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CLASSICAL PARTICLE IN PRESENCE OF MAGNETIC FIELD, HYPERBOLIC LOBACHEVSKY AND SPHERICAL RIEMANN MODELS

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In the paper exact solutions for classical problem of a **particle in magnetic field** on the background of hyperbolic **Lobachevsky** H_3 **and spherical Riemann** S_3 space models will be constructed explicitly.

1. These both are extensions for a well-known problem in theoretical physics.

2. They can be used to describe behavior of charged particles in macroscopic magnetic field in the context of astrophysics.

3. Earlier, the quantum-mechanical variant (Shrödinger equation) of the problem has been solved as well and generalized formulas for **Landau levels in the models** H_3 **and** S_3 have been produced:

Bogush A.A., Red'kov V.M., Krylov G.G.. Schrödinger particle in magnetic and electric fields in Lobachevsky and Riemann spaces. // Nonlinear Phenomena in Complex Systems. 2008. Vol. 11. no 4, P. 403 – 416.

CONTENT

PART I. PARTICLE IN THE MODEL H_3

PART II. PARTICLE IN THE MODEL S_3

Motion of a classical particle in external electromagnetic and gravitational fields is described by

$$mc^2\left(\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\sigma} \frac{dx^\beta}{ds} \frac{dx^\sigma}{ds}\right) = e\, F^{\alpha\rho}\, U_\rho \,,$$

or Lagrangian $$L = -mc^2 \sqrt{1 - \frac{V^2}{c^2}} + \frac{e}{c} A_\beta\, U^\beta$$

Lobachevsky and Riemann models have **nontrivial only 3-space structure**:

$$ds^2 = (dx^0)^2 + \mathbf{g_{jk}(x^1, x^2, x^3)}\, dx^j dx^k \,,$$

In the **model** H_3 we have used special cylindric coordinates (ρ is the curvature radius.)

$$dS^2 = c^2 dt^2 - \left(\text{ch}^2 \frac{z}{\rho}\, dr^2 + \rho^2 \;\text{ch}^2 \frac{z}{\rho}\, \text{sh}^2 \frac{r}{\rho}\, d\phi^2 + dz^2 \right) ,$$

$$z \in (-\infty, +\infty), \qquad r \in [0, +\infty), \qquad \phi \in [0, 2\pi] \, .$$

Space H_3 can be realized as a surface in 4-space (it simplifies symmetry description in H_3):

$$\mathbf{u_0^2} - \mathbf{u_1^2} - \mathbf{u_2^2} - \mathbf{u_3^2} = \rho^2 \, , \qquad \mathbf{u_0} = + \sqrt{\rho^2 + \vec{\mathbf{u}}^{\,2}} \, ,$$

$$u_1 = \rho \, \text{ch} \frac{z}{\rho} \, \text{sh} \frac{r}{\rho} \; \cos\phi \, , \qquad u_2 = \rho \, \text{ch} \frac{z}{\rho} \, \text{sh} \frac{r}{\rho} \; \sin\phi \, ,$$

$$u_3 = \rho \, \text{sh} \frac{z}{\rho} \, , \qquad u_0 = \rho \, \text{ch} \frac{z}{\rho} \, \text{ch} \, \frac{r}{\rho} \, .$$

We are to extend **the concept of a uniform magnetic field** to model H_3.

It should be a solution of Maxwell equations in H_3, and it is given by

$$\mathbf{A}_\phi = -\rho^2 \mathbf{B} \, (\, \text{ch} \frac{\mathbf{r}}{\rho} - \mathbf{1}) \, , \qquad \mathbf{F}_{\phi \mathbf{r}} = \mathbf{B} \rho \, \text{sh} \frac{\mathbf{r}}{\rho} \, ;$$

correct flat space limit: $$\left(\rho \to \infty \, , \qquad A_\phi = - \frac{B r^2}{2} \, , \qquad F_{\phi r} = B r \right) .$$

Additional **arguments for that terminology** will be given below

In the similar manner, for the **model** S_3 we have used special cylindric coordinates

$$dS^2 = c^2 dt^2 - \left(\cos^2\frac{z}{\rho}\, dr^2 + \rho^2\ \cos^2\frac{z}{\rho}\ \sin^2\frac{r}{\rho}\, d\phi^2 + dz^2 \right) ,$$

$$\mathbf{z} \in [-\pi/\mathbf{2}, +\pi/\mathbf{2}], \qquad \mathbf{r} \in [\mathbf{0}, +\pi], \qquad \phi \in [\mathbf{0}, \mathbf{2}\pi] .$$

Riemann space can be realized as a surface in 4-space (it simplifies symmetry description in S_3):

$$\mathbf{u_0^2} + \mathbf{u_1^2} + \mathbf{u_2^2} + \mathbf{u_3^2} = \rho^{\mathbf{2}} ,$$

$$u_1 = \rho\ \cos\frac{z}{\rho}\ \sin\frac{r}{\rho}\ \cos\phi , \qquad u_2 = \rho\ \cos\frac{z}{\rho}\ \sin\frac{r}{\rho}\ \sin\phi ,$$

$$u_3 = \rho\ \sin\frac{z}{\rho} , \qquad u_0 = \rho\ \cos\frac{z}{\rho}\ \cos\ \frac{r}{\rho} .$$

