional representations of a certain finite dimensional algebra of global dimension two. As an application of our approach we show that the dimension of the bounded derived category of coherent sheaves on a rational projective curve with only nodal or cuspidal singularities is at most two. In the case of the Kodaira cycles of projective lines, the corresponding tilted algebras belong to a well-known class of gentle algebras. We work out in details the tilting equivalence in the case of the Weierstrass nodal curve $zy^2 = x^3 + x^2z$.

Contents

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1. Introduction

By a result of Beilinson [6] it is known that the derived category of representations of the Kronecker quiver $\vec{Q} = \bullet \rightrightarrows \bullet$ over a field k is equivalent to the derived category of coherent sheaves on the projective line $\mathbb{P}^1_k$. This article grew up from an attempt to find a similar "geometric" interpretation of other finite-dimensional algebras.

Let X be a singular reduced projective curve of arbitrary geometric genus, having only nodes or cusps as singularities. We introduce a certain sheaf of $\mathcal{O}_X$–orders (called the Auslander sheaf) $\mathcal{A} = \mathcal{A}_X$ and study the category $\mathsf{Coh}(\mathcal{A})$ of coherent left modules on the ringed space $(X, \mathcal{A}_X)$. It turns out that the global dimension of $\mathsf{Coh}(\mathcal{A})$ is equal to *two* and the original category of coherent sheaves $\mathsf{Coh}(X)$ can be embedded into $\mathsf{Coh}(\mathcal{A})$ in two natural but different ways. Namely, we construct a pair of fully faithful functors $\mathbb{F}, \mathbb{I} : \mathsf{Coh}(X) \to \mathsf{Coh}(\mathcal{A})$, where $\mathbb{F}$ is right exact and $\mathbb{I}$ is left exact such that their images in $\mathsf{Coh}(\mathcal{A})$ are closed under extensions.

The functor $\mathbb{F}$ has a right adjoint functor $\mathbb{G} : \mathsf{Coh}(\mathcal{A}) \to \mathsf{Coh}(X)$, which is exact. We show that $\mathbb{G}$ yields an equivalence between the category $\mathsf{VB}(\mathcal{A})$ of the locally projective coherent $\mathcal{A}$–modules and the category $\mathsf{TF}(X)$ of the torsion free coherent sheaves on X. Moreover, we prove that $\mathsf{Coh}(X)$ is equivalent to the localization of $\mathsf{Coh}(\mathcal{A})$ with respect to a certain bilocalizing subcategory of torsion $\mathcal{A}$–modules and $\mathbb{G}$ can be identified with the canonical localization functor.

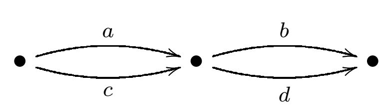
It turns out that the derived functor $\mathbb{LF} : D^-\big(\mathsf{Coh}(X)\big) \to D^-\big(\mathsf{Coh}(\mathcal{A})\big)$ is also fully faithful. Moreover, if X is a *rational* curve of arbitrary arithmetic genus then the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ has a *tilting complex*. The corresponding tilted algebra $\Gamma_X = \mathsf{End}_{D^b(\mathcal{A})}(\mathcal{H}^\bullet)$ is precisely the algebra described in [15, Appendix A.4]. In particular, Γ_X has global dimension equal to two and the derived categories $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ and $D^b\big(\mathsf{mod} - \Gamma_X\big)$ are equivalent. As a corollary we show that the dimension of the derived category $D^b\big(\mathsf{Coh}(X)\big)$ is at most two, confirming a conjecture posed by Rouquier [38]. Combining this derived equivalence with the embedding $\mathbb{LF}$, we obtain an exact fully faithful functor $\mathsf{Perf}(X) \to D^b\big(\mathsf{mod} - \Gamma_X\big)$.

If X is either a chain or a cycle of projective lines, then the corresponding tilted algebra Γ_X belongs to the class of the so-called *gentle* algebras. They are known to be of *derived-tame* representation type. This gives an alternative proof of the tameness of $\mathsf{Perf}(X)$, obtained for the first time in [13]. If X is a Kodaira cycle of projective lines, then the image of the triangulated category $\mathsf{Perf}(X)$ inside of $D^b\big(\mathsf{mod} - \Gamma_X\big)$ can be characterized in a very simple way. Let

$$\mathbb{L}\nu : D^b\big(\mathsf{mod} - \Gamma_X\big) \longrightarrow D^b\big(\mathsf{mod} - \Gamma_X\big)$$

be the derived Nakayama functor then the image of $\mathsf{Perf}(X)$ is equivalent to the full subcategory of the complexes $P^\bullet$ satisfying the property $\mathbb{L}\nu(P^\bullet) \cong P^\bullet[1]$.

If $E \subseteq \mathbb{P}^2$ is a singular Weierstraß cubic curve over an algebraically closed field k given by the equation $zy^2 = x^3 + x^2 z$, then the corresponding algebra Γ_E is the path algebra of the following quiver with relations:



$ba = 0, \quad dc = 0.$

We explicitly describe the objects of $D^b\big(\mathsf{mod} - \Gamma_E\big)$ corresponding to the line bundles of degree zero and to the structure sheaves of the regular points under the embedding $\mathsf{Perf}(E) \to D^b\big(\mathsf{mod} - \Gamma_E\big)$.

We hope that the results of this article will find applications to the homological mirror symmetry for degenerations of elliptic curves [25] and to the theory of integrable systems, in particular to the study of solutions of Yang-Baxter equations [32, 14].

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2. Auslander sheaf of orders

In this section we introduce an interesting class of non-commutative ringed spaces supported on some algebraic curves over an algebraically closed field k. Let X be a reduced algebraic curve over k having only *nodes* or *cusps* as singularities. This means that for any point x from the singular locus $\mathsf{Sing}(X)$ the completion of the local ring $\widehat{\mathcal{O}}_x$ is isomorphic to $k[\![u,v]\!]/uv$ (node) or to $k[\![u,v]\!]/(u^2 - v^3)$ (cusp).

Let $\mathcal{O} = \mathcal{O}_X$ be the structure sheaf of X and $\mathcal{K} = \mathcal{K}_X$ the sheaf of rational functions on X. Consider the ideal sheaf $\mathcal{I}$ of the singular locus $\mathsf{Sing}(X)$ (with respect to the reduced scheme structure) and the sheaf $\mathcal{F} = \mathcal{I} \oplus \mathcal{O}$.

Definition 2.1. The *Auslander sheaf* on a curve X with only nodes or cusps as singularities is the sheaf of $\mathcal{O}$–algebras $\mathcal{A} = \mathcal{A}_X := \mathcal{E}nd_X(\mathcal{F})$.

Proposition 2.2. *In the notations as above, let $\widetilde{X} \xrightarrow{\pi} X$ be the normalization of X and $\widetilde{\mathcal{O}} := \pi_*(\mathcal{O}_{\widetilde{X}})$. Then we have:*

- *The ideal sheaf $\mathcal{I}$ of the singular locus of X is equal to the* conductor ideal sheaf $\mathcal{A}nn_X(\widetilde{\mathcal{O}}/\mathcal{O})$. *In particular, $\mathcal{I}$ is an ideal sheaf in $\widetilde{\mathcal{O}}$.*
- *Moreover, we have: $\mathcal{I} \cong \mathcal{H}om_X(\widetilde{\mathcal{O}}, \mathcal{O})$, $\widetilde{\mathcal{O}} \cong \mathcal{E}nd_X(\mathcal{I})$ and*
$$\mathcal{A} = \mathcal{E}nd_X(\mathcal{F}) \cong \begin{pmatrix} \widetilde{\mathcal{O}} & \mathcal{J} \\ \widetilde{\mathcal{O}} & \mathcal{O} \end{pmatrix} \subseteq \mathsf{Mat}_2(\mathcal{K}) = \begin{pmatrix} \mathcal{K} & \mathcal{K} \\ \mathcal{K} & \mathcal{K} \end{pmatrix}.$$
*In other words, $\mathcal{A}$ is a sheaf of $\mathcal{O}$–*orders.

Proof. Let $\mathcal{J} = \mathcal{A}nn_X(\widetilde{\mathcal{O}}/\mathcal{O})$ be the conductor ideal. Then the sheaf $\mathcal{O}/\mathcal{J}$ is supported at the set of points of X where X is not normal. Since X is a reduced curve, it is exactly the singular locus of X, hence $\mathcal{J} \subseteq \mathcal{I}$. Moreover, $x \in X$ is either a nodal or cuspidal point, then $\big(\mathcal{O}/\mathcal{J}\big)_x \cong k_x$. This implies that $\mathcal{J} = \mathcal{I}$.

By the general properties of the conductor ideal it follows that the morphism of $\mathcal{O}$–modules $\mathcal{I} \to \mathcal{H}om_X(\widetilde{\mathcal{O}}, \mathcal{O})$ mapping a local section $r \in H^0(U, \mathcal{I})$ to the morphism $\varphi_r \in \mathsf{Hom}_U(\widetilde{\mathcal{O}}, \mathcal{O})$, given by the rule $\varphi_r(b) = rb$, is an isomorphism. In a similar way, the morphism $\widetilde{\mathcal{O}} \longrightarrow \mathcal{E}nd_X(\mathcal{I})$, given on the level of local sections by the rule $b \mapsto \psi_b$, where $\psi_b(a) = ba$, is an isomorphism too. □

In what follows we shall use the following standard results on the category of coherent sheaves over a sheaf of orders on a quasi-projective algebraic variety.

Theorem 2.3. *Let X be a connected quasi-projective algebraic variety over an algebraically closed field k of Krull dimension n and $\mathcal{A}$ be a sheaf of orders on X. Let $\mathsf{QCoh}(\mathcal{A})$ be the category of quasi-coherent left $\mathcal{A}$-modules and $\mathsf{Coh}(\mathcal{A})$ be the category of coherent left $\mathcal{A}$-modules. Then we have:*

1. *The category $\mathsf{Qcoh}(\mathcal{A})$ is a locally Noetherian Grothendieck category and $\mathsf{Coh}(\mathcal{A})$ is its subcategory of Noetherian objects.*
2. *For any quasi-coherent $\mathcal{A}$–modules $\mathcal{H}'$ and $\mathcal{H}''$ we have the following local-to-global spectral sequence: $H^p\big(X, \mathcal{E}xt^q_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')\big) \Longrightarrow \mathsf{Ext}^{p+q}_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')$.*
3. *If the variety X is projective, then $\mathsf{Coh}(\mathcal{A})$ is Ext–finite over k.*
4. *Assume the ringed space $(X, \mathcal{A})$ has "isolated singularities", i.e. $\mathcal{A}_x \cong \mathsf{Mat}_{n\times n}(\mathcal{O}_x)$ for all but finitely many $x \in X$. Then we have:*

$$\mathsf{gl.\,dim}\big(\mathsf{QCoh}(\mathcal{A})\big) = \mathsf{gl.\,dim}\big(\mathsf{Coh}(\mathcal{A})\big) = \sup_{x\in X} \mathsf{gl.\,dim}\big(\mathcal{A}_x - \mathsf{mod}\big) = \sup_{x\in X} \mathsf{gl.\,dim}\big(\widehat{\mathcal{A}}_x - \mathsf{mod}\big),$$

   where $\widehat{\mathcal{A}}_x := \varprojlim \mathcal{A}_x/\mathfrak{m}_x^t\mathcal{A}_x$ is the radical completion of the ring $\mathcal{A}_x$ and $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$.

5. *The canonical functor $D^*\big(\mathsf{Coh}(\mathcal{A})\big) \to \mathsf{D}^*_{\mathsf{coh}}\big(\mathsf{QCoh}(\mathcal{A})\big)$ is an equivalence of triangulated categories, where $\mathsf{D}^*_{\mathsf{coh}}\big(\mathsf{QCoh}(\mathcal{A})\big)$ is the full subcategory of $D^*\big(\mathsf{QCoh}(\mathcal{A})\big)$ consisting of complexes with coherent cohomologies and $* \in \{\emptyset, b, +, -\}$.*

Comment on the proof. (1) We refer to [16, Chapitre IV] for the definition and general properties of the locally Noetherian categories. Let $\mathcal{H}'$ and $\mathcal{H}''$ be quasi-coherent $\mathcal{A}$–modules. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')$ is a quasi-coherent $\mathcal{O}$–module. Moreover, we have isomorphisms of bifunctors

$$\mathsf{Hom}_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'') \cong H^0\big(X, \mathcal{H}om_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')\big) \cong \mathsf{Hom}_X\big(\mathcal{O}, \mathcal{H}om_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')\big).$$

The spectral sequence (2) is just the spectral sequence of the composition of two left exact functors. Note that $\mathcal{E}xt^q_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')$ is a coherent $\mathcal{O}$–module for all $q \geq 0$ provided both sheaves $\mathcal{H}'$ and $\mathcal{H}''$ are coherent $\mathcal{A}$–modules. If X is projective, this implies that $H^p\big(X, \mathcal{E}xt^q_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')\big)$ is finite dimensional over k. Hence, $\mathsf{Coh}(\mathcal{A})$ is Ext–finite in this case, what proves (3).

Let A be a left Noetherian ring. By a result of Auslander [3, Theorem 1]

$$\mathsf{gl.\,dim}(A) = \sup\big\{\mathsf{pr.dim}(A/I) \,|\, I \subseteq A \text{ is a left ideal}\big\}.$$

In particular, the global dimension of the category of all A–modules is equal to the global dimension of the category of Noetherian A–modules. Quite analogous observations show that $\mathsf{gl.\,dim}\big(\mathsf{QCoh}(\mathcal{A})\big) = \mathsf{gl.\,dim}\big(\mathsf{Coh}(\mathcal{A})\big)$. If $(X, \mathcal{A})$ has isolated singularities, then for any coherent $\mathcal{A}$–modules $\mathcal{H}'$ and $\mathcal{H}''$ we have:

$$\mathsf{kr.\,dim}\big(\mathsf{Supp}(\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{H}', \mathcal{H}''))\big) \leq \max\big\{0, n-i\big\}.$$

This implies that $\mathsf{Ext}^{n+j}_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'') \cong H^0\bigl(\mathcal{E}xt^{n+j}_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')\bigr)$. Let A be an order over a Noetherian local ring O and r be the radical of A. Then we have:

$$\mathsf{gl.dim}(A) = \mathsf{pr.dim}(A/r) = \sup\bigl\{m \ge 0 \mid \mathsf{Ext}^m_A(A/r, A/r) \ne 0\bigr\},$$

just as in [3, § 3] or [41, Chapter IV.C]. In particular, $\mathsf{gl.dim}(A) = \mathsf{gl.dim}(\widehat{A})$. For (5), we refer to [8, Theorem VI.2.10 and Proposition VI.2.11], [23, 1.7.11] or [21].

The following well-known lemma plays a key role in our theory of non-commutative ringed spaces.

Lemma 2.4. *Let A be a commutative Noetherian ring, $T = A \oplus M$ be a finitely generated A-module and $\Gamma = \mathsf{End}_A(T)$. Then the A-module T is a projective left Γ-module and the canonical ring homomorphism $A \to \mathsf{End}_\Gamma(T)$, $a \mapsto \bigl(t \mapsto at\bigr)$ is an isomorphism. In other words, T has the* double centralizer property.

Proof. First note that Γ has a matrix presentation

$$\Gamma = \begin{pmatrix} A & M^\vee \\ M & \mathsf{End}_A(M) \end{pmatrix}$$

and $1_\Gamma = 1_A + 1_M = e_A + e_M$. In particular, $T \cong \Gamma \cdot e_A$ as a left projective Γ-module. Hence, $\mathsf{Hom}_\Gamma(\Gamma e_A, \Gamma e_A) \cong e_A \Gamma e_A = A$, and the canonical morphism $A \to \mathsf{End}_\Gamma(T)$ is an isomorphism. □

Next, we shall frequently use the following standard result from category theory.

Lemma 2.5. *Let $\mathbb{F} : \mathsf{C} \longrightarrow \mathsf{D}$ be a functor admitting a right adjoint functor $\mathbb{G} : \mathsf{D} \longrightarrow \mathsf{C}$. Let $\phi : \mathbb{1}_\mathsf{C} \to \mathbb{G} \circ \mathbb{F}$ be the natural transformation given by the adjunction. Then for any objects $X, Y \in \mathsf{Ob}(\mathsf{C})$ the following diagram is commutative:*

$$\begin{array}{ccc} \mathsf{Hom}_\mathsf{C}(X, Y) & \xrightarrow{\quad\mathbb{F}\quad} & \mathsf{Hom}_\mathsf{D}\bigl(\mathbb{F}(X), \mathbb{F}(Y)\bigr) \\ & {\scriptstyle (\phi_Y)_*}\searrow \qquad \nearrow{\scriptstyle \cong} & \\ & \mathsf{Hom}_\mathsf{C}\bigl(X, \mathbb{G}\mathbb{F}(Y)\bigr) & \end{array}$$

In particular, $\mathbb{F}$ is fully faithful if and only if the natural transformation ϕ is an isomorphism.

The main result of this section is the following theorem.

Theorem 2.6. *Let X be a reduced algebraic curve over k having only nodal or cuspidal singularities, $\mathcal{F} = \mathcal{I} \oplus \mathcal{O}$ and $\mathcal{A} = \mathcal{E}nd_X(\mathcal{F})$. Then the following properties hold.*

(1) *The functor $\mathbb{G} = \mathcal{H}om_\mathcal{A}(\mathcal{F}, -\,) : \mathsf{Coh}(\mathcal{A}) \to \mathsf{Coh}(X)$ is exact and has the left adjoint functor $\mathbb{F} = \mathcal{F} \otimes_\mathcal{O} -$ and the right adjoint functor $\mathbb{H} = \mathcal{H}om_\mathcal{O}(\mathcal{F}^\vee, -\,)$.*
(2) *We have: $\mathsf{gl.dim}\bigl(\mathsf{Coh}(\mathcal{A})\bigr) = 2$.*
(3) *The functors $\mathbb{G}$ and $\mathbb{H}$ yield mutually inverse equivalences between the category $\mathsf{VB}(\mathcal{A})$ of locally projective coherent left $\mathcal{A}$-modules and the category of coherent torsion free sheaves $\mathsf{TF}(X)$. In particular, the derived functor $\mathbb{R}\mathbb{G} : D^b\bigl(\mathsf{Coh}(\mathcal{A})\bigr) \to D^b\bigl(\mathsf{Coh}(X)\bigr)$ is essentially surjective.*

(4) *The canonical transformation of functors* $\phi : \mathbb{1}_{\mathsf{Coh}(X)} \to \mathbb{G} \circ \mathbb{F}$ *is an isomorphism. In particular, the functor* $\mathbb{F}$ *is fully faithful.*
(5) *Moreover, the canonical morphism* $\psi : \mathbb{1}_{D^-(\mathsf{Coh}(X))} \to \mathbb{RG} \circ \mathbb{LF}$ *is an isomorphism and the derived functors* $\mathbb{LF} : D^-\big(\mathsf{Coh}(X)\big) \longrightarrow D^-\big(\mathsf{Coh}(\mathcal{A})\big)$ *and* $\mathbb{LF} : \mathsf{Perf}(X) \longrightarrow D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ *are fully faithful.*

Proof. (1) First note that $\mathcal{F}$ is endowed with a natural left module structure over the sheaf of algebras $\mathcal{E}nd_X(\mathcal{F})$. Moreover, consider the global sections

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

of the sheaf $\mathcal{A}$. Then we have: $\mathcal{F} \cong \mathcal{A} \cdot e_2$, hence $\mathcal{F}$ is locally projective, viewed as a left $\mathcal{A}$–module. This implies that the functor $\mathbb{G}$ is exact.

The fact that the functor $\mathbb{F}$ is left adjoint to $\mathbb{G}$ is obvious. To see that $\mathbb{G}$ possesses a right adjoint functor, consider the coherent sheaf $\mathcal{F}^\vee \cong \widetilde{\mathcal{O}} \oplus \mathcal{O}$ and the sheaf of algebras $\mathcal{B} = \mathcal{E}nd_X(\mathcal{F}^\vee)$. Since $\mathcal{F}$ is a torsion free coherent $\mathcal{O}$–module, it is locally Cohen-Macaulay. Moreover, X is a Gorenstein curve, hence the contravariant functor $\mathsf{TF}(X) \to \mathsf{TF}(X)$, mapping a torsion free sheaf $\mathcal{H}$ to $\mathcal{H}^\vee$, is an anti-equivalence. This shows that

$$\mathcal{B} = \mathcal{A}^{\mathsf{op}} = \begin{pmatrix} \widetilde{\mathcal{O}} & \widetilde{\mathcal{O}} \\ \mathcal{I} & \mathcal{O} \end{pmatrix}$$

and $\mathcal{B}^{\mathsf{op}} = \mathcal{A}$. In particular, the category of the right $\mathcal{B}$-modules is isomorphic to the category of the left $\mathcal{A}$-modules. Let $\mathsf{Coh}(\mathcal{B}^{\mathsf{op}})$ be the category of coherent *right* $\mathcal{B}$-modules, then we have a functor $\mathcal{H}om_X(\mathcal{F}^\vee, -\,) : \mathsf{Coh}(X) \to \mathsf{Coh}(\mathcal{B}^{\mathsf{op}})$ having a left adjoint functor $\mathcal{F}^\vee \otimes_{\mathcal{B}} - \, : \mathsf{Coh}(\mathcal{B}^{\mathsf{op}}) \to \mathsf{Coh}(\mathcal{A})$. The sheaf $\mathcal{F}^\vee$ is locally projective as a right $\mathcal{B}$-module. Moreover, one easily sees (for instance, as in Proposition 5.3) that $\mathcal{F}^\vee \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$, so we have the following commutative diagram of categories of functors:

$$\begin{array}{ccc} \mathsf{Coh}(\mathcal{B}^{\mathsf{op}}) & \xrightarrow{\quad = \quad} & \mathsf{Coh}(\mathcal{A}) \\ & {\scriptstyle \mathcal{F}^\vee \otimes_{\mathcal{B}} -} \searrow \quad \swarrow {\scriptstyle \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, -)} & \\ & \mathsf{Coh}(X) & \end{array}$$

Hence, the functor $\mathbb{H}$ is right adjoint to $\mathbb{G}$.

(2) By part (4) of Theorem 2.3, the global dimension of $\mathsf{Coh}(\mathcal{A})$ can be computed locally. It is clear that $\mathcal{A}_x \cong \mathsf{Mat}_2(\mathcal{O}_x)$ for a smooth point $x \in X$. Note that $\widehat{\mathcal{O}_x} \cong k[\![u, v]\!]/uv$ if the singularity is a node and $k[\![u, v]\!]/(u^2 - v^3)$ if it is a cusp. By a result of Bass [5, Corollary 7.3], the indecomposable maximal Cohen-Macaulay modules over the ring $O = k[\![u, v]\!]/xy$ are $k[\![u]\!]$, $k[\![v]\!]$ and O itself; whereas for $O = k[\![u, v]\!]/(u^2 - v^3) \cong k[\![t^2, t^3]\!]$ they are O and $k[\![t]\!]$.

