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On Relativistic Non-linear Quantum Mechanics

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Abstract

The possibility that quantum mechanics itself could be non-linear has run up against difficulties with relativistic covariance. Most of the schemes proposed up to now engender superluminal communication, and those that don't, have been equivalent to linear theories. We show in a simplified model that a proposal based on the consistent histories approach to quantum mechanics avoids the usual difficulties and a relativistic quantum theory with non-linearly defined histories is possible.

1 Introduction

There is presently a growing interest in non-linear quantum mechanics resulting from a variety of motivations: fundamental speculation, presence of gravity, string theory, representations of current algebras, etc. Although apparently well motivated, it became apparent that non-linear theories suffer from some *prima facie* serious difficulties. These are of various types, but the most notable is conflict with relativity or causality. N. Gisin [1, 2] and G. Svetlichny [3] pointed out that non-linearity allows us to use EPR-type correlations and the instantaneous nature of state-vector collapse to send a signal across a space-like interval. Analyzing further, one finds that one has in fact a contradiction with relativity [4]. Certain progress has been made in overcoming these difficulties. One of the proposals is based on the idea that since the difficulty stems from the instantaneous state-vector collapse in measurement, a modification of measurement algorithms could allow for non-linear processes without superluminal signals. G. A. Goldin, H.-D. Doebner and P. Nattermann [5, 6, 7] have argued that non-linearity *per se* does not lead to superluminal signals (this was also pointed out by Svetlichny [4]), as with the *prima facie* reasonable assumptions that all measurements are in the end expressible in terms of measurements of position, certain non-linear Schrödinger equations are then observationally equivalent, via a non-linear "gauge transformation", to the free linear equation. We shall call these the GDN theories. Unfortunately, these theories, and others studied by these authors, are non-relativistic, and we are still far from understanding the true relation of linearity to relativity.

Here we adopt an even more radical view and reconsider the question from the point of view of a quantum theory without measurements, as the complete absence of the "measurement process" will eliminate any obstruction to non-linearities from the manifest non-covariance of this process. Of the several "measurementless" theories, the one most adaptable to relativistic considerations is the consistent histories approach already widely discussed in the literature [8, 9].

We argue that in the histories approach, non-linearity and relativistic covariance can indeed coexist peacefully and present a simple model to support this view. Such a model cannot yet be taken as a proposal for a realistic theory but does establish the logical point and suggests where one should look for experimental evidence.

2 Linear and non-linear histories

Let $\mathcal{H}$ be a Hilbert space and $\Psi \in \mathcal{H}$ a normalized vector. For each $i = 1, \ldots, n$, let $P_j^{(i)}$ where $j = 1, 2, \ldots, n_i$, be a finite resolution of the identity. We call each state vector of the form

$$P_\alpha \Psi = P_{\alpha_n}^{(n)} \cdots P_{\alpha_j}^{(j)} \cdots P_{\alpha_2}^{(2)} P_{\alpha_1}^{(1)} \Psi \tag{1}$$

a *history*. Let

$$p_\alpha = ||P_\alpha \Psi||^2. \tag{2}$$

One interpretation of the above quantities is that Ψ is a Heisenberg state and that $P_j^{(i)}$ is the spectral resolution of a Heisenberg observable $A^{(i)} = \sum \lambda_j^{(i)} P_j^{(i)}$ at time t_i where $t_1 < t_2 < \cdots < t_{n-1} < t_n$. In this case, p_α is the joint probability of getting the sequence of outcomes $\lambda_{\alpha_1}^{(1)}, \ldots, \lambda_{\alpha_n}^{(n)}$ in a sequence of measurements that correspond to the observables $A^{(1)}, \ldots, A^{(n)}$. The coherent histories interpretation of quantum mechanics however goes beyond this viewpoint and in certain special cases interprets p_α as the probability of the history Q_α even if no actual measurements are made. It is a way of assigning probabilities to alternate views of the quantum state Ψ, corresponding to the possible different sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$. Such an attitude is maintained only if a condition, called *consistency*, or even a stronger condition called *decoherence*, is satisfied by the set of alternative histories.

