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On Some Generalization of Schubert's Varieties

Yuriy A. Drozd [a,1], Pavlo S. Kolomiets [a]
[a] *Kyiv State University of T.Shevchenko, Mechanical-mathematical faculty, Department of Algebra*

Abstract. One generalization of Shubert's varieties is presented based on use of two towers of "general"(very natural) form with no subspace from one of the towers belonging totally to some subspace from another tower. These varieties are proved to be reducible, their general positions are described. The number of general positions and dimensions of corresponding orbits are calculated also.

1. Introduction of Generalization

Here we will introduce some generalization of Schubert's varieties. As it is known, Schubert's varieties are built by means of any tower of vector subspaces in some affine vector space. Namely, given a tower $V_1 \subset V_2 \subset \ldots \subset V_d \subset \mathbb{A}^h$, Schubert's variety is defined as all of the d-dimension vector subspaces $U \subset \mathbb{A}^h$ such that

$$dim(U \bigcap V_1) \geqslant 1, dim(U \bigcap V_2) \geqslant 2, \ldots, dim(U \bigcap V_d) \geqslant d. \tag{1}$$

As it is known, this variety is irreducible and for dimensions of a tower $a_1, a_2, \ldots, a_d$ has a dimension, calculated by a formula:

$$\sum_{i=1}^{d} d_i - \frac{1}{2}d(d+1). \tag{2}$$

We introduce the next generalization.

We consider some partially ordered set of d subspaces, for example:

Then, in grassmanian, we consider subvariety of all of the d-dimension vector subspaces $U \subset \mathbb{A}^h$ which satisfy the following conditions:

$$\begin{array}{l} dim(U \bigcap \text{every subspace of 1st layer}) \geqslant 1 \\ dim(U \bigcap \text{every subspace of 2nd layer}) \geqslant 2 \\ \qquad\qquad\qquad \ldots \end{array} \tag{3}$$

[1]Yuriy A.Drozd, Kyiv State University of T.Shevchenko, Mechanical-mathematical faculty, Academica Glushkova 6, Kyiv, Ukraine; E-mail: yuriy_nelli@yahoo.com.

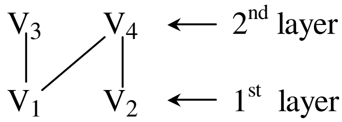

Figure 1. Example of subspaces' set.

This is a variety, closed in the grassman variety of all d-dimension subspaces. And, of course, we are interested in knowing whether it is irreducible, and if not, how many general positions it has and what are their dimensions. The most interesting case for us of mutual arrangement of defining towers, which will be considered in this article, is the case of two towers of general form with no subspace from one of the towers belonging totally to some subspace from another tower. The meaning of "general form" will be given later.

Suppose we have two towers, the first one consisting of m subspaces and the second consisting of n ($d = m + n$), where $m > n$, for the purposes of convenience:

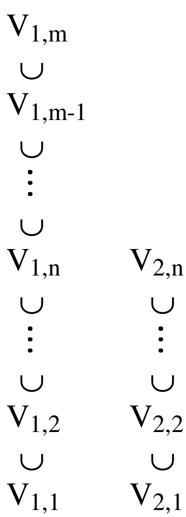

Figure 2. The towers.

The dimension of $V_{i,j}$ denote as $d_{i,j}$.

These $m+n$ subspaces $V_{i,j} \subset \mathbb{A}^h$ break $\mathbb{A}^h$ at $(m+1) \times (n+1)$ subsets in general case. They can be conveniently presented in the form of a table:

Here

$$\bigcup_{i=1}^{r} \bigcup_{j=1}^{n+1} S_{i,j} = V_{1,r}, r = \overline{1,m} \tag{4}$$

$S_{1,n+1}$	$S_{2,n+1}$	...	$S_{m,n+1}$	$S_{m+1,n+1}$
$S_{1,n}$	$S_{2,n}$	...	$S_{m,n}$	$S_{m+1,n}$
...	...	...	...	...
$S_{1,2}$	$S_{2,2}$	...	$S_{m,2}$	$S_{m+1,2}$
$S_{1,1}$	$S_{2,1}$	...	$S_{m,1}$	$S_{m+1,1}$

Figure 3. The table.

and

$$\bigcup_{j=1}^{s} \bigcup_{i=1}^{m+1} S_{i,j} = V_{2,s}, s = \overline{1,n}. \tag{5}$$

Every square

$$Sq(r,s) := \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} S_{i,j}, r = \overline{1,m}, s = \overline{1,n} \tag{6}$$

of subsets $S_{i,j}$ is a vector subspace, because

$$Sq(r,s) = V_{1,r} \bigcap V_{2,s} \tag{7}$$

evidently.