We are to extend **the concept of a uniform magnetic field** to model S_3:

$$\mathbf{A}_\phi = \rho^{\mathbf{2}}\mathbf{B}\ (\ \cos\frac{\mathbf{r}}{\rho} - \mathbf{1}) , \qquad \mathbf{F}_{\phi\mathbf{r}} = \mathbf{B}\rho\ \sin\frac{\mathbf{r}}{\rho} ;$$

correct flat space limit: $$\left(\rho \rightarrow \infty , \qquad A_\phi = -\frac{Br^2}{2} , \qquad F_{\phi r} = Br \right) .$$

In **Lobachevsky model** H_3, Lagrangian of the system is given by

$$L = -mc^2 \sqrt{1 - \frac{V^2}{c^2}} + \frac{eB\rho^2}{c} \, (\, \mathrm{ch}\frac{r}{\rho} - 1) \, (\frac{d\phi}{dt}) \; ;$$

$$V^2 = \mathrm{ch}^2\frac{z}{\rho} \, \left[\, (\frac{dr}{dt})^2 + \rho^2 \mathrm{sh}^2\frac{r}{\rho}(\frac{d\phi}{dt})^2 \, \right] + (\frac{dz}{dt})^2 \; .$$

Equations of motion look as follows:

$$\frac{d^2 r}{dt^2} + 2 \, \mathrm{th}\frac{z}{\rho} \, \frac{dz}{dt} \, \frac{dr}{dt} - \rho \, \mathrm{sh}\frac{r}{\rho} \, \left[\mathrm{ch}\frac{r}{\rho} \, \frac{d\phi}{dt} + \frac{\omega}{\mathrm{ch}^2(z/\rho)} \right] \frac{d\phi}{dt} = 0 \; ,$$

$$\frac{d}{dt} \left[\rho^2 \, \mathrm{sh}^2\frac{r}{\rho} \, \mathrm{ch}^2\frac{z}{\rho} \, \frac{d\phi}{dt} + \omega\rho^2 \, (\, \mathrm{ch}\frac{r}{\rho} - 1) \right] = 0 \; ,$$

$$\frac{d^2 z}{dt^2} - \frac{1}{\rho} \, \mathrm{ch}\frac{z}{\rho} \, \mathrm{sh}\frac{z}{\rho} \left[(\frac{dr}{dt})^2 + \rho^2 \, \mathrm{sh}^2\frac{r}{\rho} \, (\frac{d\phi}{dt})^2 \right] = 0 \; .$$

The squared velocity is conserved quantity: $V^2 =$ **const.**

In **Riemann model** S_3, Lagrangian of the system is given by

$$L = -mc^2 \sqrt{1 - \frac{V^2}{c^2}} - \frac{eB\rho^2}{c} \left(\cos\frac{r}{\rho} - 1 \right) \left(\frac{d\phi}{dt}\right) ;$$

$$V^2 = \cos^2\frac{z}{\rho} \left[\left(\frac{dr}{dt}\right)^2 + \rho^2 \sin^2\frac{r}{\rho} \left(\frac{d\phi}{dt}\right)^2 \right] + \left(\frac{dz}{dt}\right)^2 .$$

Equations of motion look

$$\frac{d^2 r}{dt^2} + 2\, \mathrm{tg}\frac{z}{\rho}\, \frac{dz}{dt}\, \frac{dr}{dt} - \rho \sin\frac{r}{\rho} \left[\cos\frac{r}{\rho}\, \frac{d\phi}{dt} + \frac{\omega}{\cos^2(z/\rho)} \right] \frac{d\phi}{dt} = 0 ,$$

$$\frac{d}{dt} \left[\rho^2 \sin^2\frac{r}{\rho}\, \cos^2\frac{z}{\rho}\, \frac{d\phi}{dt} - \omega \rho^2 \left(\cos\frac{r}{\rho} - 1 \right) \right] = 0 ,$$

$$\frac{d^2 z}{dt^2} + \frac{1}{\rho} \cos\frac{z}{\rho}\, \sin\frac{z}{\rho} \left[\left(\frac{dr}{dt}\right)^2 + \rho^2 \sin^2\frac{r}{\rho}\, \left(\frac{d\phi}{dt}\right)^2 \right] = 0 .$$

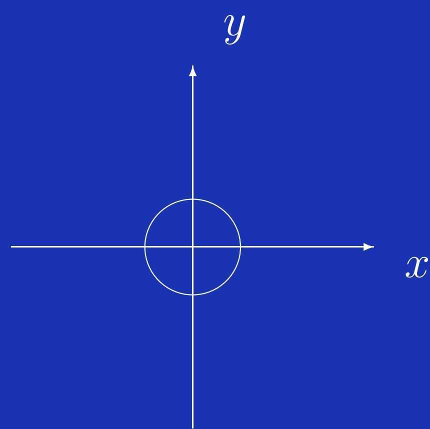
The squared velocity is conserved quantity: $V^2 =$ **const.**