The notion of consistent histories forms the basis of a new interpretation of quantum mechanics that in a certain sense transcends at the same time the Copenhagen interpretation and the Everett many-worlds one. As such it has attracted the attention of cosmologists. Its main feature that makes it attractive to the present case is that it does not rely on the notion of measurement nor on the collapse of the wave-function. Thus even though (1) can be interpreted as a sequence of evolutions and collapses, this is not essential, and (2) can be viewed as merely a formula for a joint probability. In a more generalized setting, the evolution-collapse picture is not even possible for some sets of histories. This view of quantum mechanics thus transcends the notions of instantaneous state, its evolution, and its collapse, which means that it is well suited for formulating theories in which these notions are troublesome, such as non-linear quantum mechanics.

The most naive way to adapt the consistent histories approach to non-linear quantum mechanics is to replace in (1) the linear projectors $P_j^{(i)}$ by non-linear operators $B_j^{(i)}$ and so introduce the *non-linear histories*

$$B_\alpha \Psi = B_{\alpha_n}^{(n)} \cdots B_{\alpha_j}^{(j)} \cdots B_{\alpha_2}^{(2)} B_{\alpha_1}^{(1)} \Psi,$$

with the corresponding probability function

$$b_\alpha = ||B_\alpha \Psi||^2.$$

Such expressions are in fact the correct ones for a succession of measurements for the GDN theories.

The most primitive property that the operators B should satisfy is that $\sum_\alpha b_\alpha = 1$. This is true in particular if one has $\sum_j ||B^{(i)}_j \Phi||^2 = 1$ for every i and all Φ , which is the case of GDN.

To complete the rest of the program and have an *interpretation* of this non-linear quantum mechanics, similar to the consistent histories approach of linear quantum mechanics, one needs to address the notions of consistency in the nonlinear context. There is no *a priori* difficulty in formulating such a notion, though the stronger notion of decoherence may not survive the passage to nonlinearity.

We shall not address the interpretational issues in this paper and only limit ourselves to showing that one can pass on to non-linearity while maintaining Lorentz covariance. Again, the most naive way to envisage Lorentz covariance is to assume that there is a unitary representation $U(g)$ of the Poincaré group along with an action ϕ_g of the same on suitable non-linear operators such that it makes sense to talk of the transformed histories

$$\tilde{B}_\alpha \tilde{\Psi} = \tilde{B}^{(n)}_{\alpha_n} \cdots \tilde{B}^{(j)}_{\alpha_j} \cdots \tilde{B}^{(2)}_{\alpha_2} \tilde{B}^{(1)}_{\alpha_1} \tilde{\Psi}.$$

where $\tilde{B} = \phi_g(B)$ and $\tilde{\Psi} = U(g)\Psi$. Lorentz covariance would then be expressed through the statement $\tilde{b}_\alpha = b_\alpha$. Such a scheme holds in the GDN case for Euclidean and Galileian covariances.

Now it should not be very hard to implement the above scheme without further constraints, but for an interesting theory, one should require a locality condition that would preclude superluminal signals. It would only be then that one could say that one has overcome the relativistic objections to non-linear theories. This is the concern of the next sections.