Now, we want to set such the basis of $\mathbb{A}^h$ that all the subspaces $V_{1,r}$, $V_{2,s}$ will be generated by some vectors from this basis.

We'll accomplish this in the following way. Passing the columns in the preceding table from the left to the right and each column from the bottom to the top (starting from the subset $S_{1,1}$) execute the next routine for each subset $S_{i,j}$:
If the basis of $Sq(i-1,j)$ together with the basis of $Sq(i,j-1)$ (which we have selected already) generate $Sq(i,j)$ then proceed to the next subset.
Otherwise create a direct complement of subspace generated by them to $Sq(i,j)$. This complement we can choose to be orthogonal. Denote the vectors of this complement as $B_{i,j}$ and its dimension as $D_{i,j}$:

$$\begin{aligned} &B_{i,j} := \text{orthogonal basis of direct complement of} \\ &\langle (V_{1,i-1} \bigcap V_{2,j}), (V_{1,i} \bigcap V_{2,j-1}) \rangle \text{ to } (V_{1,i} \bigcap V_{2,j}), \end{aligned} \tag{8}$$

$$D_{i,j} := \text{number of vectors in } B_{i,j}, \tag{9}$$

where

$$V_{1,m+1} \text{ and } V_{2,n+1} \text{ denote } \mathbb{A}^h \tag{10}$$

and

$$i = \overline{1, m+1}, j = \overline{1, n+1}. \quad (11)$$

Here, it's time to define the meaning of condition "general form".
Condition "of general form" means that:

$$\begin{array}{l}(1)\ D_{i,j} \geqslant 1, i = \overline{1, m+1}, j = \overline{1, n+1} \\ (2) \text{ the whole space is big enough, } D_{m+1,n+1} \geqslant n.\end{array} \quad (12)$$

This is very natural condition, it's only a little stronger then that all the intersections of any two subspaces from towers are different.

So, the matrix of each subspace $V_{1,r}$, $V_{2,s}$ in the so selected basis can be chosen to have only one "1" in each column and maximum one "1" in each row. For subspace U from variety, consider the matrix of vectors that generate it. It's rows which correspond to the nonzero rows of basis vectors of $B_{i,j}$ denote as $K_{i,j}$ for every $i = \overline{1, m+1}, j = \overline{1, n+1}$.

For each $V_{1,r}$ write down it's matrix on the left from the matrix M of U and denote this new matrix as $MV_{1,r}$. Let's consider then one of the conditions (3):

$$dim(U \bigcap V_{1,r}) \geqslant r. \quad (13)$$

It is equivalent to

$$dim(U \bigcup V_{1,r}) \leqslant d + d_{1,r} - r, \quad (14)$$

this one being equivalent to:

"all the minors having dimensions ($d + d_{1,r} - r + 1$) in matrix $MV_{1,r}$"

"equal to 0". (15)

This condition is equivalent correspondingly to:

"all the minors of the matrix $\bigcup_{i=r+1}^{m+1} \bigcup_{j=1}^{n+1} K_{i,j}$ having dimensions ($d - r + 1$)"

"equal to 0", (16)

or

"the rank of the matrix $\bigcup_{i=r+1}^{m+1} \bigcup_{j=1}^{n+1} K_{i,j} \leqslant d - r$". (17)

For subspace $V_{2,s}$ the corresponding condition is:

"the rank of the matrix $\bigcup_{i=1}^{m+1} \bigcup_{j=s+1}^{n+1} K_{i,j} \leqslant d - s$". (18)

So, instead of subspaces U the matrices M of h rows and d columns which satisfy the preceding conditions can be considered.

We'll use also tables $\tilde{M}$ of $m+1$ columns and $n+1$ rows corresponding to matrices M very extensively.

