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New Prorerty of PDE and Exact Solutions in Parametric Form

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Group properties of non-linear transfer equations were studied 30 years ago [1]. Twenty six families of solutions were discovered which are corresponding to the infinitesimal operators [1].

The classical group analysis of symmetries gives disappointing results, as there are many simple symmetries, but only few of them can be used in the applications. In this work only the practical solutions were constructed, these are the solutions of boundary value problem $Z|_{x->-\infty} = a_o, \quad Z|_{x->\infty} = a_1$ with an arbitrary radical $F(a_o) = 0, \quad F(a_1 = 0$.

A new method of construction of exact solutions for Partial Differential Equations (PDE) is proposed in this article. The classical authors in mathematics used change of variables for classification of linear PDE. However, they did not notice an important property of broad class of PDE, which was discovered in the Volosov's articles, listed in the Reference as [3]-[6][2]. This property gives the possibility of expressing PDE as $AX = b$. This is a linear algebraic equations system with regards to old variables on new variables. This equations system has the unique solution. New identity was obtained which follows from solvability conditions of the system. The methods of the above calculations and their consequence are described in this article. Let's consider the equation:

$$Z'_t - (K(Z)Z'_x)'_x + F(Z) = 0, \tag{1}$$

[1]This study was performed under the guidance of Doctor of Physics and Mathematics, Professor, E.M.Vorob'ev.

[2]Literature reference [5] is available for review on www.eqworld.ipmnet.ru

Let's describe the proposed method based on the equation (1). The proposed algorithm works providing that all functions are continuously differentiable functions.

Let's make an arbitrary replacement of variables

$$Z(x,t)|_{x=x(\xi,\ \delta),t=t(\xi,\delta)} = U(\xi,\delta). \tag{2}$$

The inverse replace of variables defines the function $Z(x,t)$ of equation (1) from the function $U(\xi,\delta)$:

$$Z(x,t) = U(\xi,\delta)|_{\xi=\xi(x,t),\ \delta=\delta(x,t)}. \tag{3}$$

We note that

$detJ = x'_{\xi}t'_{\delta} - t'_{\xi}x'_{\delta} \neq 0$ is none zero and not interminable, where

$$J = \begin{pmatrix} x'_{\xi} & t'_{\xi} \\ x'_{\delta} & t'_{\delta} \end{pmatrix}.$$

An inverse transformation exists, at least locally:

$\xi = \xi(x,t), \delta = \delta(x,t)$.

The derivatives of the old independent variables on the new variables are determind as follows:

$$\begin{gathered} \frac{\partial x}{\partial \xi} = detJ\frac{\partial \delta}{\partial t},\ \frac{\partial t}{\partial \xi} = -detJ\frac{\partial \delta}{\partial x}, \\ \frac{\partial x}{\partial \delta} = -detJ\frac{\partial \xi}{\partial t},\ \frac{\partial t}{\partial \delta} = detJ\frac{\partial \xi}{\partial x}. \end{gathered} \tag{4}$$

Let us introduce the relation:

$$\begin{gathered} K(Z)\frac{\partial Z}{\partial x}|_{x=x(\xi,\ \delta),\ t=t(\xi,\ \delta)} = Y(\xi,\delta), \\ K(Z)\frac{\partial Z}{\partial t}|_{x=x(\xi,\ \delta),\ t=t(\xi,\ \delta)} = T(\xi,\delta). \end{gathered} \tag{5}$$

Using (4) and (5), we obtain the formulas:

$$K(U(\xi,\delta))\left(\frac{\partial U}{\partial \xi}\frac{\partial t}{\partial \delta}-\frac{\partial U}{\partial \delta}\frac{\partial t}{\partial \xi}\right)=Y(\xi,\delta)[x'_{\xi}t'_{\delta}-t'_{\xi}x'_{\delta}],$$
$$K(U(\xi,\delta))\left(-\frac{\partial U}{\partial \xi}\frac{\partial x}{\partial \delta}+\frac{\partial U}{\partial \delta}\frac{\partial x}{\partial \xi}\right)=$$
$$T(\xi,\delta)[x'_{\xi}t'_{\delta}-t'_{\xi}x'_{\delta}]. \tag{6}$$

Equation (1) takes the form

$$T(\xi,\delta)-K(U)\left(\frac{\partial Y}{\partial \xi}\frac{\partial t}{\partial \delta}-\frac{\partial Y}{\partial \delta}\frac{\partial t}{\partial \xi}\right)/[x'_{\xi}t'_{\delta}-t'_{\xi}x'_{\delta}]+$$
$$K(U)F(U)=0. \tag{7}$$

Let us rewrite (5) in the form

$\frac{\partial Z(x,t)}{\partial x}=Y(\xi,\delta)/K(U)|_{\xi=\xi(x,t),\ \delta=\delta(x,t)},$

$\frac{\partial Z(x,t)}{\partial t}=T(\xi,\delta)/K(U)|_{\xi=\xi(x,t),\ \delta=\delta(x,t)}.$

As Z is continuously differentiable function, with the necessary of: $\frac{\partial}{\partial t}Z'_{x}=\frac{\partial}{\partial x}Z'_{t}$ in the variables ξ,δ.