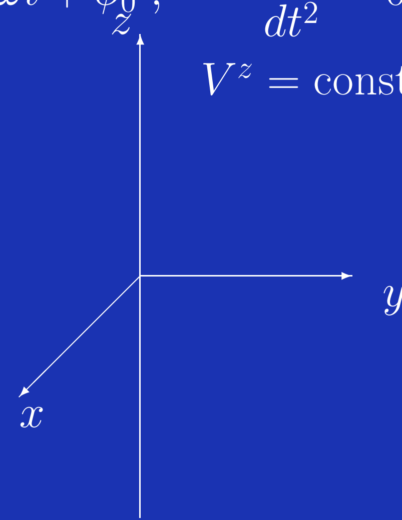
In **flat space** E_3, solutions are well-known:

$$r = r_0 = \text{const} \,, \qquad \phi(t) = \omega t + \phi_0 \,, \qquad \frac{d^2 z}{dt^2} = 0$$

$$V^z = \text{const}$$

$$x = r\cos\phi$$
$$y = r\sin\phi$$





There exist many other **SHIFTED IN PLANE** (x, y) **trajectories**,

they all are in essence the same.

In the first place, the task is to construct **their analogues** in models H_3 and S_3.

It is convenient to introduce **dimensionless** coordinates and parameters:

$$t \Longleftarrow \frac{ct}{\rho} \,, \qquad r \Longleftarrow \frac{r}{\rho} \,, \qquad z \Longleftarrow \frac{z}{\rho} \,,$$

$$B \qquad \Longleftarrow \qquad \frac{e}{m}\, \frac{\rho B}{c} \sqrt{1 - \frac{V^2}{c^2}} \,,$$

then EQUATIONS ARE MUCH **SIMPLIFIED** (no redundant elements):

In H_3 model

$$\frac{d^2r}{dt^2} + 2\,\mathrm{th} z\,\frac{dz}{dt}\,\frac{dr}{dt} - \mathrm{sh}\,r\,\left[\mathrm{ch}\,r\,\frac{d\phi}{dt} + \frac{B}{\mathrm{ch}^2\,z}\right]\frac{d\phi}{dt} = 0\,,$$

$$\frac{d}{dt}\left[\mathrm{sh}^2 r\,\mathrm{ch}^2 z\,\frac{d\phi}{dt} + B\,(\,\mathrm{ch}\,r - 1)\right] = 0\,, \qquad I = \mathrm{const}\,,$$

$$\frac{d^2z}{dt^2} - \mathrm{ch}\,z\,\mathrm{sh}\,z\,\left[(\frac{dr}{dt})^2 + \mathrm{sh}^2\,r\,(\frac{d\phi}{dt})^2\right] = 0\,.$$

In S_3 model

$$\frac{d^2r}{dt^2} + 2\,\mathrm{tg}\,z\,\frac{dz}{dt}\,\frac{dr}{dt} - \sin r\,\left[\cos r\,\frac{d\phi}{dt} + \frac{B}{\cos^2 z}\right]\frac{d\phi}{dt} = 0\,,$$

$$\frac{d}{dt}\left[\sin^2 r\,\cos^2 z\,\frac{d\phi}{dt} - B\,(\,\cos r - 1)\right] = 0\,, \qquad I = \mathrm{const}\,,$$

$$\frac{d^2z}{dt^2} + \cos z\,\sin z\,\left[(\frac{dr}{dt})^2 + \sin^2 r\,(\frac{d\phi}{dt})^2\right] = 0\,.$$

In H_3,

let $r = r_0 = \text{const}$, then eqs. reduce to

$$\frac{d\phi}{dt} = \frac{\alpha}{\text{ch}^2 z} , \qquad \frac{dV^z}{dt} = A\,\frac{\text{sh}\ z}{\text{ch}^3 z} ,$$
$$\alpha = -\,\frac{B}{\text{ch}\ r_0} , \qquad A = B^2\ \text{th}^2 r_0\ > 0 \tag{1}$$

There exist **effective repulsion** to both sides **from the center $z = 0$.**

One can simplify (translate 2-nd order to 1-st order) equation the second equation to

$$\frac{A}{\text{ch}^2 z} = \text{const} - (\frac{dz}{dt})^2 .$$

const must be identified as $\epsilon = V^2$:

$$\frac{A}{\text{ch}^2\ z} = \epsilon - (\frac{dz}{dt})^2 ,$$

In the limit of flat space A corresponds to a transversal squared velocity $V_\perp^2$.

