'Homotopy' of Prandtl and Nadai solutions

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Preliminaries

Actually, the mathematical theory of plasticity is one of the detailed parts of solid mechanics. The study of the plane ideal plasticity is of a fundamental importance in mechanical and civil engineering, because it serves as a model problem to calculate different technological processes.

A systematic method of determining stress fields in ideal plastic bodies obeying the Saint-Venant – Mises' yield criterion in plane strain was developed in the 1920s by Prandtl, Hencky, Mises and others. This method, generally known as the slip line theory, is based on an analysis of characteristic curves (known in the mathematical plasticity theory as slip lines) of the hyperbolic system of plane plasticity.

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As for exact closed-form solutions of the system, there are a few of them:

- the Prandtl solution [L. Prandtl, 1923] to describe stresses of a rectangular block of plastic-rigid material compressed between rigid parallel plates which are assumed to be rough;
- the solution for a cavity of circular form, stressed by uniform pressure;
- Nadai solutions: a) for the stresses in the plastic region around a circular cavity loaded by a constant shear stress and b) solution for the channel with straight line borders [A. Nadai, 1924];
- the spiral-symmetrical solution for the channel with logarithmic spiral borders [B. Annin, 1985].

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Prandtl solution, being the first one, has obtained numerous generalizations both theoretically for the three-dimension [Ishlinskii, 1988] and plane cases, and for some practical applications.

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The talk is structured as follows:

1. we provide some known results for the system of plane ideal plasticity;
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Plane plasticity system

Two equilibrium equations and strongly nonlinear Saint-Venant – Mises' yield criterion (condition on the second invariant of the stress tensor):

$$\begin{gathered}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0,\\ (\sigma_x-\sigma_y)^2+4\tau_{xy}^2=4k^2,\end{gathered} \tag{1}$$

$\sigma_x, \sigma_y, \tau_{xy}$ are components of a stress tensor, k is a constant of plasticity.

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change of variables by Lévy

$$\sigma_x = \sigma - k \sin 2\theta,$$
$$\sigma_y = \sigma + k \sin 2\theta,$$
$$\tau_{xy} = k \cos 2\theta,$$

system (1) $\Rightarrow$ quasilinear one:

$$\frac{\partial \sigma}{\partial x} - 2k\left(\frac{\partial \theta}{\partial x}\cos 2\theta + \frac{\partial \theta}{\partial y}\sin 2\theta\right) = 0, \qquad (2)$$
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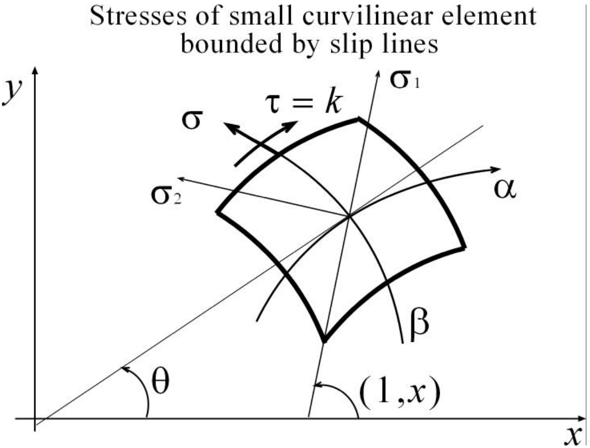

σ is hydrostatic pressure, $\theta + \pi/4$ is the angle between the first principal direction of a stress tensor and the *ox*-axis.