Hence, in both cases we have: $\mathsf{CM}(O) = \mathsf{add}\big(\mathfrak{m} \oplus O\big)$. By a result of Auslander and Roggenkamp [4], the algebra $\widehat{A} = \mathsf{End}_O(\mathfrak{m} \oplus O)$ has global dimension *two*, which proves the claim. See also Remark 2.7.

(3) Let $x \in X$ be an arbitrary point of the curve X, $O = \mathcal{O}_x$ and $F = \mathcal{F}_x$ and $B = \mathsf{End}_O(F)$. Note that $F \cong F^\vee$. Then we have a pair of adjoint functors $\bar{\mathbb{H}} = \mathsf{Hom}_O(F, -\,) :$

$\mathsf{mod}-O \to \mathsf{mod}-B$ and $\bar{\mathbb{G}} = F \otimes_B - : \mathsf{mod}-B \to \mathsf{mod}-O$. By a result of Bass [5, Corollary 7.3], any maximal Cohen-Macaulay O-module is a direct summand of F^n for some $n \geq 0$, so the functors $\bar{\mathbb{H}}$ and $\bar{\mathbb{G}}$ induce mutually inverse equivalences between the category $\mathsf{CM}(O)$ of the maximal Cohen-Macaulay modules and the category $\mathsf{pro}(B)$ of projective right B-modules. In particular, the canonical transformations of functors given by the adjunction $\bar{\zeta} : \mathbb{1}_{\mathsf{pro}(B)} \to \bar{\mathbb{H}} \circ \bar{\mathbb{G}}$ and $\bar{\xi} : \bar{\mathbb{G}} \circ \bar{\mathbb{H}} \to \mathbb{1}_{\mathsf{CM}(O)}$ are isomorphisms.

Since the functors $\mathbb{G}$ and $\mathbb{H}$ form an adjoint pair, we have natural transformations of functors $\zeta : \mathbb{1}_{\mathsf{VB}(\mathcal{B})} \to \mathbb{H} \circ \mathbb{G}$ and $\xi : \mathbb{G} \circ \mathbb{H} \to \mathbb{1}_{\mathsf{TF}(X)}$. For any torsion free sheaf $\mathcal{H}$ on the curve X we have a morphism of $\mathcal{O}$-modules $\xi_{\mathcal{H}} : \mathbb{G} \circ \mathbb{H}(\mathcal{H}) \to \mathcal{H}$. Moreover, for any point $x \in X$ we have: $(\xi_{\mathcal{H}})_x = \bar{\xi}_{\mathcal{H}_x}$, hence $\xi_{\mathcal{H}}$ is an isomorphism for any $\mathcal{H}$. This implies that ξ is an isomorphism of functors. In a similar way, ζ is an isomorphism of functors, too. Hence, the categories $\mathsf{TF}(X)$ and $\mathsf{VB}(\mathcal{A})$ are equivalent.

In order to show that $\mathbb{G}$ is essentially surjective, first note that any object in $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ has a finite resolution by a complex whose terms are locally projective $\mathcal{A}$-modules. Since the locally Cohen-Macauly $\mathcal{O}$–modules are precisely the torsion free $\mathcal{O}$–modules, any object in $D^b\big(\mathsf{Coh}(X)\big)$ is quasi-isomorphic to a bounded complex whose terms are torsion free coherent $\mathcal{O}$–modules. Since $\mathbb{G}$ establishes an equivalence between $\mathsf{VB}(\mathcal{A})$ and $\mathsf{TF}(X)$ and is exact, any object in $D^b\big(\mathsf{Coh}(X)\big)$ has a pre-image in $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$.

(4) Let $\phi : \mathbb{1}_{\mathsf{Coh}(X)} \to \mathbb{G} \circ \mathbb{F}$ be the natural transformation of functors given by the adjunction. Since $\mathbb{G}$ is exact and $\mathbb{F}$ right exact, the composition $\mathbb{G} \circ \mathbb{F}$ is right exact, too. By Lemma 2.4 we know that the canonical morphism of sheaves of $\mathcal{O}$–algebras $\phi_{\mathcal{O}} : \mathcal{O} \to \mathcal{E}nd_{\mathcal{A}}(\mathcal{F})$ is an isomorphism. It implies that for any locally free coherent $\mathcal{O}$–module $\mathcal{E}$ the canonical morphism $\phi_{\mathcal{E}} : \mathcal{E} \to \mathbb{G}\mathbb{F}(\mathcal{E})$ is an isomorphism.

Let $\mathcal{N}$ be a coherent sheaf on X. Since X is quasi-projective, we have a presentation $\mathcal{E}_1 \xrightarrow{f} \mathcal{E}_0 \to \mathcal{N} \to 0$, where $\mathcal{E}_0$ and $\mathcal{E}_1$ are locally free. This gives a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathcal{E}_1 & \xrightarrow{\quad f \quad} & \mathcal{E}_0 & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \phi_{\mathcal{E}_1}} & & \Big\downarrow{\scriptstyle \phi_{\mathcal{E}_0}} & & \Big\downarrow{\scriptstyle \phi_{\mathcal{N}}} & & \\
\mathbb{G} \circ \mathbb{F}(\mathcal{E}_1) & \xrightarrow{\mathbb{G}\circ\mathbb{F}(f)} & \mathbb{G} \circ \mathbb{F}(\mathcal{E}_0) & \longrightarrow & \mathbb{G} \circ \mathbb{F}(\mathcal{N}) & \longrightarrow & 0,
\end{array}$$

where $\phi_{\mathcal{E}_0}$ and $\phi_{\mathcal{E}_1}$ are isomorphisms. Hence, $\phi_{\mathcal{N}}$ is an isomorphism for any coherent sheaf $\mathcal{N}$ on the curve X and ϕ is an isomorphism of functors. The fact that $\mathbb{F}$ is fully faithful follows from Lemma 2.5.

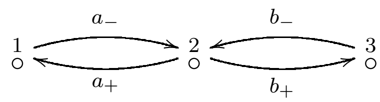
(5) The derived functors $\mathbb{LF}$ and $\mathbb{RF}$ form again an adjoint pair, see for example [24, Lemma 15.6]. Since $\mathbb{G}$ is exact, the adjunction morphism $\psi : \mathbb{1}_{D^-(\mathsf{Coh}(X)} \to \mathbb{RG} \circ \mathbb{LF}$ coincides with $\mathbb{L}\phi$. Since ϕ is an isomorphism, the natural transformation $\mathbb{L}\phi$ is an isomorphism, too. Lemma 2.5 implies that the derived functor $\mathbb{LF} : D^-\big(\mathsf{Coh}(X)\big) \longrightarrow D^-\big(\mathsf{Coh}(\mathcal{A})\big)$ and its restriction on the category of perfect complexes $\mathbb{LF} : \mathsf{Perf}(X) \longrightarrow D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ are fully faithful. □

Remark 2.7. Let O be either $k[\![x,y]\!]/xy$ or $k[\![x,y]\!]/(y^2 - x^3)$, R be the normalization of O and $I = \mathfrak{m}$ be the maximal ideal of O, which in this case is the conductor ideal.

Consider the following O–order:

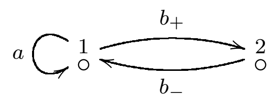
$$A = \mathsf{End}_O(I \oplus O) = \begin{pmatrix} R & I \\ R & O \end{pmatrix},$$

which is the *Auslander algebra* of the ring O. Then A is isomorphic to the completion of the path algebra of the following quiver with relations:



$$b_+a_- = 0, \quad a_+b_- = 0$$

if the singularity is a node and to the completion of the path algebra



$$a^2 = b_-b_+$$

if the singularity is a cusp. Since $\mathsf{gl.dim}(A) = \mathsf{pr.dim}\big(A/\mathsf{rad}(A)\big)$, to compute the global dimension of A it suffices to compute the projective dimension of the simple A–modules.

- If A is nodal, the projective resolutions of the simple A–modules are
  - $0 \to P_1 \oplus P_3 \xrightarrow{(a_+ b_+)} P_2 \to S \to 0$,
  - $0 \to P_3 \xrightarrow{b_+} P_2 \xrightarrow{a_-} P_1 \to S_1 \to 0$,
  - $0 \to P_1 \xrightarrow{a_+} P_2 \xrightarrow{b_-} P_3 \to S_3 \to 0$.
- If A is cuspidal, the projective resolutions of the simple A–modules are
  - $0 \to P_1 \xrightarrow{b_+} P_2 \to S_2 \to 0$,
  - $0 \to P_1 \xrightarrow{\binom{b_-}{a}} P_2 \oplus P_1 \xrightarrow{(b_+\ a)} P_1 \to S_1 \to 0$.

We conclude this section by the following easy observation.

Proposition 2.8. *Let D be the full subcategory of the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ consisting of the complexes $\mathcal{G}^\bullet$ such that for any point $x \in X$ the localization $\mathcal{G}^\bullet_x \in \mathsf{Ob}\big(D^b(\mathcal{A}_x - \mathsf{mod})\big)$ has a finite projective resolution by objects from $\mathsf{add}(\mathcal{F}_x)$. Then D is triangulated, idempotent complete and $\mathbb{F} : \mathsf{Perf}(X) \longrightarrow \mathsf{D}$ is an equivalence of categories.*

Proof. First note that the image of $\mathsf{Perf}(X)$ under functor $\mathbb{F} : D^-\big(\mathsf{Coh}(X)\big) \to D^-\big(\mathsf{Coh}(\mathcal{A})\big)$ belongs to D. Consider the pair of the natural transformations $\mathbb{1}_{D^-(X)} \xrightarrow{\phi} \mathbb{G} \circ \mathbb{F}$ and $\mathbb{F} \circ \mathbb{G} \xrightarrow{\psi} \mathbb{1}_{D^-(\mathcal{A})}$. By Theorem 2.6, the natural transformation ϕ is an isomorphism. Moreover, $\psi|_{\mathsf{D}}$ is an isomorphism, too. Hence, $\mathbb{F}$ and $\mathbb{G}$ are quasi-inverse equivalences between the categories $\mathsf{Perf}(X)$ and D. $\square$

3. Auslander–Reiten translation and τ-periodic complexes

We first recall the following important result about the existence of the Serre functor in the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$, where $\mathcal{A}$ is the Auslander sheaf of orders attached to a reduced projective curve X with at most nodal and cuspidal singularities.

Theorem 3.1. *Let X be a reduced projective curve having only nodes or cusps as singularities, $\mathcal{F} = \mathcal{I} \oplus \mathcal{O}$ and $\mathcal{A} = \mathcal{E}nd_X(\mathcal{F})$. Consider the $\mathcal{A}$–bimodule*

$$\omega_{\mathcal{A}} := \mathcal{H}om_X(\mathcal{A}, \omega_X) \cong \begin{pmatrix} \mathcal{I} & \mathcal{I} \\ \widetilde{\mathcal{O}} & \mathcal{O} \end{pmatrix} \otimes \omega_X,$$

where ω_X is the canonical sheaf of X. Then the endofunctor $\tau : \mathcal{G}^\bullet \mapsto \omega_{\mathcal{A}} \overset{\mathbb{L}}{\otimes} \mathcal{G}^\bullet$ is the Auslander–Reiten translation in the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$. This means that for any pair of objects $\mathcal{G}^\bullet, \mathcal{H}^\bullet$ of $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ we have bifunctorial isomorphisms

$$\mathsf{Hom}_{D^b(\mathcal{A})}\big(\mathcal{H}^\bullet, \mathcal{G}^\bullet\big) \cong \mathbb{D}\big(\mathsf{Ext}^1_{D^b(\mathcal{A})}\big(\mathcal{G}^\bullet, \tau(\mathcal{H}^\bullet)\big)\big),$$

where $\mathbb{D} = \mathsf{Hom}_k(\,-\,, k)$ is the duality over the base field.

Proof. Since X is a Gorenstein curve and the sheaf $\mathcal{A}$ is Cohen-Macaulay as an $\mathcal{O}$–module, we have a quasi-isomorphism of the following complexes of $\mathcal{A}$–bimodules:

$$\mathbb{R}\mathcal{H}om_X(\mathcal{A}, \omega_X) \cong \omega_{\mathcal{A}}.$$

Hence, Theorem 3.1 is a special case of [28, Theorem A.4] and [42, Proposition 6.14]. □

Corollary 3.2. *Let X be a Kodaira cycle of projective lines, $\mathcal{O} \xrightarrow{w} \omega_X$ be an isomorphism given by a no-where vanishing regular differential 1-form $w \in H^0(\omega_X)$. Then we have an injective morphism of $\mathcal{A}$–bimodules $\theta = \theta_w : \omega_{\mathcal{A}} \to \mathcal{A}$ yielding a natural transformation of exact endofunctors*

$$\theta : \tau_{D^b(\mathcal{A})} \longrightarrow \mathbb{1}_{D^b(\mathcal{A})}$$

of the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$. In particular, the category D_θ defined as the full subcategory of $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ consisting of all objects $\mathcal{G}^\bullet$ such that $\theta_{\mathcal{G}^\bullet}$ is an isomorphism, is a triangulated subcategory of $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$.

Proof. Let $\mathcal{G}_1^\bullet \xrightarrow{f} \mathcal{G}_2^\bullet \xrightarrow{g} \mathcal{G}_3^\bullet \xrightarrow{h} \mathcal{G}_1^\bullet[1]$ be a distinguished triangle in $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathcal{A})$. Since θ is a natural transformation of exact functors, we have a commutative diagram

$$\begin{array}{ccccccc}
\tau(\mathcal{G}_1^\bullet) & \xrightarrow{\tau(f)} & \tau(\mathcal{G}_2^\bullet) & \xrightarrow{\tau(g)} & \tau(\mathcal{G}_3^\bullet) & \xrightarrow{\tau(h)} & \tau(\mathcal{G}_1^\bullet)[1] \\
\downarrow{\scriptstyle \theta_{\mathcal{G}_1^\bullet}} & & \downarrow{\scriptstyle \theta_{\mathcal{G}_2^\bullet}} & & \downarrow{\scriptstyle \theta_{\mathcal{G}_3^\bullet}} & & \downarrow{\scriptstyle \theta_{\mathcal{G}_1^\bullet[1]}} \\
\mathcal{G}_1^\bullet & \xrightarrow{f} & \mathcal{G}_2^\bullet & \xrightarrow{g} & \mathcal{G}_3^\bullet & \xrightarrow{h} & \mathcal{G}_1^\bullet[1].
\end{array}$$

This implies that if $\mathcal{G}_1^\bullet$ and $\mathcal{G}_2^\bullet$ are objects of D_θ, then $\mathcal{G}_3^\bullet$ belongs to D_θ, too. □

The main goal of this section is to establish other descriptions of the category D_θ. To do this, we consider the local case first.

Lemma 3.3. *Let O be a nodal singularity, R its normalization and $I = \mathsf{ann}_O(R/O)$ the conductor ideal. Let*

$$A = \mathsf{End}_O(I \oplus O) = \begin{pmatrix} R & I \\ R & O \end{pmatrix}, \ \omega_A = \mathsf{Hom}_O(A, \omega_O) = \begin{pmatrix} I & I \\ R & O \end{pmatrix}, \ F = A \cdot e_2 = \begin{pmatrix} I \\ O \end{pmatrix}$$

be the Auslander algebra of O, its dualizing module and the indecomposable projective module corresponding to the simple A–module of projective dimension one. Let $P^\bullet$ be an object of the derived category $D^b(A-\mathsf{mod})$, then the following conditions are equivalent:

(1) *we have an isomorphism* $\omega_A \overset{\mathbb{L}}{\otimes}_A P^\bullet \cong P^\bullet$.
(2) *the complex* $P^\bullet$ *is quasi-isomorphic to a bounded complex of modules with entries from* $\mathsf{add}(F)$.
(3) *we have an isomorphism* $(A/\omega_A) \overset{\mathbb{L}}{\otimes}_A P^\bullet \cong 0$.

Proof. We first consider the case when O is complete. Then $O = k[\![x,y]\!]/xy$ and $R = k[\![x]\!] \times k[\![y]\!]$. By [43, Lemma 6.4.1], the functor $\tau = \omega_A \overset{\mathbb{L}}{\otimes}_A -$ is an auto-equivalence of $D^b(A-\mathsf{mod})$. Next, the category $D^b(A-\mathsf{mod})$ is Krull–Schmidt and there are exactly three indecomposable A–modules:

$$P_x = \begin{pmatrix} k[\![x]\!] \\ k[\![x]\!] \end{pmatrix}, \; P_y = \begin{pmatrix} k[\![y]\!] \\ k[\![x]\!] \end{pmatrix} \text{ and } F = \begin{pmatrix} I \\ O \end{pmatrix}.$$

From the exact sequence $0 \to \omega_A \xrightarrow{\theta} A \to A/\omega_A \to 0$, for any complex $P^\bullet$ from $D^b(A-\mathsf{mod})$ we get a distinguished triangle

$$\omega_A \overset{\mathbb{L}}{\otimes}_A P^\bullet \xrightarrow{\theta_{P^\bullet}} P^\bullet \longrightarrow (A/\omega_A) \overset{\mathbb{L}}{\otimes}_A P^\bullet \longrightarrow \omega_A \overset{\mathbb{L}}{\otimes}_A P^\bullet[1].$$

This implies that $\tau|_{\mathsf{Hot}^b(\mathsf{add}(F))} \xrightarrow{\theta} \mathbb{1}_{\mathsf{Hot}^b(\mathsf{add}(F))}$ is an isomorphism of functors. On the other hand, we have:

$$\tau(P_x) = I_x := \begin{pmatrix} xk[\![x]\!] \\ k[\![x]\!] \end{pmatrix} \quad \text{and} \quad \tau(P_y) = I_y := \begin{pmatrix} yk[\![y]\!] \\ k[\![y]\!] \end{pmatrix}.$$

Note that we have the following short exact sequences:

$$0 \longrightarrow P_y \xrightarrow{y} F \longrightarrow I_x \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow P_x \xrightarrow{x} F \longrightarrow I_y \longrightarrow 0.$$

For a complex $P^\bullet$ from $D^b(A-\mathsf{mod})$ we define its defect $d(P^\bullet)$ as follows:

$$d(P^\bullet) = \sup\left\{ n \in \mathbb{Z} \,\middle|\, H^n\big(A/\omega_A \overset{\mathbb{L}}{\otimes}_A P^\bullet\big) \ne 0 \right\}.$$

In particular, $d(P^\bullet) = -\infty$ if and only if $P^\bullet \in \mathsf{Ob}\big(\mathsf{Hot}^b\big(\mathsf{add}(F)\big)\big)$. By the definition of the functor τ it is clear that $d(P^\bullet) \ge d\big(\tau(P^\bullet)\big)$ and $d(P^\bullet) = d\big(\tau(P^\bullet)\big)$ if and only if $P^\bullet \in \mathsf{Ob}\big(\mathsf{Hot}^b\big(\mathsf{add}(F)\big)\big)$. Since $D^b(A-\mathsf{mod})$ is a Krull–Schmidt category, this shows the equivalence (1) $\Longleftrightarrow$ (2). The equivalence (2) $\Longleftrightarrow$ (3) easily follows from existence of minimal projective resolutions over A.

Now we consider the general case, when O is not necessary complete. The implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) are clear in this case as well. In order to show the implication (1) $\Longrightarrow$ (2), it is sufficient to show that a complex $P^\bullet \in \mathsf{Ob}\big(D^b(A-\mathsf{mod})\big)$ is quasi-isomorphic to a bounded complex of modules with entries from $\mathsf{add}(F)$ if and only if $\widehat{P}^\bullet \in \mathsf{Ob}\big(D^b(\widehat{A}-\mathsf{mod})\big)$ is quasi-isomorphic to a bounded complex of modules with entries from $\mathsf{add}(\widehat{F})$.

By a result of Bass, see [5, Corollary 7.3], any indecomposable torsion free O–module is isomorphic either to O or to a direct summand of R^m for some $m \ge 1$. Since the category of torsion free O–modules is equivalent to the category of projective A–modules, any indecomposable projective module is isomorphic either to F or to a direct summand of P^m for some $m \geq 1$, where $P = A \cdot e_1$. We assume all entries of the complex $P^\bullet$ are projective. Then we have decompositions: $P^n = P_1^n \oplus P_2^n$ for all $n \in \mathbb{Z}$, where $P_1^n \in \mathsf{add}(F)$ and $P_2^n \in \mathsf{add}(P)$. In the set of complexes of projective modules homotopic to $P^\bullet$, consider a representative $Q^\bullet$ with the smallest possible number $\sum_{n\in\mathbb{Z}}\bigl(\mathrm{rk}_O(P_2^n)\bigr)$. Let n be the biggest index for which $P_2^n \neq 0$. Then we have:

$$Q^\bullet = \left(\dots \longrightarrow Q_1^{n-1} \oplus Q_2^{n-1} \xrightarrow{\begin{pmatrix}\alpha_{n-1} & \beta_{n-1}\\ \gamma_{n-1} & \delta_{n-1}\end{pmatrix}} Q_1^n \oplus Q_2^n \xrightarrow{(\alpha_n\ \beta_n)} Q_1^{n+1} \longrightarrow \dots\right).$$

The category of complexes over $\widehat{A}$ is Krull-Schmidt. If $\widehat{Q}^\bullet$ is quasi-isomorphic to a complex, whose entries belong to $\mathsf{add}(\widehat{F})$ then the morphism $\hat{\delta}_{n-1} : \widehat{Q}_2^{n-1} \to \widehat{Q}_2^n$ is surjective. Since the completion functor is faithfully flat, the morphism $\delta_{n-1} : Q_2^{n-1} \to Q_2^n$ is surjective, too. Since both modules are projective, δ_{n-1} is the projection on a direct summand. This implies that the complex $Q^\bullet$ is homotopic to a complex of the form

$$\bigl(\dots \longrightarrow Q_1^{n-1} \oplus \bar{Q}_2^{n-1} \longrightarrow Q_1^n \longrightarrow Q_1^{n+1} \longrightarrow \dots\bigr).$$

Contradiction. Hence, $Q^\bullet$ belongs to $\mathsf{Hot}^b\bigl(\mathsf{add}(F)\bigr)$, as wanted. □

From now on, let X be a Kodaira cycle of projective lines. We fix a no-where vanishing regular differential form $w \in H^0(\omega_X)$ identifying $\omega_{\mathcal{A}}$ with a sheaf of two-sided $\mathcal{A}$-ideals. Hence, we have a short exact sequence of $\mathcal{A}$–bimodules $0 \to \omega_{\mathcal{A}} \to \mathcal{A} \to \mathcal{A}/\omega_{\mathcal{A}} \to 0$ and for any complex $\mathcal{G}^\bullet$ from $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathcal{A})$ there is a distinguished triangle

$$\tau(\mathcal{G}^\bullet) \xrightarrow{\theta_{\mathcal{G}^\bullet}} \mathcal{G}^\bullet \longrightarrow (\mathcal{A}/\omega_{\mathcal{A}}) \overset{\mathbb{L}}{\otimes} \mathcal{G}^\bullet \longrightarrow \tau(\mathcal{G}^\bullet)[1].$$

The following theorem is the main result of this section.

