Symmetry in Nonlinear Mathematical Physics 1997, V.2, 450–454.

Conditional Symmetry and Exact Solutions of the Kramers Equation

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Abstract

We research some Q- and Q_1, Q_2-conditional symmetry properties of the Kramers equation. Using that symmetry, we have constructed the well-known Boltzmann solution.

1 Q-conditional symmetry

Let us consider the Kramers equation which describes the motion of a particle in a fluctuating medium [1]

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(V'(x)u) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right), \tag{1}$$

where $u = u(t, x, y)$ is the probability density, γ is a constant and $V(x)$ is an external potential.

The group properties of equation (1) for various potentials were investigated in detail by means of the Lie method [2].

Theorem 1. *The maximal invariance local group of the Kramers equation (1) is 1) a six-dimensional Lie group, when $V'(x) = kx + c$ (k, c are constants), $k \neq -\frac{3}{4}\gamma^2, \frac{3}{16}\gamma^2$; 2) an eight-dimensional Lie group, when $V'(x) = kx + c$, $k = -\frac{3}{4}\gamma^2, \frac{3}{16}\gamma^2$; 3) a two-dimensional Lie group generated by the operators $P_0 = \partial_t$ and I, when $V'(x) \neq kx + c$.*

Investigation of the conditional invariance allows us to obtain new classes of the potential $V(x)$ for which we can find exact solutions of equation (1) [3, 4, 5]. Let us consider an infinitesimal operator of the form

$$Q = \xi^0(t, x, y, u)\frac{\partial}{\partial t} + \xi^1(t, x, y, u)\frac{\partial}{\partial x} + \xi^2(t, x, y, u)\frac{\partial}{\partial y} + \eta(t, x, y, u)\frac{\partial}{\partial u}. \tag{2}$$

We say that equation (1) is Q–conditionally invariant if the system of equations (1) and

$$Qu(t, x, y) = 0 \tag{3}$$

is invariant under the action of operator (2) [5].

Remark. If Q is a Q-conditional operator for a PDE, then the equation is $\bar{Q}$-conditionally invariant under $\bar{Q} = f(t,x,y,u)Q$, where $f(t,x,y,u)$ is an arbitrary function of independent and dependent variables. We say that Q and $\bar{Q}$ are equivalent operators. Then we may consider $\xi^0 = 1$ in (2) if $\xi^0 \not\equiv 0$.

Let us consider equation (2) with the potential

$$V' = kx^{-1/3} + \frac{3}{16}\gamma^2 x, \quad k \neq 0 \tag{4}$$

The operator

$$Q = \partial_t - \frac{3}{4}\gamma x \partial_x + \left(\frac{3}{8}\gamma^2 - \frac{1}{4}\gamma y\right)\partial_y + \gamma u \partial_u \tag{5}$$

is not a Lie symmetry operator of equations (1), (3) as follows from Theorem 1. However the operator (5) gives the invariant solution (ansatz)

$$u(t,x,y) = \exp(\gamma t)\varphi(\omega_1,\omega_2), \quad \omega_1 = \exp\left(\frac{3}{4}\gamma t\right)x, \quad \omega_2 = x^{-1/3}y + \frac{3}{4}\gamma x^{2/3}, \tag{6}$$

which reduces the equation. Indeed, substituting (6) into (1), (3), we obtain the reduced equation

$$\omega_2\varphi_1 - \left(\frac{\omega_2^2}{2} + k\right)\omega_1^{-1}\varphi_2 - \gamma\omega_1^{-1}\varphi_{22} = 0, \quad \varphi_i = \frac{\partial\varphi}{\partial\omega_i}, \quad \varphi_{22} = \frac{\partial^2\varphi}{\partial\omega_2\partial\omega_2}.$$

This equation may be integrated, in particular, when $\varphi = \varphi(\omega_2)$.

Theorem 2. *All Q-conditional operators of equation (1) with $V' \neq kx^{-1/3} + \frac{3}{16}\gamma^2 x$ are eqivalent to ∂_t, I.*

Theorem 3. *Equations (1) and (3) have the following Q-conditional symmetry operators*

$$Q = \partial_t + F(t)x\partial_x + \left(\frac{1}{3}F(t)y + F'(t)x + \frac{2}{3}F^2 x\right)\partial_y + fu\partial_u,$$

where $F(t)$ is an arbitrary solution of the equation

$$\begin{aligned}
&F'' + 2FF' + \frac{4}{9}F^3 - \frac{1}{4}\gamma^2 F = 0,\\
&2\gamma f = -y^2\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) - yx\left(\gamma F' + \frac{2}{3}\gamma F^2 + \frac{1}{2}\gamma^2 F\right) -\\
&\qquad x^2\left(\frac{3}{8}\gamma^2 F' + \frac{1}{4}\gamma^2 F^2 + \frac{3}{16}\gamma^3 F\right) - x^{2/3}k\left(2F' + \frac{4}{3}F^2 + \gamma F\right) +\\
&\qquad C\exp\left\{-\tfrac{2}{3}\int F dt\right\} - \tfrac{4}{3}\gamma F + \gamma^2,
\end{aligned}$$

where $C =$ const.