Taking into consideration on (4),(5), we can write this equality in the form

$$-\frac{\partial x}{\partial \delta}\frac{\partial}{\partial \xi}\left[\frac{Y}{K(U)}\right]+\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \delta}\left[\frac{Y}{K(U)}\right]-\frac{\partial t}{\partial \delta}\frac{\partial}{\partial \xi}\left[\frac{T}{K(U)}\right]+\frac{\partial t}{\partial \xi}\frac{\partial}{\partial \delta}\left[\frac{T}{K(U)}\right]=0. \tag{8}$$

System (6)-(8) will be analyzed in two stages. At the first stage, we consider system (6)-(8) as a nonlinear algebraic system regarding the derivatives $x'_{\xi},\ x'_{\delta},\ t'_{\xi},\ t'_{\delta}$.

Theorem 1. The implicit linear algebraic system (6)-(8) with regarding the derivatives $x'_{\xi},\ x'_{\delta},\ t'_{\xi},\ t'_{\delta}$, has the unique solution

$$\frac{\partial x}{\partial \xi}=\Psi_1(\xi,\delta),\quad \frac{\partial x}{\partial \delta}=\Psi_2(\xi,\delta), \tag{9}$$

$$\frac{\partial t}{\partial \xi}=\Psi_3(\xi,\delta),\quad \frac{\partial t}{\partial \delta}=\Psi_4(\xi,\delta), \tag{10}$$

where

$$\Psi_1(\xi,\delta) \stackrel{\text{def}}{=} (K[-FKU'_{\xi}(U'_{\delta}T'_{\xi} - T'_{\delta}U'_{\xi}) - (-TU'_{\delta}Y'^{2}_{\xi} - TT'_{\delta}U'^{2}_{\xi} + TU'_{\delta}T'_{\xi}U'_{\xi} - YY'_{\delta}T'_{\xi}U'_{\xi} + TY'_{\delta}Y'_{\xi}U'_{\xi} + YT'_{\delta}U'_{\xi}Y'_{\xi})])/P_1(\xi,\delta), \quad (11)$$

$$\Psi_2(\xi,\delta) \stackrel{\text{def}}{=} K[-FKU'_{\delta}[U'_{\delta}T'_{\xi} - T'_{\delta}U'_{\xi}] - TT'_{\xi}U'^{2}_{\delta} + YY'_{\delta}T'_{\xi}U'_{\delta} + TT'_{\delta}U'_{\xi}U'_{\delta} - YT'_{\delta}Y'_{\xi}U'_{\delta} + TY'_{\delta}Y'_{\xi}U'_{\delta} - TY'^{2}_{\delta}U'_{\xi}]/P_1(\xi,\delta), \quad (12)$$

$$\Psi_3(\xi,\delta) \stackrel{\text{def}}{=} K[-YY'_{\xi} + FKU'_{\xi} + TU'_{\xi}][U'_{\delta}Y'_{\xi} - Y'_{\delta}U'_{\xi}]]]/P_1(\xi,\delta), \quad (13)$$

$$\Psi_4(\xi,\delta) \stackrel{\text{def}}{=} K[-YY'_{\delta} + FKU'_{\delta} + TU'_{\delta}][U'_{\delta}Y'_{\xi} - Y'_{\delta}U'_{\xi}]/P_1(\xi,\delta), \quad (14)$$

where

$$P_1(\xi,\delta) = FK[(TY'_{\xi} - T'_{\xi}Y)U'_{\delta} + (YT'_{\delta} - TY'_{\delta})U'_{\xi}] + TY[-U'_{\delta}T'_{\xi} + U'_{\xi}T'_{\delta}] + Y^2[Y'_{\delta}T'_{\xi} - T'_{\delta}Y'_{\xi}] + T^2[U'_{\delta}Y'_{\xi} - Y'_{\delta}U'_{\xi}], \quad (15)$$

and moreover

$$detJ = \frac{K(U)^2(Y'_{\delta}U'_{\xi} - U'_{\delta}Y'_{\xi})^2}{P_1(\xi,\delta)}. \quad (16)$$

□

Proof. The equation (8) is linear. Let's divide the first equation (6) by Y and divide second equation by T and deduct second equation from first equation and obtain linear equation. Analogously, the linear equation arise from first equation (6) and equation (7)

We can express any three derivatives x'_{ξ}, x'_{δ}, t'_{ξ}, by means of one derivative. E.g. we can express them by means of t'_{δ}.

We can substitute it to (7) and obtain a linear algebraic equation for t'_{δ} (!) . This is valid for any PDE of the second order. This is the substance of the newly discovered property of PDE.

We obtained $\mathbf{AX} = \mathbf{b}$.

Matrix **A** has the form:

$$\mathbf{A} = \begin{pmatrix} YU'_{\delta} & -YU'_{\xi} & TU'_{\delta} & -TU'_{\xi} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}.$$