Theorem 3.4. *Let X be a Kodaira cycle of projective lines, $\mathcal{I}$ be the ideal sheaf of the singular locus of X, $\mathcal{F} = \mathcal{I} \oplus \mathcal{O}$ and $\mathcal{A} = \mathcal{E}nd_X(\mathcal{F})$ be the Auslander sheaf of X. For an object $\mathcal{G}^\bullet$ of the derived category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathcal{A})$ the following conditions are equivalent:*

1. *we have: $\tau(\mathcal{G}^\bullet) \cong \mathcal{G}^\bullet$;*
2. *the morphism $\theta_{\mathcal{G}^\bullet}$ is an isomorphism;*
3. *we have: $(\mathcal{A}/\omega_{\mathcal{A}}) \overset{\mathbb{L}}{\otimes} \mathcal{G}^\bullet \cong 0$;*
4. *$\mathcal{G}^\bullet$ is an object of the category D_θ introduced in Corollary 3.2;*
5. *$\mathcal{G}^\bullet$ is an object of the category D introduced in Proposition 2.8.*

Proof. The equivalences (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) and the implication (2) $\Longrightarrow$ (1) are obvious.

Note that $(\mathcal{A}/\omega_{\mathcal{A}}) \overset{\mathbb{L}}{\otimes} \mathcal{G}^\bullet \cong 0$ in $D^b\bigl(\mathsf{Coh}(\mathcal{A})\bigr)$ if and only if $(\mathcal{A}/\omega_{\mathcal{A}})_x \overset{\mathbb{L}}{\otimes} \mathcal{G}^\bullet_x \cong 0$ in $D^b(\mathcal{A}_x-\mathsf{mod})$ for all $x \in X$. Hence, the equivalence (3) $\Longleftrightarrow$ (5) follows from Lemma 3.3. The implication (1) $\Longrightarrow$ (3) can be shown in a similar way. □

Combining Proposition 2.8 and Theorem 3.4, we obtain the following corollary.

Corollary 3.5. *Let X be a Kodaira cycle of projective lines and $\mathcal{A}$ be its Auslander sheaf. Then the image of the functor $\mathbb{F} : \mathsf{Perf}(X) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathcal{A})$ is the category of complexes $\mathcal{G}^\bullet$ such that $\tau(\mathcal{G}^\bullet) \cong \mathcal{G}^\bullet$, where $\tau = \omega_{\mathcal{A}} \overset{\mathbb{L}}{\otimes} -$ is the Auslander-Reiten translate in $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathcal{A})$.*

4. Serre quotients and perpendicular categories

Let X be a reduced curve over k having only nodal singularities and $\mathcal{A}$ be its Auslander sheaf. The main goal of this section is to construct two *different* but natural embeddings of $\mathsf{Coh}(X)$ into $\mathsf{Coh}(\mathcal{A})$ such that its image will be closed under extensions.

In Theorem 2.6 it was shown that the functor $\mathbb{F} : \mathsf{Coh}(X) \to \mathsf{Coh}(\mathcal{A})$ is fully faithful. The next proposition characterizes the image of the functor $\mathbb{F}$.

Proposition 4.1. *Let $\mathcal{M}$ be a coherent left $\mathcal{A}$–module. Then there exists a coherent $\mathcal{O}$–module $\mathcal{N}$ such that $\mathbb{F}(\mathcal{N}) \cong \mathcal{M}$ if and only if $\mathcal{M}$ has a locally projective presentation*

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_1 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_0 \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\mathcal{E}_0$ and $\mathcal{E}_1$ are locally free coherent $\mathcal{O}$–modules.

Proof. One direction is clear: if $\mathcal{N}$ is a coherent $\mathcal{O}$–module then it has a locally free presentation $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{N} \to 0$ inducing a locally projective presentation $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_1 \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_0 \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N} \to 0$. Other way around, let $\mathcal{E}_0$ and $\mathcal{E}_1$ be locally free coherent $\mathcal{O}$-modules such that

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_1 \xrightarrow{f} \mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_0 \longrightarrow \mathcal{M} \to 0$$

is an exact sequence of coherent left $\mathcal{A}$-modules. Since the functor $\mathbb{F}$ is fully faithful, there exists a morphism of $\mathcal{O}$–modules $g : \mathcal{E}_1 \to \mathcal{E}_0$ such that $f = \mathbb{F}(g)$. Put $\mathcal{N} := \mathsf{coker}(g)$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_1 & \xrightarrow{\mathbb{F}(g)} & \mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N} & \longrightarrow & 0 \\
\cong\downarrow & & \downarrow\cong & & & & \\
\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_1 & \xrightarrow{f} & \mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_0 & \longrightarrow & \mathcal{M} & \longrightarrow & 0,
\end{array}$$

implying that $\mathcal{M} \cong \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N}$. □

Proposition 4.2. *The category $\mathsf{Im}(\mathbb{F})$ is closed under extensions in $\mathsf{Coh}(\mathcal{A})$.*

Proof. Let $\mathcal{M}'$ and $\mathcal{M}''$ be two coherent left $\mathcal{A}$ modules belonging to the image of $\mathbb{F}$ and

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

be an exact sequence in $\mathsf{Coh}(\mathcal{A})$. Then $\mathcal{M}$ also belongs to the image of $\mathbb{F}$. Indeed, by the assumption there exists coherent $\mathcal{O}_X$-modules $\mathcal{N}'$ and $\mathcal{N}''$ such that $\mathcal{M}' \cong \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N}'$ and $\mathcal{M}'' \cong \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N}''$. Take any locally free presentation $\mathcal{E}_1' \to \mathcal{E}_0' \to \mathcal{N}' \to 0$ of the coherent sheaf $\mathcal{N}'$. By Serre's vanishing theorems (see [20, Section III.5]) there exists an ample line bundle $\mathcal{L}$ and a natural number $n \gg 0$ such that

- the evaluation morphism $f'' = \mathrm{ev} : \mathsf{Hom}_X(\mathcal{L}^{\otimes -n}, \mathcal{N}'') \overset{k}{\otimes} \mathcal{L}^{\otimes -n} \to \mathcal{N}''$ is an epimorphism;
- we have the vanishing: $H^1\big(\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N}') \otimes \mathcal{L}^{\otimes n}\big) = 0$.

Set $\mathcal{E}_0'' := \mathsf{Hom}_X(\mathcal{L}^{\otimes -n}, \mathcal{N}'') \overset{k}{\otimes} \mathcal{L}^{\otimes -n}$ and observe that the coherent left $\mathcal{A}$-module $\mathcal{F} \otimes \mathcal{E}_0''$ is locally projective. Hence, $\mathcal{E}xt^i_{\mathcal{A}}\big(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}_0'', -\big) = 0$ for $i \geq 1$. Moreover, for any coherent left $\mathcal{A}$-module $\mathcal{G}$ we have an isomorphism of functors $\mathsf{Hom}_{\mathcal{A}}(\mathcal{G}, -\,) \cong H^0\big(\mathcal{H}om_{\mathcal{A}}(\mathcal{G}, -)\big)$

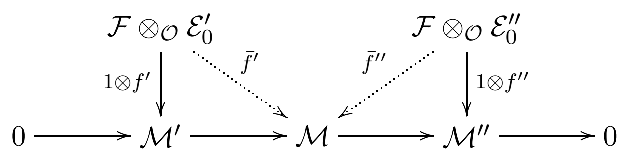
from the category of coherent $\mathcal{A}$-modules to the category of finite dimensional vector spaces over k, where $\mathcal{H}om_{\mathcal{A}}(\mathcal{G}, -) : \mathsf{Coh}(\mathcal{A}) \to \mathsf{Coh}(X)$. This induces the short exact sequence:

$$0 \longrightarrow H^1\big(\mathcal{H}om_{\mathcal{A}}(\mathcal{G},\mathcal{G}')\big) \longrightarrow \mathsf{Ext}^1_{\mathcal{A}}(\mathcal{G},\mathcal{G}') \longrightarrow H^0\big(\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{G},\mathcal{G}')\big)$$

coming from the standard local-to-global spectral sequence $H^p\big(\mathcal{E}xt^q_{\mathcal{A}}(\mathcal{G},\mathcal{G}')\big) \Longrightarrow \mathsf{Ext}^{p+q}_{\mathcal{A}}(\mathcal{G},\mathcal{G}')$. Since the $\mathcal{A}$–module $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}''_0$ is locally projective, it implies that

$$\mathsf{Ext}^1_{\mathcal{A}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}''_0, \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N}') \cong H^1\big(\mathcal{H}om_{\mathcal{A}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}''_0, \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N}')\big) \cong$$
$$\cong H^1\big(\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}} \mathcal{N}') \otimes_{\mathcal{O}} \mathcal{E}''^{\vee}_0\big) = 0.$$

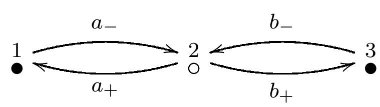
Hence, we can lift the epimorphism $1 \otimes f'' : \mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}''_0 \to \mathcal{M}''$ to a morphism $\bar{f}'' : \mathcal{F} \to \mathcal{M}$, yielding a morphism $\bar{f} = (\bar{f}', \bar{f}'') : \mathcal{F} \otimes_{\mathcal{O}} (\mathcal{E}'_0 \oplus \mathcal{E}''_0) \to \mathcal{M}$:



By 5-lemma, $\bar{f}$ is an epimorphism. In a similar way, we construct a presentation $\mathcal{E}''_1 \to \mathcal{E}''_0 \to \mathcal{N}'' \to 0$, inducing a presentation $\mathcal{F} \otimes_{\mathcal{O}} (\mathcal{E}'_1 \oplus \mathcal{E}''_1) \to \mathcal{F} \otimes_{\mathcal{O}} (\mathcal{E}'_0 \oplus \mathcal{E}''_0) \to \mathcal{M} \to 0$. □

Summing up, Theorem 2.6 and Proposition 4.2 imply that the category of coherent sheaves $\mathsf{Coh}(X)$ is equivalent to a full subcategory of $\mathsf{Coh}(\mathcal{A})$, which is closed under taking cokernels and extensions. It turns out that if the category $\mathsf{Coh}(X)$ can be embedded into $\mathsf{Coh}(\mathcal{A})$ in a completely different way.

Recall that for any singular point $x \in X$ the algebra $A = \widehat{\mathcal{A}}_x$ is isomorphic to the radical completion of the path algebra of the following quiver with relations:

(2) 

$b_+a_- = 0, \quad a_+b_- = 0$

Let T be the full subcategory of the category of torsion coherent $\mathcal{A}$–modules, supported at the singular locus of X and corresponding to the simple A–modules labeled by bullets. Then T is a semi-simple abelian category. Moreover, T is a Serre subcategory of $\mathsf{Coh}(\mathcal{A})$ and for any $\mathcal{T}', \mathcal{T}'' \in \mathsf{Ob}(\mathsf{T})$ we have: $\mathsf{Ext}^1_{\mathcal{A}}(\mathcal{T}', \mathcal{T}'') = 0$.

Remark 4.3. Although the category T is semi-simple, the second extension group $\mathsf{Ext}^2_{\mathcal{A}}(\mathcal{T}', \mathcal{T}'')$ is not necessarily zero for a pair of objects $\mathcal{T}', \mathcal{T}'' \in \mathsf{Ob}(\mathsf{T})$. Indeed, for the two simple A–modules S_1 and S_3 from T we have: $\mathsf{Ext}^2_A(S_1, S_3) = k = \mathsf{Ext}^2_A(S_3, S_1)$.

Definition 4.4. Following Geigle and Lenzing [18], the *perpendicular category* $\mathsf{T}^{\perp}$ of the Serre subcategory T is defined as follows:

$$\mathsf{T}^{\perp} = \big\{\mathcal{G} \in \mathsf{Ob}\big(\mathsf{Coh}(\mathcal{A})\big) \,\big|\, \mathsf{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{G}) = 0 = \mathsf{Ext}^1_{\mathcal{A}}(\mathcal{T}, \mathcal{G}) \quad \text{for all} \quad \mathcal{T} \in \mathsf{Ob}(\mathsf{T})\big\}.$$

In particular, $\mathsf{T}^{\perp}$ is closed under taking kernels and extensions inside of $\mathsf{Coh}(\mathcal{A})$.

Proposition 4.5. *Let $\mathsf{T}_{\mathrm{sim}}$ be the set of the simple objects of T. The perpendicular category $\mathsf{T}^{\perp}$ has the following description:*

$$\mathsf{T}^{\perp} = \bigl\{\mathcal{G} \in \mathsf{Ob}\bigl(\mathsf{Coh}(\mathcal{A})\bigr) \,\big|\, \mathcal{H}om_{\mathcal{A}}(\mathcal{T},\mathcal{G}) = 0 = \mathcal{E}xt^1_{\mathcal{A}}(\mathcal{T},\mathcal{G}) \text{ for all } \mathcal{T} \in \mathsf{T}_{\mathrm{sim}}\bigr\}.$$

Next[1]*, the category $\mathsf{VB}(\mathcal{A})$ of locally projective left $\mathcal{A}$–modules is a full subcategory of $\mathsf{T}^{\perp}$.*

Proof. First note that for any objects $\mathcal{G} \in \mathsf{Ob}(\mathsf{Coh}(\mathcal{A}))$, $\mathcal{T} \in \mathsf{Ob}(\mathsf{T})$ and $i \in \mathbb{Z}$, the sheaves $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{T},\mathcal{G})$ are torsion $\mathcal{O}$–modules. In particular, we have the isomorphisms: $\mathsf{Ext}^i_{\mathcal{A}}(\mathcal{T},\mathcal{G}) \cong H^0\bigl(\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{T},\mathcal{G})\bigr)$. Hence, for any $i \in \mathbb{Z}$, the vanishing of $\mathsf{Ext}^i_{\mathcal{A}}(\mathcal{T},\mathcal{G})$ is equivalent to the vanishing of $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{T},\mathcal{G})$. By induction on the length we get:

$$\mathcal{G} \in \mathsf{Ob}(\mathsf{T}^{\perp}) \iff \mathcal{H}om_{\mathcal{A}}(\mathcal{T},\mathcal{G}) = 0 = \mathcal{E}xt^1_{\mathcal{A}}(\mathcal{T},\mathcal{G}) \text{ for all } \mathcal{T} \in \mathsf{T}_{\mathrm{sim}}\,.$$

This implies the first part of the statement.

In order to show that $\mathsf{VB}(\mathcal{A})$ is a full subcategory of $\mathsf{T}^{\perp}$, it is sufficient to prove that for any indecomposable projective A–module P and any simple module $S \in \mathsf{T}_{\mathrm{sim}}$ we have:

$$\mathsf{Hom}_A(S,P) = 0 = \mathsf{Ext}^1_A(S,P).$$

This vanishing easily follows from the explicit form of the projective resolution of S given in Remark 2.7. □

For a coherent $\mathcal{A}$–module $\mathcal{H}$, consider its maximal subobject $\mathsf{t_D}(\mathcal{H})$ and the canonical short exact sequence $0 \to \mathsf{t_D}(\mathcal{H}) \to \mathcal{H} \to \bar{\mathcal{H}} \to 0$. Next, let $\bar{\mathcal{H}} \to \mathcal{I}^\bullet_{\bar{\mathcal{H}}}$ be an injective resolution of $\bar{\mathcal{H}}$. Choose a distinguished triangle in $D^b\bigl(\mathsf{Coh}(\mathcal{A})\bigr)$ determined by the canonical evaluation morphism of complexes of sheaves $\mathsf{ev}_{\bar{\mathcal{H}}}$:

$$\bar{\mathcal{H}} \xrightarrow{u} \widetilde{\mathcal{H}} \xrightarrow{v} \bigoplus_{\mathcal{T}\in\mathsf{T}_{\mathrm{sim}}} \mathsf{Hom}_{D^b(\mathcal{A})}\bigl(\mathcal{T},\mathcal{I}^\bullet_{\bar{\mathcal{H}}}[1]\bigr) \otimes_k \mathcal{T} \xrightarrow{\mathsf{ev}_{\bar{\mathcal{H}}}} \mathcal{I}^\bullet_{\bar{\mathcal{H}}}[1]. \tag{3}$$

Obviously, this triangle corresponds to a representative of the class of the universal extension sequence

$$0 \longrightarrow \bar{\mathcal{H}} \xrightarrow{u} \widetilde{\mathcal{H}} \xrightarrow{v} \bigoplus_{\mathcal{T}\in\mathsf{T}_{\mathrm{sim}}} \mathsf{Ext}^1_{\mathcal{A}}(\mathcal{T},\bar{\mathcal{H}}) \otimes_k \mathcal{T} \longrightarrow 0. \tag{4}$$

Inspired by the work of Geigle and Lenzing [18, Section 2], we get the following result.

Theorem 4.6. *The correspondence $\mathcal{H} \mapsto \widetilde{\mathcal{H}}$ can be extended to a functor $\mathbb{J} : \mathsf{Coh}(\mathcal{A}) \to \mathsf{T}^{\perp}$. This functor $\mathbb{J}$ is left adjoint to the inclusion $\mathbb{I} : \mathsf{T}^{\perp} \to \mathsf{Coh}(\mathcal{A})$. Moreover, the functor $\mathsf{T}^{\perp} \to \mathsf{Coh}(\mathcal{A})/\,\mathsf{T}$ defined as the composition $\mathsf{T}^{\perp} \xrightarrow{\mathbb{I}} \mathsf{Coh}(\mathcal{A}) \xrightarrow{\mathbb{P}} \mathsf{Coh}(\mathcal{A})/\,\mathsf{T}$ is an equivalence of categories. Here $\mathsf{Coh}(\mathcal{A})/\,\mathsf{T}$ is the Serre quotient category and $\mathbb{P}$ is the corresponding projection functor. In particular, the perpendicular category $\mathsf{T}^{\perp}$ is abelian and the functor $\mathbb{J}$ is right exact.*

Proof. Denote $\mathcal{E}_{\mathsf{T}}(\bar{\mathcal{H}}) := \oplus_{\mathcal{T}\in\mathsf{T}_{\mathrm{sim}}} \mathsf{Ext}^1_{\mathcal{A}}(\mathcal{T},\bar{\mathcal{H}}) \otimes_k \mathcal{T}$. We check first that for any coherent $\mathcal{A}$–module $\mathcal{H}$, the corresponding $\mathcal{A}$–module $\widetilde{\mathcal{H}}$ given by the universal extension sequence (4) belongs to $\mathsf{T}^{\perp}$. Let $\mathcal{S}$ be an arbitrary element of $\mathsf{T}_{\mathrm{sim}}$ and $f : \mathcal{S} \to \widetilde{\mathcal{H}}$ be a non-zero morphism. Since $\mathsf{Hom}_{\mathcal{A}}(\mathcal{S},\bar{\mathcal{H}}) = 0$, the morphism $g := v_*(f) = vf$ is non-zero too. Since

[1]The first-named author would like to thank Catharina Stroppel for drawing his attention to this fact.

the category T is semi-simple, the morphism $g : \mathsf{T} \to \mathcal{E}_{\mathsf{T}}(\bar{\mathcal{H}})$ can be identified with the inclusion morphism of a direct summand. By the definition of the evaluation morphism, $(\mathsf{ev}_{\bar{\mathcal{H}}}) \circ g \neq 0$. But on the other hand, $(\mathsf{ev}_{\bar{\mathcal{H}}}) \circ g = (\mathsf{ev}_{\bar{\mathcal{H}}}) \circ v \circ f = 0$. Contradiction.

Since $\mathsf{Hom}_{\mathcal{A}}(\mathcal{S}, \bar{\mathcal{H}}) = \mathsf{Hom}_{\mathcal{A}}(\mathcal{S}, \widetilde{\mathcal{H}}) = \mathsf{Ext}^1_{\mathcal{A}}\big(\mathcal{S}, \mathcal{E}_{\mathsf{T}}(\bar{\mathcal{H}})\big) = 0$, the short exact sequence (4) induces an exact sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathcal{A}}(\mathcal{S}, \mathcal{E}_{\mathsf{T}}(\bar{\mathcal{H}})) \longrightarrow \mathsf{Ext}^1_{\mathcal{A}}(\mathcal{S}, \bar{\mathcal{H}}) \xrightarrow{u_*} \mathsf{Ext}^1_{\mathcal{A}}(\mathcal{S}, \widetilde{\mathcal{H}}) \longrightarrow 0.$$

Since the category T is semi-simple, we have: $\mathsf{dim}_k\big(\mathsf{Hom}_{\mathcal{A}}(\mathcal{S}, \mathcal{E}_{\mathsf{T}}(\bar{\mathcal{H}}))\big) = \mathsf{dim}_k\big(\mathsf{Ext}^1_{\mathcal{A}}(\mathcal{S}, \bar{\mathcal{H}})\big)$. From the dimension reasons we conclude that $\mathsf{Ext}^1_{\mathcal{A}}(\mathcal{S}, \widetilde{\mathcal{H}}) = 0$. Hence, $\widetilde{\mathcal{H}}$ belongs to $\mathsf{T}^{\perp}$ as stated. The functor $\mathsf{T}^{\perp} \to \mathsf{Coh}(\mathcal{A})/\mathsf{T}$ is fully faithful by [16, Chapitre III]. Moreover, by [18, Proposition 2.2] this functor is an equivalence of categories. In particular, the category $\mathsf{T}^{\perp}$ is abelian.