In Lobachevsky model **transversal motion** should vanish (to be frozen) **when $z \to \pm\infty$.**

The signs $\pm$ correspond to motion along axis z in opposite directions. Behavior of $z(t)$:

$$I. \qquad \epsilon > \mathbf{A}\ , \qquad \mathbf{z} \in (-\infty, +\infty)\ ,$$

$$\mathrm{sh}\ z(t) = \pm\sqrt{1 - A/\epsilon}\ \mathrm{sh}\ \sqrt{\epsilon}\, t\ , \qquad z_0 = 0\ ;$$

Trajectories run through $z = 0$.

$$II. \qquad \epsilon < \mathbf{A}\ , \qquad \mathrm{sh}^{\mathbf{2}}\mathbf{z} > \frac{\mathbf{A}}{\epsilon} - \mathbf{1}\ ,$$

$$\mathrm{sh}\ z(t) = \pm\sqrt{\frac{A}{\epsilon} - 1}\ \mathrm{ch}\ \sqrt{\epsilon}\, t\ .$$

The particle is rejected at the points $t = 0$. Such an effect does not exist in flat space model

(For brevity we will omit a very peculiar case at $\epsilon = A$.)

Now we are to find $\phi(t)$ (no need to distinguish between I and II)

$$A \neq \epsilon\ , \qquad \phi - \phi_0 = \frac{\alpha}{\sqrt{A}}\ \mathrm{arcth}\ \Big(\sqrt{\frac{A}{\epsilon}}\ \mathrm{th}\ \sqrt{\epsilon} t\Big)\ .$$

When $t \to +\infty$ we obtain a finite value for total rotation angle **(rotation freezing)**:

$$(\phi - \phi_0)|_{t\to\infty} = \frac{\alpha}{\sqrt{A}}\ \mathrm{arcth}\ \sqrt{\frac{A}{\epsilon}}\ .$$

In S_3,

let $r = r_0 =$ const, then eqs. reduce to

$$\frac{d\phi}{dt} = \frac{\alpha}{\cos^2 z}\,, \qquad \frac{dV^z}{dt} = -A\,\frac{\sin z}{\cos^3 z}\,,$$

$$\alpha = -\,\frac{B}{\cos r_0}\,, \qquad A = B^2\ \mathrm{tg}^2 r_0\ > 0$$

There exist **effective attraction to the center $z = 0$.**

One can simplify (2-nd order to 1-st order) equation the second equation to

$$\frac{A}{\cos^2 z} = \mathrm{const} - (\frac{dz}{dt})^2\,,$$

const must be identified as ϵ

$$\epsilon = \frac{A}{\cos^2\ z} + (\frac{dz}{dt})^2\,,$$

In contrast to Lobachevsky model, now only one possibility is realized: $\epsilon > A$): **No rotation freezing effect** exist here, instead the **motion must be finite**, and there must arise **turning points** in z variable. Therefore motion must be **periodical.**

Analytical formulas are

(signs ($\pm$) correspond to motions in opposite direction along z):

$$\begin{aligned} r = r_0 = \text{const}\ , \qquad \epsilon > A\ , \\ \sin z(t) = \pm\sqrt{1-\frac{A}{\epsilon}}\ \sin\sqrt{\epsilon}\ t\ , \\ \phi-\phi_0 = \frac{\alpha}{\sqrt{A}}\ \text{arctg}\ (\sqrt{\frac{A}{\epsilon}}\ \text{tg}\ \sqrt{\epsilon}t)\ . \end{aligned} \tag{2}$$

Distinctive feature of the motion is its **periodicity and its closed** character.

The period T is determined by

$$T = \frac{\pi}{\sqrt{\epsilon}} \qquad \left(\text{in usual units}\ \ T = \rho\,\frac{\pi}{V}\right)\ .$$

Special case $\epsilon = A$**:**

$$z(t) = 0\ \ ,\ \ \phi(t) = \phi_0 + \alpha\ t\ ,$$

rotation with constant angular velocity on the circle $r = r_0$ in absence any motion along z.

Space shifts and gauge symmetry of the uniform magnetic field in H_3

Now the question is on the role of the $SO(3.1)$ symmetry in the model H_3. In the first place we are interested in shift transformations.

Let us turn to a pair of coordinate systems in space H_3:

$$u_1 = \text{ch}\, z\, \text{sh}\, r \cos\phi\ , \qquad u_2 = \text{ch}\, z\, \text{sh}\, r \sin\phi\ , \qquad u_3 = \text{sh}\, z\ , \qquad u_0 = \text{ch}\, z\, \text{ch}\, r\ ;$$

$$u_1' = \text{ch}\, z'\, \text{sh}\, r' \cos\phi'\ , \qquad u_2' = \text{ch}\, z'\, \text{sh}\, r' \sin\phi'\ , \qquad u_3' = \text{sh}\, z'\ , \qquad u_0' = \text{ch}\, z'\, \text{ch}\, r'\ ,$$