The table $\tilde{M}$ corresponding to the matrix M is defined as here:
It's columns are enumerated from left to right starting from 1, it's rows are enumerated from bottom to top starting from 1. It's cells are denoted as $\tilde{K}_{i,j}$.
If rows $K_{i,j}$ of M have the row with non zeroes standing in positions $l_1, \ldots, l_k$, then in cell $\tilde{K}_{i,j}$ the sum $l_1 + l_2 + \ldots + l_k$ stands. If there are several non zero rows in $K_{i,j}$, the corresponding sums in $\tilde{K}_{i,j}$ are separated by comma. The order of numbers in any sum and order of sums in any cell are insignificant.

This form of writing down elements of our variety is very similar to matrices but more convenient.

Here are some useful denotements for matrices:

$$M_i = \bigcup_{j=1}^{n+1} K_{i,j}, \qquad N_j = \bigcup_{i=1}^{m+1} K_{i,j}, \qquad S_k = \bigcup_{i=k}^{m+1} M_i, \qquad Z_k = \bigcup_{j=k}^{n+1} N_j \quad (19)$$

and correspondingly for tables:

$$\tilde{M}_i = \bigcup_{j=1}^{n+1} \tilde{K}_{i,j}, \qquad \tilde{N}_j = \bigcup_{i=1}^{m+1} \tilde{K}_{i,j}, \qquad \tilde{S}_k = \bigcup_{i=k}^{m+1} \tilde{M}_i, \qquad \tilde{Z}_k = \bigcup_{j=k}^{n+1} \tilde{N}_j \quad (20)$$

2. Investigation of Properties

Now we consider an algebraical group of all the transformations of these matrices which do not violate conditions laid on them. Then we consider orbits created by this group. We can easily see that such transformations are the next ones:

(1) Adding a row from $K_{r,s}, r \in [1, \ldots, m+1], s \in [1, \ldots, n+1]$ multiplied by some number to any row from $\bigcup_{i=1}^{r} \bigcup_{j=1}^{s} K_{i,j}$. (21)
(2) Adding a column multiplied by some number to any other column.

On tables and matrices we'll do the following three operations:

(1) If in M there is a row in $K_{r,s}$ with only nonzero element at l's position, then we make zeroes at this position in all rows of all $K_{i,j}, i = \overline{1,r}, j = \overline{1,s}$, except this row.

In $\tilde{M}$, therefore, there is sum l in cell $\tilde{K}_{r,s}$, and after this operation all numbers l in all sums standing in cells $\bigcup_{i=1}^{r} \bigcup_{j=1}^{s} \tilde{K}_{i,j}$, except this sum will be eliminated.

This operation doesn't change the orbit of our algebraical group. Denote it as $Op1$ with l.

(2) If in M there is a row in $K_{r,s}$ with some number of non zeroes, two of them standing at positions l_1,l_2, then, we can eliminate non zero at position l_2 in this row. In every other row with non zero at position l_1 some non zero could appear at position l_2.

In $\tilde{M}$, respectively, $\tilde{K}_{r,s}$ has sum $l_1 + l_2 + \ldots + l_k$ and after the operation l_2 will be eliminated in it. And in all other sums with number l_1, l_2 in the same sum could appear.

This operation doesn't change the orbit of our algebraical group also.

(3) If there is some zero row in matrix M then we can change it into as little non zero row as we wish if this operation doesn't make ranks of corresponding sub matri-

ces exceeding their maximal allowed values. Then, multiplying this row by some value, which is the operation of our algebraical group, we can make it as large as we wish.

Making this operation, we can change the orbit of this matrix, but we don't change the closure of the orbit.

So, we've done all the preparations and are ready for the first theorem.

2.1. Reducibility

Theorem 1 *Every orbit is a degeneration of some orbit from the set of orbits $O_{m,n}$ which is defined as follows:*
Orbit $O \in O_{m,n}$ if it has element (denote it e_O) with corresponding table $\tilde{M}$ satisfying the following conditions:
(1) all sums in all cells consist from only one number, id.est., there are $d = m + n$ numbers in the table each number appearing once and only once.
(2) each cell can have at most 1 number except of the top-rightmost one which can have from 0 to n numbers.

$$|\tilde{K}_{i,j}| \leqslant 1,\ i = \overline{1, m+1},\ j = \overline{1, n+1}; \qquad |\tilde{K}_{m+1,n+1})| \leqslant n \tag{22}$$

(3) each column has 1 number except of the rightmost one which has n numbers.

$$|\tilde{M}_i| = 1,\ i = \overline{1, m}; \qquad |\tilde{M}_{m+1}| = n \tag{23}$$

(4) each row has 1 number except of the topmost one which has m numbers.

$$|\tilde{N}_j| = 1,\ j = \overline{1, n}; \qquad |\tilde{N}_{n+1}| = m \tag{24}$$

$\triangleleft$ Let's suppose we have element of our variety with corresponding table $\tilde{M}$. If it doesn't satisfy the conditions above then we will transform it to one which satisfies them using three operations on tables described previously.