Now we check that the assignment $\mathcal{H} \mapsto \widetilde{\mathcal{H}}$ can be extended to a functor $\mathsf{Coh}(\mathcal{A}) \to \mathsf{T}^{\perp}$. Let $f : \mathcal{H}' \to \mathcal{H}''$ be a morphism in $\mathsf{Coh}(\mathcal{A})$. It is easy to see that f maps $\mathsf{t}_{\mathsf{D}}(\mathcal{H}')$ to $\mathsf{t}_{\mathsf{D}}(\mathcal{H}'')$. For any object $\mathcal{H}$ we fix a representative of the cokernel $\mathcal{H} \xrightarrow{w} \mathcal{H}/\mathsf{t}_{\mathsf{D}}(\mathcal{H})$. Then we obtain the induced map $\bar{f} : \bar{\mathcal{H}}' \to \bar{\mathcal{H}}''$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{t}_{\mathsf{D}}(\mathcal{H}') & \longrightarrow & \mathcal{H}' & \xrightarrow{w'} & \bar{\mathcal{H}}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \\
0 & \longrightarrow & \mathsf{t}_{\mathsf{D}}(\mathcal{H}'') & \longrightarrow & \mathcal{H}'' & \xrightarrow{w''} & \bar{\mathcal{H}}'' & \longrightarrow & 0.
\end{array}$$

Moreover, the assignment $f \mapsto \bar{f}$ is functorial: $\overline{gf} = \bar{g}\bar{f}$ and $\bar{\mathbb{1}}_{\mathcal{H}} = \mathbb{1}_{\bar{\mathcal{H}}}$. Next, functoriality of the evaluation morphism and axioms of triangulated categories imply there exists a morphism $\tilde{f} : \widetilde{\mathcal{H}}' \to \widetilde{\mathcal{H}}''$ making the following diagram commutative:

$$\begin{array}{ccccccc}
\bar{\mathcal{H}}' & \xrightarrow{u'} & \widetilde{\mathcal{H}}' & \xrightarrow{v'} & \mathcal{E}_{\mathsf{T}}(\bar{\mathcal{H}}') & \xrightarrow{\mathsf{ev}_{\bar{\mathcal{H}}'}} & \bar{\mathcal{H}}'[1] \\
{\scriptstyle \bar{f}}\downarrow & & \vdots{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \bar{f}_*} & & \downarrow{\scriptstyle \bar{f}[1]} \\
\bar{\mathcal{H}}'' & \xrightarrow{u''} & \widetilde{\mathcal{H}}'' & \xrightarrow{v''} & \mathcal{E}_{\mathsf{T}}(\bar{\mathcal{H}}'') & \xrightarrow{\mathsf{ev}_{\bar{\mathcal{H}}'}} & \bar{\mathcal{H}}''.
\end{array}$$

Since $\mathsf{Hom}_{\mathcal{A}}(\mathcal{S}, \widetilde{\mathcal{H}}'') = 0$ for all $\mathcal{S} \in \mathsf{T}_{\mathrm{sim}}$, such a morphism $\tilde{f}$ is unique. So, we obtain a functor $\mathbb{J} : \mathsf{Coh}(\mathcal{A}) \to \mathsf{T}^{\perp}$.

It is easy to see that we have an isomorphism of functors $\xi : \mathbb{J} \circ \mathbb{I} \to \mathbb{1}_{\mathsf{T}^{\perp}}$. Moreover, there is a natural transformation $\zeta : \mathbb{1}_{\mathsf{Coh}(\mathcal{A})} \to \mathbb{I} \circ \mathbb{J}$, where for a coherent sheaf $\mathcal{H}$ the morphism $\zeta_{\mathcal{H}}$ is defined to be the composition $\mathcal{H} \xrightarrow{w} \bar{\mathcal{H}} \xrightarrow{u} \widetilde{\mathcal{H}}$. From the short exact sequences defining the sheaves $\bar{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$ we conclude that for any object $\mathcal{H}''$ from the perpendicular category $\mathsf{T}^{\perp}$ the morphisms w'_* and u'_* are the isomorphisms. Hence, the morphism $\mathsf{Hom}_{\mathcal{A}}\big(\mathbb{J}(\mathcal{H}'), \mathcal{H}''\big) \to \mathsf{Hom}_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')$ given by the composition

$$\mathsf{Hom}_{\mathcal{A}}(\widetilde{\mathcal{H}}', \mathcal{H}'') \xrightarrow{w'_*} \mathsf{Hom}_{\mathcal{A}}(\bar{\mathcal{H}}', \mathcal{H}'') \xrightarrow{u'_*} \mathsf{Hom}_{\mathcal{A}}(\mathcal{H}', \mathcal{H}'')$$

is an isomorphism. This shows that $\mathbb{J}$ is left adjoint to the embedding $\mathbb{I} : \mathsf{T}^{\perp} \to \mathsf{Coh}(\mathcal{A})$. Since the category $\mathsf{T}^{\perp}$ is abelian, the functor $\mathbb{J}$ is right exact. $\square$

Remark 4.7. By [18, Proposition 2.2] we also know that the Serre subcategory T is *localizing*. This means that the canonical functor $\mathbb{P} : \mathsf{Coh}(\mathcal{A}) \to \mathsf{Coh}(\mathcal{A})/\,\mathsf{T}$ has a right adjoint functor $\widetilde{\mathbb{J}} : \mathsf{Coh}(\mathcal{A})/\,\mathsf{T} \to \mathsf{Coh}(\mathcal{A})$. Theorem 4.8 implies $\mathbb{P}$ has also a left adjoint functor, hence T is even a *bilocalizing* subcategory. Moreover, the image of $\widetilde{\mathbb{J}}$ belongs to the perpendicular category $\mathsf{T}^{\perp}$ and there is an isomorphism of functors $\widetilde{\mathbb{J}} \circ \mathbb{P} \cong \mathbb{J}$.

Note that the exact functor $\mathbb{G} = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, -\,) : \mathsf{Coh}(\mathcal{A}) \to \mathsf{Coh}(X)$ vanishes on the category T. Using the universal property of the Serre quotient category, we obtain an exact functor

$$\mathbb{E} : \mathsf{Coh}(\mathcal{A})/\,\mathsf{T} \longrightarrow \mathsf{Coh}(X)$$

such that $\mathbb{E} \circ \mathbb{P} = \mathbb{G}$. The main result of this section is the following theorem.

Theorem 4.8. *The functor* $\mathbb{E} : \mathsf{Coh}(\mathcal{A})/\,\mathsf{T} \to \mathsf{Coh}(X)$ *is an equivalence of abelian categories. Moreover, the functors* $\mathbb{G} \circ \mathbb{I} : \mathsf{T}^{\perp} \longrightarrow \mathsf{Coh}(X)$ *and* $\mathbb{J} \circ \mathbb{F} : \mathsf{Coh}(X) \longrightarrow \mathsf{T}^{\perp}$ *are mutually quasi-inverse equivalences of abelian categories.*

Proof. By Remark 4.7, the second statement implies the first one.

Next, recall that we have two adjoint pairs of functors:

$$\mathsf{T}^{\perp} \underset{\mathbb{J}}{\overset{\mathbb{I}}{\rightleftarrows}} \mathsf{Coh}(\mathcal{A}) \underset{\mathbb{F}}{\overset{\mathbb{G}}{\rightleftarrows}} \mathsf{Coh}(X).$$

Let $\mathbb{F} \circ \mathbb{G} \xrightarrow{\eta} \mathbb{1}_{\mathsf{Coh}(\mathcal{A})}, \mathbb{1}_{\mathsf{Coh}(X)} \xrightarrow{\phi} \mathbb{G} \circ \mathbb{F}, \mathbb{1}_{\mathsf{Coh}(\mathcal{A})} \xrightarrow{\zeta} \mathbb{I} \circ \mathbb{J}$ and $\mathbb{J} \circ \mathbb{I} \xrightarrow{\xi} \mathbb{1}_{\mathsf{T}^{\perp}}$ be the morphisms given by the adjunction. Then the functors $\mathbb{J} \circ \mathbb{F}$ and $\mathbb{G} \circ \mathbb{I}$ also form an adjoint pair, whose adjunction morphisms are:

$$\mu : \mathbb{1}_{\mathsf{Coh}(X)} \xrightarrow{\phi} \mathbb{G}\mathbb{F} \xrightarrow{\mathbb{G}(\zeta)\mathbb{F}} \mathbb{G}\mathbb{I} \circ \mathbb{J}\mathbb{F} \quad \text{and} \quad \nu : \mathbb{J}\mathbb{F} \circ \mathbb{G}\mathbb{I} \xrightarrow{\mathbb{J}(\eta)\mathbb{I}} \mathbb{J}\mathbb{I} \xrightarrow{\xi} \mathbb{1}_{\mathsf{T}^{\perp}}.$$

In order to show that $\mathbb{J}\mathbb{F}$ and $\mathbb{G}\mathbb{I}$ are mutually inverse equivalences of categories, it is sufficient to show the morphisms $\mu_{\mathcal{G}}$ and $\nu_{\mathcal{H}}$ are isomorphisms for arbitrary objects $\mathcal{G} \in \mathsf{Ob}\big(\mathsf{Coh}(X)\big)$ and $\mathcal{H} \in \mathsf{Ob}(\mathsf{T}^{\perp})$. From the construction of the functors $\mathbb{F}, \mathbb{G}, \mathbb{I}$ and $\mathbb{J}$ it is clear that the morphism of $\mathcal{O}_x$–modules $\big(\mu_{\mathcal{G}}\big)_x$ and the morphism of $\mathcal{A}_x$–modules $\big(\nu_{\mathcal{H}}\big)_x$ are isomorphisms provided x is a smooth point of the curve X.

Let $x \in X$ be a singular point. Recall that the completion functor is faithfully flat (see for example [2, Theorem 10.17]), hence a morphism of $\mathcal{O}_x$ modules $M \xrightarrow{f} N$ is an isomorphism if and only if the morphism $\widehat{M} \xrightarrow{\hat{f}} \widehat{N}$ is an isomorphism of $\widehat{\mathcal{O}}_x$–modules. Let $O = \widehat{\mathcal{O}}_x = k[\![u,v]\!]/uv$, A be the Auslander algebra of O, $F = \widehat{\mathcal{F}}_x$ and $\bar{\mathsf{T}}$ be full subcategory of $A-\mathsf{mod}$ whose objects correspond to the torsion sheaves from the category T supported at x. Let $\bar{\mathbb{F}} = F \otimes_O - : O - \mathsf{mod} \to A - \mathsf{mod}$, $\bar{\mathbb{G}} = \mathsf{Hom}_A(F, \ ) : A - \mathsf{mod} \to O - \mathsf{mod}$, $\bar{\mathbb{I}} : \bar{\mathsf{T}} \to A - \mathsf{mod}$ be the inclusion functors and $\bar{\mathbb{J}}$ be the right adjoint to $\mathbb{I}_x$. Then we have a commutative diagram of categories and functors

$$
(5)\qquad
\begin{array}{ccccc}
\mathsf{T}^{\perp} & \underset{\mathbb{J}}{\overset{\mathbb{I}}{\rightleftarrows}} & \mathsf{Coh}(\mathcal{A}) & \underset{\mathbb{F}}{\overset{\mathbb{G}}{\rightleftarrows}} & \mathsf{Coh}(X). \\
\downarrow & & \downarrow & & \downarrow \\
\bar{\mathsf{T}}^{\perp} & \underset{\bar{\mathbb{J}}}{\overset{\bar{\mathbb{I}}}{\rightleftarrows}} & A-\mathsf{mod} & \underset{\bar{\mathbb{F}}}{\overset{\bar{\mathbb{G}}}{\rightleftarrows}} & O-\mathsf{mod}.
\end{array}
$$

The vertical arrows correspond to the composition of the localization functor with the functor of the radical completion. Thus, we have to show the natural transformations of functors

$$\bar{\mu} : \mathbb{1}_{O-\mathsf{mod}} \longrightarrow \bar{\mathbb{G}}\bar{\mathbb{I}} \circ \bar{\mathbb{J}}\bar{\mathbb{F}} \quad \text{and} \quad \bar{\nu} : \bar{\mathbb{J}}\bar{\mathbb{F}} \circ \bar{\mathbb{G}}\bar{\mathbb{I}} \longrightarrow \mathbb{1}_{\bar{\mathsf{T}}^{\perp}}.$$

are isomorphisms. By Lemma 2.4, the canonical map

$$O = \mathsf{Hom}_O(O, O) \longrightarrow \mathsf{Hom}_A\big(\bar{\mathbb{F}}(O), \bar{\mathbb{F}}(O)\big) = \mathsf{Hom}_A(F, F)$$

is an isomorphism of algebras. By Proposition 4.5, the module F belongs to $\bar{\mathsf{T}}^{\perp}$. Our next goal is to show F is a projective generator of $\bar{\mathsf{T}}^{\perp}$. Indeed, by [18, Proposition 2.2] we know the category $\bar{\mathsf{T}}^{\perp}$ is equivalent to the Serre quotient category $A-\mathsf{mod}/\bar{\mathsf{T}}$. Let $\bar{\mathbb{P}} : A-\mathsf{mod} \longrightarrow A-\mathsf{mod}/\bar{\mathsf{T}}$ be the canonical functor. Then $\bar{\mathbb{P}}(F)$ is a generator of $A-\mathsf{mod}/\bar{\mathsf{T}}$, i.e. any object in $A-\mathsf{mod}/\bar{\mathsf{T}}$ is a quotient of an object from $\mathsf{add}\big(\bar{\mathbb{P}}(F)\big)$. To prove this, it is sufficient to show that for any projective A–module Q the object $\mathbb{P}(Q)$ is the quotient of an object from $\mathsf{add}\big(\bar{\mathbb{P}}(F)\big)$. In the notations of Remark 2.7, we have the following short exact sequence in $A-\mathsf{mod}$:

$$0 \longrightarrow P_3 \xrightarrow{b_+} P_2 \xrightarrow{a_-} P_1 \longrightarrow S_1 \longrightarrow 0,$$

yielding the short exact sequence $0 \to \mathbb{P}(P_3) \to \mathbb{P}(P_2) \to \mathbb{P}(P_1) \to 0$ in the quotient category $A-\mathsf{mod}/\mathsf{T}$. In the same way, we have an exact sequence $0 \to \mathbb{P}(P_1) \to \mathbb{P}(P_2) \to \mathbb{P}(P_3) \to 0$.

Finally, we check that $\bar{\mathbb{P}}(F)$ is projective in $A-\mathsf{mod}/\bar{\mathsf{T}}$. Indeed, assume $\bar{\mathbb{P}}(X) \xrightarrow{f} \bar{\mathbb{P}}(F)$ is an epimorphism in $A-\mathsf{mod}/\bar{\mathsf{T}}$. By the definition of the Serre quotient category [16], such a morphism is represented by a diagram in $A-\mathsf{mod}$

$$
\begin{array}{ccc}
Y & \xrightarrow{\quad g \quad} & Q \\
{\scriptstyle i}\big\downarrow & & \big\uparrow{\scriptstyle p} \\
X & \overset{f}{\dashrightarrow} & F
\end{array}
$$

where i is a monomorphism with cokernel belonging to $\bar{\mathsf{T}}$, p is an epimorphism whose kernel belongs to $\bar{\mathsf{T}}$ and g is a morphism in $A-\mathsf{mod}$.

Since F has no subobjects from $\bar{\mathsf{T}}$, the morphism p is an isomorphism. A morphism $\bar{\mathbb{P}}(g) : \bar{\mathbb{P}}(Y) \to \bar{\mathbb{P}}(F)$ is an epimorphism in $A-\mathsf{mod}/\bar{\mathsf{T}}$ if and only if the cokernel of g belongs to $\bar{\mathsf{T}}$. But F has no proper quotients belonging to $\bar{\mathsf{T}}$. Hence, g is an epimorphism in $A-\mathsf{mod}$. Since F is projective, the morphism g splits, i.e there exists $j : F \to Y$ such that $gj = \mathbb{1}_F$. But then $f \circ \mathbb{P}(j) = \mathbb{1}_{\bar{\mathbb{P}}(F)}$, hence $\bar{\mathbb{P}}(F)$ is projective, as wanted.

This implies that $F = \bar{\mathbb{J}}\bar{\mathbb{F}}(O)$ is a projective generator in $\bar{\mathsf{T}}^{\perp}$, hence the functor $\bar{\mathbb{J}}\bar{\mathbb{F}} : O-\mathsf{mod} \to \bar{\mathsf{T}}^{\perp}$ is a exact. Moreover, the canonical morphism $\mathsf{Hom}_O(O,O) \longrightarrow \mathsf{Hom}_A(F,F)$ is an isomorphism. Hence, $\bar{\mathbb{J}}\bar{\mathbb{F}}$ is an equivalence of categories and its adjoint functor $\bar{\mathbb{G}}\bar{\mathbb{I}}$ is an equivalence, too. □

Remark 4.9. Although $\mathsf{T}^{\perp}$ and $\mathsf{Coh}(X)$ are *equivalent* abelian categories and the functors $\mathbb{I}$ and $\mathbb{F}$ are fully faithful, the full subcategories $\mathbb{I}(\mathsf{T}^{\perp})$ and $\mathbb{F}\big(\mathsf{Coh}(X)\big)$ of the category $\mathsf{Coh}(\mathcal{A})$ are *different*. To show this, it is sufficient to consider the local situation. Let $O = k[\![u,v]\!]/uv$ and A be the corresponding Auslander algebra. Consider the O–module $O_u = k[\![u]\!]$. It has a presentation $O \xrightarrow{v} O \to O_u \to 0$. The functor $\mathbb{F}$ is right exact, moreover, it induced an equivalence between the category $\mathsf{add}(O)$ and the category $\mathsf{add}(P_2) = \mathsf{add}(F)$. This implies that $X_u = \mathbb{F}(O_u)$ is given by the presentation $P_2 \xrightarrow{b_+b_-} P_2 \to X_u \to 0$. It is then easy to see that $\mathsf{Hom}_A(S_3, X_u) = k$. Hence, X_u does not belong to $\bar{\mathsf{T}}^{\perp}$.

5. Tilting on rational projective curves with nodal and cuspidal singularities

Let X be a reduced *rational* projective curve with only nodes or cusps as singularities and $\mathcal{A}$ be its Auslander sheaf of orders on X. The main goal of this section is to show that the derived category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathcal{A})$ has a *tilting complex* and is equivalent to the derived category of finite dimensional right modules over a certain finite dimensional algebra Γ_X.

5.1. **Construction of a tilting complex.** Let $X = \bigcup_{i=1}^{n} X_i$, where all components X_i are irreducible and $\widetilde{X} \xrightarrow{\pi} X$ be the normalization map. Then $\widetilde{X} = \bigcup_{i=1}^{n} \widetilde{X_i}$, where $\widetilde{X_i} \cong \mathbb{P}^1$ is the normalization of X_i and we have: $\widetilde{\mathcal{O}} = \bigoplus_{i=1}^{n} \widetilde{\mathcal{O}_i}$, where $\widetilde{\mathcal{O}_i} = \pi_*(\mathcal{O}_{\widetilde{X_i}})$, $1 \le i \le n$.

Lemma 5.1. *In the notations as above, consider the locally projective $\mathcal{A}$–module $\mathcal{P} = \mathcal{A} \cdot e_1$, where $e_1 \in H^0(\mathcal{A})$ is the idempotent given by the equation* (1)*. Then for any pair of line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on the curve X of the same multidegree we have: $\mathcal{P} \otimes_{\mathcal{O}} \mathcal{L}_1 \cong \mathcal{P} \otimes_{\mathcal{O}} \mathcal{L}_2$.*

Proof. Since $\widetilde{X}$ is a union of projective lines, we have: $\pi^*\mathcal{L}_1 \cong \pi^*\mathcal{L}_2$. The projection formula implies that $\pi_*\pi^*\mathcal{L}_1 \cong \widetilde{\mathcal{O}} \otimes_X \mathcal{L}_1 \cong \widetilde{\mathcal{O}} \otimes_X \mathcal{L}_2 \cong \pi_*\pi^*\mathcal{L}_2$, hence $\mathcal{P} \otimes_{\mathcal{O}} \mathcal{L}_1 \cong \mathcal{P} \otimes_{\mathcal{O}} \mathcal{L}_2$. □

In what follows, we shall use the notation

$$\mathcal{P} = \begin{pmatrix} \widetilde{\mathcal{O}} \\ \widetilde{\mathcal{O}} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathcal{O}}_1 \\ \widetilde{\mathcal{O}}_1 \end{pmatrix} \oplus \begin{pmatrix} \widetilde{\mathcal{O}}_2 \\ \widetilde{\mathcal{O}}_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \widetilde{\mathcal{O}}_n \\ \widetilde{\mathcal{O}}_n \end{pmatrix} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n.$$

For a vector $\underline{m} = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$ and a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ of multi-degree $\underline{m}$ we denote $\mathcal{P}(\underline{m}) = \mathcal{P} \otimes_X \mathcal{L} \cong \mathcal{P}_1(m_1) \oplus \mathcal{P}_2(m_2) \oplus \cdots \oplus \mathcal{P}_n(m_n)$. For $\underline{m} = (m, m, \dots, m)$ we shall use the notation: $\mathcal{P}(\underline{m}) = \mathcal{P}(m)$.

Let $\mathcal{H}$ be a coherent left $\mathcal{A}$–module and $e_1, e_2 \in H^0(\mathcal{A})$ be the idempotents given by (1). Then, as $\mathcal{O}$–module, $\mathcal{H}$ it splits into the direct sum $\mathcal{H} = e_1 \cdot \mathcal{H} \oplus e_2 \cdot \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ is an $e_1\mathcal{A}e_1 = \widetilde{\mathcal{O}}$–module with the induced $\mathcal{O}$–module structure and $\mathcal{H}_2$ is an $e_2\mathcal{A}e_2 = \mathcal{O}$–module. Using these notations, we write $\mathcal{H} = \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}$. Obviously, a left $\mathcal{A}$–module $\mathcal{H}$ is torsion free as an $\mathcal{O}$–module if and only if both $\mathcal{O}$–modules $\mathcal{H}_1$ and $\mathcal{H}_2$ are torsion free.

Next, we shall need the following standard technique from the theory of lattices over orders. Let O be a reduced local ring and $Q = Q_1 \times Q_2 \times \cdots \times Q_n$ be its total ring of fractions, where Q_i is a field for all $1 \le i \le n$. Let A be an order over O, then we have:

$$\begin{aligned} Q(A) := Q \otimes_O A &\cong \mathsf{Mat}_{s_1\times s_1}(Q_1) \times \mathsf{Mat}_{s_2\times s_2}(Q_2) \times \cdots \times \mathsf{Mat}_{s_n\times s_n}(Q_n) \\ &:= Q_1(A) \times Q_2(A) \times \cdots \times Q_n(A). \end{aligned}$$

Recall that the ring $\mathsf{Mat}_{s_i\times s_i}(Q_i)$ is Morita-equivalent to Q_i for all $1 \le i \le n$. For an A–module M consider the $Q(A)$–module $Q(M) = Q \otimes_O M$. We say M is torsion free if the canonical morphism of A–modules $M \to Q(M)$ is injective. In that case, we identify M with its image in $Q(M)$.