System is a hyperbolic one and has two families of characteristic curves defined from equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\cot\theta.$$

with corresponding Riemann invariants:

$$\xi = \sigma/(2k) - \theta, \ \eta = \sigma/(2k) + \theta.$$

by means of applying hodograph transformation $x = x(\sigma, \theta)$, $y = y(\sigma, \theta)$ one can obtain the corresponding linear system ($J \neq 0$):

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$$x = u \cos \theta - v \sin \theta, \ \ y = u \sin \theta + v \cos \theta,$$

and taking ξ, η as a new independent ones:

$$\frac{\partial u}{\partial \xi} + \frac{v}{2} = 0, \ \ \frac{\partial v}{\partial \eta} + \frac{u}{2} = 0.$$

Linear system

$$\frac{\partial u}{\partial \xi} + \frac{v}{2} = 0, \ \frac{\partial v}{\partial \eta} + \frac{u}{2} = 0.$$

is integrated by the method of Riemann and generally is expressed in therms of Bessel function of zero order [Geiringer, 1958].

$(x, y) \Longleftrightarrow (\sigma, \theta)$:

1. if $J_1 = \partial(\sigma, \theta)/\partial(x, y) = 0$ we could not linearize (simple stress state).
2. if $J_2 = \partial(x, y)/\partial(\sigma, \theta) = 0$ we couldn't regress.

[H.Geiringer, 1958]: if a family of slip lines has an envelope ($J_2 = 0$), then it well be a natural boundary for the analytic solution.

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Admitted symmetries

[Senashov, 1988]: Lie algebra L of point transformations is formed by:

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - \frac{\partial}{\partial \theta},\ X_3 = \frac{\partial}{\partial \sigma},$$

$$X_4 = \xi_1(x,y,\sigma,\theta)\frac{\partial}{\partial x} + \xi_2(x,y,\sigma,\theta)\frac{\partial}{\partial y} - 4k\theta\frac{\partial}{\partial \sigma} - \frac{\sigma}{k}\frac{\partial}{\partial \theta},$$

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where

$$\xi_1 = x\cos 2\theta + y\sin 2\theta + y\frac{\sigma}{k},\ \xi_2 = x\sin 2\theta - y\cos 2\theta - x\frac{\sigma}{k},$$

and (x_0, y_0) is an arbitrary solution of linearized system.

- X_1 scales in the plane xy: $x' = e^{a_1}x,\ y' = e^{a_1}y$;
- X_2 rotation group:

$$x' = x\cos a_2 + y\sin a_2,\ y' = -x\sin a_2 + y\cos a_2,\ \theta' = \theta + a_2;$$

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One parametric group of X_4:

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x' &= u e^{a_4} \cos\theta' - v e^{-a_4} \sin\theta', \\
y' &= u e^{a_4} \sin\theta' + v e^{-a_4} \cos\theta', \\
\sigma' &= 2k\left(\frac{\sigma}{2k}\cosh 2a_4 - \theta \sinh 2a_4\right), \\
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where u and v are Mikhlin variables:

$$u = x\cos\theta + y\sin\theta,\;\; v = -x\sin\theta + y\cos\theta.$$

X_4 acts over $u(\xi,\eta)$, $v(\xi,\eta)$ as a scales:

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so for x, y, σ, θ we can call them ***quasi-scales***.

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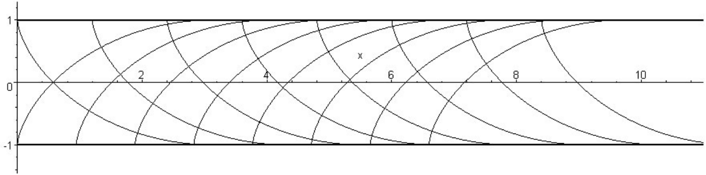
Prandtl solution

In terms of variables σ, θ has the form:

$$\sigma = -p_1 - k\frac{x}{h} + k\sqrt{1 - \frac{y^2}{h^2}}, \;\; y = h\cos 2\theta,$$

where $2h = \text{const}$ is the height of a block, $p_1 = \text{const}$ is a value of the pressure on the plate when $x = 0$. Boundary conditions:

$$\theta\big|_{y=h} = \pi n, \;\; n \in \mathbb{Z}, \;\; \sigma\big|_{y=h} = -p_1 - k\frac{x}{h}.$$



The slip lines families are the parts of cycloids:

$$x = h(\mp 2\theta - \sin 2\theta) - h\left(2C_i + p_1/k\right), \;\; y = h\cos 2\theta, \quad i = 1, 2,$$

have two envelopes $y = \pm h$.