related by the shift $(0-1)$

$$\left| \begin{array}{c} u_0' \\ u_1' \end{array} \right| = \left| \begin{array}{cc} \text{ch}\, \beta & \text{sh}\, \beta \\ \text{sh}\, \beta & \text{ch}\, \beta \end{array} \right| \left| \begin{array}{c} u_0 \\ u_1 \end{array} \right| \ , u_2' = u_2\ , \quad u_3' = u_3\ .$$

or in cylindric coordinates

$$\underline{0-1}, \qquad z' = z\ , \qquad \text{sh}\, r'\ \sin\phi' = \text{sh}\, r\ \sin\phi\ ,$$

$$\text{sh}\, r'\ \cos\phi' = \text{sh}\beta\, \text{ch}\, r + \text{ch}\, \beta\, \text{sh}\, r\ \cos\phi\ ,$$

$$\text{ch}\, r' = \text{ch}\, \beta\, \text{ch}\, r + \text{sh}\, \beta\, \text{sh}\, r\ \cos\phi\ \ ;$$

With respect to that change $(r, \phi) \Longrightarrow (r', \phi')$ magnetic field transforms according to

$$F_{\phi' r'} = \frac{\partial x^\alpha}{\partial \phi'} \frac{\partial x^\beta}{\partial r'} F_{\alpha\beta} = \left(\frac{\partial \phi}{\partial \phi'} \frac{\partial r}{\partial r'} - \frac{\partial r}{\partial \phi'} \frac{\partial \phi}{\partial r'}\right) F_{\phi r} \; , \qquad F_{\phi r} = B \text{ sh } r \; ;$$

so the **magnetic field transforms** with the help of Jacobian:

$$F_{\phi' r'} = J \; F_{\phi r} \; , \qquad J = \begin{vmatrix} \frac{\partial r}{\partial r'} & \frac{\partial r}{\partial \phi'} \\ \frac{\partial \phi}{\partial r'} & \frac{\partial \phi}{\partial \phi'} \end{vmatrix} \; , \qquad F_{\phi r} = B \text{ sh } r \; .$$

After calculation, the Jacobian of the shift $(0-1)$ reads

$$J = \frac{\text{sh } r'}{\text{sh } r} \; ;$$

and therefore this shift $(0-1)$ leaves **invariant** the uniform magnetic field under consideration

$$F_{\phi r} = B \; \textbf{sh} \; r \; , \qquad F_{\phi' r'} = B \; \textbf{sh} \; r' \; .$$

By symmetry reason we can conclude the same result for shifts of the type $(0-2)$. However, shifts of the type $(0-3)$ **result in different things:** the uniform magnetic field in the space H_3 is not invariant with respect to the shifts $(0-3)$.

Electromagnetic field in terms of 4-potential in H_3

The rule to transform the field with respect to the shift $(0-1)$ looks

$$A_\phi = -B\,(\text{ch}\ r - 1) \qquad \Longrightarrow \qquad A'_{\phi'} = \frac{\partial \phi}{\partial \phi'}\,A_\phi\ , \qquad A'_{r'} = \frac{\partial \phi}{\partial r'}\,A_\phi\ ;$$

In flat space, the shift $\vec{r}\,' = \vec{r} + \vec{b}$ generates a definite gauge transformation:

$$\vec{A}(\vec{r}\,) = \frac{1}{2}\,\vec{B}\times\vec{r}, \qquad \vec{A}\,'(\vec{r}\,') = \frac{1}{2}\vec{B}\times\vec{r}\,' + \nabla_{\vec{r}\,'}\,\Lambda\ ,\ \boldsymbol{\Lambda} = -\,\frac{\mathbf{bB}}{\mathbf{2}}\,\mathbf{y}'\ .$$

By analogy reason one could expect something similar in Lobachevsky space as well:

$$A'_{\phi'} = \frac{\partial \phi}{\partial \phi'}\,A_\phi = -B\,(\text{ch}\ r' - 1) + \frac{\partial}{\partial \phi'}\,\Lambda\ ,$$
$$A'_{r'} = \frac{\partial \phi}{\partial r'}\,A_\phi = \frac{\partial}{\partial r'}\,\Lambda\ \ .$$

It is indeed so – and the **gauge function** has been found:

$$\boldsymbol{\Lambda}(\mathbf{r}', \phi') = +2B\ \text{arctg}\ \left(\frac{(\text{ch}\ \beta - 1)(\text{ch}\ r' - 1) - \text{sh}\ \beta\ \text{sh}\ r'\ \cos\phi'}{\text{sh}\ \beta\ \text{sh}\ r'\ \sin\phi'}\right) - 2B\phi' + \lambda_0\ .$$

Space shifts and gauge symmetry of the uniform magnetic field in S_3

Now the question is on the role of the $SO(4)$ symmetry in the model S_3.