Theorem 4. *All Q-conditional symmetry operators ($\xi^0 = 1$) of equation (1) with $V' = kx$, $k \neq -\frac{1}{4}\gamma^2, \frac{3}{16}\gamma^2$ are equivalent to the Lie symmetry operators.*

Theorem 5. *All $Q-$conditional symmetry operators ($\xi^0 = 1$) of equation (1) with $V' = kx$, $k = -\frac{1}{4}\gamma^2, \frac{3}{16}\gamma^2$ (including those equivalent to Lie symmetry operators) have the following form*

$$Q = \partial_t + (F(t)x + G(t))\partial_x + \left[\frac{1}{3}F(t)y + \left(F'(t) + \frac{2}{3}F(t)^2\right)x + G'(t) + \frac{2}{3}F(t)G(t)\right]\partial_y + f(t,x,y)u\partial_u.$$

Here, the functions $F(t)$, $G(t)$ satisfy the following equations

$$F'' + 2FF' + \frac{4}{9}F^3 + \left(\frac{4}{5}k - \frac{2}{5}\gamma^2\right)F = 0,$$
$$G'' + \frac{4}{3}FG' + \frac{2}{3}F'G + \gamma G' + \frac{2}{3}\gamma FG + \frac{4}{9}F^2G + kG = h(t),$$

where $h(t)$ satisfies the equation

$$h'' + \left(\frac{4}{3}F - \gamma\right)h' + \left(k + \frac{2}{3}F' + \frac{4}{9}F^2 - \frac{2}{3}\gamma F\right)h = 0.$$

The function $f(t,x,y)$ is

$$a)\ k = -\frac{3}{4}\gamma^2,\quad 2\gamma f = -y^2\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) - y\left[x\left(\gamma F' + \frac{2}{3}\gamma F^2\right) + h\right] + \frac{3}{4}\gamma^3x^2F + x\left(h' - \gamma h + \frac{2}{3}Fh\right) + s(t),$$

$$b)\ k = \frac{3}{16}\gamma^2,\quad 2\gamma f = -y^2\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) - y\left[x\left(\gamma F' + \frac{2}{3}\gamma F^2 + \frac{1}{2}\right) + h\right] - x^2\left(\frac{3}{16}\gamma^3F + \frac{3}{8}\gamma^2F' + \frac{1}{4}\gamma^2F^2\right) + x\left(h' - \gamma h + \frac{2}{3}Fh\right) + s(t),$$

where in both a) and b) the function $s(t)$ satisfies the equation

$$s' + \frac{2}{3}Fs = -2\gamma\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) + \frac{4}{3}\gamma^2F.$$

2 Q_1,Q_2-conditional symmetry

Let we consider two operators Q_1 and Q_2 which have the form (2).

Definition [6]. *We say that equation (1) is Q_1,Q_2-conditionally invariant if the system*

$$\begin{aligned} &Q_1u = 0,\\ &Q_2u = 0,\\ &\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(V'(x)u) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right) \end{aligned} \tag{7}$$

is invariant under the operators Q_1 and Q_2.

Here, we restrict the form of the operators:

$$
\begin{aligned}
&Q_1 = \frac{\partial}{\partial x} - \eta^1(t,x,y,u)\frac{\partial}{\partial u},\\
&Q_2 = \frac{\partial}{\partial y} - \eta^2(t,x,y,u)\frac{\partial}{\partial u}.
\end{aligned}
\tag{8}
$$

System (7) for operators (8) can be written as

$$
\begin{aligned}
&\frac{\partial u}{\partial x} = \eta^1(t,x,y,u),\\
&\frac{\partial u}{\partial y} = \eta^2(t,x,y,u),\\
&\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(V'(x)u) + \gamma\frac{\partial}{\partial y}(yu + \frac{\partial u}{\partial y}).
\end{aligned}
\tag{9}
$$

Following the Lie's algorithm [7], we find that

$$
\begin{aligned}
&\frac{\partial \eta^1}{\partial t} + \gamma u\frac{\partial \eta^1}{\partial u} - V''(x)\eta^2 - (\gamma y + V'(x))\frac{\partial \eta^1}{\partial y} + y\frac{\partial \eta^1}{\partial x} - \gamma\eta^1 -\\
&\qquad \gamma\frac{\partial^2 \eta^1}{\partial y \partial y} - 2\gamma\eta^2\frac{\partial^2 \eta^1}{\partial y \partial u} = 0,\\
&\frac{\partial \eta^2}{\partial t} + \gamma u\frac{\partial \eta^2}{\partial u} - \gamma\eta^2 - (\gamma y + V'(x))\frac{\partial \eta^2}{\partial y} + y\frac{\partial \eta^2}{\partial x} - \gamma\eta^2 -\\
&\qquad \gamma\frac{\partial^2 \eta^2}{\partial y \partial y} - 2\gamma\eta^2\frac{\partial^2 \eta^2}{\partial y \partial u} = 0.
\end{aligned}
\tag{10}
$$

It is easy to show that $\eta^1 = -V'(x)u$, $\eta^2 = -yu$ is a solution of system (10). According to the algorithm [5] using the operators

$$
Q_1 = \frac{\partial}{\partial x} - V'(x)u\frac{\partial}{\partial u}, \qquad Q_2 = \frac{\partial}{\partial y} - yu\frac{\partial}{\partial u},
\tag{11}
$$

we find the ansatz invariant under operators (8)

$$
u = \varphi(t)\exp\left\{-V(x) - y^2/2\right\}.
\tag{12}
$$

Substitution of (12) into (1) gives $\varphi'(t) = 0$. So, we find the solution which is the Boltzmann distribution

$$
u(x,y) = N\exp\left\{-V(x) - y^2/2\right\}
$$

(N is a normalization constant). It is a stationary solution.

From the above example we see that the further work on finding Q- and Q_1, Q_2-conditional symmetry operators is of great interest.

One of the author (SS) is grateful to the DFFD of Ukraine (project 1.4/356) for the financial support.

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