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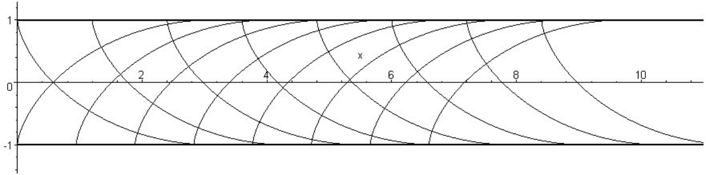

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Is invariant solution for subalgebra $< X_3 + \gamma X_5 >$.
Acting by quasi-scales X_4 we obtain «reproduced» solution:

$$-\frac{x}{h} = e^{a_4}\sin\theta\cos\theta' + e^{-a_4}\cos\theta\sin\theta' + \\ + \frac{\sigma' + p_1}{k}(e^{a_4}\sin\theta\sin\theta' + e^{-a_4}\cos\theta\cos\theta'),$$

$$\frac{y}{h} = e^{a_4}\cos\theta\cos\theta' - e^{-a_4}\sin\theta\sin\theta' + \\ + \frac{\sigma' + p_1}{k}(e^{a_4}\cos\theta\sin\theta' - e^{-a_4}\sin\theta\cos\theta'),$$

where $\theta = \dfrac{\sigma'}{2k}\sinh 2a_4 + \theta'\cosh 2a_4$ and parametric equations for «deformed» slip lines (θ' is parameter):

$$x = -\frac{h}{k}\left(2k(K_1 + \theta') + p_1\right)\left(\cosh a_4\cos(\theta - \theta') - \sinh a_4\cos(\theta + \theta')\right) - \\ - h\left(\sinh a_4\sin(\theta - \theta') + \cosh a_4\sin(\theta + \theta')\right),$$

$$y = -\frac{h}{k}\left(2k(K_1 + \theta') + p_1\right)\left(\cosh a_4\sin(\theta - \theta') - \sinh a_4\sin(\theta + \theta')\right) - \\ - h\left(-\sinh a_4\cos(\theta - \theta') - \cosh a_4\cos(\theta + \theta')\right), \ \theta = K_1\sinh 2a_4 + \theta' e^{2a_4},$$

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$$+\frac{\sigma' + p_1}{k}(e^{a_4}\cos\theta\sin\theta' - e^{-a_4}\sin\theta\cos\theta'),$$

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$$x = -\frac{h}{k}\left(2k(K_1 + \theta') + p_1\right)\left(\cosh a_4\cos(\theta - \theta') - \sinh a_4\cos(\theta + \theta')\right) -$$
$$- h\left(\sinh a_4\sin(\theta - \theta') + \cosh a_4\sin(\theta + \theta')\right),$$
$$y = -\frac{h}{k}\left(2k(K_1 + \theta') + p_1\right)\left(\cosh a_4\sin(\theta - \theta') - \sinh a_4\sin(\theta + \theta')\right) -$$
$$- h\left(-\sinh a_4\cos(\theta - \theta') - \cosh a_4\cos(\theta + \theta')\right), \ \theta = K_1\sinh 2a_4 + \theta' e^{2a_4};$$