3 Free quantum fields

This section is based on the suggestion presented in [4]. Consider a free neutral scalar relativistic quantum field. For each limited space-time region $\mathcal{O}$, let $\mathcal{A}(\mathcal{O})$ be the von-Neuman algebra of observables associated to $\mathcal{O}$. Consider now a set of limited space-time regions $\mathcal{O}_1, \ldots \mathcal{O}_n$ which are so disposed that for any two, either all points of one are space-like in relation to all points of the other, or they are time-like. Assume the regions are numbered so that whenever one is in the time-like future of another, then the first one has a greater index. Let $P_i \in \mathcal{A}(\mathcal{O}_i)$ be orthogonal projections that correspond to outcomes of measurements made in the corresponding regions. Let Ψ represent a Heisenberg state in some reference frame and prior to all measurements. According to the usual rules, the probability to obtain all the outcomes represented by the projections is: $||P_n \cdots P_2 P_1 \Psi||^2$. We will modify this formula by replacing P_i by B_i, a possibly non-linear operator, likewise somehow associated to the region $\mathcal{O}_i$, whenever there is a region $\mathcal{O}_j$ that is time-like past to the given one. This effectively differentiates between space-like and time-like conditional probabilities. For this to be consistent, relativistic, and causal, the (in general non-linear) operators B_i have to satisfy certain constraints. There are several ambiguities in the above construction. The relative order of the $\mathcal{O}_i$ is not determined except for the case of time-like separation. Presumably, the ambiguity of the order corresponds to the possible choices of the time-like reference direction. Even for some of these, two space-like separated regions may not be separated by the time coordinate, in such cases we shall suppose that the relative order does not matter. This means that we should allow certain permutations of the sequence $\mathcal{O}_1, \ldots, \mathcal{O}_n$ and the choice of such a permutation must not affect the final assignment of probabilities. We shall call each allowed permutation an *admissible sequence*. There is also the ambiguity in the relation $P_i \in \mathcal{O}_i$, resulting from the inclusion $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ whenever $\mathcal{O}_1 \subset \mathcal{O}_2$. This too must not affect the final assignment. The operators B_i can depend on several aspects of the construction. In principle, each B_i can depend on the full set of constituents $\{\Psi, P_1, \mathcal{O}_1, \ldots, P_n, \mathcal{O}_n\}$ and even, in a self-consistent manner, on the choices of the other B_j. Such generality leaves little room for insight. Given that, we are trying to establish here a point of principle, that non-linear relativistic quantum mechanics *is* possible, and not propose what would be a realistic theory, we shall look for the simplest type of modification. For a typical datum $P \in \mathcal{A}(\mathcal{O})$ in an admissible sequence, we shall at times write $B_{\mathcal{O}}$ for the corresponding B.

1. If $\mathcal{O}_i$ and $\mathcal{O}_j$ are space-like separated, then
$$[B_i, P_j] = [P_i, B_j] = [B_i, B_j] = 0 \tag{3}$$

2. If $\mathcal{O} \subset \mathcal{O}'$ and $P \in \mathcal{A}(\mathcal{O})$ is a projector, then in any sequence in which P and $\mathcal{O}$ take place for which changing $\mathcal{O}$ to $\mathcal{O}'$ results in a new admissible sequence (with, of course, identical spatial-temporal relation between the regions), one has $B_{\mathcal{O}'} = B_{\mathcal{O}}$

3. If $U(g)$ is a unitary operator representing the element g of the Poincaré group, then $B_{g\mathcal{O}} = U(g) B_{\mathcal{O}} U(g)^*$

4. For all resolutions of identity $P_j \in \mathcal{A}(\mathcal{O})$ one has $\sum ||B_j \Phi||^2 = ||\Phi||^2$ for all states Φ.

The bracket in (3) is a commutator, for example, $[B_i, P_j] = B_i P_j - P_j B_i$ and not the Lie bracket of the two operators interpreted as vector fields on Hilbert space, which for non-linear operators would be different.

We leave the problem of finding operators B_i to the next section and first discuss some of the consequences.

Let us now pick in each space-time region $\mathcal{O}_i$ a finite resolution of the identity $P_j^{(i)}$, where $j = 1, 2, \ldots, n_i$ with $P_j^{(i)} \in \mathcal{A}(\mathcal{O}_i)$. We have $\sum_j P_j^{(i)} = I$.

Consider now the expression
$$p_\alpha = ||B_{\alpha_1}^{(n)} \cdots B_{\alpha_j}^{(j)} \cdots B_{\alpha_2}^{(2)} B_{\alpha_1}^{(1)} \Psi||^2, \tag{4}$$
where $B_j^{(i)} = P_j^{(i)}$ if there is no region $\mathcal{O}_k$ to the time-like past of $\mathcal{O}_j$ and a possibly different operator if there is. Expression (4) is to be interpreted as the joint probability distribution of alternate histories as discussed in the previous section. From property (4) it follows that $\sum_\alpha p_\alpha = 1$ so that the interpretation as a joint probability is consistent. Property (1) assures us that the mentioned ambiguity in the temporal order of space-like separated regions does not affect the numerical values of the probabilities p_α, but only the way they are labeled. Property (3) assures us that the mentioned ambiguity in associating