Where is $a_{21} = -K(U)Y'_{\delta} + YK'(U)U'_{\delta}$, $a_{22} = K(U)Y'_{\xi} - YK'(U)U'_{\xi}$, $a_{23} = -K(U)T'_{\delta} + TK'(U)U'_{\delta}$, $a_{24} = K(U)T'_{\xi} - TK'(U)U'_{\xi}$, $a_{33} = -YY'_{\delta} + F(U)K(U)U'_{\delta} + TU'_{\delta}$, $a_{34} = YY'_{\xi} - F(U)K(U)U'_{\xi} - TU'_{\xi}$, $a_{44} = P_1(\xi,\delta)$. Vector X, b have the form $\mathbf{X} = (\mathbf{x}'_{\xi}, \mathbf{x}'_{\delta}, \mathbf{t}'_{\xi}, \mathbf{t}'_{\delta})^{\mathbf{T}}$, $\mathbf{b} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{b_4})^{\mathbf{T}}$, $b_4 = \Psi_4(\xi,\delta)P_1(\xi,\delta)$. Vector symbol T means conjugation. Eigen values of matrix **A** have the form:

$$\lambda_1 = -YY'_{\delta} + FKU'_{\delta} + TU'_{\delta},$$

$$\lambda_2 = Y^2[T'_{\delta}Y'_{\xi} - Y'_{\delta}T'_{\xi}] - T[FK+T][U'_{\delta}Y'_{\xi} - Y'_{\delta}U'_{\xi}] +$$
$$+Y[FK+T][U'_{\delta}T'_{\xi} - T'_{\delta}U'_{\xi}],$$

$$\lambda_{3,4} = \tfrac{1}{2}[M \pm \sqrt{D}], \quad M = KY'_{\xi} + Y(U'_{\delta} - K'(U)U'_{\xi}),$$

$$D = 4YK(Y'_{\delta}U'_{\xi} - U'_{\delta}Y'_{\xi}) + [KY'_{\xi} + Y(U'_{\delta} - K'(U)U'_{\xi})]^2.$$

The author proposes the alternative classification for PDE solutions on Eigen values.

Eigen pair can be discovered easily. We are not going to discuss here their interesting properties.

At the second stage, consider the new first-order system (9),(10) with the functions $x = x(\xi,\delta), t = t(\xi,\delta)$.

It is well known that the solvability of a system of this type is verified by calculating the second mixed derivatives of the functions $x = x(\xi,\delta)$ and $t = t(\xi,\delta)$ on the arguments ξ and δ :

$$x''_{\xi\delta} = x''_{\delta\xi}, \quad t''_{\xi\delta} = t''_{\delta\xi}. \tag{17}$$

The central result of this article is as follows [3]-[6]:

Theorem 2.

1) We have the new identity

$$(\frac{\partial\Psi_1}{\partial\delta}-\frac{\partial\Psi_2}{\partial\xi})/T \ \equiv \ (\frac{\partial\Psi_3}{\partial\delta}-\frac{\partial\Psi_4}{\partial\xi})/Y,$$

where the functions $\Psi_i, i=1,...,4$ have the from (11)-(15).

2) Two solvability conditions (17) of system (9),(10) have multiply coefficient (or record monomial factor) of arbitrary functions $U, \ Y, \ T$

$$\frac{\partial}{\partial\delta}\Psi_3-\frac{\partial}{\partial\xi}\Psi_4=0, \tag{18}$$

where Ψ_3, Ψ_4 are the right-parts in (13), (14).

Corollary 1. If some free functions $U, \ Y, \ T$ satisfy the condition (18), than systems (9),(10) and (6)-(8) are solvable.

This property (Theorems 1, 2) of second-order partial differential equations was not known before.

The specific ways of satisfying the conditions of (18) are discussed below:

Example 1. Let's consider the Zel'dovich equation, which is well known in combustion theory [2],(Kolmogorov, Petrovsky, Piskunov [8]-[10]):

$$Z'_t-Z''_{xx}-Z^2(1-Z)=0. \tag{19}$$

Let's consider equation (1) with $K(Z)=1$, and $F(Z)=Z^2(1-Z)$.

Suppose that $G(\xi,U)$ is a function of two variables and $w(U)$, $v(U)$ are functions of one variable. We seek the functions $Y(\xi,\delta), \ T(\xi,\delta)$ in (5) in the form $Y(\xi,\delta)=G(\xi,U)+h(U), \ \ T(\xi,\delta)=w(U)+v(U)G(\xi,U)$, where $U=U(\xi,\delta)$.

In the articles [5],[6] the case $h(U)=0$ was considered.

Theorem 3 Let us $G(\xi,U)\in C^2[R\otimes R], \ \ w(U), \ h(U), \ v(U)\in C^2[R]$.

Condition (18) takes the form

$$[v''(U)]G^3+[2(h'-v)v'(U)+vh''-2hv''-w''(U)]G^2+$$
$$+[(3U^2(-1+U)+2hv+2w-4hh')v'(U)-2hvh''+h^2v''+2hw''(U)]G+$$
$$U(-2+3U)hv+U(2-3U)w+(1-U)U^2vh'+2h(-U^2+U^3+w-hh')v'+$$
$$(U-1)U^2w'-h^2vh''+h^2w''=0. \tag{20}$$

□

We can try to satisfy this equation (20) by making equal the coefficients of the powers of G to zero. We obtain a system of four equations for the two functions w, v:

$$v''(U)=0,$$
$$2(h'-v)v'(U)+vh''-2hv''-w''(U)=0,$$
$$(3U^2(-1+U)+2hv+2w-4hh')v'(U)-2hvh''+h^2v''+2hw''(U)=0,$$
$$U(-2+3U)hv+U(2-3U)w+(1-U)U^2vh'+2h(-U^2+U^3+w-$$
$$-hh')v'+(U-1)U^2w'-h^2vh''+h^2w''=0$$

It turns out that in a number of interesting cases, all the four equations can be solved. Moreover, the function $G(\xi,U)$, as well as U (!), remains arbitrary.