We will use the term "free" cell which means for $\tilde{K}_{m+1,n+1}$ that it has less then n sums. For all other cells this term means empty, id.est. the cells have no sums.

(1) Consider path in $\tilde{M}$ which starts in top-rightmost cell and passes rows from right to left and columns from top to bottom, passing the rows firstly.

$$\tilde{K}_{m+1,n+1} \to \tilde{K}_{m,n+1} \to \ldots \to \tilde{K}_{1,n+1} \to \tilde{K}_{m+1,n} \to \ldots \to \tilde{K}_{1,1}. \tag{25}$$

Suppose that going this path we arrived in cell $\tilde{K}_{i,j}$ which has some sum and suppose that this sum has number l which we didn't meet before on the path. Then eliminate all other numbers in this sum by l. In $\tilde{M}$ some cells can change but all the cells on the already passed path will not change as they haven't number l in their sums by our supposition. So, instead of the sum we have number l. Perform $Op1$ with this number. Now, if not in current cell then it can appear only in rows below. Denote it as an "original" number and it's position(cell) as corresponding "original" position.

If the sum consists only from numbers met before then leave it as it is. Only note that these numbers are from rows above, not from N_j.

Now proceed with all other sums from this cell and then all other cells. After doing this we will have $m + n$ "original" numbers and sums of "original" numbers from rows above. Notice that the upper row has only "original" numbers.

(2) Now let's make all sums to consist only from one number and these numbers to be unique in the table.

Mathematical Induction:

Consider path in $\tilde{M}$

$$\tilde{K}_{m+1,n+1} \to \tilde{K}_{m+1,n} \to \ldots \to \tilde{K}_{m+1,1} \to \tilde{K}_{m,n+1} \to \ldots$$
$$\ldots \to \tilde{K}_{m,1} \to \ldots \to \tilde{K}_{1,1}. \tag{26}$$

Let's suppose that we consider cell $\tilde{K}_{r,s}$ on the path, and suppose that all previous cells on the path have only "original" numbers in them and these numbers appear once in $\tilde{M}$. Then we'll do the same for this cell.

For the first cell on the path this is evidently true.

Suppose in $\tilde{K}_{r,s}$ some "original" number stays, then it can appear also only in cells to the left and to the bottom. Perform $Op1$ with this "original" number and we'll have it unique in $\tilde{M}$ which is the purpose of the point.

Suppose in $\tilde{K}_{r,s}$ the sum of "original" numbers from $\tilde{Z}_{s+1}$ (they are not the numbers from previous cells on the path) stays. Let these numbers be a set $\tilde{\beth}_{n+1}$ of "original" numbers from $\tilde{N}_{n+1}, \ldots, \tilde{\beth}_{s+1}$ of "original" numbers from $\tilde{N}_{s+1}$ (some sets can be empty). For every nonempty set $\tilde{\beth}_k$ choose the "original" number l which stays in the leftmost cell. Then, in our sum, eliminate all other "original" numbers of the set $\tilde{\beth}_k$ by l. Our "original" number l in its "original" position will become a sum of "original" numbers of set $\tilde{\beth}_k$. We can eliminate all of them except l by corresponding "original" numbers from "original" positions. Do it. All other "original" numbers in "original" positions will not change, and all sums of "original" numbers from corresponding $\tilde{Z}_*$ will remain to be sums from them.

Thus, each $\tilde{\beth}_k$ consists only from one "original" number now. Take the lowest $\tilde{\beth}_y$ with "original" number l in its "original" cell $\tilde{K}_{x,y}$ ($x < r, y > s$ with our suppositions).