To construct the envelope for the family of characteristics $x = x(\theta', K_i)$, $y = y(\theta', K_i)$ use necessary condition of existence:

$$\frac{\partial x}{\partial K_i}\frac{\partial y}{\partial \theta'} - \frac{\partial y}{\partial K_i}\frac{\partial x}{\partial \theta'} = 0, \quad i = 1, 2,$$

due to relations along characteristics gives for K_i:

$$\frac{\partial x}{\partial K_1} - \frac{\partial y}{\partial K_1}\cot\theta' = 0, \quad \frac{\partial x}{\partial K_2} + \frac{\partial y}{\partial K_2}\tan\theta' = 0,$$

therefore

$$K_1 = -\theta' - p_1/(2k) + \left(e^{2a_4}/\sinh 2a_4 - 1/2\right)\tan\theta', \;\; a_4 \neq 0$$
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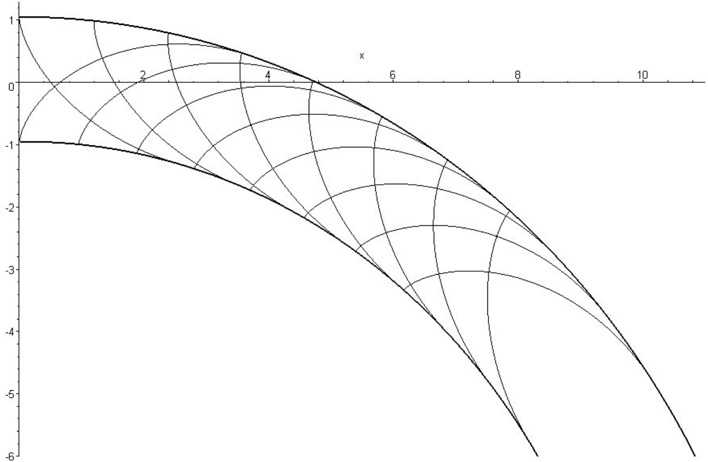
Slip line field looks as shown



describes the block of plastic-rigid material compressed between rigid plates of specific form.

Group foliation

Quasilinear plasticity system is automorphic one with respect to the group

$$X_5 = x_0(\sigma,\theta)\frac{\partial}{\partial x} + y_0(\sigma,\theta)\frac{\partial}{\partial y}$$

$$x' = x + a_5 x_0(\sigma,\theta),$$
$$y' = y + a_5 y_0(\sigma,\theta),$$

since any nonsingular solution (with Jacobian $\neq 0$) can be moved to another nonsingular solution by group transformation.

Let $\chi_1 = (x_1(\sigma,\theta), y_1(\sigma,\theta))$, $\chi_2 = (x_2(\sigma,\theta), y_2(\sigma,\theta))$ are two solutions of linearized system, define implicitly two solutions U_1 и U_2 of quasilinear system. Let us take in X_5:

$$x_0 = x_1 - x_2, \quad y_0 = y_1 - y_2 \Rightarrow$$

$$x' = x_2 + a_5 x_0 = a_5 x_1 + (1 - a_5) x_2,$$
$$y' = y_2 + a_5 y_0 = a_5 y_1 + (1 - a_5) y_2,$$

that gives the linear combination of two solutions and defines the family of reproduced solutions:

$$\sigma = \sigma(x, y, a_5), \quad \theta = \theta(x, y, a_5).$$

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One can relate two nonsingular solutions U_1, U_2, represented in the form χ_1, χ_2.

The linear combination of this form can be called «homotopy» of solution χ_1, χ_2.

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Nadai solution [Nadai, 1924]

in therms of the functions σ, θ can be written:

$$\sigma = -kc\left[\ln\left(x^2+y^2\right) + \ln\left\{c + \sin\left(2\theta - 2\arctan\frac{y}{x}\right)\right\}\right] + A,$$

$$\theta = \arctan\frac{y}{x} - \frac{\pi}{4} + \arctan\left\{\sqrt{\frac{c-1}{c+1}}\tan\frac{\sqrt{c^2-1}}{c}\left(\theta + \frac{\pi}{4}\right)\right\},$$

satisfied boundary conditions:

$$\theta|_{\varphi=\alpha} = \alpha, \quad \sigma|_{\varphi=\alpha} = -kc\ln\left(x^2+y^2\right) + A.$$

