THE ELECTROMAGNETIC DIRAC-FOCK-PODOLSKY PROBLEM AND ITS ANALYSIS WITHIN THE SYMPLECTIC REDUCTION THEORY

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Abstract. Symplectic structures associated to connection forms on certain types of principal fiber bundles are constructed via analysis of reduced geometric structures on fibered manifolds invariant under naturally related symmetry groups. This approach is then applied to nonstandard Hamiltonian analysis of of dynamical systems of Maxwell and Yang-Mills type. A symplectic reduction theory of the classical Maxwell equations is formulated so as to naturally include the Lorentz condition (ensuring the existence of electromagnetic waves), thereby solving the well known Dirac -Fock - Podolsky problem. Symplectically reduced Poissonian structures and the related classical minimal interaction principle for the Yang-Mills equations are also considered.

1. Introduction

When investigating dynamical systems, which are invariant under symmetry group actions, on canonical symplectic manifolds, additional mathematical structures often arise. Analysis of these structures almost invariably produces important dynamical insights about the systems. For example, the Cartan connection on an associated principal fiber bundle leads to a more detailed understanding of the reductions of the dynamical system on invariant submanifolds and quotient manifolds.

Problems related to the investigation of properties of reduced dynamical systems on symplectic manifolds were studied, e.g., in [1, 15, 14, 23, 22], where the relationship between a symplectic structure on the reduced space and the connection on a principal fiber bundle was explicitly formulated. Other aspects of dynamical systems related to properties of reduced symplectic structures were studied in [16, 17, 18] where, in particular, the reduced symplectic structure was completely described within the framework of the classical Dirac scheme, and several applications to nonlinear (including celestial) dynamics were given.

It is well known [5, 3, 9, 12, 13, 11] that the Hamiltonian formulation of Maxwell's electromagnetic field equations involves a very important classical problem; namely, to intrinsically introduce the Lorentz condition, which guarantees the wave structure of propagating quanta and the positivity of energy. Unfortunately, in spite of extensive classical studies by Dirac, Fock and Podolsky [10], the problem remains open. Consequently, the Lorentz condition is usually imposed in modern electrodynamics as an external constraint rather than arising naturally from the Hamiltonian (or Lagrangian) theory. Moreover, it was shown by Pauli, Dirac, Bogolubov and Shirkov and others [5, 11, 9, 6] that the quantum Lorentz condition is incompatible with existing quantization approaches for electromagnetic field theory, except in an average sense. These difficulties stimulated our study of this problem using symplectic reduction theory, which allows a systematic introduction of the external charge and current conditions into the Hamiltonian formalism, and actually leads to the solution to the Lorentz condition problem described herein.

Some applications of the method to Yang-Mills type equations interacting with a point charged particle are presented. In particular, by analyzing reduced geometric structures on fibered manifolds invariant under the action of a symmetry group, we construct the symplectic structures associated with connection forms on suitable principal fiber bundles. We begin with a brief description of the mathematical preliminaries of the related Poissonian structures on the corresponding reduced symplectic manifolds, which are often used [1, 21, 20] in various problems of dynamics in modern mathematical physics. These methods are then applied to studying the nonstandard Hamiltonian properties of Maxwell and Yang-Mills type dynamical systems.

Date: present.

1991 *Mathematics Subject Classification.* Primary 34A30, 34B05 Secondary 34B15 .

Key words and phrases. Maxwell equations, Hamiltonian system, canonical reduction, symplectic structures, connections, principal fiber bundles, Yang-Mills gauge fields .

Our main contribution here is a novel formulation of a symplectic reduction theory for the classical Maxwell electromagnetic field equations that provides a means of naturally including the Lorentz condition (ensuring [5, 6] the existence of electromagnetic waves) in the associated Hamiltonian structure - thereby solving the Dirac-Fock-Podolsky [10] problem mentioned above. In addition, we also use our symplectic reduction theory to investigate the Poissonian structures and the classical minimal interaction principle related to Yang-Mills equations.

2. Symplectic structures and reduction on manifolds: preliminaries

In this section, we shall outline the basic elements of symplectic structures and reduction on manifolds employed in the sequel.