a region to a projector leaves the resulting probabilities the same. Finally, property (3) assures us that if we replace the data

$$\{\Psi, P_1, \mathcal{O}_1, \ldots, P_n, O_n\}$$

by

$$\{U(g)\Psi, U(g)P_1U(g)^*, g(\mathcal{O}_1), \ldots, U(g)P_nU(g)^*, g(O_n)\},$$

the resulting probabilities don't change, that is, the theory is relativistically covariant. One still does not know how to compute joint probabilities for events in regions that are neither space-like nor time-like to each other, nor exactly how to interpret the formalism based on the consistent histories approach. We leave this question for posterior investigation. In any case, when only space-like separated regions occur in the histories, then the situation is the same as in conventional quantum mechanics.

To show that joint probabilities as defined above do not lead to superluminal signals, suppose one region $\mathcal{O}$ is space-like separated in relation to all the others. Then by the considerations above, one can label the regions so that $\mathcal{O} = \mathcal{O}_1$. The probability of observing the event that corresponds to $P_j^{(1)}$ is, if no other observations are made, given by $||P_j^{(1)}\Psi||^2$, and by condition (4) it is also $\sum_{\alpha_2,\ldots,\alpha_n} p_{\alpha_n,\ldots,\alpha_2,j}$ that is, the marginal probability if the other observations *are* made. This means that the probability of an event is independent of what happens in space-like separated regions, and so no signals using long-range correlations are possible, just as in the linear case.

One would thus have a non-linear relativistic quantum mechanics if conditions (1–4) can be realized.

4 A simple explicit model

Free fields can be realized in appropriate Fock spaces. One has $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_0 = \mathbb{C}$ is the subspace spanned by the vacuum, and each $\mathcal{H}_n$ is the n-particle subspace. We consider a free scalar field $\phi(x)$ of mass m. In configuration space, H_n now consists of symmetric functions $\Phi(x_1, \ldots x_n)$ of n space-time coordinates which obey a Klein-Gordon equation in each space-time variable and contain only positive-frequency Fourier components in each momentum variable.

A very simple way of satisfying (1–4) is to set in the time-like case $B_j^{(i)} = BP_j^{(i)}B^{-1}$, where B is an invertible not-necessarily linear operator that is Poincaré invariant, $U(g)BU(g)^* = B$. In this case, it is easily seen that all the conditions are automatically satisfied except possibly for the case of condition (1) which involves a modified and a non-modified projector, that is,

$$[BPB^{-1}, Q] = 0 \tag{5}$$

if P and Q are orthogonal projectors belonging to space-like separated regions, and condition (4) which would be satisfied if B were norm-preserving $||B\Psi|| = ||\Psi||$. If B is a real homogeneous operator $Br\Psi = rB\Psi$ for real r, then one can define a new operator $(||\Psi||B\Psi)/||B\psi||$ which is now norm-preserving and continues to satisfy all the other desired properties, so we shall not concern ourselves anymore with norm-preservation.

A stronger condition than (5) would be to assume that

$$[BAB^{-1}, C] = 0 \tag{6}$$

if A and C are operators belonging to von-Neuman algebras of space-like separated regions. Such a condition may seem somewhat strong given that B is supposed to be non-linear, but one sees similar situations in the GDN theories, in which the non-linear operators are in fact linear on spaces generated by functions with disjoint supports. In the GDN theories such a property follows essentially from homogeneity and the local character of differential operators. As such we can hope to achieve it in our case also. In particular, one should have for smeared fields

$$[B\phi(f)B^{-1}, \phi(g)] = 0 \tag{7}$$

for f and g with space-like separated supports.

We must now face the task of finding an appropriate B. Now it is not hard to find Poincaré invariant non-linear operators. As an example, for each n, let M_n be a permutation and Lorentz invariant non-linear differential operator acting on a function $g(x_1, \ldots, x_n)$ of n space-time points. One can apply M_n to the n-particle component Φ_n of a vector in a Fock space. Now $M_n\Phi_n$ is not necessarily a positive-frequency solution of the Klein-Gordon equation, but we can then convolute it with an appropriate Green's function. Define the operator C by $(C\Phi)_n = \Delta^{(+)\otimes n} \star M_n\Phi_n$ where $\Delta^{(+)\otimes n}$ is the n-fold tensor product of $\Delta^{(+)}(x)$, the positive frequency invariant Green's function for the Klein-Gordon equation, and $\star$ denotes convolution. Much more elaborate operators in which the various n-particle sectors get coupled can also be constructed.