Lemma 5.2. *In the notations as above, let M and N be two Noetherian torsion free A–modules,*

$$Q(M) = \begin{pmatrix} Q_1 \\ Q_1 \\ \vdots \\ Q_1 \end{pmatrix}^{\oplus m_1} \oplus \begin{pmatrix} Q_2 \\ Q_2 \\ \vdots \\ Q_2 \end{pmatrix}^{\oplus m_2} \oplus \cdots \oplus \begin{pmatrix} Q_n \\ Q_n \\ \vdots \\ Q_n \end{pmatrix}^{\oplus m_n}$$

and

$$Q(N) = \begin{pmatrix} Q_1 \\ Q_1 \\ \vdots \\ Q_1 \end{pmatrix}^{\oplus l_1} \oplus \begin{pmatrix} Q_2 \\ Q_2 \\ \vdots \\ Q_2 \end{pmatrix}^{\oplus l_2} \oplus \cdots \oplus \begin{pmatrix} Q_n \\ Q_n \\ \vdots \\ Q_n \end{pmatrix}^{\oplus l_n}.$$

Then there is an isomorphism of O–modules $c : S(M,N) \to \mathsf{Hom}_A(M,N)$, where $S(M,N)$ is defined as follows:

$$\Big\{ f = (f_1, f_2, \dots, f_n) \in \mathsf{Mat}_{l_1\times m_1}(Q_1) \times \mathsf{Mat}_{l_2\times m_2}(Q_2) \times \cdots \times \mathsf{Mat}_{l_n\times m_n}(Q_n) \Big| f \cdot M \subseteq N \Big\}$$

and $c(f) \cdot m = f \cdot m$ for $m \in M$ and $f \in S(M,N)$.

Proof. First note that the morphism c is well-defined and injective. To prove surjectivity, note that $Q(A)$ is injective as A–module. By [3, Theorem 1] it is sufficient to show that for any exact sequence $0 \to I \xrightarrow{\alpha} A$ and any morphism $\beta : I \longrightarrow Q(A)$ there exists a morphism $\gamma : A \to Q(A)$ such that $\gamma\beta = \alpha$. To prove it note that $Q \otimes_O Q \cong Q$ and $Q \otimes_O -$ is an exact functor, hence the morphism α factorizes through $Q(I)$ and we have a commutative diagram

$$\begin{array}{ccc} I & \xhookrightarrow{\ \alpha\ } & A \\ \downarrow & & \downarrow \\ Q(I) & \hookrightarrow & Q(A) \\ \downarrow & \swarrow & \\ Q(A) & & \end{array} \qquad \beta : I \to Q(A)$$

Since the category of Q–modules is semi-simple, we get the factorization we need.

Hence, for any A–module N the module $Q(N)$ is injective over A. Let $g : M \to N$ be a morphism of A–modules. By the injectivity of $Q(N)$ there exists a morphism $\bar{g} : Q(M) \to Q(N)$ making the following diagram commutative:

$$\begin{array}{ccc} M & \longrightarrow & Q(M) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\ N & \longrightarrow & Q(N). \end{array}$$

But then $\bar{g} \in S(M, N)$ and $g = c(\bar{g})$. □

Our next aim is to transfer this technique to the case of sheaves of A–modules, where A is the Auslander order attached to a projective curve with only nodal or cuspidal singularities.

Let $\mathcal{K}_i$ be the sheaf of rational functions on the irreducible component X_i and $\mathcal{K}$ be the sheaf of rational functions on X. Then we have: $\mathcal{K} \cong \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$. Let Q_i be the field of rational functions on the component X_i, then the category of coherent $\mathcal{K}_i$–modules is equivalent to the category of finite dimensional vector spaces over Q_i. Let $\mathcal{A}$ be the Auslander sheaf of X and $\mathcal{H}$ be a torsion free coherent $\mathcal{A}$–module. Then the canonical morphism of $\mathcal{A}$–modules

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}' \\ \mathcal{H}'' \end{pmatrix} \longrightarrow \mathcal{K}(\mathcal{H}) := \mathcal{K} \otimes_{\mathcal{O}} \mathcal{H} = \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_1 \end{pmatrix}^{m_1} \oplus \begin{pmatrix} \mathcal{K}_2 \\ \mathcal{K}_2 \end{pmatrix}^{m_2} \oplus \cdots \oplus \begin{pmatrix} \mathcal{K}_n \\ \mathcal{K}_n \end{pmatrix}^{m_n}$$

is a monomorphism. In what follows, we consider a torsion free $\mathcal{A}$–module $\mathcal{H}$ as a submodule of $\mathcal{K}(\mathcal{H})$.

Proposition 5.3. *In the notations as above, let $\mathcal{G}$ be a torsion free $\mathcal{A}$–module and*

$$\mathcal{K}(\mathcal{G}) = \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_1 \end{pmatrix}^{l_1} \oplus \begin{pmatrix} \mathcal{K}_2 \\ \mathcal{K}_2 \end{pmatrix}^{l_2} \oplus \cdots \oplus \begin{pmatrix} \mathcal{K}_n \\ \mathcal{K}_n \end{pmatrix}^{l_n}.$$

Consider the sheaf $\mathcal{S}(\mathcal{H}, \mathcal{G})$ associated with the following presheaf:

$$U \mapsto \Big\{ f = \big(f_1, \dots, f_n\big) \in \mathsf{Mat}_{l_1 \times m_1}\big(\mathcal{K}_1(U)\big) \oplus \cdots \oplus \mathsf{Mat}_{l_n \times m_n}\big(\mathcal{K}_n(U)\big) \;\Big|\; f \cdot \mathcal{H}(U) \subseteq \mathcal{G}(U) \Big\}.$$

Then the canonical morphism of $\mathcal{O}$–modules $c : \mathcal{S}(\mathcal{H}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{H}, \mathcal{G})$ is an isomorphism.

Proof. First note the morphism c is well-defined. Moreover, for any point $x \in X$ its stalk c_x coincides with the morphism from Lemma 5.2 applied to the $\mathcal{O}_x$–order $\mathcal{A}_x$. Hence, c is an isomorphism of $\mathcal{O}$–modules. □

Corollary 5.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be a pair of coherent torsion free left $\mathcal{A}$–modules and $\mathcal{G}$ be locally projective. Then for any $n \geq 0$ we have an isomorphism of vector spaces:*

$$\mathsf{Ext}^n_{\mathcal{A}}(\mathcal{G}, \mathcal{H}) = H^n\bigl(\mathcal{S}(\mathcal{G}, \mathcal{H})\bigr).$$

Proof. Indeed, since $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{G}, \mathcal{H}) = 0$ for all $i \geq 1$, the local-to-global spectral sequence implies that $\mathsf{Ext}^i_{\mathcal{A}}(\mathcal{G}, \mathcal{H}) \cong H^0\bigl(\mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{H})\bigr) \cong H^0\bigl(\mathcal{S}(\mathcal{G}, \mathcal{H})\bigr)$. □

Corollary 5.5. *Let X be a curve with nodal or cuspidal singularities, $\mathcal{A}$ be its Auslander sheaf, $\mathcal{G}$ and $\mathcal{H}$ be two torsion free coherent $\mathcal{A}$–modules such that*

$$\mathcal{K}(\mathcal{G}) = \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_1 \end{pmatrix} \oplus \begin{pmatrix} \mathcal{K}_2 \\ \mathcal{K}_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \mathcal{K}_n \\ \mathcal{K}_n \end{pmatrix} \cong \mathcal{K}(\mathcal{H}).$$

Then the $\mathcal{O}$–module $\mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{H})$ is isomorphic to the sheaf associated with the presheaf

$$U \mapsto \left\{ f = \bigl(f_1, \ldots, f_n\bigr) \in \mathcal{K}_1(U) \oplus \cdots \oplus \mathcal{K}_n(U) \;\middle|\; \begin{array}{l} f \cdot \mathcal{H}'(U) \subseteq \mathcal{G}'(U) \\ f \cdot \mathcal{H}''(U) \subseteq \mathcal{G}''(U) \end{array} \right\}.$$

In particular, for the locally projective $\mathcal{A}$–modules $\mathcal{P} = \mathcal{A} \cdot e_1$ and $\mathcal{F} = \mathcal{A} \cdot e_2$, where $e_1, e_2 \in H^0(\mathcal{A})$ are the idempotents given by (1), we have:

$$\widetilde{\mathcal{O}} \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \mathcal{P}),\; \mathcal{O} \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{F}),\; \widetilde{\mathcal{O}} \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{P}) \text{ and } \mathcal{I} \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \mathcal{F}).$$

The following proposition plays the key role in our approach to non-commutative rational projective curves.

Proposition 5.6. *Let X be a rational reduced projective curve with only nodal or cuspidal singularities and $\mathcal{A}$ be its Auslander sheaf of orders. Consider the torsion $\mathcal{A}$–module $\mathcal{S}$ given by its locally free resolution*

$$0 \longrightarrow \begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{I} \\ \mathcal{O} \end{pmatrix} \longrightarrow \mathcal{S} \longrightarrow 0.$$

Note that the first term of this short exact sequence is isomorphic to $\mathcal{P}(-2)$, the middle term is $\mathcal{F}$ and $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_t$, where the torsion module $\mathcal{S}_i$ is supported at the singular point $x_i \in X$ and corresponds to the unique simple $\widehat{\mathcal{A}}_{x_i}$–module of projective dimension one. Then the complex

$$\begin{aligned} \mathcal{H}^\bullet &:= \mathcal{S}[-1] \oplus \mathcal{P}(-1) \oplus \mathcal{P} = \\ &= \bigl(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_t\bigr)[-1] \oplus \bigl(\mathcal{P}_1(-1) \oplus \mathcal{P}_1\bigr) \oplus \bigl(\mathcal{P}_2(-1) \oplus \mathcal{P}_2\bigr) \oplus \ldots \bigl(\mathcal{P}_n(-1) \oplus \mathcal{P}_n\bigr) \end{aligned}$$

is rigid *in the derived category of coherent sheaves $D^b\bigl(\mathsf{Coh}(\mathcal{A})\bigr)$, i.e. for all $i \neq 0$ we have*

$$\mathsf{Hom}_{D^b(\mathcal{A})}\bigl(\mathcal{H}^\bullet, \mathcal{H}^\bullet[i]\bigr) = 0.$$

Proof. By Corollary 5.4 we have:

$$\begin{aligned}\mathsf{Ext}^i_{\mathcal{A}}\big(\mathcal{P}(-1)\oplus\mathcal{P},\,\mathcal{P}(-1)\oplus\mathcal{P}\big) = H^i\Big(\mathcal{S}\big(\mathcal{P}(-1)\oplus\mathcal{P},\mathcal{P}(-1)\oplus\mathcal{P}\big)\Big) = \\ = H^i\big(X,\tilde{\mathcal{O}}(-1)\oplus\tilde{\mathcal{O}}^{\oplus 2}\oplus\tilde{\mathcal{O}}(1)\big) = H^i\big(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-1)\oplus\mathcal{O}_{\widetilde{X}}^{\oplus 2}\oplus\mathcal{O}_{\widetilde{X}}(1)\big) = 0\end{aligned}$$

for all $i \ne 0$.

Now we check the torsion $\mathcal{A}$–module $\mathcal{S}$ is also exceptional. Again, using the local-to-global spectral sequence, we have: $\mathsf{Ext}^i_{\mathcal{A}}(\mathcal{S},\mathcal{S}) = H^0\big(X,\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{S},\mathcal{S})\big)$, $i \ge 0$. Hence, the vanishing of $\mathsf{Ext}^i_{\mathcal{A}}(\mathcal{S},\mathcal{S})$ can be checked locally. Using the projective resolution of the simple $\widehat{\mathcal{A}}_{x_i}$–module $\widehat{\mathcal{S}}_{x_i}$ given in Remark 2.7, we get the desired vanishing.

Next, $\mathcal{S}$ is torsion and $\mathcal{P}(n)$ is torsion free, hence we get: $\mathsf{Hom}_{\mathcal{A}}\big(\mathcal{S},\mathcal{P}(n)\big) = 0$ for all $n \in \mathbb{Z}$. Since $\mathcal{S}$ has a locally projective resolution of length one, we have: $\mathcal{E}xt^i_{\mathcal{A}}\big(\mathcal{S},\mathcal{P}(n)\big) = 0$ for $i \ne 1$. The local-to-global spectral sequence implies that $\mathsf{Ext}^i_{\mathcal{A}}\big(\mathcal{S},\mathcal{P}(n)\big) = 0$ for all $i \in \mathbb{Z}$. Finally, it remains to note that $\mathcal{E}xt^i_{\mathcal{A}}\big(\mathcal{P}(n),\mathcal{S}\big) = 0$ for all $n \in \mathbb{Z}$ and $i \ge 0$, so the local-to-global spectral sequence implies again $\mathsf{Ext}^i_{\mathcal{A}}\big(\mathcal{P}(n),\mathcal{S}\big) = 0$. □

Let D be a triangulated category admitting all set-indexed coproducts. Recall that an object $X \in \mathsf{Ob}(\mathsf{D})$ is called *compact* if for an arbitrary family $\big\{Y_i\big\}_{i\in I}$ of objects of D the canonical map

$$\oplus_{i\in I}\,\mathsf{Hom}_{\mathsf{D}}(X,Y_i) \longrightarrow \mathsf{Hom}_{\mathsf{D}}(X,\oplus_{i\in I}Y_i)$$

is an isomorphism. An object X compactly generates D if it is compact and

$$X^{\perp} := \Big\{Y \in \mathsf{Ob}(\mathsf{D}) \,\Big|\, \mathsf{Hom}_{\mathsf{D}}(X,Y[n]) = 0\,\forall\, n \in \mathbb{Z}\Big\} = 0.$$

Recall the following result of Keller [22].

Theorem 5.7. *Let* D *be an algebraic triangulated category admitting all set-indexed coproducts and* X *be a compact generator of* D *such that* $\mathsf{Hom}_{\mathsf{D}}\big(X,X[n]\big) = 0$ *for all* $n \in \mathbb{Z}\setminus\{0\}$. *Let* $\Gamma = \mathsf{End}_{\mathsf{D}}(X)$ *and* $\mathsf{Mod}-\Gamma$ *be the category of* all *right* Γ*–modules. Then there exists an exact equivalence of triangulated categories* $\mathbb{T} : \mathsf{D} \longrightarrow D\big(\mathsf{Mod}-\Gamma\big)$ *such that for an arbitrary object* $Y \in \mathsf{Ob}(\mathsf{D})$ *we have:* $H^n\big(\mathbb{T}(Y)\big) = \mathsf{Hom}_{\mathsf{D}}(X,Y[n])$, *where* $\mathsf{Hom}_{\mathsf{D}}(X,Y[n])$ *is endowed with the natural structure of a right* $\Gamma = \mathsf{End}_{\mathsf{D}}(X)$*–module. Such an object* X *is called* tilting *and its endomorphism algebra* Γ *is the corresponding* tilted algebra.

In order to restrict the equivalence $\mathbb{T}$ on the derived category of Noetherian objects of a Grothendieck abelian category, we use the following result of Krause [26, Proposition 2.3].

Theorem 5.8. *Let* A *be a locally Noetherian Grothendieck category of* finite global dimension *and* N *be its full subcategory of Noetherian objects. Let* $\mathsf{D}_c(\mathsf{A})$ *be the category of compact objects of* $D(\mathsf{A})$. *Then the image of the canonical functor* $D^b(\mathsf{N}) \to D(\mathsf{A})$ *is equivalent to* $\mathsf{D}_c(\mathsf{A})$.

The following result was explained to the first-named author by Daniel Murfet.

Proposition 5.9. *Let A be a locally Noetherian Grothendieck category of finite global dimension, N be its full subcategory of Noetherian objects and $\mathsf{D} = D(\mathsf{A})$ be the derived category of A. Let $\mathbb{I} : D^-(\mathsf{N}) \to \mathsf{D}$ be the canonical functor. Then an object X of the category D belongs to the image of $\mathbb{I}$ if and only if for every family of objects $\{Y\}_{i\in I}$ of objects of D such that $\oplus_{i\in I} Y_i$ has bounded below cohomology, the canonical map*

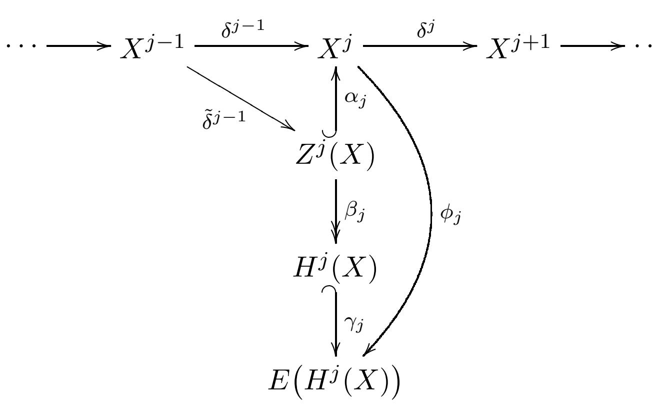
$$\bigoplus_{i\in I} \mathsf{Hom}_{\mathsf{D}}\big(X, Y_i\big) \longrightarrow \mathsf{Hom}_{\mathsf{D}}\big(X, \bigoplus_{i\in I} Y_i\big)$$

is an isomorphism.

Proof. We follow the main steps of the proof of [26, Lemma 4.1]. First check that any object X of the category $D^-(\mathsf{N})$ has the stated property. Let $\{Y_i\}_{i\in I}$ be a family of complexes from A with common lower bound for the non-vanishing cohomology. Then there exists $n \in \mathbb{Z}$ such that for all $j < n$ and $i \in I$ we have: $H^j(Y_i) = 0$. Let $\tau_{\geq n}(X)$ be the truncation of X. By the assumption, the complex $\tau_{\geq n}(X)$ has bounded Noetherian cohomology. Since A has finite global dimension, the complex $\tau_{\geq n}(X)$ is compact in D and we have canonical isomorphisms

$$\bigoplus_{i\in I} \mathsf{Hom}_{\mathsf{D}}\big(X, Y_i\big) \cong \bigoplus_{i\in I} \mathsf{Hom}_{\mathsf{D}}\big(\tau_{\geq n}(X), Y_i\big) \cong \mathsf{Hom}_{\mathsf{D}}\big(\tau_{\geq n}(X), \bigoplus_{i\in I} Y_i\big) \cong \mathsf{Hom}_{\mathsf{D}}(X, \bigoplus_{i\in I} Y_i).$$

Now, let X be an object of D such that the map $\oplus_{i\in I} \mathsf{Hom}_{\mathsf{D}}(X, Y_i) \longrightarrow \mathsf{Hom}_{\mathsf{D}}(X, \oplus_{i\in I} Y_i)$ is an isomorphism for an arbitrary family of objects with a common lower bound for the non-vanishing cohomology. First we check there exists $n \in \mathbb{Z}$ such that for all $m \geq n$ we have: $H^m(X) = 0$. Let $X = \big(\cdots \to X^{j-1} \xrightarrow{\delta^{j-1}} X^j \xrightarrow{\delta^j} X^{j+1} \to \dots\big)$ and assume $H^j(X) \neq 0$. Let $\mathsf{ker}(\delta^j) := \big(Z^j(X), \alpha_j\big)$, $Z^j(X) \to H^j(X)$ be the canonical epimorphism and $H^j(X) \xrightarrow{\gamma_j} E\big(H^j(X)\big)$ be the injective envelope of $H^j(X)$. Note that the composition $\gamma_j\beta_j$ is non-zero. Moreover, since $E\big(H^j(X)\big)$ is injective, there exists a morphism φ_j : $X^j \to E\big(H^j(X)\big)$ such that $\varphi_j\alpha_j = \gamma_j\beta_j$. Next, by the universal property of the kernel, there exists a morphism $\tilde{\delta}^{j-1} : X^{j-1} \to Z^j(X)$ such that $\alpha_j\tilde{\delta}^{j-} = \delta^{j-1}$.



Note that $\varphi_j\delta^{j-1} = \gamma_j\beta_j\tilde{\delta}^{j-1} = 0$. As a result, we get a morphism $X \to E\big(H^j(X)\big)[-j]$ inducing a non-zero map in cohomology. Hence, if X has unbounded cohomology to the right, the morphism $X \to \oplus_{j\in\mathbb{Z}_+} E\big(H^j(X)\big)[-j]$ can not factor through a finite set of indices in $\mathbb{Z}_+$. This implies that the canonical map

$$\bigoplus_{j\in\mathbb{Z}_+} \mathsf{Hom}_{\mathsf{D}}\Big(X, E\big(H^j(X)\big)[-j]\Big) \longrightarrow \mathsf{Hom}_{\mathsf{D}}\Big(X, \bigoplus_{j\in\mathbb{Z}_+} E\big(H^j(X)\big)[-j]\Big)$$

is not an isomorphism. Contradiction.

In remains to show that X has coherent cohomology. Let $\{E_i\}_{i\in I}$ be an arbitrary family of injective objects in A. Then we have:

$$\mathsf{Hom}_{\mathsf{D}}\big(X, \bigoplus_{i\in I} E_i[-n]\big) \cong \mathsf{Hom}_{\mathsf{A}}\big(H^n(X), \bigoplus_{i\in I} E_i\big) \cong \bigoplus_{i\in I} \mathsf{Hom}_{\mathsf{A}}\big(H^n(X), \bigoplus_{i\in I} E_i\big)$$

A result of Rentschler [35] allows to conclude that $H^n(X)$ is Noetherian. □

The following theorem is the main result of our article.

Theorem 5.10. *Let X be a reduced rational projective curve with nodal or cuspidal singularities, $\mathcal{A}$ be its Auslander sheaf of orders and $\mathcal{H}^\bullet = \mathcal{S}[-1]\oplus\mathcal{P}(-1)\oplus\mathcal{P}$ be the rigid complex from Proposition 5.6. Then $\mathcal{H}^\bullet$ is a tilting complex in the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$.*

Proof. By Proposition 5.6, the complex $\mathcal{H}^\bullet$ is rigid. In order to apply Theorem 5.7, we have to show that the right orthogonal of $\mathcal{H}^\bullet$ in the unbounded derived category $D\big(\mathsf{Qcoh}(\mathcal{A})\big)$ is zero. Let $\mathsf{C} = \mathsf{D}(\mathcal{H}^\bullet)$ be the smallest triangulated subcategory of $D\big(\mathsf{Qcoh}(\mathcal{A})\big)$ containing $\mathcal{H}^\bullet$. Our goal is to show that the right orthogonal of C inside of $D\big(\mathsf{Qcoh}(\mathcal{A})\big)$ is zero.

First observe that the Euler sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(m-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(m)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(m+1) \longrightarrow 0$$

in the category of coherent sheaves on $\mathbb{P}^1$ induces the short exact sequences

$$0 \longrightarrow \mathcal{P}_i(m-1) \longrightarrow \mathcal{P}_i(m)^{\oplus 2} \longrightarrow \mathcal{P}_i(m+1) \longrightarrow 0$$

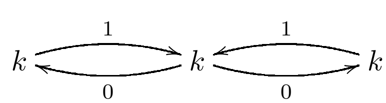
in the category $\mathsf{Coh}(\mathcal{A})$ for all $1 \le i \le n$ and $m \in \mathbb{Z}$. This implies that all locally projective $\mathcal{A}$-modules $\mathcal{P}_i(m)$ belong to the triangulated category C. In particular, the locally projective $\mathcal{A}$–module $\mathcal{P}(-2)$ belongs to the category C. The short exact sequence

$$0 \longrightarrow \mathcal{P}(-2) \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_t \longrightarrow 0$$

implies that the locally projective $\mathcal{A}$–module $\mathcal{F}$ belongs to C.