Let us turn to a pair of coordinate systems in space S_3:

$$u_1 = \cos z \ \sin r \cos\phi \ , \qquad u_2 = \cos z \ \sin r \sin\phi \ , \qquad u_3 = \sin z \ , \qquad u_0 = \cos z \ \cos r \ ;$$

$$u'_1 = \cos z' \ \sin r' \cos\phi' \ , \qquad u'_2 = \cos z' \ \sin r' \sin\phi' \ , \qquad u'_3 = \sin z' \ , \qquad u'_0 = \cos z' \ \sin r' \ ,$$

related by the shift $(0-1)$

$$\left| \begin{array}{c} u'_0 \\ u'_1 \end{array} \right| = \left| \begin{array}{cc} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{array} \right| \left| \begin{array}{c} u_0 \\ u_1 \end{array} \right| \ , u'_2 = u_2 \ , \quad u'_3 = u_3 \ .$$

or in cylindric coordinates

$$\underline{0-1}, \qquad z' = z \ , \qquad \sin r' \ \sin\phi' = \sin r \ \sin\phi \ ,$$

$$\sin r' \ \cos\phi' = \sin\beta \ \cos r + \cos\beta \ \sin r \ \cos\phi \ ,$$

$$\cos r' = \cos\beta \ \cos r - \sin\beta \ \sin r \ \cos\phi \ ;$$

With respect to that change $(r,\phi) \Longrightarrow (r',\phi')$ magnetic field transforms according to

$$F_{\phi' r'} = \frac{\partial x^{\alpha}}{\partial \phi'} \frac{\partial x^{\beta}}{\partial r'} F_{\alpha\beta} = \left(\frac{\partial \phi}{\partial \phi'} \frac{\partial r}{\partial r'} - \frac{\partial r}{\partial \phi'} \frac{\partial \phi}{\partial r'}\right) F_{\phi r} \; , \qquad F_{\phi r} = B \; \sin r \; ;$$

so the **magnetic field transforms** with the help of Jacobian:

$$F_{\phi' r'} = J \; F_{\phi r} \; , \qquad J = \begin{vmatrix} \frac{\partial r}{\partial r'} & \frac{\partial r}{\partial \phi'} \\ \frac{\partial \phi}{\partial r'} & \frac{\partial \phi}{\partial \phi'} \end{vmatrix} \; , \qquad F_{\phi r} = B \; \text{sh} \; r \; .$$

After calculation, the Jacobian of the shift $(0-1)$ reads

$$J = \frac{\sin r'}{\sin r} \; ;$$

and therefore this shift $(0-1)$ leaves **invariant** the uniform magnetic field under consideration

$$F_{\phi r} = B \; \sin r \; , \qquad F_{\phi' r'} = B \; \sin r' \; .$$

By symmetry reason we can conclude the same result for shifts of the type $(0-2)$. However, shifts of the type $(0-3)$ **behave differently:** the uniform magnetic field in the space H_3 is not invariant with respect to the shifts $(0-3)$.

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, then The rule to transform the field with respect to the shift $(0-1)$ looks

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In flat space, the shift $\vec{r}\,' = \vec{r} + \vec{b}$ generates a definite gauge transformation:

$$\vec{A}(\vec{r}\,) = \frac{1}{2}\,\vec{B}\times\vec{r}, \qquad \vec{A}\,'(\vec{r}\,') = \frac{1}{2}\vec{B}\times\vec{r}\,' + \nabla_{\vec{r}\,'}\,\Lambda\,, \qquad \mathbf{\Lambda} = -\,\frac{\mathbf{bB}}{\mathbf{2}}\,\mathbf{y}'\,.$$

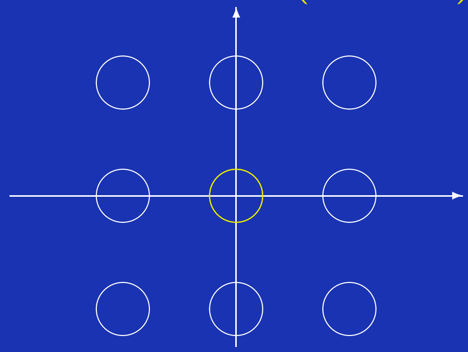
By analogy reason one could expect something similar in Lobachevsky space as well:

$$A'_{\phi'} = \frac{\partial \phi}{\partial \phi'}\,A_\phi = B\,(\cos r' - 1) + \frac{\partial}{\partial \phi'}\,\Lambda\,,$$
$$A'_{r'} = \frac{\partial \phi}{\partial r'}\,A_\phi = \frac{\partial}{\partial r'}\,\Lambda\;\;.$$

It is indeed so and the **gauge function** has been found:

$$\mathbf{\Lambda}(\mathbf{r}',\phi') = -2B\;\mathrm{arctg}\;\left(\frac{(\cos\beta - 1)(\cos r' - 1) - \sin\beta\;\sin r'\;\cos\phi'}{\sin\beta\;\sin r'\;\sin\phi'}\right) + 2B\phi' + \lambda_0\,.$$