Consider now the path

$$\tilde{K}_{r,y} \to \tilde{K}_{r,y+1} \to \ldots \to \tilde{K}_{r,n+1} \to \tilde{K}_{r+1,n+1} \to \ldots \to \tilde{K}_{m+1,n+1}. \tag{27}$$

There is at least one "free" cell on it. If there is no one, then according to our suppositions all of them have at least one "original" number appearing only once in $\tilde{M}$ except of $\tilde{K}_{m+1,n+1}$ which has at least n "original" numbers. Then, $rank(\tilde{S}_{m+1}) \geqslant n$, $rank(\tilde{S}_m) \geqslant n + m + 1 - (m)$, $\ldots$, $rank(\tilde{S}_{r+1}) \geqslant n + m + 1 - (r + 1)$, and $rank(\tilde{S}_r) \geqslant n + m + 1 - (r) + 1$. But this last value is greater by one than the maximal allowable rank for $\tilde{S}_r$ which is equal to $n + m - (r) + 1$ (see Cond.17), so we come to contradiction.

Thus, there is at least one "free" cell on the path. Insert our sum into the first "free" cell.

We will not raise the rank of $\tilde{Z}_y$ because the sum consists of "original" numbers from "original" positions equal or higher than row y.

We'll not violate any restrictions laid on ranks of all $\tilde{S}_k$, $\tilde{Z}_k$.
Really, suppose this is $\tilde{K}_{r,t}$, $t \geqslant y \Longrightarrow$ ranks of all $\tilde{S}_k$ don't change, and

$$\begin{cases} rank(\tilde{Z}_p) \leqslant m+n+1-p-1, p=\overline{y+1,t}; \\ rank(\tilde{Z}_y) \leqslant m+n+1-y, \quad . \end{cases} \tag{28}$$

$\Rightarrow$ inserting our sum we can raise some ranks at most at 1 and, so

$$\begin{cases} rank(\tilde{Z}_f), f > t & \text{don't change;} \\ rank(\tilde{Z}_f), y < f \leqslant t, & \text{change within the allowable limits;} \\ rank(\tilde{Z}_y), & \text{don't change.} \end{cases} \tag{29}$$

Suppose the first "free" is $\tilde{K}_{t,n+1}$, $t \geqslant r \Longrightarrow$

$$\begin{cases} rank(\tilde{Z}_p) \leqslant m+n+1-p-1, p=\overline{y+1,n+1}; \\ rank(\tilde{Z}_y) \leqslant m+n+1-y \end{cases} \tag{30}$$

so, inserting our sum we can raise some ranks at most at 1, id.est. within the allowable limits.

$$\begin{cases} rank(\tilde{S}_p) \leqslant n+m+1-p-1, p=\overline{r+1,t}; \\ rank(\tilde{S}_r) \leqslant n+m+1-r \end{cases} \tag{31}$$

$\Rightarrow$ inserting our sum we can raise some ranks at most at 1 and, so

$$\begin{cases} rank(\tilde{S}_f), f > r & \text{don't change;} \\ rank(\tilde{S}_f), r < f \leqslant t, & \text{change within the allowable limits;} \\ rank(\tilde{S}_r), & \text{don't change.} \end{cases} \tag{32}$$

So, we can insert our sum.

Now, if the sum consists not only of l but of other "original" numbers then eliminate all of them in it by l. After this, instead of l in $\tilde{K}_{r,s}$ we'll have a sum of the same numbers. In all other cells on the path of math.induction some sums can also change, but they are out of interest at this moment.

Perform $Op1$ with l, then it will appear only in the inserted cell.

So, following our path, we either keep "original" number in its "original" position (the highest cell with this number), then it don't appear elsewhere, or transfer some "original" number from its "original" cell (not on the already followed path) to some cell on the already followed path, making it unique in the table.

After this, there is one and only one nonzero in each column of M and at most one nonzero in each row of M. In $\tilde{M}$, correspondingly, there are $m+n$ unique numbers from 1 to $m+n$.

(3) Now, let's make the ranks of all $\tilde{S}_{m+1}, \ldots, \tilde{S}_1$ be the maximal allowed, id.est. $n, n+1, \ldots, n+m$.

(a) If $rank(\tilde{S}_{m+1}) = n$ then proceed to the next point. Otherwise, suppose $\tilde{M}_i$, $i \leqslant m$ is the rightmost nonempty column and $\tilde{K}_{i,j}$ is the topmost nonempty cell in $\tilde{M}_i$. So, the number l stays in $\tilde{K}_{i,j}$.