Consider the torsion $\mathcal{A}$–module $\mathcal{T}$, which is the cokernel of the canonical inclusion morphism $\mathcal{F} \to \mathcal{P}$. Since $\mathcal{F}$ and $\mathcal{P}$ belong to C, it follows that $\mathcal{T}$ is an object of C, too. Moreover, $\mathcal{T}$ is supported at the singular locus of X, hence $\mathcal{T} \cong \bigoplus_{j=1}^{t} \mathcal{T}_{p_j}$. Let T_j be the finite-length module over $\widehat{\mathcal{A}}_{p_j}$ corresponding to the torsion sheaf $\mathcal{T}_{p_j}$. We have:

- if p_j is a node then T_j is given by



- if p_j is a cusp then T_j is given by the quiver representation

$$\begin{pmatrix}0&1\\0&0\end{pmatrix} \circlearrowright k^2 \underset{\binom{1}{0}}{\overset{(0\,1)}{\rightleftarrows}} k$$

If the point p_j is nodal then for any $\lambda \in k^*$ we have a short exact sequence

$$0 \longrightarrow S_j \longrightarrow U_j(\lambda) \longrightarrow T_j \longrightarrow 0,$$

where $U_j(\lambda)$ is the module

$$k \underset{(1\,0)}{\overset{\binom{0}{1}}{\rightleftarrows}} k^2 \underset{(1\,0)}{\overset{\binom{0}{\lambda}}{\leftrightarrows}} k\ .$$

Similarly, for a cuspidal point p_j the module $U_j(\lambda)$ is given by the representation

$$\begin{pmatrix}0&1\\0&0\end{pmatrix} \circlearrowright k^2 \underset{\begin{pmatrix}\lambda&1\\0&0\end{pmatrix}}{\overset{\begin{pmatrix}0&0\\0&1\end{pmatrix}}{\rightleftarrows}} k^2$$

for some $\lambda \in k$. In particular, for any choice of $\lambda_1, \lambda_2, \dots, \lambda_t \in k^*$, the torsion sheaf $\mathcal{U}$ corresponding to the module $\bigoplus_{j=1}^{t} U_j(\lambda_j)$ belongs to the category C.

Let $\mathcal{L} \in \operatorname{Pic}(X)$ be a line bundle of multidegree $(1, 1, \dots, 1)$. Then for any $m \geq 1$ there exists an exact sequence

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}^{-m} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}^{-m+1} \longrightarrow \bigoplus_{j=1}^{t} \mathcal{U}_j(\lambda_j) \longrightarrow 0.$$

In particular, for all $m \geq 0$ the sheaf $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}^{-m}$ belongs to C. Hence, for any $m \geq 0$ the locally free $\mathcal{A}$–module $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}^{-m}$ belongs to C, too.

Let $\mathcal{G}$ be an arbitrary coherent $\mathcal{A}$–module. Then it is also a coherent $\mathcal{O}$-module. Since $\mathcal{L}$ is an ample line bundle on X, by a theorem of Serre there exists $m \geq 0$ such that the evaluation morphism $\mathsf{Hom}_X(\mathcal{L}^{-m}, \mathcal{G}) \overset{k}{\otimes} \mathcal{L}^{-m} \xrightarrow{\mathsf{ev}} \mathcal{G}$ is surjective. In particular, there exists $N \geq 0$ such that there exists an epimorphism of $\mathcal{O}$–modules $(\mathcal{L}^{-m})^N \to \mathcal{G}$. Since the functor $\mathcal{A} \otimes_{\mathcal{O}} -$ is right exact, we have epimorphisms $\mathcal{A} \otimes_{\mathcal{O}} (\mathcal{L}^{-m})^N \xrightarrow{1 \otimes \mathsf{ev}} \mathcal{A} \otimes_{\mathcal{O}} \mathcal{G} \xrightarrow{\mathsf{can}} \mathcal{G}$.

Let $\mathcal{G}^\bullet = \big(\cdots \to \mathcal{G}^{n-1} \xrightarrow{\delta^{n-1}} \mathcal{G}^n \xrightarrow{\delta^n} \mathcal{G}^{n+1} \to \dots\big)$ be a complex from the category $D\big(\mathsf{Qcoh}(\mathcal{A})\big)$. If $\mathcal{G}^\bullet \ncong 0$ then there exists $n \in \mathbb{Z}$ such that $\mathcal{H}^n(\mathcal{G}^\bullet) \neq 0$. Let $\mathcal{K}_n = \mathsf{ker}(\delta^n)$. Since any quasi-coherent $\mathcal{A}$–module is a direct limit of its coherent submodules, there exists a *coherent* submodule $\mathcal{N}_n$ of $\mathcal{K}_n$ such that it is not a subobject of the sheaf $\mathsf{im}(\delta^{n-1}) \subseteq \mathcal{K}_n$.

Consider the morphism $\big(\mathcal{A} \otimes \mathcal{L}^{-m}\big)^N[-n] \xrightarrow{f} \mathcal{G}^\bullet$ in the derived category $D\big(\mathsf{Qcoh}(\mathcal{A})\big)$ defined as the composition

$$\big(\mathcal{A} \otimes \mathcal{L}^{-m}\big)^N[-n] \longrightarrow \mathcal{N}_n[-n] \longrightarrow \mathcal{K}_n[-n] \longrightarrow \mathcal{G}^\bullet.$$

Since $\mathcal{H}^n(f) \neq 0$, the morphism f is non-zero in $D^b(\mathsf{Qcoh}(\mathcal{A}))$, too. This shows that $\mathsf{C}^\perp = 0$, hence $\mathcal{H}^\bullet$ is a tilting complex. □

Corollary 5.11. *Theorem 5.10, Theorem 5.7 and Proposition 5.9 imply that there exists an equivalence of triangulated categories* $\mathbb{T} : D(\mathsf{Qcoh}(X)) \longrightarrow D(\mathsf{Mod} - \Gamma_X)$ *inducing equivalences of triangulated categories*

$$D^-(\mathsf{Coh}(\mathcal{A})) \longrightarrow D^-(\mathsf{mod} - \Gamma_X) \text{ and } D^b(\mathsf{Coh}(\mathcal{A})) \longrightarrow D^b(\mathsf{mod} - \Gamma_X).$$

In particular, we have exact fully faithful functors

$$D^-(\mathsf{Coh}(X)) \longrightarrow D^-(\mathsf{mod} - \Gamma_X) \text{ and } \mathsf{Perf}(X) \longrightarrow D^b(\mathsf{mod} - \Gamma_X).$$

5.2. **Description of the tilted algebra.** Our next goal is to describe the tilted algebra $\mathsf{End}_{D^b(\mathcal{A})}(\mathcal{H}^\bullet)$ as the path algebra of some quiver with relations. Recall our notation. Let X be a rational projective curve with only nodes and cusps as singularities, $\pi : \widetilde{X} \to X$ its normalization, $\widetilde{X} = \bigcup_{i=1}^n \widetilde{X}_i$, where all $\widetilde{X}_i \cong \mathbb{P}^1$. Let

$$\mathsf{Sing}(X) = \{p_1, p_2, \dots, p_r, p_{r+1}, \dots p_{r+s}\}$$

be the singular locus of X, where $p_1, \dots, p_r$ are nodes and $p_{r+1}, \dots, p_{r+s}$ are cusps. Choose homogeneous coordinates $(u_i : v_i)$ on each irreducible component $\widetilde{X}_i$ and for any pair (i, j) such that $1 \le i \le n$ and $1 \le j \le r + s$ consider the set of points

$$\pi^{-1}(p_j) \cap \widetilde{X}_i = \{q_{ij}^{(k)} = (\alpha_{ij}^{(k)} : \beta_{ij}^{(k)}) \,|\, 1 \le k \le m_{ij}\},$$

where $m_{ij} = 0, 1$ or 2. We additionally assume the coordinates are chosen in such a way that $(\alpha_{ij}^{(k)} : \beta_{ij}^{(k)}) \neq (1 : 0)$ for all indices i, j, k such that p_j is a cusp.

Definition 5.12. The algebra Γ_X attached to a rational projective curve X with nodal or cuspidal singularities, is the path algebra of the following quiver with relations:

- It has $2n + r + s$ vertices: for each index $1 \le i \le n$ we have two points a_i and b_i and for each index $1 \le j \le r + s$ we have one point c_j.
- The arrows of Γ_X are as follows.
  - For any index $1 \le i \le n$ we have two arrows $u_i, v_i : a_i \to b_i$;
  - For any index $1 \le j \le r$ (nodal point) we have m_{ij} arrows $w_{ij}^{(k)} : c_j \to a_i$;
  - For any index $r + 1 \le j \le r + s$ (cuspidal point) and the unique index i with $m_{ij} = 1$ we have two arrows $w_{ij}, w'_{ij} : c_j \to a_i$
- The relations are as follows: for any $1 \le i \le n$, $1 \le j \le r$ (nodal point) and $1 \le k \le m_{ij}$ we have:

  $$\big(\beta_{ij}^{(k)} u_i - \alpha_{ij}^{(k)} v_i\big) w_{ij}^{(k)} = 0$$

  and for any $1 \le i \le n$, $r + 1 \le j \le r + s$, $m_{ij} = 1$ (cuspidal point) we have:

  $$\big(\beta_{ij}^{(1)} u_i - \alpha_{ij}^{(1)} v_i\big) w'_{ij} = 0, \quad \big(\beta_{ij}^{(1)} u_i - \alpha_{ij}^{(1)} v_i\big) w_{ij} = v_i w'_{ij}.$$

Remark 5.13. The algebra Γ_X is exactly the algebra defined in [15, Appendix A]. Since all paths in Γ_X have the length at most two, we have: $\mathsf{gl.dim}(\Gamma_X) = 2$.

Proposition 5.14. *The endomorphism algebra of the tilting complex* $\mathcal{H}^\bullet$ *from Theorem 5.10 is isomorphic to the algebra* Γ_X *introduced in Definition 5.12.*

Proof. A choice of homogeneous coordinates $(u_i : v_i)$ on $\widetilde{X}_i$ yields a pair of distinguished sections $z_i^0, z_i^\infty \in H^0\big(\mathcal{O}_{\widetilde{X}_i}(1)\big)$, where $z_i^0(0:1) = 0$ and $z_i^\infty(1:0) = 0$. They correspond to a pair of distinguished morphisms $u_i, v_i \in \mathsf{Hom}_{\mathcal{A}}\big(\mathcal{P}_i(-1), \mathcal{P}_i\big)$, $\quad 1 \le i \le n$ and form a basis of this morphism space. In the course of the proof of Proposition 5.6 we have seen that the only non-trivial contributions to $\mathsf{End}_{D^b(\mathsf{Coh}(\mathcal{A}))}(\mathcal{H}^\bullet)$ come from:

$$\mathsf{Hom}_{\mathcal{A}}\big(\widetilde{\mathcal{P}}_i(-1), \mathcal{P}_i\big) \cong \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}\big) = k^2$$

and

$$\mathsf{Ext}^1_{\mathcal{A}}\big(\mathcal{S}_j, \mathcal{P}_i(-1)\big) \cong H^0\Big(\mathcal{E}xt^1_{\mathcal{A}}\big(\mathcal{S}_j, \mathcal{P}_i(-1)\big)\Big) \cong k^2 \cong \mathsf{Ext}^1_{\mathcal{A}}\big(\mathcal{S}_j, \mathcal{P}_i\big).$$

Since the spaces $\mathsf{Ext}^1_{\mathcal{A}}\big(\mathcal{S}_j, \mathcal{P}_i(-1)\big)$ can be computed locally, we carry out calculations over the complete ring $\widehat{\mathcal{A}}_{p_j}$, following the notations of Remark 2.7.

<u>1-st case</u>. Assume $p_j \in \mathsf{Sing}(X)$ is nodal and its both preimages

$$\pi^{-1}(p_j) = \big\{q_{ij}^{(1)}, q_{ij}^{(2)}\big\} = \big\{(\alpha_{ij}^{(1)} : \beta_{ij}^{(1)}), (\alpha_{ij}^{(2)} : \beta_{ij}^{(2)})\big\}$$

belong to the same irreducible component $\widetilde{X_i}$. Recall that in this case, the algebra $\widehat{\mathcal{A}_{p_j}}$ is given by the completion of the following quiver with relations:

$$\overset{1}{\bullet} \underset{a_j^+}{\overset{a_j^-}{\rightleftarrows}} \overset{2}{\bullet} \underset{b_j^+}{\overset{b_j^-}{\leftrightarrows}} \overset{3}{\bullet} \qquad b_j^+ a_j^- = 0, \quad a_j^+ b_j^- = 0.$$

Moreover, we have: $\widehat{(\mathcal{P}_i)}_{p_j} \cong P_j^{(1)} \oplus P_j^{(3)}$, $\widehat{\mathcal{F}}_{p_j} \cong P_j^{(2)}$. Recall that the simple module $S_j = S_j^{(2)}$ has a projective resolution

$$0 \longrightarrow P_j^{(1)} \oplus P_j^{(3)} \xrightarrow{(a_j^+ b_j^+)} P_j^{(2)} \longrightarrow S_j \longrightarrow 0.$$

Hence, $\mathsf{Ext}^1_{\widehat{\mathcal{A}}_{p_j}}\big(S_j, P_j^{(1)} \oplus P_j^{(3)}\big) \cong k^2 = \big\langle w_{ij}^{(1)}, w_{ij}^{(2)}\big\rangle$, where $w_{ij}^{(1)}$ is given by the following morphism in the homotopy category:

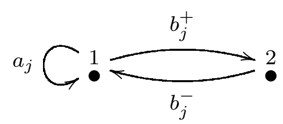
$$\begin{array}{ccccc} 0 & \longrightarrow & P_j^{(1)} \oplus P_j^{(3)} & \longrightarrow & P_j^{(2)} \\ & & \Big\downarrow {\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \\ 0 & \longrightarrow & P_j^{(1)} \oplus P_j^{(3)} & \longrightarrow & 0 \end{array}$$

and the morphism $w_{ij}^{(2)}$ is defined in a similar way. This implies that any morphism from $\mathcal{S}_j[-1]$ to $\mathcal{P}_i$ in the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ factors through $\mathcal{P}_i(-1)$. Next, note that the section $\beta_{ij}^{(k)} u_i - \alpha_{ij}^{(k)} v_i$ vanishes only at the point $q_{ij}^{(k)} = (\alpha_{ij}^{(k)} : \beta_{ij}^{(k)})$, $k = 1, 2$. Hence, we have the equalities:

$$(6) \qquad (\beta_{ij}^{(1)} u_i - \alpha_{ij}^{(1)} v_i) w_{ij}^{(1)} = 0, \quad (\beta_{ij}^{(2)} u_i - \alpha_{ij}^{(2)} v_i) w_{ij}^{(2)} = 0$$

in the morphism space $\mathsf{Hom}_{(\mathcal{A})}\big(\mathcal{S}_j[-1], \mathcal{P}_i\big)$. Moreover, the morphisms $(\beta_{ij}^{(1)} u_i - \alpha_{ij}^{(1)} v_i) w_{ij}^{(2)}$ and $(\beta_{ij}^{(2)} u_i - \alpha_{ij}^{(2)} v_i) w_{ij}^{(1)}$ are linearly independent. Since $\mathsf{Ext}^1_{\widehat{\mathcal{A}}_{p_j}}\big(S_j, P_j^{(1)} \oplus P_j^{(3)}\big) \cong k^2$, there are no other relations between $w_{ij}^{(k)}$, u_i and v_i but those described in (6).

2-nd case. Assume the point p_j is cuspidal and $\pi^{-1}(p_j) = q_{ij} = (\alpha_{ij} : \beta_{ij}) \in \widetilde{X}_i$. In this case, $\widehat{\mathcal{A}}_{p_j}$ is isomorphic to the completion of the path algebra



$$a_j^2 = b_j^- b_j^+.$$

We have: $\big(\mathcal{P}_i(-1)\big)_{p_j} \cong (\mathcal{P}_i)_{p_j} \cong P_j^{(1)}$ and $0 \to P_j^{(1)} \xrightarrow{b_j^+} P_j^{(2)} \to S_j \to 0$ is a projective resolution of the *rigid* simple module S_j. Hence, $\mathsf{Ext}^1_{\mathcal{A}}(\mathcal{S}_j, \mathcal{P}_i) \cong H^0\big(\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{S}_j, \mathcal{P}_i)\big) = k^2$. Since the homogeneous coordinates $(u_i : v_i)$ are chosen in such a way that $q_{ij} \neq (1 : 0)$, the morphisms v_i and $\beta_{ij} u_i - \alpha_{ij} v_i : \mathcal{P}_i(-1) \to \mathcal{P}_i$ are linearly independent. Let $w'_{ij} := (\beta_{ij} u_i - \alpha_{ij} v_i)_{p_j}$ and $w_{ij} := (v_i)_{p_j}$ be the induced $\widehat{\mathcal{A}}_{p_j}$–linear morphisms of the projective module $P_j^{(1)}$. Denote by the same letters the induced morphisms of complexes from $\big(P_j^{(1)} \xrightarrow{b_j^+} P_j^{(2)}\big)[-1]$ to $P_j^{(1)}$ in the homotopy category of projective $\widehat{\mathcal{A}}_{p_j}$–modules. Then w_{ij} and w'_{ij} form a basis of $\mathsf{Ext}^1_{\mathcal{A}}\big(\mathcal{S}_j, \mathcal{P}_i(-1)\big) = k^2$ and we obtain the relations

$$\begin{cases} (\beta_{ij} u_i - \alpha_{ij} v_i) w'_{ij} & = & 0 \\ (\beta_{ij} u_i - \alpha_{ij} v_i) w_{ij} & = & v_i w'_{ij} \end{cases}$$

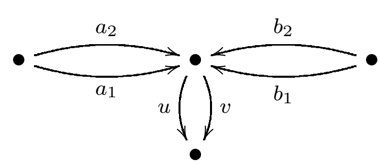
in the morphism space $\mathsf{Ext}^1_{\mathcal{A}}(\mathcal{S}_j, \mathcal{P}_i)$. As in the Case 1, it follows that there are no other relations between u_i, v_i, w_{ij} and w'_{ij}.

3-rd case. The case when p_j is nodal and its preimages belong to different components of $\widetilde{X}$ is completely similar to the first case and is therefore left to the reader. □

Example 5.15. Let X be an irreducible nodal rational projective curve of arithmetic genus two, p_1 and p_2 its singular points, $\mathbb{P}^1 \xrightarrow{\pi} X$ its normalization. Assume that coordinates on $\mathbb{P}^1$ are chosen in such a way that

$$\pi^{-1}(p_1) = \big\{0 = (0:1), \infty = (1:0)\big\} \quad \text{and} \quad \pi^{-1}(p_2) = \big\{(1:1), (\lambda:1)\big\}$$

with $\lambda \in k \setminus \{0, 1\}$. Then the algebra Γ_X is the path algebra of the following quiver



subject to the relations $ua_1 = 0$, $va_2 = 0$, $(u - v)b_1 = 0$ and $(u - \lambda v)b_2 = 0$. It seems to be an interesting problem to study compactified moduli spaces of vector bundles on X in terms of representations of the algebra Γ_X.

5.3. **Dimension of the derived category of a rational projective curve.** As a consequence of our approach, we obtain an upper bound of the dimension of the derived category of coherent sheaves of reduced rational projective curve with only nodal or cuspidal singularities.

Let C be an idempotent complete triangulated subcategory and A, B be its two idempotent complete full subcategories closed under shifts. Following Rouquier [38], we denote by $\mathsf{A} * \mathsf{B}$ the full subcategory of C, whose objects are those objects X of C for which there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \mathsf{Ob}(\mathsf{A})$ and $B \in \mathsf{Ob}(\mathsf{B})$. For an object $X \in \mathsf{Ob}(\mathsf{C})$ we denote by $\langle X \rangle$ the smallest full subcategory of C, closed under taking shifts, direct sums and direct summands. Next, for any positive integer n we define subcategories $\langle X \rangle_n$ by the following rule:

$$\langle X \rangle_1 = \langle X \rangle, \quad \langle X \rangle_{n+1} = \big\langle \langle X \rangle_1 * \langle X \rangle_n \big\rangle.$$

An object $X \in \mathsf{Ob}(\mathsf{C})$ is a *strong generator* if $\langle X \rangle_n = \mathsf{C}$ for some positive integer n. Rouquier suggested the following definition of the dimension of a triangulated category C:

$$\mathsf{dim}(\mathsf{C}) = \inf \Big\{ n \in \mathbb{Z}_+ \,\Big|\, \exists\, X \in \mathsf{Ob}(\mathsf{C}) : \langle X \rangle_{n+1} = \mathsf{C} \Big\}.$$

He has also proven that the dimension of the derived category of coherent sheaves of a separated scheme X of finite type over a perfect field k is always finite, see [38, Theorem 7.38]. Moreover, if X is smooth of dimension n then $n \le \mathsf{dim}\big(D^b\big(\mathsf{Coh}(X)\big)\big) \le 2n$, see [38, Proposition 7.9 and Proposition 7.16]. By a recent result of Orlov [30], for a *smooth* projective curve X over a field k we have:

$$\mathsf{dim}\big(D^b\big(\mathsf{Coh}(X)\big)\big) = 1.$$

The case of the singular projective curves still remains open. However, our technique allows to deduce the following result.

Theorem 5.16. *Let X be a reduced rational projective curve with only nodal or cuspidal singularities. Let $S = \{p_1, p_2, \dots, p_t\}$ be the singular locus of X and $X_1, X_2, \dots, X_n$ be the irreducible components of X. Let $\mathcal{O}_i = \mathcal{O}_{X_i}$ be the structure sheaf of X_i, $1 \le i \le n$ and $\mathcal{O}_i(-1) = \mathcal{O}_i \otimes \mathcal{O}_X(-q_i)$, where $q_i \in X_i$ is a smooth point. Consider the coherent sheaf*

$$\mathcal{G} = \bigoplus_{i=1}^{n} \Big(\mathcal{O}_i(-1) \oplus \mathcal{O}_i \Big) \oplus \bigoplus_{j=1}^{t} k_{p_j}.$$

Then we have: $\langle \mathcal{G} \rangle_3 = D^b\big(\mathsf{Coh}(X)\big)$. In particular, $\mathsf{dim}\big(D^b\big(\mathsf{Coh}(X)\big)\big) \le 2$.