2.1. Symplectic reduction on cotangent fiber bundles with symmetry. Consider an n-dimensional smooth manifold M and the cotangent vector fiber bundle $T^*(M)$. We equip (see [2], Chapter VII) the cotangent space $T^*(M)$ with the canonical Liouville 1-form $\lambda(\alpha^{(1)}) := pr_M^*\alpha^{(1)} \in \Lambda^1(T^*(M))$, where $pr_M : T^*(M) \to M$ is the canonical projection and

$$\alpha^{(1)}(u) = \sum_{j=1}^{n} v_j du^j, \tag{2.1}$$

where $(u, v) \in T^*(M)$ are the corresponding canonical local coordinates on $T^*(M)$. Thus, any group of diffeomorphisms of the manifold M naturally lifted to the fiber bundle $T^*(M)$ preserves the invariance of the canonical 1-form $\lambda(\alpha^{(1)}) \in \Lambda^1(T^*(M))$. In particular, if a smooth action of a Lie group G is given on the manifold M, then every element $a \in \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of the Lie group G, generates the vector field $k_a \in T(M)$ in a natural manner. Furthermore, since the group action on M, i.e.,

$$\varphi : G \times M \to M, \tag{2.2}$$

generates a diffeomorphism $\varphi_g \in Diff\ M$ for every element $g \in G$, this diffeomorphism lifts naturally to the corresponding diffeomorphism $\varphi_g^* \in Diff\ T^*(M)$ of the cotangent fiber bundle $T^*(M)$, which also leaves the canonical 1-form $pr_M^*\alpha^{(1)} \in \Lambda^1(T^*(M))$ invariant; namely,

$$\varphi_g^*\lambda(\alpha^{(1)}) = \lambda(\alpha^{(1)}) \tag{2.3}$$

holds [1, 2, 15] for every 1-form $\alpha^{(1)} \in \Lambda^1(M)$. Thus, we can define on $T^*(M)$ the corresponding vector field $K_a : T^*(M) \to T(T^*(M))$ for every element $a \in \mathcal{G}$. Then condition (2.3) can be rewritten in the following form for all $a \in \mathcal{G}$:

$$L_{K_a} \cdot pr_M^*\alpha^{(1)} = pr_M^* \cdot L_{k_a}\alpha^{(1)} = 0,$$

where L_{K_a} and L_{k_a} are the ordinary Lie derivatives on $\Lambda^1(T^*(M))$ and $\Lambda^1(M)$, respectively.

The canonical symplectic structure on $T^*(M)$ is defined as

$$\omega^{(2)} := d\lambda(\alpha^{(1)}) \tag{2.4}$$

and is invariant, i.e., $L_{K_a}\omega^{(2)} = 0$ for all $a \in \mathcal{G}$.

For any smooth function $H \in D(T^*(M))$, a Hamiltonian vector field $K_H : T^*(M) \to T(T^*(M))$ such that

$$i_{K_H}\omega^{(2)} = -dH \tag{2.5}$$

is defined, and vice versa, because the symplectic 2-form (2.4) is nondegenerate. Using (2.5) and (2.4), we easily establish that the Hamiltonian function $H := H_K \in D(T^*(M))$ is given as $H_K = pr_M^*\alpha^{(1)}(K_H) = \alpha^{(1)}(pr_M^* K_H) = \alpha^{(1)}(k_H)$, where $k_H \in T(M)$ is the corresponding vector field on the manifold M, whose lift to the fiber bundle $T^*(M)$ coincides with the vector field $K_H : T^*(M) \to T(T^*(M))$. For $K_a : T^*(M) \to T(T^*(M))$, where $a \in \mathcal{G}$, it is easy to establish that the corresponding Hamiltonian function $H_a = \alpha^{(1)}(k_a) = pr_M^*\ \alpha^{(1)}(K_a)$ for $a \in \mathcal{G}$ defines [1, 15, 14] a linear momentum mapping $l : T^*(M) \to \mathcal{G}^*$ according to the rule

$$H_a :=< l, a >, \tag{2.6}$$

where $<\cdot,\cdot>$ is the corresponding convolution on $\mathcal{G}^* \times \mathcal{G}$. By virtue of definition (2.6), the momentum mapping $l : T^*(M) \rightarrow \mathcal{G}^*$ is invariant under the action of any invariant Hamiltonian vector field $K_b : T^*(M) \rightarrow T(T^*(M))$ for any $b \in \mathcal{G}$. Indeed, $L_{K_b} < l, a >= L_{K_b} H_a = -L_{K_a} H_b = 0$, because, by definition, the Hamiltonian function $H_b \in D(T^*(M))$ is invariant under the action of any vector field $K_a : T^*(M) \rightarrow T(T^*(M))$, $a \in \mathcal{G}$.