Let's consider the path

$$\tilde{K}_{m+1,j} \to \tilde{K}_{m+1,j+1} \to \ldots \to \tilde{K}_{m+1,n+1}. \tag{33}$$

There is a "free" cell on it, otherwise

$$rank(\tilde{S}_{m+1}) \geqslant rank(\tilde{K}_{m+1,n+1}) \geqslant n \tag{34}$$

- contradiction.

Insert l in the first "free". As in the previous point doing this we either will not change ranks of $\tilde{S}_k$, $\tilde{Z}_k$ or change them within the allowable limits.

And $rank(\tilde{S}_{m+1})$ becomes greater at 1.

After finite number of steps $rank(\tilde{S}_{m+1}) = n$.

(b) Now, look for the rightmost nonempty $\tilde{K}_{i,j}$, $i \leqslant m$. If $i = m \Rightarrow$ there is only one number in $\tilde{M}_m$ and we may proceed further. If $i < m$ then transfer this number from $\tilde{K}_{i,j}$ to $\tilde{K}_{m,j}$.

In the same way transfer all numbers from $\tilde{K}_{i,j}$, $i < m$ to the right as much as possible $\Rightarrow$ The goal of this point is accomplished. In addition, we haven't changed the ranks of all $\tilde{Z}_k$.

Now, $\tilde{M}_1, \ldots, \tilde{M}_m$ have 1 number, and $\tilde{M}_{m+1}$ has n numbers.

(4) Now, let's make the ranks of all $\tilde{Z}_{n+1}, \ldots, \tilde{Z}_1$ be the maximal allowed, id.est. $m, m+1, \ldots, m+n$.

Suppose $rank(\tilde{Z}_{n+1}) < m \Rightarrow$ look for the topmost nonempty $\tilde{K}_{i,j}$, $j \leqslant n$ and transfer l from $\tilde{K}_{i,j}$ to $\tilde{K}_{i,n+1}$. We can do this always, because, if $i \leqslant m$ then l is the only one number in $\tilde{M}_i$, and if $i = m+1$ then $\tilde{K}_{m+1,n+1}$ can contain all the numbers from $\tilde{M}_{m+1}$.

So, lift all the numbers from $\tilde{K}_{i,j}$, $j \leqslant n$ as much as possible analogically to point 3b. Ranks of all $\tilde{S}_k$ haven't changed and $\tilde{N}_1, \ldots, \tilde{N}_n$ have 1 number, and $\tilde{N}_{n+1}$ has m numbers.

So, our proof is finished. $\triangleright$

Theorem 2 *None of the orbits from $O_{m,n}$ is a degeneration of another orbit from $O_{m,n}$.*

$\triangleleft$ Let's define for each orbit O an ordered set of numbers G_O:

$$T_{m+1,n+1}; T_{m+1,n}; \ldots; T_{m+1,1}; T_{m,n+1}; \ldots; T_{m,1}; \ldots; T_{1,1}. \tag{35}$$

Orbits are identified uniquely by these sets of numbers. Really, all allowed algebraical transformation of:
(1) rows - don't change them evidently, because to the rows of some matrix corresponding to $T_{\cdot,\cdot}$ the rows of the same matrix can be added only.
(2) columns don't change rank of any matrix altogether.

It's obvious that if we can go from some element of some orbit to another orbit by algebraical transformation and then by infinitely small change then we can do this by some other infinitely small change.

Thus, to prove the statement of the theorem we may prove that from e_O, $O \in O_{m,n}$ we can go to no other orbit $O' \in O_{m,n}$ by all infinitely small changes.

So, suppose we have table $\tilde{M}$ and matrix M of e_O and suppose we've done some infinitely small change. Some infinitely small numbers appeared in zero and nonzero rows of M. For matrix M of e_O, writing number in nonzero row is equivalent to adding one column to another. Wherefrom, the infinitely small change we've done is equivalent to allowed column's transformations first and then infinitely small change which add small numbers in zero rows $\Rightarrow$ we can consider only such infinitely small changes.