The difficulty in (7) is of course the presence of B^{-1}. We shall try to overcome this by assuming that B is a part of a one-parameter group $B(r)$ generated by a non-linear operator K. Thus, the equation $\frac{d}{dr}\Phi(r) = K\Phi(r)$ is solved by $\Phi(r) = B(r)\Phi(0)$. We assume $B = B(1)$ and that (7) holds for each $B(r)$. To the first order, one then has

$$[[K, \phi(f)], \phi(g)] = 0 \tag{8}$$

for f and g with space-like separated supports. This equation imposes a recursive series of constraints on the n-particle operators K_n for which, however, there are no formal obstructions. We shall not here go into a full analysis of these constraints as the typical situation already arises when we apply (8) to the vacuum state. We assume that the K_n operators do not change the number of particles. One must have $K_0 = 0$ as the vacuum is the unique Lorentz invariant state. For a one particle function $g(x)$, let $\tilde{g} = i\Delta^{(+)} \star g$, and let $\hat{\otimes}$ denote the symmetric tensor product. One then derives from (8), applying $[[K, \phi(f)], \phi(g)]$ to the vacuum, that if f and g have space-like separate supports, then

$$K_2\tilde{f}\hat{\otimes}\tilde{g} = (K_1\tilde{f})\hat{\otimes}\tilde{g} + \tilde{f}\hat{\otimes}(K_1\tilde{g}), \tag{9}$$

$$0 = (g, K_1\tilde{f}) + (f, K_1\tilde{g}), \tag{10}$$

where $(g, f) = \int g(x)f(x)\,dx$. Now (9) defines K_2 on the symmetric tensor product of two functions in terms of K_1. This is similar to the tensor derivation property for separating non-linear Schrödinger equations [10]. We can thus assume that the hierarchy K_n is in fact a tensor derivation with respect to the symmetric tensor product. Equation (10) must now hold for functions with space-like separated supports. This will hold, just as it does in the linear free quantum field theory if K_1 does not change the support of a function on which it acts, which is not hard to achieve. The conclusion now is that in fact, at least formally, a causal non-linear relativistic quantum mechanics of the type described in the initial sections of this paper is possible.

5 Conclusions

The previous two sections have argued the logical point that, indeed, relativistic nonlinear histories without superluminal signals are possible. Because of it's *ad hoc* nature, the model presented above cannot be considered realistic. In particular, if the space-time regions involved constitute a time-like chain, then the joint probabilities in the non-relativistic limit would not be of the type governed by a non-linear Schrödinger equation. Thus, we have not yet shown that the GDN theories can be obtained as non-relativistic limits. All the schemes based on non-linear histories which differentiate between space-like and time-like joint-probabilities in principle should exhibit physical effects as one crosses the light cone. Thus, in a typical photon correlation experiment, one can delay the light-ray on one side so that at a certain point the detector events become time-like. In crossing the light cone, an effect should be present that was not forseen by the linear theory. This happens in the models above as $||PQ\Phi||^2$ suddenly becomes $||BQ\Phi||^2$. There are strong plausibility arguments [4, 11] that theories that do not suffer such discontinuities at the future light-cone are necessarily linear, so a true verification of non-linearity would involve light-cone experiments. Theories of the type here considered are thus *light cone singular* and, for such theories, the notion of non-relativistic limit has to be modified. Whereas in usual theories the non-relativistic regime is one for which all relevant velocities are small compared to the velocity of light, in light-cone singular theories one must add the requirement that all relevant space-time intervals be time-like. This further requirement removes the paradox that a causal relativistic theory may have a non-relativistic limit that seems manifestly acausal by allowing instantaneous signals through long-range correlations, and may thus remove a major objection to formulation of non-linear quantum mechanics.

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