Constant $c > 1$ is related to channel angle 2α in the following way:

$$\alpha + \pi/4 = \frac{c}{\sqrt{c^2-1}}\arctan\sqrt{(c+1)/(c-1)}, \quad \alpha \in (0, \pi/2).$$

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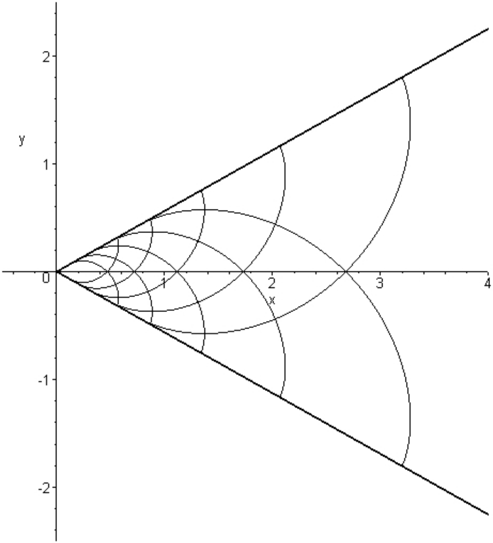
The sides of the channel are rough and it is supposed that the frictional stress is constant.



flow of plastic material through the wedge-shaped converging channel (total angle 2α)

The Nadai solution for the linearized system has the form (index N):

$$x_N = \pm \exp\left(\frac{A-\sigma}{2kc}\right) S^{-1}(\theta), \quad y_N = \pm x_N T(\theta),$$

$$T(\theta) = \tan\left[\theta + \pi/4 - \right.$$

$$\left. - \arctan\left\{\sqrt{\frac{c-1}{c+1}} \tan\frac{\sqrt{c^2-1}}{c}\left(\theta + \frac{\pi}{4}\right)\right\}\right],$$

$$S(\theta) = \sqrt{c + cT^2(\theta) + (1 - T^2(\theta))\sin 2\theta - 2T(\theta)\cos 2\theta}.$$

The Prandtl solution of the linearized system (index P):

$$x_P = -\sigma h/k - p_1 h/k - h\sin 2\theta, \quad y_P = h\cos 2\theta.$$

Homotopy of two solutions:

$$x = ax_N + (1-a_5)x_P, \; y = a_5 y_N + (1-a_5)y_P.$$

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$$\Gamma_1 : x = (a_5 - 1)h\left(\sin 2\theta - 2c\ln\left(2hc\frac{a_5 - 1}{a_5}\frac{S(\theta)}{1 - T(\theta)\cot\theta}\right)\right)$$
$$- \frac{2(1 - a_5)hc}{1 - T(\theta)\cot\theta} - \frac{(1 - a_5)h}{k}(A + p_1), \ \theta \in (0, \alpha),$$
$$y = (1 - a_5)h\cos 2\theta - \frac{2(1 - a_5)hc}{1 - T(\theta)\cot\theta}T(\theta);$$

$$\Gamma_2 : x = (a_5 - 1)h\left(\sin 2\theta - 2c\ln\left(2hc\frac{a_5 - 1}{a_5}\frac{S(\theta)}{1 + T(\theta)\tan\theta}\right)\right)$$
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$$y = (1 - a_5)h\cos 2\theta - \frac{2(1 - a_5)hc}{1 + T(\theta)\tan\theta}T(\theta),$$
$$\theta \in (-\alpha - \pi/2, -\pi/2).$$

Note, that envelope Γ_1 is transformed to envelope Γ_2 through the change of θ for $-\pi/2 - \theta$.

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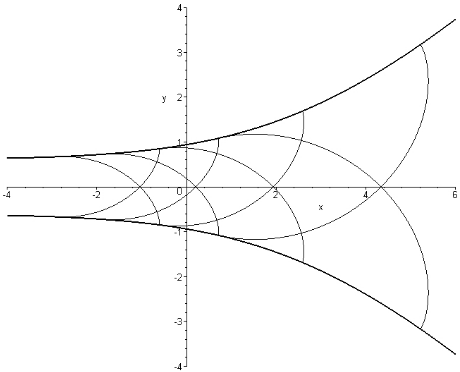
Note, that envelope Γ_1 is transformed to envelope Γ_2 through the change of θ for $-\pi/2 - \theta$.