Eq.(20) can be solved by setting $v(U)=\frac{3U}{\sqrt{2}}-\frac{1}{\sqrt{2}}$, $w(U)=3U^2(1-U)/2+\frac{(-1+3U)h(U)}{\sqrt{2}}$.

System (9),(10) has the form

$$\frac{\partial x}{\partial \xi}=((w+vG)G'_{\xi}-(Fv+vw+v^2G+G^2v'+Gw'-(w+vG)G'_{\xi})U'_{\xi})/P_1, \tag{21}$$

$$\frac{\partial x}{\partial \delta}=((Gv+w)G'_{\delta}-[Fv+v^2G+vw+G^2v'+Gw'-(w+vG)G'_{U}]U'_{\delta})/P_1, \tag{22}$$

$$\frac{\partial t}{\partial \xi} = (-GG'_{\xi} + (F + Gv + w - GG'_{U})U'_{\xi})/P_1,$$
$$\frac{\partial t}{\partial \delta} = (F + w + Gv - GG'_{U})U'_{\delta})/P_1,$$
(23)

where $P_1 = Fw + w^2 + vwG - G^2(Gv' + w')$. The Jacobian has the form $J = \frac{G'_{\xi}U'_{\delta}}{P_1}$.

The exact solutions of the PDE is already constructed, since relation (17), (18) was satisfied. However, usually it required more detailed formulas. Let's extend reviewed situation.

Let's choose the function $G(\xi, U)$ so that the last two equations (23) of the system take the elementary form

$$t(\xi, \delta) = \xi, \ t'_{\xi} = 1, \ \ t'_{\delta} = 0.$$

Then, we have the bellow system of two equations for $G(\xi, U)$, :

$$G'_{U} = [(1-U)U^2 + \sqrt{2}(3U-1)G - \sqrt{2}h + 3\sqrt{2}Uh - 2Gh' - 2hh']/(2(G+h)),$$
$$G'_{\xi} = 3[\sqrt{2}U(U-1) - 2G - 2h]^2(2G + 2h - \sqrt{2}U^2)/(8\sqrt{2}(G+h))\quad(24)$$

Integrating this system, we obtain the:

$$C_1 + 3\xi/2 = \frac{\sqrt{2}(1-U)}{\sqrt{2}U(U-1) - 2G - 2h} - \ln(\sqrt{2}U(U-1) - 2G - 2h) + \ln(\sqrt{2}U^2 - 2G - 2h). \quad (25)$$

We still need to analyze the first pair of equations (22) for the function $x(\xi, \delta)$.

It turn out that, if we can do the variables replacement $x(\xi, \delta) = \chi(U, R)$, $U = U(\xi, \delta)$, $R = G(\xi, U)$, then system (21)-(23)was integrated. After that, we can return to the variables ξ, δ,which have the form

$$x = \frac{2}{3}(C_2 + \sqrt{2}\ln[U-1] - \sqrt{2}\ln[U]) + [4(-\ln[U-1] + \ln[U] +$$
$$+\ln(-2G + \sqrt{2}(U-1)U - 2h) -$$
$$\ln(-2G + \sqrt{2}U^2 - 2h))((1-3U)^2 + 2\sqrt{2}(1-3U)h^{'}(U) + 2(h^{'})^2)]/(3(\sqrt{2}(1-$$
$$3U)^2 + 4(1-3U)h^{'}(U) + 2\sqrt{2}(h^{'})^2)) -$$
$$[2\sqrt{2}(U-1)((-1+3U)^3 - 3\sqrt{2}(1-3U)^2h^{'}(U) + 6(-1+3U)(h^{'})^2 -$$
$$2\sqrt{2}(h^{'})^3)]/[3(2G - \sqrt{2}(U-1)U + 2h)(1 - 3U + \sqrt{2}h^{'})(\sqrt{2}(1-3U)^2 +$$
$$4(1-3U)h^{'}(U) + 2\sqrt{2}(h^{'})^2)]. \quad (26)$$

We have (25),(26) exact solution equation (19) in parametric form.

The authors believe that the solution (26) can not be constructed based on the classic technique of the group analysis. However, if we consider $h(U) = 0$ we have the possibility to return to the original variables x, t. This simplified solution can be constructed by other methods, e.g. by the method of Satsuma-Hirota.

We can came back to the variables $x,\ t$. Let's introduce the notation

$$t = \xi,\ x(\xi, \delta) = X(t, s(x,t)),\ \delta = s(x,t),\ G = r(t, U(t, s(x,t))),$$

and $h(U) = 0$.

Let's consider the equation (19) and the change of variables

$$Z(x,t) = U(t, s(x,t)). \quad (27)$$

Let a $t > 0$, and $U(t,s) \in C^2[R_+ \otimes R]$, has that the relation for the function $r(t,U) = r(t, U(t, s(x,t)))$ takes the form

$$C_1 + t = \frac{2\sqrt{2}(1-U)}{3\sqrt{2}U(U-1) - 6r} - \frac{2}{3}\ln(\sqrt{2}U(U-1) - 2r) + \frac{2}{3}\ln(\sqrt{2}U^2 - 2r), \quad (28)$$

(see (25), $h = 0$). Also relation (26) takes form

$$X(t,s) = C_1 + \frac{2}{3}[\frac{1-U}{-2r+\sqrt{2}(U-1)U} +$$
$$+\sqrt{2}\ \ln(2r - \sqrt{2}\ (U-1)U) - \sqrt{2}\ \ln(2r - \sqrt{2}\ U^2)]. \qquad (29)$$

We have a system of two equations for $X(t,s)$,

$$\frac{\partial X}{\partial t} = \frac{1}{\sqrt{2}} - \frac{3U}{\sqrt{2}} + \frac{3U^2(U-1)}{2r} + \frac{U'_t}{r}, \quad \frac{\partial X}{\partial s} = \frac{U'_s}{r}. \qquad (30)$$

This is system (21)-(22) for $h = 0$.