So, suppose in $\tilde{M}$ we wrote $l_1 + \ldots l_k$ somewhere, for example in $\tilde{K}_{r,s}$. Suppose, $(r,s) \nleqslant (x_{l_i}, y_{l_i})$ (where sign means component wise relation) for some $i \Rightarrow$ some ranks of $\tilde{S}_\cdot$, $\tilde{Z}_\cdot$ changed $\Rightarrow (r,s) \leqslant (x_{l_i}, y_{l_i})$, $\forall i = \overline{1,k} \Rightarrow$ this change is equivalent to allowed algebraical transformation of rows, which don't change the set of ranks G_O as we proved already. And so on for each infinitely small sum added.

Thus, we can't come from one orbit in $O_{m,n}$ to another. ▷

2.2. *The Number and Dimensions of Orbits*

So, we can easily calculate the number of orbits in $O_{m,n}$ as the number of tables defined in *Theorem1*.

Proposition 1 *The number of orbits from $O_{m,n}$ equals to*

$$N_{m,n} = |O_{m,n}| = \sum_{k=0}^{n} C_n^k \cdot A_m^k. \tag{36}$$

◁ To define some table $\tilde{M}$ for e_O of orbit $O \in O_{m,n}$ we can define the position of numbers in all columns except the rightmost one. The number of such possibilities equals to the number of ordered selections of m numbers from the set of vertical positions

$$\{1, 2, \ldots, n, \underbrace{n+1, \ldots, n+1}_{m}\}$$

So, suppose we select k numbers less than $n+1 \Rightarrow$ for k we have $C_n^k \cdot A_m^k$ variants $\Rightarrow$ summing over all possible $k = \overline{0,n}$ we'll have the needed number. ▷

Now we'll calculate the ranks of orbits from $O_{m,n}$.

Theorem 3 *Suppose table $\tilde{M}$ of some element of orbit $O \in O_{m,n}$ looks like: number l stays in $\tilde{K}_{x_l,y_l}$, $l = \overline{1, m+n}$, $1 \leqslant x_l \leqslant m+1$, $1 \leqslant y_l \leqslant n+1$. Then, it's dimension is calculated by the formula:*

$$rank(O) = \sum_{l=1}^{m+n} r(l) + \sum_{l=1}^{m+n} max_{\tilde{K}_{i,j} \in \Sigma_{x_l,y_l}} r(i,j) - (m+n)^2, \tag{37}$$

where $r(l)$, $r(i,j)$, $\Sigma_{i,j}$ are defined as here

$$r(l) = \sum_{i=1}^{x_l} \sum_{j=1}^{y_l} D_{i,j}; \quad r(i,j) = \sum_{r=i}^{m+1} \sum_{s=j}^{n+1} |\tilde{K}_{r,s}|; \quad \Sigma_{i,j} = \tilde{S}_{i+1} \bigcup \tilde{Z}_{j+1}. \tag{38}$$

$\lhd$ As number in cell $\tilde{K}_{x,y}$ can operate on each cell $\tilde{K}_{x',y'}$ with $x' \leqslant x$, $y' \leqslant y$, then the dimension which is introduced by number l to dimension of orbit O equals to

$$r(l) := \sum_{i=1}^{x_l} \sum_{j=1}^{y_l} D_{i,j}. \tag{39}$$

$$L := \sum_{l=1}^{m+n} r(l) \tag{40}$$

can be calculated in another way.

Cell $\tilde{K}_{i,j}$ can be operated on by all numbers which stay in cells $\tilde{K}_{r,s}$ with $r \geqslant i$, $s \geqslant j$. The number of the latter is

$$r(i,j) = \sum_{r=i}^{m+1} \sum_{s=j}^{n+1} |\tilde{K}_{r,s}|. \tag{41}$$

Then,

$$L = \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} r(i,j). \tag{42}$$

Further, let's calculate ranks which are introduced by each column of matrix $\widehat{M}$ with zero elements.

$\widehat{M}$ is a matrix obtained from matrix M by inserting "1" in each place, where we can do it by our allowed algebraical transformations of rows over M.

Suppose we are considering column l of $\widehat{M}$ which has zero elements. As column l can be expressed in other nonzero rows, then the rank introduced by column l equals to

$$max_{\tilde{K}_{i,j} \in \Sigma_{x_l, y_l}} r(i,j) \tag{43}$$

with $\Sigma_{i,j}$ defined before.

As we can freely permutate the rows of M by our allowed algebraical transformations, then, we have to subtract $(m+n)^2$ from already obtained dimension. $\rhd$

References

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