For $a_5 \in (0,1)$ the homotopy solution is an exact implicit solution of plasticity system. It describes the stresses for the block with borders Γ_1, Γ_2.



$$a_5 = 0.4, c = 1.4, A = 0, h = p_1 = k = 1$$

Boundary conditions:

$$\sigma|_{\Gamma_1} = A - 2kc \ln\left(-2hc\frac{1-a_5}{a_5}\frac{S(\theta)}{1-T(\theta)\cot\theta}\right),\ \theta \in (0,\alpha);$$

$$\sigma|_{\Gamma_2} = A - 2kc \ln\left(2hc\frac{a_5-1}{a_5}\frac{S(\theta)}{1+T(\theta)\tan\theta}\right),\ \theta \in (-\pi/2-\alpha, -\pi/2).$$

Nadai solution for a circular cavity

[A. Nadai, 1924] for the plastic zone around a circular cavity of the radius R, subjected to a constant shear stress ($\neq 0$) in addition to uniform pressure can be expressed as follows:

$$\sigma = -k \ln \tan(\beta + \pi/4) - p, \ \ \theta = \varphi - \pi/2 + \beta, \ \ \cos 2\beta = R^2/r^2 > 0$$

(r, φ) are polar coordinates. Boundary conditions:

$$\sigma|_{r=R} = -p, \quad \theta|_{r=R} = \varphi - \pi/2.$$

Corresponding solution for linearized system is (index NC):

$$x_{NC} = -R\left(\sin\theta \cosh\frac{\sigma + p}{2k} + \cos\theta \sinh\frac{\sigma + p}{2k}\right),$$

$$y_{NC} = -R\left(\sin\theta \sinh\frac{\sigma + p}{2k} - \cos\theta \cosh\frac{\sigma + p}{2k}\right).$$

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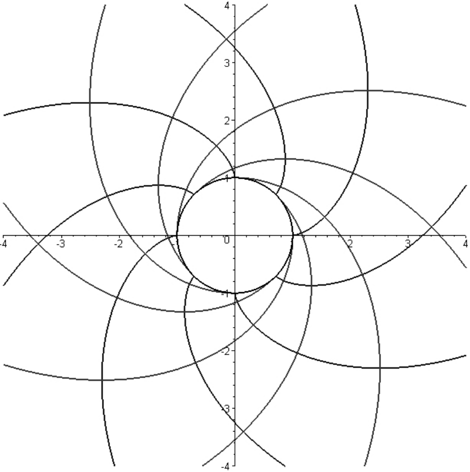
Equations for characteristics:

$$x = -R\left(\sin\theta\cosh\left(\pm\theta + C_i + p/(2k)\right) + \cos\theta\sinh\left(\pm\theta + C_i + p/(2k)\right)\right),$$
$$y = -R\left(\sin\theta\sinh\left(\pm\theta + C_i + p/(2k)\right) - \cos\theta\cosh\left(\pm\theta + C_i + p/(2k)\right)\right).$$



$r = R$ is an envelope

Homotopy with Prandtl solution:

$$x = a_5 x_P + (1 - a_5) x_{NC}, \quad y = a_5 y_P + (1 - a_5) y_{NC}.$$

Equation of envelope for corresponding family of slip-lines looks:

$$\Gamma : x = a_5 h(p - p_1)/k - 2a_5 h \operatorname{arsinh} \frac{2a_5 h \sin\theta}{R(a_5 - 1)} -$$
$$- \sin\theta \sqrt{4a_5^2 h^2 \sin^2\theta + R^2(1 - a_5)^2}, \; a_5 \neq 1,$$
$$y = a_5 h + \cos\theta \sqrt{4a_5^2 h^2 \sin^2\theta + R^2(1 - a_5)^2}.$$