Theorem 4. Suppose that (28),(29)is hold, and $h(U) = 0$. In this case, Eq.(19) can be solved . Exact solution equation (19) have the form $Z(x,t) = U(t, s(x,t))$,

$$Z(x,t) = \frac{2\exp(t/2) - 2\exp(x/\sqrt{2})}{2\exp(t/2) + \exp(x/\sqrt{2})(-2 + 2t + x\sqrt{2})}. \qquad (31)$$

□

Proof. Let's express from (29) $\ln(\sqrt{2}U(U-1) - 2r)$ and substitute in (28): $r(t,U) = (1-U)(-C_1U + \sqrt{2}(1+tU) + UX)/(\sqrt{2}C_1 - 2t - \sqrt{2}X)$. We shall substitute r in (29) and raise it to power $\frac{3}{2\sqrt{2}}$ and proceed to exponential functions. Here have put a constant of shift on X (29),$C_1 = 0$. It always can be restored.

Consider the case $h(U) \neq 0$.

Theorem 5. Suppose $h(U) \neq 0$. Exact solution equation (19) have the form (25),(26).

□

This is a new real family of solution to (19).

Example 2. Consider the Fitz–Hugh–Nagumo–Semenov semilinear parabolic partial differential equation, which is well known in biophysics theory [7],[8]:

$$Z'_t - Z''_{xx} - Z(1 - Z^2) = 0. \qquad (32)$$

Here we show one more way of integration of system (21)-(23). Suppose that $G(\xi, U) = \xi$ (!), $w(U)$, $v(U)$ are functions of one variable. We seek the functions $Y(\xi,\delta)$, $T(\xi,\delta)$ in (5) in the form $Y(\xi,\delta) = \xi$, $T(\xi,\delta) = w(U) + v(U)\xi$, where $U = U(\xi,\delta)$.

In this case, equation analogous Eq.(20) can be solved by setting $v(U) = 3U/\sqrt{2}$, $w(U) = 3U(1-U^2)/2$. To avoid any doubts, let's express (21)-(23) in an obvious form:

$$x'_{\xi} = [2U(1-U^2+\sqrt{2}\xi) + (-\sqrt{2}U^2+\sqrt{2}U^4-2\xi-2\sqrt{2}\xi^2)U'_{\xi}]/P_1,$$

$$x'_{\delta} = (-\sqrt{2}U^2+\sqrt{2}U^4-2\xi-2\sqrt{2}\xi^2)U'_{\delta}/P_1,$$

$$t'_{\xi} = [-2(2\xi+U(-1+U^2-3\sqrt{2}\xi)U'_{\xi})]/(3P_1),$$

$$t'_{\delta} = [2(U-U^3+3\sqrt{2}U\xi)U'_{\delta}]/(3P_1).$$

Suppose $G = \xi$ and calculate integrals (!).

The denominator in system of ODE (22)-(23) can be presented in the form of six multipliers

$$P_1 = Fw + w^2 + vwG - G^2(Gv' + w') = U^2 - 2U^4 + U^6 + 3\sqrt{2}U^2\xi - 3\sqrt{2}U^4\xi - 2\xi^2 + 6U^2\xi^2 - 2\sqrt{2}\xi^3 = (U + \sqrt{\sqrt{2}+2\xi}/2^{1/4})(U - \sqrt{\sqrt{2}+2\xi}/2^{1/4})(U - (-1 - \sqrt{1+4\sqrt{2}\xi})/2)(U - (1-\sqrt{1+4\sqrt{2}\xi})/2)(U - (-1 + \sqrt{1+4\sqrt{2}\xi})/2)(U - (1+\sqrt{1+4\sqrt{2}\xi})/2).$$

Then the exact solution of Eq. (32) can be written in parametric form as

$$x(\xi,\delta)=\frac{1}{\sqrt{2+8\sqrt{2}\xi}}\left(\frac{\sqrt{2}+8\xi+\sqrt{2+8\sqrt{2}\xi}}{\sqrt{-1-2\sqrt{2}\xi-\sqrt{1+4\sqrt{2}\xi}}}\times\right.$$
$$\times\arctan\left[\frac{\sqrt{2}U}{\sqrt{-1-2\sqrt{2}\xi-\sqrt{1+4\sqrt{2}\xi}}}\right]+$$
$$+\frac{-\sqrt{2}-8\xi+\sqrt{2+8\sqrt{2}\xi}}{\sqrt{-1-2\sqrt{2}\xi+\sqrt{1+4\sqrt{2}\xi}}}\times$$
$$\left.\times\arctan\left[\frac{\sqrt{2}U}{\sqrt{-1-2\sqrt{2}\xi+\sqrt{1+4\sqrt{2}\xi}}}\right]\right).$$

$$t(\xi,\delta)=-\frac{2}{3}\ln[U-\sqrt{\sqrt{2}+2\xi}/2^{1/4}]-\frac{2}{3}\ln[U+\sqrt{\sqrt{2}+2\xi}/2^{1/4}]+$$
$$\frac{1}{3}\ln[U+(-1+\sqrt{1+4\sqrt{2}\xi})/2]+\frac{1}{3}\ln[U+(1+\sqrt{1+4\sqrt{2}\xi})/2]+$$
$$\frac{1}{3}\ln[U-(-1+\sqrt{1+4\sqrt{2}\xi})/2]+\frac{1}{3}\ln[U-(1+\sqrt{1+4\sqrt{2}\xi})/2]. \quad (33)$$