Analytical description of the **all (shifted) trajectories in H_3**



is given through constructing **3 conserved quantities**

$$\epsilon = \text{ch}^2 z \left[(\frac{dr}{dt})^2 + \text{sh}^2 r (\frac{d\phi}{dt})^2 \right] + (\frac{dz}{dt})^2 , \qquad 0 < \epsilon < 1 , \qquad \epsilon = \text{const} ,$$

$$I = \text{sh}^2 r \ \text{ch}^2 z \ \frac{d\phi}{dt} + B(\text{ ch } r - 1) , \qquad \mathbf{I} = \text{const} ,$$

$$A = \text{ch}^4 z \left[(\frac{dr}{dt})^2 + \text{sh}^2 r \ (\frac{d\phi}{dt})^2 \right] , \qquad A > 0 , \qquad \mathbf{A} = \text{const} ,$$

they permit to reduce the task to calculating the integrals **(NO MORE DETAILS):**

$$\frac{dz}{\pm\sqrt{\epsilon - A/\text{ch}^2 z}} = dt \qquad \Longrightarrow \qquad \mathbf{z} = \mathbf{z}(\mathbf{t}) ,$$

$$\frac{d \text{ ch } r}{\pm \sqrt{A \ (\text{ch}^2 r - 1) - [\ I - B \ (\text{ ch } r - 1 \ ) \ ]^2}} = \frac{dt}{\text{ch}^2 z(t)} \qquad \Longrightarrow \qquad \mathbf{r} = \mathbf{r}(\mathbf{t}) ,$$

$$d\phi = \frac{1}{\text{ch}^2 z(t)} \ \frac{I - B \ [\text{ ch } r(t) - 1]}{\text{ch}^2 r(t) - 1} \ dt \qquad \Longrightarrow \qquad \phi = \phi(\mathbf{t}) .$$

Trajectory equation $F(r,\phi)=0$, the role of Lorentz $SO(3,1)$ shifts in H_3

Now, let us consider the trajectory equation $F(r,\phi)$

$$\frac{[\,(I+B)-B\ \text{ch}\ r\,]\ dr}{\text{sh}\ r\ \sqrt{A\ \text{sh}^2 r-[\,(I+B)-B\ \text{ch}\ r\,]^2}}=d\phi\ \Longrightarrow$$

$$\mathbf{F}(\mathbf{r},\phi)=\mathbf{0}:\qquad (I+B)\ \text{ch}\ \mathbf{r}-\sqrt{(I+B)^2+(A-B^2)}\ \ \text{sh}\ \mathbf{r}\ \ \cos\phi=\mathbf{B}\ .$$

This is the most general form of trajectory equation $F(r,\phi)=0$.

Trajectory equation $F(r,\phi)=0$ translated to coordinate (r',ϕ') looks

$$\mathbf{F}(\mathbf{r}',\phi')=\mathbf{0}:\qquad \Big[\ \text{ch}\ \beta\ (I+B)+\text{sh}\beta\ \sqrt{(I+B)^2+(A-B^2)}\ \Big]\ \text{ch}\ \mathbf{r}'-$$
$$-\Big[\ \text{sh}\ \beta\ (I+B)\ +\text{ch}\ \beta\ \sqrt{(I+B)^2+(A-B^2)}\ \Big]\ \text{sh}\ \mathbf{r}'\ \cos\phi'=\mathbf{B}\ , \qquad (3)$$

They are of the same form if parameters transform according to Lorentz shift

$$I'+B=\text{ch}\ \beta\ (I+B)+\text{sh}\ \beta\ \sqrt{(I+B)^2+(A-B^2)}\ ,$$
$$\sqrt{(I'+B)^2+(A'-B^2)}=\text{sh}\ \beta\ (I+B)+\text{ch}\ \beta\ \sqrt{(I+B)^2+(A-B^2)}\ . \qquad (4)$$

These Lorentz shifts leave invariant the following combination in parametric space:

$$\text{inv} = (I+B)^2 - \left(\sqrt{(I+B)^2 + (A-B^2)} \ \right)^2 \qquad \Longrightarrow \qquad \mathbf{A}' = \mathbf{A} \ . \tag{5}$$

This means that **Lorentz shifts vary only parameter** I.

It has sense to introduce **new parameters** J, C:

$$\mathbf{J} = I + B \ , \qquad \mathbf{C} = \sqrt{(I+B)^2 + (A-B^2)} \tag{6}$$

then (4) read

$$J' = \text{ch}\ \beta\ J + \text{sh}\ \beta\ C \ , \qquad C' = \text{sh}\ \beta\ J + \text{ch}\ \beta\ C \tag{7}$$

and **invariant form of trajectory equation** $F(r,\phi) = 0$ can be presented as

$$\mathbf{J}\ \text{ch}\ \mathbf{r} - \mathbf{C}\ \text{sh}\ \mathbf{r} \cos\phi = \mathbf{B} \ , \tag{8}$$

in any other shifted reference frame it looks

$$\mathbf{J}'\ \text{ch}\ \mathbf{r}' - \mathbf{C}'\ \text{sh}\ \mathbf{r}' \cos\phi' = \mathbf{B} \ .$$