Along boundary line Γ function σ takes values:

$$\sigma|_\Gamma = -p + 2k \operatorname{arsinh} \frac{2a_5 h \sin\theta}{R(a_5 - 1)}.$$

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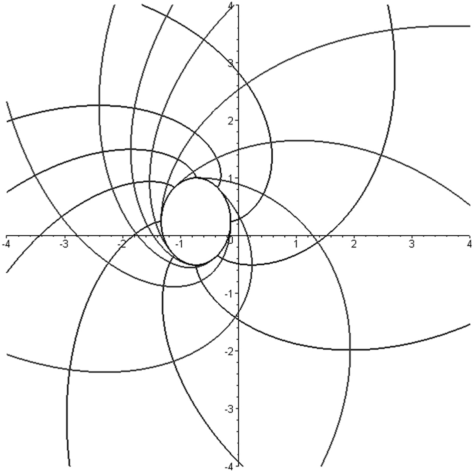
Slip-lines field for homotopy looks as follows:



Note, that homotopy solution describes a stress state around the cavity of the form Γ when $a_5 < R/(2h + R)$, because only for these values of a_5 the boundary line is non-self-intersecting.

In particular case, when the constant shear stress is equal to zero, Nadai solution takes the form:

$$x_{NC} = Re^{\frac{p_2 - k}{2k}} \cos(\theta - \pi/4)\, e^{\frac{\sigma}{2k}}, \;\; y_{NC} = Re^{\frac{p_2 - k}{2k}} \sin(\theta - \pi/4)\, e^{\frac{\sigma}{2k}},$$

with boundary conditions along $r = R$: $\sigma = -p_2 + k, \;\; \theta = \phi + \pi/4$.

For homotopy solution, taking equivalent boundary conditions one can obtain the boundary line:

$$r = -2ah\cos\phi + (1 - a)Re^{\frac{p_2 - p_1}{2k}},$$

which is a limacon of Pascal. This result is similar to the solution obtained in [Senashov and Yakhno, 2007].

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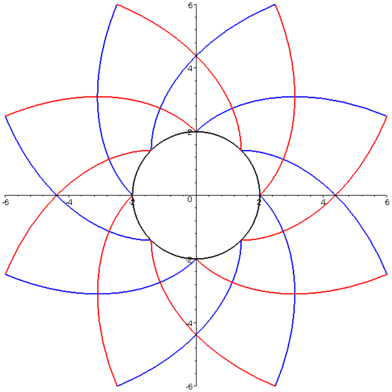
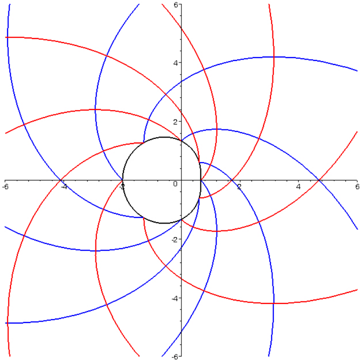


Initial slip-lines.



Transformed slip-lines for limacon of Pascal.

In left figure one can see two families of characteristic curves (spirals) for the circular solution with $p_2 = k$ for the circular cavity of the radius $R = 2$. The deformed slip-lines are presented in right figure for a limacon of Pascal $(h = 1,\ p_1 = p_2)$.

Conclusions

The action of Lie group of point transformations not only over the set of known solutions, but over the families of characteristic curves permits to find out efficiently the suitable boundary conditions for reproduced solutions.

Some families of exact solutions for the system of plane ideal plasticity as a result of homotopy of well-known exact solutions of A. Nadai and L. Prandtl are constructed. By means of homotopy parameter, one can relate any two known solutions of plane plasticity system, if it is possible to express them in the form of solutions for the corresponding linearized system.

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