Theorem 6. Suppose that (33) is hold. Exact solution equation (32) have the form $Z(x,t)=U(\xi,\delta)|_{\delta=\delta(x,t),\xi=\xi(x,t)}$. □

After that, we came back to the variable ξ from the (33) takes the form $\xi=\frac{-U^2+\exp(3t)(U^2-1)\pm\sqrt{U^2+\exp(3t)(1-U^2)}}{\sqrt{2}(-1+\exp(3t))}$. $\delta-$ is arbitrary parameter.

Let's substitute this expression in the first parity (33) $x=x(\xi,U)$ and we shall receive the solution in the parametrical form. This is a new real family of solution to (32).

Example 3. In the three-dimensional case, consider the semilinear parabolic equation and the change of variables

$$Z'_t-Z''_{xx}-Z''_{yy}+f(t,Z)=0,$$
$$Z(x,y,t)|_{x=x(\xi,\delta,\tau),y=y(\xi,\delta,\tau),t=t(\xi,\delta,\tau)}=U(\xi,\delta,\tau). \quad (34)$$

Suppose that the Jacobian $J \neq 0$. Suppose also that there exists (at least locally) the inverse transformation

$\xi = \xi(x, y, t),\ \delta = \delta(x, y, t),\ \tau = \tau(x, y, t)$, and the derivatives an related as

$$\frac{\partial \tau}{\partial t} = (y'_{\delta} x'_{\xi} - x'_{\delta} y'_{\xi})/J,\ \ \frac{\partial \tau}{\partial y} = (x'_{\delta} t'_{\xi} - t'_{\delta} x'_{\xi})/J, \ldots \text{ (see [4])}.$$

We set

$$\frac{\partial Z}{\partial x}|_{x=x(\xi,\delta,\tau),y=y(\xi,\delta,\tau),t=t(\xi,\delta,\tau)} = Y(\xi, \delta, \tau),$$
$$\frac{\partial Z}{\partial y}|_{x=x(\xi,\delta,\tau),y=y(\xi,\delta,\tau),t=t(\xi,\delta,\tau)} = M(\xi, \delta, \tau),$$
$$\frac{\partial Z}{\partial t}|_{x=x(\xi,\delta,\tau),y=y(\xi,\delta,\tau),t=t(\xi,\delta,\tau)} = T(\xi, \delta, \tau). \tag{35}$$

The functions $Y,\ M,\ T$ are unknown.

Let's supplement the relations written above by the equalities of mixed derivatives

$$Z_{x,y} = Z_{y,x},\ Z_{x,t} = Z_{t,x},\ Z_{y,t} = Z_{t,y} \tag{36}$$

in the variables ξ, δ, τ.

The nonlinear algebraic system of seven equations in the variables ξ, δ, τ is similar to system (5)-(7), which follows from system (34)-(36) with respect to the nine variable-derivatives

$x'_{\xi},\ x'_{\delta},\ x'_{\tau},\ y'_{\xi},\ y'_{\delta},\ y'_{\tau},\ t'_{\xi},\ t'_{\delta},\ t'_{\tau}$, are underdetermined. Hence, there is much arbitrariness in the choice of functions. In this case we used "method C" from [4]. Solutions of the equations (36) have the form

$$y'_{\xi} = \frac{-[T(\xi, \delta, \tau)t'_{\xi} - U'_{\xi} + Y(\xi, \delta, \tau)x'_{\xi}]}{M},$$
$$y'_{\delta} = \frac{-[T(\xi, \delta, \tau)t'_{\delta} - U'_{\delta} + Y(\xi, \delta, \tau)x'_{\delta}]}{M},$$
$$y'_{\tau} = \frac{-[T(\xi, \delta, \tau)t'_{\tau} - U'_{\tau} + Y(\xi, \delta, \tau)x'_{\tau}]}{M}. \tag{37}$$

By integrating the all resulting relations (36) and equation analogous (7) we have the theorem.

Theorem 7 Suppose that $G(\xi,\delta,\tau,U)$ is a twice continuously differentiable function of four variables and $h(t,U)$, $w(t,U)$, $r(t,U)$ is a twice continuously differentiable function of two variables, where $U=U(\xi,\delta,\tau)$.