Proof. Let $\mathcal{A}$ be the Auslander sheaf of X. By Theorem 5.10, the derived category $D^b\big(\mathsf{Coh}(\mathcal{A})\big)$ is equivalent to $D^b(\mathsf{mod} - \Gamma_X)$. Moreover, $\mathsf{gl.dim}(\Gamma_X) = 2$ and the equivalence $\mathbb{T}$ maps the tilting complex $\mathcal{H}^\bullet$ to the regular module Γ_X. By [38, Lemma 7.1] it is known that $\langle \Gamma_X \rangle_3 = D^b(\mathsf{mod} - \Gamma_X)$. This implies that $\langle \mathcal{H}^\bullet \rangle_3 = D^b\big(\mathsf{Coh}(\mathcal{A})\big)$.

Consider now the exact functor $\mathbb{G} : \mathsf{Coh}(\mathcal{A}) \to \mathsf{Coh}(X)$. By Theorem 2.6, the derived functor $\mathbb{G} : D^b\big(\mathsf{Coh}(\mathcal{A})\big) \to D^b\big(\mathsf{Coh}(X)\big)$ is essentially surjective, hence $\big\langle \mathbb{G}(\mathcal{H}^\bullet) \big\rangle_3 =$

$D^b(\mathsf{Coh}(X))$. To conclude the proof, it remains to note that

$$\mathbb{G}(\mathcal{H}^\bullet) \cong \bigoplus_{i=1}^{n} \Big(\mathcal{O}_i(-1) \oplus \mathcal{O}_i\Big) \oplus \bigoplus_{j=1}^{t} k_{p_j}[-1].$$

□

6. Coherent sheaves on Kodaira cycles and gentle algebras

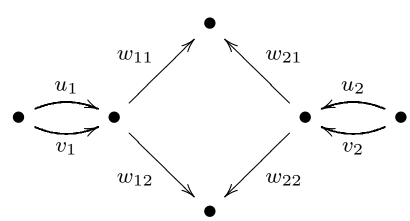
In this section we discuss some corollaries from the results obtained in the previous section, in the case of Kodaira cycles of projective lines. To deal with *left modules*, we prefer to replace the tilted algebra Γ_X by its opposite $\Lambda_X = \Gamma_X^{\mathrm{op}}$.

Proposition 6.1. *Let $E = E_n$ be a Kodaira cycle of n projective lines (in the case $n = 1$ it is an irreducible plane nodal cubic curve), $\mathcal{A}$ be the Auslander sheaf and $\Lambda = \Lambda_E$ be the opposite algebra of the corresponding tilted algebra. Then we have:*

1. *The algebra Λ_E is gentle, see* [1].
2. *The categories* $\mathsf{Perf}(E)$ *and* $\mathsf{Coh}(E)$ *are tame in the "pragmatic sense"*[2].

Proof. The fact that the algebra Λ is gentle, follows from Proposition 5.14 and the definition of the gentle algebras. Moreover, the gentle algebras are derived-tame [31, 37]. Since we have fully faithful functors $\mathsf{Perf}(E) \to D^b(\mathsf{Coh}(\mathcal{A})) \xrightarrow{\sim} D^b(\Lambda - \mathsf{mod})$, and $\mathsf{Coh}(X) \to \mathsf{Coh}(\mathcal{A}) \to D^b(\mathsf{Coh}(\mathcal{A})) \xrightarrow{\sim} D^b(\Lambda - \mathsf{mod})$, the categories $\mathsf{Perf}(E)$ and $\mathsf{Coh}(E)$ are equivalent to full subcategories of a representation-tame category $D^b(\Lambda - \mathsf{mod})$. This precisely means they are pragmatic-tame. □

Example 6.2. Let $E = E_2$ be a Kodaira cycle of two projective lines. Then the algebra Λ_E is the path algebra of the following quiver



subject to the relations: $w_{11}v_1 = 0$, $w_{22}v_2 = 0$, $w_{12}u_1 = 0$ and $w_{21}u_2 = 0$. In particular, there exists a fully faithful functor $\mathsf{Perf}(E) \to D^b(\Lambda_E - \mathsf{mod})$.

Let Λ be a finite-dimensional algebra over a field k. Then the Nakayama functor

$$\nu := \mathbb{D}\,\mathsf{Hom}_\Lambda(\,-\,,\Lambda) : \Lambda - \mathsf{mod} \longrightarrow \Lambda - \mathsf{mod}$$

is right exact. Moreover, if $\Lambda = k\vec{Q}/\rho$ is the path algebra of a finite quiver with relations then $\nu(P_i) = I_i$, where P_i and I_i are the indecomposable projective and injective modules corresponding to the vertex $i \in Q_0$. If $\mathsf{gl.dim}(\Lambda) < \infty$ then by a result of Happel [19], the derived functor

$$\mathbb{S} := \mathbb{L}\nu : D^b(\Lambda - \mathsf{mod}) \longrightarrow D^b(\Lambda - \mathsf{mod})$$

[2] A more precise result implying the tameness in a "strict sense" was obtained in [13].

is the Serre functor of the category $D^b(\Lambda - \mathsf{mod})$ and the functor $\tau_\Lambda = \mathbb{S}[-1]$ is the Auslander-Reiten translate in $D^b(\Lambda - \mathsf{mod})$ (see also [34]).

Corollary 6.3. *Let E be a Kodaira cycle of projective lines, $\mathcal{A}$ be its Auslander sheaf and Λ be the opposite algebra of the corresponding tilted algebra. Consider the category $\mathsf{Band}(\Lambda)$, which is the full subcategory of $D^b(\Lambda - \mathsf{mod})$ whose objects are the complexes $P^\bullet$ such that $\tau_\Lambda(P^\bullet) \cong P^\bullet$. Then $\mathsf{Band}(\Lambda)$ is triangulated and idempotent complete. Moreover, it is triangle equivalent to the category of perfect complexes $\mathsf{Perf}(E)$.*

Proof. Since $\mathbb{T} : D^b\bigl(\mathsf{Coh}(\mathcal{A})\bigr) \to D^b(\Lambda - \mathsf{mod})$ is an equivalence of categories, we have an isomorphism of functors $\mathbb{T} \circ \tau_{\mathcal{A}} \cong \tau_\Lambda \circ \mathbb{T}$. This implies that the category $\mathsf{Band}(\Lambda)$ is equivalent to the category D_θ introduced in Corollary 3.2. By Theorem 3.4 and Corollary 3.5 we get that $\mathsf{Band}(\Lambda)$ is equivalent to $\mathsf{Perf}(E)$. In particular, it is idempotent-complete. □

Remark 6.4. Corollary 6.3 implies the following result on the shape of the Auslander–Reiten quiver of the gentle algebra Λ, attached to a cycle of projective lines E. Let $P^\bullet$ be an indecomposable object of $D^b(\Lambda - \mathsf{mod})$ such that $\tau_\Lambda^n(P^\bullet) \cong P^\bullet$. Then $n = 1$ and $P^\bullet$ is an object of $\mathsf{Band}(\Lambda)$. In other words, the Auslander-Reiten quiver of $D^b(\Lambda - \mathsf{mod})$ does not contain tubes of length bigger than one.

Note that all indecomposable *band complexes* of $D^b(\Lambda - \mathsf{mod})$ in the sense of the work of Butler and Ringel [9] (see also [37]) are τ–periodic. However, their definition of bands and strings of the derived category of a gentle algebra is purely combinatorial. In particular, in certain cases algebra automorphisms map band complexes to string complexes. Moreover, it is not completely clear that their notion of bands and strings coincide with the corresponding notion of bands and strings used in our previous paper [12].

In the particular situation of a gentle algebra Λ which is the tilted algebra attached to a Kodaira cycle of projective lines, we say that an indecomposable object of $D^b(\Lambda - \mathsf{mod})$ is a *band* if and only if it belongs to $\mathsf{Band}(\Lambda)$. The indecomposable objects of $D^b(\Lambda - \mathsf{mod})$ which are not bands will be called *strings*. The description of indecomposable objects in $D^b(\Lambda - \mathsf{mod})$ [31, 37, 12] implies that strings do not have continuous moduli and are classified by discrete parameters. The same concerns the indecomposable objects of the derived category $D^b\bigl(\mathsf{Coh}(E)\bigr)$ which do not belong to $\mathsf{Perf}(E)$, see [13].

The interplay between various categories occurring in our construction can be explained by the following diagram:

$$\begin{array}{ccccc}
D^-\bigl(\mathsf{Coh}(E)\bigr) & \xrightarrow{\mathbb{F}} & D^-\bigl(\mathsf{Coh}(\mathcal{A}_E)\bigr) & \xrightarrow{\mathbb{T}} & D^-\bigl(\Lambda_E - \mathsf{mod}\bigr) \\
\uparrow & \nearrow & \uparrow & & \uparrow \\
D^b\bigl(\mathsf{Coh}(E)\bigr) & & D^b\bigl(\mathsf{Coh}(\mathcal{A}_E)\bigr) & \xrightarrow{\mathbb{T}} & D^b\bigl(\Lambda_E - \mathsf{mod}\bigr) \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{Perf}(E) & \xrightarrow{\sim} & \mathsf{D}_\theta & \xrightarrow{\sim} & \mathsf{Band}(\Gamma_E).
\end{array}$$

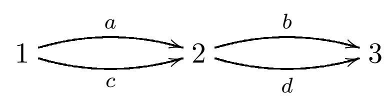
Summing up, we get that the triangulated category $\mathsf{Perf}(E)$ is a full subcategory of two different derived categories. From one side, it is a subcategory of the derived category of coherent sheaves $\mathsf{Coh}(E)$, whose global dimension is *infinity*. From another side, it is a subcategory of the derived category of representations of the algebra Γ_E, whose global dimension is *two*. The "complement" of $\mathsf{Perf}(E)$ in both categories is "small": it consists of direct sums of complexes described by discrete parameters.

7. Tilting exercises on a Weierstrass nodal cubic curve

Let $E \subseteq \mathbb{P}^2$ be a singular Weierstraß cubic curve over an algebraically closed field k given by the equation $zy^2 = x^3 + x^2z$. By Corollary 5.11, there exists a fully faithful functor

$$\mathsf{Perf}(E) \xrightarrow{\mathbb{F}} D^b\big(\mathsf{Coh}(\mathcal{A})\big) \xrightarrow{\mathbb{T}} D^b\big(\Lambda - \mathsf{mod}\big),$$

where $\mathcal{A} = \mathcal{A}_E$ is the Auslander sheaf of orders and $\Lambda = \Lambda_E$ is the path algebra of the following quiver with relations:



$$ba = 0 = dc.$$

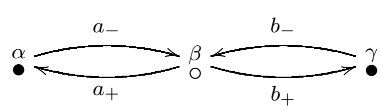
The algebra Λ is interesting from various perspectives. First of all, it is gentle, hence derived-tame. Next, by a work of Seidel [40, Section 3], it is related with the directed Fukaya category of a certain Lefschetz pencil.

Our next goal is to compute the complexes in $D^b\big(\Lambda - \mathsf{mod}\big)$ corresponding to the images of certain perfect coherent sheaves on E under the functor $\mathbb{T} \circ \mathbb{F}$.

Let $\pi : \mathbb{P}^1 \to E$ be the normalization of E and $s = (0 : 0 : 1) \in E$ be the singular point. Choose coordinates on $\mathbb{P}^1$ in such a way that $\pi^{-1}(s) = \{0, \infty\}$, where $0 = (0 : 1)$ and $\infty = (1 : 0)$. This choice yields two distinguished sections $z_0, z_\infty \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}\big) = H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$ such that $z_0(0) = 0$ and $z_\infty(\infty) = 0$. Recall that

$$A := \widehat{\mathcal{A}}_s \cong \begin{pmatrix} k[\![u]\!] \times k[\![v]\!] & (u, v) \\ k[\![u]\!] \times k[\![v]\!] & k[\![u, v]\!]/(uv) \end{pmatrix}$$

is isomorphic to the radical completion of the following quiver with relations:



$$b_+a_- = 0, \quad a_+b_- = 0.$$

Let

$$P_\alpha = \begin{pmatrix} k[\![u]\!] \\ k[\![u]\!] \end{pmatrix}, \; P_\gamma = \begin{pmatrix} k[\![v]\!] \\ k[\![v]\!] \end{pmatrix} \text{ and } P_\beta = F = \begin{pmatrix} (u, v) \\ k[\![u, v]\!]/(uv) \end{pmatrix}$$

be the indecomposable projective A–modules. We distinguish two locally projective $\mathcal{A}$–modules

$$\mathcal{P} = \begin{pmatrix} \widetilde{\mathcal{O}} \\ \widetilde{\mathcal{O}} \end{pmatrix} \text{ and } \mathcal{F} = \begin{pmatrix} \mathcal{I} \\ \widetilde{\mathcal{O}} \end{pmatrix}$$

and for any $n \in \mathbb{Z}$ we have: $\widehat{\mathcal{P}(n)_s} \cong P_\alpha \oplus P_\gamma$, whereas $\widehat{\mathcal{F}_s} \cong F$. By Lemma 5.3, there are the following canonical isomorphisms:

$$H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \cong H^0\Big(\mathcal{S}\big(\mathcal{P}(-1), \mathcal{P}\big)\Big) \cong \mathsf{Hom}_{\mathcal{A}}\big(\mathcal{P}(-1), \mathcal{P}\big).$$

Hence, we get two distinguished elements in $\mathsf{Hom}_{\mathcal{A}}\big(\mathcal{P}(-1),\mathcal{P}\big)$, which will be denoted by z_0 and z_∞. They are characterized by the property that there exists isomorphisms $t_1 : \widehat{\mathcal{P}(-1)}_s \to P_\alpha \oplus P_\gamma$ and $t_2 : \widehat{\mathcal{P}_s} \to P_\alpha \oplus P_\gamma$ making the following diagrams commutative:

$$\begin{array}{ccc} \widehat{\mathcal{P}(-1)}_s & \xrightarrow{z_0} & \widehat{\mathcal{P}_s} \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle t_2} \\ P_\alpha \oplus P_\gamma & \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & \mathsf{id} \end{smallmatrix}\right)} & P_\alpha \oplus P_\gamma \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{P}(-1)}_s & \xrightarrow{z_\infty} & \widehat{\mathcal{P}_s} \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle t_2} \\ P_\alpha \oplus P_\gamma & \xrightarrow{\left(\begin{smallmatrix} \mathsf{id} & 0 \\ 0 & y \end{smallmatrix}\right)} & P_\alpha \oplus P_\gamma. \end{array}$$

Let $\mathcal{S}_\beta$ be the torsion $\mathcal{A}$–module supported at the singular point $s \in E$ and corresponding to the simple A–module S_β. Then for any $n \in \mathbb{Z}$ canonical map

$$H^0\Big(\mathcal{E}xt^1_{\mathcal{A}}\big(\mathcal{S}_\beta, \mathcal{P}(n)\big)\Big) \longrightarrow \mathsf{Ext}^1_{\mathcal{A}}\big(\mathcal{S}_\beta, \mathcal{P}(n)\big)$$

is an isomorphism. By Remark 2.7, the simple module S_β has the following projective resolution:

$$0 \longrightarrow P_\alpha \oplus P_\gamma \xrightarrow{(\,u,v\,)} P_\beta \longrightarrow S_\beta \longrightarrow 0.$$

Hence, $\mathsf{Ext}^1_A(S_\beta, P_\alpha \oplus P_\gamma) = k^2 = \langle \xi, \eta \rangle$, where ξ and η are induced by the A–linear morphisms given by the matrices

$$\xi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \eta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : \ P_\alpha \oplus P_\gamma \longrightarrow P_\alpha \oplus P_\gamma.$$

In particular, $z_0\xi = 0$ and $z_\infty\eta = 0$. By the definition of the tilting equivalence $\mathbb{T} : D^b\big(\mathsf{Coh}(\mathcal{A})\big) \to D^b\big(\Lambda - \mathsf{mod}\big)$ given by the tilting complex $\mathcal{H}^\bullet = \mathcal{S}_\beta[-1] \oplus \mathcal{P}(-1) \oplus \mathcal{P}$, we have the following result.

Lemma 7.1. *We have:* $\mathbb{T}(\mathcal{P}) \cong P_1, \mathbb{T}\big(\mathcal{P}(-1)\big) = P_2$ *and* $\mathbb{T}(\mathcal{S}_\beta) \cong P_3[1]$, *where* P_i *is the indecomposable projective* Λ*–module corresponding to the vertex* i, $1 \le i \le 3$.

Our next goal is to compute the images of certain finite length objects in $\mathsf{Coh}(\mathcal{A})$. Let $x = (\lambda : \mu) \in \mathbb{P}^1$ be an arbitrary point. Consider the torsion $\mathcal{A}$–module $\mathcal{T}_x$ given by its locally projective resolution

$$0 \longrightarrow \mathcal{P}(-1) \xrightarrow{\mu z_0 - \lambda z_1} \mathcal{P} \longrightarrow \mathcal{T}_x \longrightarrow 0.$$

Note that $\mathcal{T}_x$ is supported at the point $\pi(\lambda : \mu) \in E$. In particular, if $x \notin \{0, \infty\}$ then $\mathcal{T}_x$ is supported at a smooth point of E. For $x \in \{0, \infty\}$ the sheaf $\mathcal{T}_x$ corresponds to the A–modules

$$T_0 = \ k \underset{0}{\overset{1}{\rightleftarrows}} k \rightleftarrows 0 \quad \text{and } T_\infty = \ 0 \rightleftarrows k \underset{0}{\overset{1}{\leftrightarrows}} k\ .$$

Since $\mathsf{Ext}^i_{\mathcal{A}}\big(\mathcal{S}_\beta, \mathcal{P}(n)\big) = 0$ for all $n \in \mathbb{Z}$ and $i \in \mathbb{Z}_{\ge 0}$, we also have the vanishing $\mathsf{Ext}^i_{\mathcal{A}}(\mathcal{S}_\beta, \mathcal{T}_x) = 0$ for all $i \ge 0$.

Next, note that $\mathsf{Hom}_{\mathcal{A}}\big(\mathcal{P}(n), \mathcal{T}_x\big) = k$ and $\mathsf{Ext}^i_{\mathcal{A}}\big(\mathcal{P}(n), \mathcal{T}_x\big) = 0$ for all $i > 0$ and $n \in \mathbb{Z}$. This implies that $H^i\big(\mathbb{T}(\mathcal{T}_x)\big) = 0$ for $i \ne 0$. In particular, the complex $\mathbb{T}(\mathcal{T}_x)$ is isomorphic in $D^b\big(\Lambda - \mathsf{mod}\big)$ to the stalk complex

$$\ldots \longrightarrow 0 \longrightarrow N_x \longrightarrow 0 \longrightarrow \ldots$$

where $N_x = H^0\big(\mathbb{T}(\mathcal{T}_x)\big)$.

Recall that for an arbitrary representation M of the quiver Λ we have: $\mathsf{Hom}_\Lambda(P_i, M) = M(i)$, where $M(i)$ is the dimension of M at the vertex i, $1 \le i \le 3$. This allows to compute the multi-dimension of the zero cohomology of $\mathbb{T}(\mathcal{T}_x)$: $\underline{\mathsf{dim}}(N_x) = \big(1,\,1,\,0\big)$. Moreover, the right Γ–module N_x endowed with the natural structure of an $\mathsf{End}_{D^b(\mathcal{A})}\big(\mathcal{H}^\bullet\big)$–module has the following projective resolution:

$$0 \longrightarrow \mathsf{Hom}_{D^b(\mathcal{A})}\big(\mathcal{H}^\bullet, \mathcal{P}(-1)\big) \xrightarrow{(\mu z_0 - \lambda z_\infty)_*} \mathsf{Hom}_{D^b(\mathcal{A})}\big(\mathcal{H}^\bullet, \mathcal{P}\big) \longrightarrow N_x \longrightarrow 0.$$

Interpreting it in the terms of quiver representations, we obtain the following result.

Proposition 7.2. *For any $x = (\lambda : \mu) \in \mathbb{P}^1$ we have: $\mathbb{T}(\mathcal{T}_x) = N_x[0]$, where N_x is the following representation of the algebra Λ:*

$$k \underset{\mu}{\overset{\lambda}{\rightrightarrows}} k \rightrightarrows 0\ .$$

As the next step, we compute the images under $\mathbb{T}$ of two other exceptional simple $\mathcal{A}$–modules.

Proposition 7.3. *Let S_α and S_γ be the simple A–modules corresponding to the vertices α and γ, and $\mathcal{S}_\alpha$ and $\mathcal{S}_\gamma$ be the corresponding torsion $\mathcal{A}$–modules. Then we have:*

$$\mathbb{T}(\mathcal{S}_\alpha) = \; k \underset{0}{\overset{1}{\rightrightarrows}} k \underset{1}{\overset{0}{\rightrightarrows}} k \quad \text{and} \quad \mathbb{T}(\mathcal{S}_\gamma) = \; k \underset{1}{\overset{0}{\rightrightarrows}} k \underset{0}{\overset{1}{\rightrightarrows}} k.$$

Proof. First note that

$$\mathsf{Ext}^i_{\mathcal{A}}(\mathcal{S}_\beta, \mathcal{S}_\alpha) = H^0\big(\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{S}_\beta, \mathcal{S}_\alpha)\big) = \begin{cases} k & \text{if} \quad i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In a similar way, for any $n \in \mathbb{Z}$ we have:

$$\mathsf{Ext}^i_{\mathcal{A}}\big(\mathcal{P}(n), \mathcal{S}_\alpha\big) \cong H^0\Big(\mathcal{E}xt^i_{\mathcal{A}}\big(\mathcal{P}(n), \mathcal{S}_\alpha\big)\Big) = \begin{cases} k & \text{if} \quad i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $\mathbb{T}(\mathcal{S}_\alpha)$ and $\mathbb{T}(\mathcal{S}_\gamma)$ are indecomposable stalk complexes. Their zero cohomology are Λ–modules M_α and M_γ, whose multi-dimension is the vector $(1, 1, 1)$. However, there are precisely two *indecomposable* Λ–modules with this multi-dimension. Since $\mathsf{Hom}_{\mathcal{A}}(\mathcal{T}_\infty, \mathcal{S}_\alpha) = 0 = \mathsf{Hom}_{\mathcal{A}}(\mathcal{T}_0, \mathcal{S}_\gamma)$ and $\mathbb{T}$ is an equivalence of categories, we have: $\mathsf{Hom}_\Lambda(N_\infty, M_\alpha) = 0 = \mathsf{Hom}_\Lambda(N_0, M_\gamma)$. This implies that M_α and M_γ are given by the quiver representations as stated above. □

Finally, we shall compute the image of the Jacobian $\mathrm{Pic}^0(E)$ in the derived category $D^b(\Lambda - \mathsf{mod})$.