Correspondingly the main invariant reads

$$\text{inv} = \mathbf{J}^2 - \mathbf{C}^2 = \mathbf{J}'^2 - \mathbf{C}'^2 = \mathbf{B}^2 - \mathbf{A} \ . \tag{9}$$

Depending on the sign of this invariant

we may reach the most simple description by means of an appropriate shift:

1) $B^2 - A > 0$ **(finite motion)**

$$J_0^2 = B^2 - A \;, \qquad C_0 = 0 \;,$$
$$\text{trajectory equation} \qquad J_0 \operatorname{ch} r = B \;; \tag{10}$$

2) $B^2 - A < 0$ **(infinite motion)**

$$J_0 = 0 \;, \qquad C_0^2 = A - B^2$$
$$\text{trajectory equation} \qquad - C_0 \operatorname{sh} r \cos\phi = B \;. \tag{11}$$

Special case exists

3)$B^2 = A$ **(infinite motion)**

$$J = I + B \;, \qquad C = I + B \;,$$
$$\text{trajectory equation} \qquad \operatorname{ch} r - \operatorname{sh} r \cos\phi = \frac{B}{I + B} \;, \tag{12}$$

By symmetry reasons, Lorentzian shifts of the type $(0-2)$ will manifest themselves analogously.

Tragectory $F(r,\phi)=0$ in the model S_3 and SO(4) symmetry

Now, let us consider tragectory in the form $F(r,\phi)=0$:

$$\int \frac{[\,I + B(\cos r - 1)\,]\,dr}{\sin r\,\sqrt{A\ \sin^2 r - [\,I + B(\cos r - 1)\,]^2}} = \phi\ .$$

After integration, **general trajectory equation** $F(r,\phi)=0$ **in the model** S_3 looks

$$(B-I)\cos \mathbf{r} + \sqrt{(A+B^2)-(I-B)^2}\,\sin \mathbf{r}\ \cos\phi = \mathbf{B}\ .$$

Let us consider behavior of this equation with respect to) shifts $(0-1)$ in space S_3:

$$[\,\cos\alpha\,(B-I)\ +\sin\alpha\,\sqrt{(A+B^2)-(I-B)^2}\,]\cos\mathbf{r}' +$$
$$+[\,-\sin\alpha\,(B-I)\ +\cos\alpha\,\sqrt{(A+B^2)-(I-B)^2}\,]\sin\mathbf{r}'\ \cos\phi' = \mathbf{B}\ .$$

we have seen invariance property of the trajectory equation if parameters transform according to

$$\mathbf{B}'-\mathbf{I}' = \cos\alpha\,(\mathbf{B}-\mathbf{I})\ +\sin\alpha\,\sqrt{(\mathbf{A}+\mathbf{B}^2)-(\mathbf{I}-\mathbf{B})^2}\ ,$$
$$\sqrt{(\mathbf{A}'+\mathbf{B}^2)-(\mathbf{I}'-\mathbf{B})^2} = -\sin\alpha\,(\mathbf{B}-\mathbf{I})\ +\cos\alpha\,\sqrt{(\mathbf{A}+\mathbf{B}^2)-(\mathbf{I}-\mathbf{B})^2}\ ;$$

With notation

$$B - I = J\ , \qquad C = \sqrt{(A + B^2) - (I - B)^2}$$

trajectory equation has the following invariant form

$$J \cos r + C\ \sin r\ \cos\phi = B \qquad \Longrightarrow \qquad J' \cos r' + C'\ \sin r'\ \cos\phi' = B\ ,$$

with respect to Euclidean shifts $(0-1)$ in S_3 parameters J, C transform according to

$$\mathbf{J}' = \cos\alpha\ \mathbf{J} + \sin\alpha\ \mathbf{C}\ , \qquad \mathbf{C}' = -\sin\alpha\ \mathbf{J} + \cos\alpha\ \mathbf{C}\ .$$

This parametric shift leaves invariant the (Euclidean) combination of two parameters:

$$\text{inv} = J^2 + C^2 = J'^2 + C'^2 = A + B^2 \qquad \Longrightarrow \qquad \mathbf{A} = \mathbf{A}' = \text{inv}\ . \tag{13}$$

By special choice of a shift one can translate the general equation **to 2 simple forms:**

$$J_0 = \sqrt{A + B^2}\ ,\ C_0 = 0 \qquad \Longrightarrow \qquad J_0 \cos r_0 = B\ ;$$

$$J_0 = 0\ ,\ C_0 = \sqrt{A + B^2} \qquad \Longrightarrow \qquad C_0\ \sin r\ \cos\phi = B\ . \tag{14}$$

CLASSICAL PARTICLE IN PRESENCE OF MAGNETIC FIELD, HYPERBOLIC LOBACHEVSKY AND SPHERICAL RIEMANN MODELS

In the paper an exact solutions for classical problem of a particle in magnetic field on the background of hyperbolic Lobachevsky H_3 and spherical Riemann S_3 space models will be constructed explicitly.

Thank You, wishing good luck

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