Then, the exact solution of equation (34) can by written in parametric form as

$$T(\xi,\delta,\tau)=c_1w(t,U)+w(t,U)\int_{Uo}^{U}\frac{w'_t ds}{w^2(t,s)},\quad Y(\xi,\delta,\tau)=c_2w(t,U),$$
$$M(\xi,\delta,\tau)=w(t,U),\quad x(\xi,\delta,\tau)=r(t,U)+h(t,U)G(\xi,\delta,\tau,U),\quad (38)$$
$$y(\xi,\delta,\tau)=-c_1t(\xi,\delta,\tau)-c_2r(t,U)-c_2G(\xi,\delta,\tau,U)h(t,U)+\int_{Uo}^{U}\frac{ds}{w(t,s)}+c_3,$$

$c_i\neq 0,\quad i=1-3.$

The function $w(t,U)$ is determined from the differential equation

$$\frac{f(t,U)}{w(t,U)}+c_1+\int_{Uo}^{U}\frac{w'_t(t,s)ds}{w^2(t,s)}-2w'_U(t,U)=0.$$

□

The twice differentiable functions

$G(\xi,\delta,\tau,U)$, $r(t,U)$, $h(t,U)$, $t(\xi,\delta,\tau)$, as well as the function U (!), remain arbitrary.

If $f=f(Z)$, in Eq.(34), then $h(U)$, $w(U)$, $r(U)$ are functions of one variable. The function $w(U)$ is determined from the first- order ordinary differential equation (known as the Abel equation)

$$f(U)=w(U)[(c_2^2+1)w'(U)-c_1].\qquad(39)$$

Example 4. Let's consider the modification equation Fisher- Kolmogorov-Petrovsky-Piskunov ($FKPP$) [10][3]

$$Z'_t-Z''_{xx}+F(Z)=0,\ Z(-\infty,t)=a_0,\ Z(\infty,t)->a_1,\qquad(40)$$

[3]This study was performed with A.K.Volosova.

where $F(Z)=\frac{2Z^3\lambda^2}{9}\pm(a_0-Z)(a_1-Z)$. In Reference [1] was parameter $\lambda=0,\ a_0=0,$

$a_1=1,\ a_1>a_0$.

We can choose $F(U)$ for (1), than $v(U)=\pm\frac{3U}{2\lambda}+U\lambda$, $w(U)=-3F(U)/2$ and the relations (18),(20) hold.

Systems (9),(10) has the form:

$$\frac{\partial x}{\partial\xi}=((w+vG)G'_\xi-(Fv+vw+v^2G+G^2v'+Gw'-(w+vG)G'_\xi)U'_\xi)/P_1, \quad (41)$$

$$\frac{\partial x}{\partial\delta}=((Gv+w)G'_\delta-[Fv+v^2G+vw+G^2v'+Gw'-(w+vG)G'_U]U'_\delta)/P_1,$$

$$\frac{\partial t}{\partial\xi}=(-GG'_\xi+(F+Gv+w-GG'_U)U'_\xi)/P_1,$$
$$\frac{\partial t}{\partial\delta}=(F+w+Gv-GG'_U)U'_\delta)/P_1. \quad (42)$$

The functions P_1, J has the form $P_1=Fw+w^2+vwG-G^2(Gv'+w')$, $J=\frac{G'_\xi U'_\delta}{P_1}$.

The exact solutions of the PDE is already constructed, since relation (18), (20) was satisfied. However, usually it required more detailed formulas.

Let's assume

$a_0=0,\ a_1=1,\ G(\xi,U)=\xi$. The systems (41),(42) has the form:

$$\frac{\partial x}{\partial\xi}=[3[-6U\lambda^2(-9+9U+2U^2\lambda^2)+18\lambda(3+2U\lambda^2)\xi+Q_1U'_\xi]]/(\lambda P_2),$$
$$\frac{\partial x}{\partial\delta}=[3[Q_1U'_\delta]]/(\lambda P_2), \quad (43)$$

$$\frac{\partial t}{\partial\xi}=-6[18\lambda\xi+Q_2U'_\xi]/P_2,$$
$$\frac{\partial t}{\partial\delta}=-3Q_2U'_\delta/P_2,$$

where $Q_1 = -27U\lambda+27U^2\lambda-18U^2\lambda^3+24U^3\lambda^3+4U^4\lambda^5-81\xi-54\lambda^2\xi-36\lambda^3\xi^2$,
$Q_2 = -9U\lambda + 9U^2\lambda + 2U^3\lambda^3 - 27\xi - 18U\lambda^2\xi$,

$P_2 = (-9 + 9U + 2U^2\lambda^2 - 6\lambda\xi)(-9U^2\lambda + 9U^3\lambda + 2U^4\lambda^3 - 27U\xi-$
$12U^2\lambda^2\xi + 18\lambda\xi^2) = (-9 + 9U + 2U^2\lambda^2 - 6\lambda\xi)*$
$(3Uk^2 + k(9U + 4U^2 - 12\lambda\xi^2))(-3Uk^2 + k(9U + 4U^2 - 12\lambda\xi^2))/(8\lambda k^2)$,
$k = \sqrt{9 + 8\lambda^2}$.

Denominator P_2 has three square multipliers. We can integration second and fourth equations (43).

Theorem 8

The exact solution of equation (40) $Z(x,t) \in [a_0, a_1]$ can be written in parametric form of system (41),(42), where $v(U) = \pm\frac{3U}{2\lambda} + U\lambda$, $w(U) = -3F(U)/2$, and if $a_0 = 0,\ a_1 = 1,\ G(\xi, U) = \xi$ that we have system (43).