Proposition 7.4. *The functor $\mathbb{T}\circ\mathbb{F}$ identifies the Jacobian $\mathrm{Pic}^0(E) \cong k^*$ with the following family of complexes in the derived category $D^b(\Lambda - \mathsf{mod})$*

$$\Big\{U^\bullet_\lambda\Big\}_{\lambda\in k^*} = \left\{ P_3 \xrightarrow{\binom{\lambda b}{d}} \underline{P_2 \oplus P_2} \xrightarrow{(a,c)} P_1 \right\}_{\lambda \in k^*},$$

where the underlined term of $U^\bullet_\lambda$ has degree zero. Moreover, for any $\lambda \in k^*$ the complex $U^\bullet_\lambda$ is spherical in the sense of Seidel and Thomas [39] *and we have:* $H^0(U^\bullet_\lambda) = S_3$, $H^1(U^\bullet_\lambda) = S_1$

Proof. Since $\mathbb{T} : D^b\big(\mathsf{Coh}(\mathcal{A})\big) \to D^b(\Lambda - \mathsf{mod})$ is an equivalence of categories, we have an isomorphism of functors $\mathbb{T} \circ \tau_{\mathcal{A}} \cong \tau_\Lambda \circ \mathbb{T}$. Next, we know that $\mathbb{T}(\mathcal{P}) = P_1$. This implies that

$$\mathcal{R} := \tau_{\mathcal{A}}(\mathcal{P}) = \begin{pmatrix} \mathcal{I} \\ \widetilde{\mathcal{O}} \end{pmatrix} \text{ and } \mathbb{T}(\mathcal{R}) = \tau_\Lambda(P_1) = S_1[-1],$$

where S_1 is the simple Λ–module corresponding to the vertex 1 (note that S_1 is injective). Consider the torsion $\mathcal{A}$–module $\mathcal{T}$ supported at the singular point $s \in E$ and defined as

$$\mathcal{T} = \begin{pmatrix} 0 \\ \widetilde{\mathcal{O}}/\mathcal{I} \end{pmatrix} = \mathsf{coker}\left[\begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{I} \\ \widetilde{\mathcal{O}} \end{pmatrix}\right].$$

Since $\mathsf{Hom}_{\mathcal{A}}\big(\mathcal{P}(n), \mathcal{S}_\beta\big) = 0$ for all $n \in \mathbb{Z}$, the canonical morphism

$$\mathsf{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{S}_\beta) \longrightarrow \mathsf{Hom}_{\mathcal{A}}(\mathcal{R}, \mathcal{S}_\beta)$$

is an isomorphism. Another canonical morphism $\mathsf{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{S}_\beta) \longrightarrow \mathsf{Hom}_E(\widetilde{\mathcal{O}}/\mathcal{I}, \mathcal{O}/\mathcal{I})$ is an isomorphism as well, hence $\mathsf{Hom}_{\mathcal{A}}(\mathcal{R}, \mathcal{S}_\beta) = k^2$. Since $(\widetilde{\mathcal{O}}/\mathcal{I})_s \cong k_s \times k_s$ and $(\mathcal{O}/\mathcal{I})_s = k_s$ are rings, the vector space $\mathsf{Hom}_E(\widetilde{\mathcal{O}}/\mathcal{I}, k_s)$ has two *distinguished* basis elements $\bar{w}_0$ and $\bar{w}_\infty$, which correspond to the non-trivial idempotents of the ring $(\widetilde{\mathcal{O}}/\mathcal{I})_s$. Let w_0 and w_∞ be the corresponding elements of $\mathsf{Hom}_{\mathcal{A}}(\mathcal{R}, \mathcal{S}_\beta)$. For any $(\lambda, \mu) \in k^2 \setminus \{0, 0\}$ consider the short exact sequence

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{R} \xrightarrow{w} \mathcal{T} \longrightarrow 0,$$

where $w = \mu w_0 - \lambda w_\infty$. The $\mathcal{A}$–module $\mathcal{X}$ only depends on the ratio $x = (\lambda : \mu) \in \mathbb{P}^1$.

We claim that $\mathcal{X}$ is a locally projective $\mathcal{A}$–module precisely when $x \notin \{0, \infty\}$. Moreover, in this case we have: $\mathcal{X}_s \cong F$. Indeed, $\mathcal{X}$ is locally projective at all smooth points of E. Let $\bar{w} = \mu\bar{w}_0 - \lambda\bar{w}_\infty : \mathcal{T} \to \mathcal{S}_\beta$. Then $\mathcal{X}_s$ is isomorphic to the middle term of the following short exact sequence:

$$0 \longrightarrow \begin{pmatrix} I \\ I \end{pmatrix} \longrightarrow \begin{pmatrix} I \\ I + (\lambda, \mu)O \end{pmatrix} \longrightarrow \mathsf{ker}(\bar{w}) \longrightarrow 0,$$

where we view $(\lambda, \mu) \in k \times k$ as an element of the normalization $k[\![u]\!] \times k[\![v]\!]$. Other way around, it is not difficult to show that for any line bundle $\mathcal{L}_x \in \mathrm{Pic}^0(E) \cong k^*$ the locally projective $\mathcal{A}$–module $\mathcal{X} := \mathcal{F} \otimes_E \mathcal{L}_x$ fits into a short exact sequence

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{R} \longrightarrow \mathcal{S}_\beta \longrightarrow 0.$$

Summing up, for all non-zero morphisms $w = \mu w_0 - \lambda w_\infty \in \mathsf{Hom}_{\mathcal{A}}(\mathcal{R}, \mathcal{S}_\beta)$, where $x = (\lambda : \mu) \notin \{0, \infty\}$, the mapping cone $\mathsf{cone}(w)[-1]$ is $\tau_{\mathcal{A}}$–periodic and isomorphic to a stalk complex $\mathcal{X}[0]$, where $\mathcal{X}$ is a locally projective $\mathcal{A}$–module. Applying the functor $\mathbb{T}$, we obtain a distinguished triangle

$$\mathbb{T}(\mathcal{X}) \longrightarrow S_1[-1] \xrightarrow{\mathbb{T}(w)} S_3[1] \longrightarrow \mathbb{T}(\mathcal{X})[1].$$

Next, we have: $\mathsf{Hom}_{D^b(\Lambda)}\big(S_1[-1], S_3[1]\big) \cong \mathsf{Ext}^2_\Lambda(S_1, S_3) = k^2$. Note that S_1 has the projective resolution

$$0 \longrightarrow P_3^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} b & 0 \\ 0 & d \end{smallmatrix}\right)} P_2^{\oplus 2} \xrightarrow{(a\ c)} P_1 \longrightarrow S_1 \longrightarrow 0.$$

Hence, any element $w \in \mathsf{Ext}^2_\Lambda(S_1, S_3)$ is given by a morphism of Λ–modules $(\lambda\mathbb{1}, \mu\mathbb{1}) : P_3^{\oplus 2} \to P_3$, where $(\lambda, \mu) \in k^2$. Moreover, one can check that the complex $\mathsf{cone}(w)$ is τ_Λ–periodic if and only if $(\lambda : \mu) \notin \{0, \infty\}$. In that case, $\mathsf{cone}(w)$ is isomorphic to the complex $U^\bullet_\nu$, where $\nu = -\frac{\mu}{\lambda}$.

□

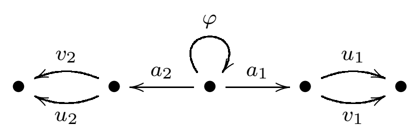
8. Some generalizations and concluding remarks

8.1. **Tilting on other degenerations of elliptic curves.** Theorem 5.10 can be generalized to the case of curves with more complicated singularities. For example, let $E \subseteq \mathbb{P}^2$ be a tachnode plane cubic curve given by the equation $y(yz - x^2) = 0$. Again, let $\pi : \widetilde{E} \to E$ be the normalization, $\widetilde{\mathcal{O}} = \pi_*(\mathcal{O}_{\widetilde{E}})$ and $\mathcal{I} = \mathcal{A}nn_E(\widetilde{\mathcal{O}}/\mathcal{O})$ be the conductor ideal.

In this case, consider again the coherent sheaf $\mathcal{F} = \mathcal{I} \oplus \mathcal{O}$ and the sheaf of $\mathcal{O}$–orders $\mathcal{A} = \mathcal{E}nd_E(\mathcal{F})$. Let $\mathcal{A} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{F}$ be the decomposition of $\mathcal{A}$ into a direct sum of indecomposable locally projective modules. Define the torsion $\mathcal{A}$–module $\mathcal{S}$ via the short exact sequence

$$0 \longrightarrow \mathcal{P}_1 \oplus \mathcal{P}_2 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow 0.$$

Then the complex $\mathcal{H}^\bullet = \mathcal{S}[-1] \oplus \big(\mathcal{P}_1(-1) \oplus \mathcal{P}_1\big) \oplus \big(\mathcal{P}_2(-1) \oplus \mathcal{P}_2\big)$ is rigid and the endomorphism algebra $\Gamma_E = \mathsf{End}_{D^b(\mathcal{A})}(\mathcal{H}^\bullet)$ is isomorphic to the path algebra



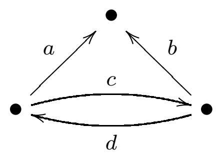
subject to the relations $\varphi^2 = 0, v_1 a_1 = 0, v_2 a_2 = 0, u_1 a_1 \varphi = 0$ and $u_2 a_2 \varphi = 0$. Note that in this case $\mathsf{gl.dim}(\Gamma_E) = \infty$. However, similar to the proof of Theorem 5.10 one can show the complex $\mathcal{H}^\bullet$ is tilting in the sense of Theorem 5.7. Thus, we have a fully faithful functor $\mathsf{Perf}(E) \to D^b(\mathsf{mod} - \Gamma_E)$.

8.2. **Tilting on chains of projective lines.** Let X be a chain of projective lines. In [10] it was shown that $D\big(\mathsf{Qcoh}(X)\big)$ has a tilting vector bundle. In particular, we have a triangle equivalence $D^*\big(\mathsf{Coh}(X)\big) \xrightarrow{\cong} D^*(A_X - \mathsf{mod})$, where $* \in \{-, b\}$ and A_X is the opposite algebra of the corresponding tilted algebra. Composing this functor with the embedding obtained in Corollary 5.11, get an interesting fully faithful functor

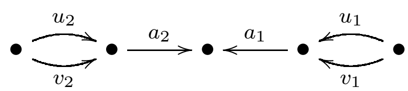
$$D^-(A_X - \mathsf{mod}) \longrightarrow D^-(\Lambda_X - \mathsf{mod})$$

which is worth to study in further details. Note that both algebras A_X and Λ_X are gentle, hence derived-tame. We hope that these geometric realizations of gentle algebras will contribute to a better understanding of the representation theory of gentle algebras.

Example 8.1. Let $X = V(x_0x_1) \subset \mathbb{P}^2$ be a chain of two projective lines. Then A_X is the path algebra of the following quiver



subject to the relations $cd = 0 = dc$. The algebra Λ_X is the path algebra of another quiver



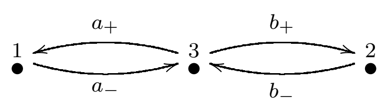
subject to the relations $u_i a_i = 0$, $i = 1, 2$.

8.3. **Non-commutative curves with nodal singularities.** Similar to constructions of Section 2 and Section 5, one can study a new class of derived-tame non-commutative curves which generalizes weighted projective lines of Geigle and Lenzing [17]. Their characteristic property is that the completion of their stalks are generically matrix algebras over $k[\![t]\!]$, whereas the *singular* stalks are either *hereditary orders* or *nodal algebras*. The last class of $k[\![t]\!]$–orders was introduced in [11].

Example 8.2. Let $X = \mathbb{P}^1$, $Z = \{0, \infty\}$ and $\mathcal{I} = \mathcal{I}_Z$ be the ideal sheaf of Z. Consider the following sheaf of $\mathcal{O}_{\mathbb{P}^1}$–orders:

$$\mathcal{A} = \begin{pmatrix} \mathcal{O} & \mathcal{I} & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \\ \mathcal{O} & \mathcal{I} & \mathcal{O} \end{pmatrix}.$$

Then for $x \neq 0, \infty$ the algebra $\widehat{\mathcal{A}_x} \cong \mathsf{Mat}_3\bigl(k[\![t]\!]\bigr)$, whereas for $x \in \{0, \infty\}$ it is the completion of the path algebra of the so-called *Gelfand quiver*:

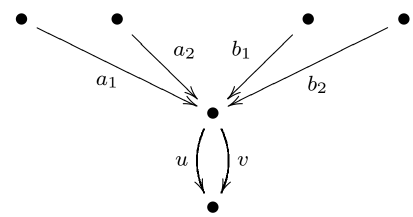
$G =$ 

$\qquad a_- a_+ = b_- b_+.$

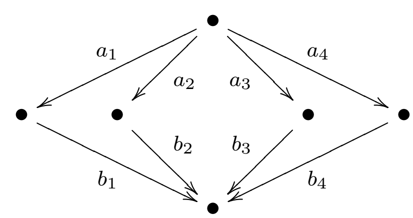
Let S_i be the simple G–module corresponding to the vertex $i = 1, 2$ and $\mathcal{S}_i^x$ be the corresponding $\mathcal{A}$–module for $x \in \{0, \infty\}$. Let $e_1 \in H^0(\mathcal{A})$ be the primitive idempotent corresponding to the left upper corner unit element and

$$\mathcal{P} = \mathcal{A} \cdot e_1 = \begin{pmatrix} \mathcal{O} \\ \mathcal{O} \\ \mathcal{O} \end{pmatrix}$$

be the corresponding locally projective $\mathcal{A}$–module. Similar to the proof of Proposition 5.6 and Theorem 5.10 one can show that the complex $\mathcal{H}^\bullet = \bigl(\mathcal{S}_1^0 \oplus \mathcal{S}_2^0 \oplus \mathcal{S}_1^\infty \oplus \mathcal{S}_2^\infty\bigr)[-1] \oplus \mathcal{P}(-1) \oplus \mathcal{P}$ is tilting in $D^b\bigl(\mathsf{Coh}(\mathcal{A})\bigr)$. Its endomorphism algebra $\Gamma = \mathsf{End}_{D^b(\mathcal{A})}(\mathcal{H}^\bullet)$ is isomorphic to the path algebra of the quiver



subject to the relations $ua_i = 0$, $vb_i = 0$, $i = 1, 2$. The algebra Γ is derived equivalent to the path algebra of the quiver



subject to the relations $b_1a_1 = b_2a_2$ and $b_3a_3 = b_4a_4$. This algebra is a degeneration of the canonical tubular algebra $T_\lambda = T(2, 2, 2, 2; \lambda)$, $\lambda \in k \setminus \{0, 1\}$, introduced by Ringel in [36].

The algebra $T(2, 2, 2, 2; \lambda)$ has the following geometric interpretation. Consider the elliptic curve $E \subseteq \mathbb{P}^2$ given by the equation $zy^2 = x(x - z)(x - \lambda z)$, where $\lambda \in k \setminus \{0, 1\}$. Then the group $\mathbb{Z}_2$ acts on E by the rule $(x : y : z) \mapsto (x : -y : z)$. By a result of Geigle and Lenzing [17, Proposition 4.1 and Example 5.8], see also [33, Corollary 1.4], there is a derived equivalence

$$D^b\bigl(\mathsf{Coh}^{\mathbb{Z}_2}(E)\bigr) \xrightarrow{\cong} D^b(\mathsf{mod} - T_\lambda).$$

Hence, the derived category $D^b\bigl(\mathsf{Coh}(\mathcal{A})\bigr)$ can be viewed as a *degeneration* of $D^b\bigl(\mathsf{Coh}^{\mathbb{Z}_2}(E)\bigr)$.

8.4. **Configuration schemes of Lunts.** Our construction of non-commutative curves attached to a nodal rational projective curve, is closely related with the category of coherent sheaves on a *configuration scheme*, introduced by Lunts in [27]. Let X be a union of projective lines intersecting transversally. Lunts introduces a category $\mathsf{Coh}(\mathcal{X})$, constructs a fully faithful exact functor $\mathsf{Perf}(X) \to D^b\bigl(\mathsf{Coh}(\mathcal{X})\bigr)$ and shows that $D^b\bigl(\mathsf{Coh}(\mathcal{X})\bigr)$ has a tilting *sheaf*, whose endomorphism algebra is isomorphic to the algebra Γ_X introduced in Definition 5.12. Moreover, his approach can be generalized to higher dimensions, in particular to the case of the singular quintic threefold $V(x_0x_1x_3x_3x_4) \subseteq \mathbb{P}^4$ important in the string theory.

The construction of Lunts seems not to generalize straightforwardly to the case of arbitrary nodal rational projective curves. Moreover, one can check that for a union of projective lines X intersecting transversally, the category $\mathsf{Coh}(\mathcal{X})$ is *not* equivalent to $\mathsf{Coh}(\mathcal{A}_X)$. In other words, $\mathsf{Coh}(\mathcal{X})$ is the heart of an interesting t-structure in the triangulated category $D^b\bigl(\mathsf{Coh}(\mathcal{A}_X)\bigr)$. An exact relation between the categories $\mathsf{Coh}(\mathcal{X})$ and $\mathsf{Coh}(\mathcal{A}_X)$ will be studied in a separate paper.

References

[1] I. Assem, A. Skowroński, *Iterated tilted algebras of type* $\tilde{A}_n$, Math. Z. **195** (1987), no. **2**, 269–290.
[2] M. Atiyah, I. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing (1969).
[3] M. Auslander, *On the dimension of modules and algebras. III. Global dimension*, Nagoya Math. J. **9** (1955), 67–77.
[4] M. Auslander, K. Roggenkamp, *A characterization of orders of finite lattice type*, Invent. Math. **17** (1972), 79–84.
[5] H. Bass, *On the ubiquity of Gorenstein rings* Math. Z. **82** (1963) 8–28.
[6] A. Beĭlinson, *Coherent sheaves on* $\mathbb{P}^n$ *and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. **3**, 68–69.
[7] L. Bodnarchuk, I. Burban, Yu. Drozd, G.-M. Greuel, *Vector bundles and torsion free sheaves on degenerations of elliptic curves*, Global aspects of complex geometry, 83–128, Springer, Berlin (2006).
[8] A. Borel et al, *Algebraic D-modules*, Perspectives in Mathematics, 2. Academic Press, Inc., (1987).
[9] M. Butler, C.-M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra **15** (1987), 145–179.
[10] I. Burban, *Derived categories of coherent sheaves on rational singular curves*, Representations of finite dimensional algebras and related topics in Lie theory and geometry, 173–188, Fields Inst. Commun., **40** (2004).
[11] I. Burban, Yu. Drozd, *Derived categories of nodal algebras*, J. Algebra **272** (2004), no. **1**, 46–94.
[12] I. Burban, Yu. Drozd, *On derived categories of certain associative algebras*, Representations of algebras and related topics, 109–128, Fields Inst. Commun. **45** (2005).
[13] I. Burban, Yu. Drozd, *Coherent sheaves on rational curves with simple double points and transversal intersections*, Duke Math. J. **121** (2004), no. **2**, 189–229.
[14] I. Burban, B. Kreußler, *Vector bundles on degenerations of elliptic curves and Yang-Baxter equations*, arXiv:0708.1685.
[15] Yu. Drozd, G.-M. Greuel, *Tame and wild projective curves and classification of vector bundles*, J. Algebra **246** (2001), no. **1**, 1–54.
[16] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962) 323–448.
[17] W. Geigle, H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles, 265–297, Lecture Notes in Mathematics **1273**, Springer (1987).
[18] W. Geigle, H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra vol. **15**, no. **2**, 273–343 (1991).
[19] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, LMS Lecture Note Series **119**, Cambridge University Press, (1988).
[20] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. **52**, Springer-Verlag, (1977).
[21] L. Illusie, *Existance de résolutions globales, Exposé II*, Théorie des intersections et théorème de Riemann-Roch, Séminaire de Géométrie Algèbrique du Bois-Marie 1966–1967 (SGA 6), Lect. Notes Math. 225, (1971) 160–221.
[22] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.
[23] M. Kaschiwara, P. Schapira, *Sheaves on Manifolds* **292**, Springer (1990).
[24] B. Keller, *Derived categories and their uses*, Handbook of algebra, vol. **1**, Elsevier (1996).
[25] M. Kontsevich, *Homological algebra of mirror symmetry*, International congress of mathematicians, ICM'94, Birkhäuser, vol. **I** (1995) 120–139.
[26] H. Krause, *The stable derived category of a Noetherian scheme* Compos. Math. **141** (2005), no. **5**, 1128–1162.
[27] V. Lunts, *Coherent sheaves on configuration schemes*, J. Algebra **244**, no. **2**, 379–406, (2001).
[28] K. de Naeghel, M. van den Bergh, *Ideal classes of three-dimensional Sklyanin algebras*, J. of Algebra vol. **276**, 515–551 (2004).
[29] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. 9 (1996), 205–236.
[30] D. Orlov, *Remarks on generators and dimensions of triangulated categories*, arXiv:0804.1163.

[31] Z. Pogorzały, A. Skowroński, *Self-injective biserial standard algebras*, J. Algebra **138** (1991), no. **2**, 491–504.
[32] A. Polishchuk, *Classical Yang-Baxter equation and the A_∞-constraint,* Adv. Math. **168** (2002), no. **1**, 56–95.
[33] A. Polishchuk, *Holomorphic bundles on 2-dimensional noncommutative toric orbifolds*, Noncommutative geometry and number theory, 341–359, Aspects Math., E37, Vieweg, (2006).
[34] I. Reiten, M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. **2**, 295–366.
[35] R. Rentschler, Sur les modules M tels que Hom(M,–) commute avec les sommes directes, C. R. Acad. Sci. Paris Ser. A **268** (1969), 930–933.
[36] C.-M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics **1099**, Springer (1984).
[37] C.-M. Ringel, *The repetitive algebra of a gentle algebra*, Bol. Soc. Mat. Mexicana (3) **3** (1997), no. **2**, 235–253.
[38] R. Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. **2**, 193–256.
[39] P. Seidel, R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001) 37–108.
[40] P. Seidel, *More about vanishing cycles and mutation*, Symplectic geometry and mirror symmetry, 429–465, World Sci. Publ., River Edge, (2001).
[41] J. -P. Serre, *Local Algebra*, Springer Monographs in Mathematics, Springer (2000).
[42] A. Yekutieli, J. Zhang, *Dualizing complexes and perverse sheaves on noncommutative ringed schemes*, Selecta Math. (N.S.) **12** (2006), no. **1**, 137–177.
[43] M. van den Bergh, *Non-commutative crepant resolutions*, The legacy of Niels Henrik Abel, 749–770, Springer, (2004).

Mathematisches Institut Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
E-mail address: burban@math.uni-bonn.de

Institute of Mathematics National Academy of Sciences, Tereschenkivska str. 3, 01601 Kyiv, Ukraine, and Max-Plank-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: drozd@imath.kiev.ua
URL: www.imath.kiev.ua/∼drozd