Exact solution of equation (40) in parametric form have the form:

$$Z(x,t) = U(\xi,\delta)|_{\xi=\xi(x,t)} = [-9E_1 - 3\eta kE_2 \pm$$
$$\pm[(9E_1 + 3\eta kE_2)^2 +$$
$$24(E_1 - 2kE_2)\lambda^2(-4k\lambda\xi E_2 + E_1(3 + 2\lambda\xi))]^{(1/2)}]/(4\lambda^2(E_1 - 2kE_2))|_{\xi=\xi(x,t)}, \tag{45}$$

where $k = \sqrt{9 + 8\lambda^2}$ и $\eta = k - 3,\ E_1 = \exp[(3kt + 4x\lambda)/(2\eta)]$,
$E_2 = \exp[(3t/(2\eta)]$.

□

Proof.

Exact solution of system (43) have the form:

$$\begin{aligned}
&t = \frac{-2}{3}\ln|-9+9U+2U^2\lambda^2-6\lambda\xi| - \\
&\frac{-k+3}{3k}\ln|-3Uk^2+k(9U+4U^2\lambda^2-12\lambda\xi| + \\
&+\frac{3+k}{3k}\ln|3Uk^2+k(9U+4U^2\lambda^2-12\lambda\xi)|, \\
&x = \frac{1}{\lambda}\ln|-9+9U+2U^2\lambda^2-6\lambda\xi| + \\
&\frac{-k+3+4\lambda^2}{2\lambda k}\ln|-3Uk^2+k(9U+4U^2\lambda^2-12\lambda\xi)| - \\
&-\frac{2\lambda(1-k)}{9+8\lambda^2-3k}\ln|3Uk^2+k(9U+4U^2\lambda^2-12\lambda\xi)|.
\end{aligned} \tag{46}$$

Hence, from (46) we have:

$$\exp[\frac{4x\lambda+3tk}{-6+2k}](-9+9U+2U^2\ \lambda^2-6\lambda\xi)+$$

$$\exp[\frac{3\ t}{-6+2\ k}][3U\ k^2-k(9U+4U^2\ \lambda^2-12\lambda\xi)]=0.$$

We were study solutions of Kolmogorov- Fokker- Plank equation with Sinizyn S.O., Iourtchenko D.V. too.

Conclusion:

The following new facts complementing PDE theory were discovered in the study of Mr. K.A.Volosov:

1. The system of four first-order equations (6)-(8) which is replacing the second -order PDE (1) is a system of linear algebraic equations regarding derivatives $x'_\xi,\ x'_\delta,\ t'_\xi,\ t'_\delta$ and has unique solution;

2. There is a new identity which allows to record monomial factor (18) under the solvability conditions. It is sometimes called closure condition;

3. Monomial factor (18) is the closure condition; the solvability condition is one between ness relation for three arbitrary functions U, Y, T. This property

was not noticed earlier. It is useful to consider the respective closure condition together with each differential equation;

4. Using this approach it is possible to construct new families of exact solutions in the parametric form, the examples of which were presented in this article. These solutions can not be constructed using the classical method of batch properties analysis;

5. It is useful to study the Eigen pair's properties for matrix A. The alternative classification of PDE solutions is possible through eigenvalues of matrix A .

Below you can find the extract from the comments to the article of

Mr.M.V.Karasev, Doctor of Physics and Mathematics, Professor, Head of Chair of Applied Mathematics of Moscow State Institute of Electronics and Mathematics, Laureate of State award of Russian Federation.

The article contains several interesting solutions and whole classes of solutions for nonlinear differential equations with partial secondary derivatives (PDE) important in applications. Sometimes in the theory of PDE the approach is used when by making certain transformation (e.g. by replacement of variables) an equation is converted in order to bring it to another equation whose solution is already known. Thus, the known solution generates nontrivial solution of initial equation. In Mr. K.A. Volosov's approach it is suggested a priori not fixing the type of replacement of variables, leaving that arbitrary on the first stage. The system derived after replacement of variables contains both required vector-function (sought after solution and its derivatives) and unknown coordinate transformation (Jacobi's matrix of variables replacements). The number of equations in the system is always less than the number of variables which opens possibilities for construction of solution by introduction of different interconnections between the components of sought after vector-function. Limitation for the choice of connections requires consistency of the equation for coordinate replacement function co-ordinate replacement function must be consistent. A condition of consistency is mandatory for the remaining independent components of sought after vector-function.

For example, in classic case, for PDE second order with two independent variables, the sought after vector-function has three components; coordinate replacement contains two more functions; and all of them together subordinate to four differential equations of first-order. It was found that all the components of Jacobi 2x2 matrix of co-ordinate replacement cab be explicitly expressed through the components of the sought after vector-function. On this stage it is not required to introduce any additional connections. Then the received formula for the Jacobi's matrix is considered the system of first-order equations relative to two functions of coordinate replacement, and the condition of this system consistency is noted. It is reduced only to one equation, but contains three components of sought after vector-function. As a result, a big freedom appears to meet the consistency condition. On this stage it is useful to introduce connections, which allows in many cases to reduce the condition of consistency to a standard differential equation or even make it explicitly solvable. After that it is necessary to return to construction of coordinate replacement, i.e. to integration of the system of first order, but this time with right part already known. To the extent this integration can be executed explicitly in quadrature, to the same extent it is possible to construct the explicit solution of the initial nonlinear equation with partial derivatives.

This algorithm is not tied to any group or symmetry attributes. The proposed transformation of nonlinear differential multi variable equations has common nature with the mechanism of integration. The coefficients and type of nonlinearity in the equation solved are not specified, but remain general functions.

This work opens very interesting line in many areas of mathematics and its applications dealing with nonlinear equations with higher partial derivatives.

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