We now fix a regular value of the momentum mapping $l(u,v) = \xi \in \mathcal{G}^*$ and consider the corresponding submanifold $\mathcal{M}_\xi := \{(u,v) \in T^*(M) : l(u,v) = \xi \in \mathcal{G}^*\}$. Owing to definition (2.1) and the invariance of the 1-form $pr_M^*\ \alpha^{(1)} \in \Lambda^1(T^*(M))$ under the action of the Lie group G on $T^*(M)$, we have

$$
\begin{aligned}
&< \ l(g \circ (u,v)), a >= pr_M^* \alpha^{(1)}(K_a)(g \circ (u,v)) = \\
&= \ pr_M^* \alpha^{(1)}(K_{Ad_{g-1}a})(u,v) := \\
&= \ < l(u,v), Ad_{g-1}a >=< Ad_{g-1}^* l(u,v), a >
\end{aligned}
\tag{2.7}
$$

for any $g \in G$ and all $a \in \mathcal{G}$ and $(u,v) \in T^*(M)$. Now it follows from (2.7) that $l(g \circ (u,v)) = Ad_{g-1}^* l(u,v)$ for every $g \in G$ and all $(u,v) \in T^*(M)$. This means that the diagram

$$
\begin{array}{ccc}
T^*(M) & \xrightarrow{l} & \mathcal{G}^* \\
g \downarrow & & \downarrow Ad_{g-1}^* \\
T^*(M) & \xrightarrow{l} & \mathcal{G}^*
\end{array}
$$

is commutative for all elements $g \in G$. The corresponding action $g : T^*(M) \rightarrow T^*(M)$ is called equivariant [1, 15].

Let $G_\xi \subset G$ denote the stabilizer of a regular element $\xi \in \mathcal{G}^*$ with respect to the related co-adjoint action. It is obvious in this case that the action of the Lie subgroup G_ξ on the submanifold $\mathcal{M}_\xi \subset T^*(M)$ is naturally defined; we assume that it is free and proper. Using this action on $\mathcal{M}_\xi$, we can define [1, 17, 18, 19, 20] a so-called reduced space $\bar{\mathcal{M}}_\xi$ by taking the factor with respect to the action of the subgroup G_ξ on $\mathcal{M}_\xi$, i.e.,

$$
\bar{\mathcal{M}}_\xi := \mathcal{M}_\xi / G_\xi. \tag{2.8}
$$

The quotient space (2.8) induces a symplectic structure $\bar{\omega}_\xi^{(2)} \in \Lambda^2(\bar{\mathcal{M}}_\xi)$ on itself, which is defined as follows:

$$
\bar{\omega}_\xi^{(2)}(\bar{\eta}_1, \bar{\eta}_2) = \omega_\xi^{(2)}(\eta_1, \eta_2), \tag{2.9}
$$

where $\bar{\eta}_1, \bar{\eta}_2 \in T(\bar{\mathcal{M}}_\xi)$ are arbitrary vectors onto which vectors $\eta_1, \eta_2 \in T(\mathcal{M}_\xi)$ are projected for at any point $(u_\xi, v_\xi) \in \mathcal{M}_\xi$. It follows from (2.8) that this projection onto the point $\bar{\mu}_\xi \in \bar{\mathcal{M}}_\xi$ is unique .

Let $\pi_\xi : \mathcal{M}_\xi \rightarrow T^*(M)$ denote the corresponding imbedding mapping into $T^*(M)$ and let $r_\xi : \mathcal{M}_\xi \rightarrow \bar{\mathcal{M}}_\xi$ be the corresponding reduction to the space $\bar{\mathcal{M}}_\xi$. Then relation (2.9) can be rewritten in the form

$$
r_\xi^* \bar{\omega}_\xi^{(2)} = \pi_\xi^* \omega^{(2)}, \tag{2.10}
$$

which is defined on vectors on the cotangent space $T^*(\mathcal{M}_\xi)$. To establish the symplecticity of the 2-form $\omega_\xi^{(2)} \in \Lambda^2(\bar{\mathcal{M}}_\xi)$, we use the corresponding non-degeneracy of the Poisson bracket $\{\cdot,\cdot\}_\xi^r$ on $\bar{\mathcal{M}}_\xi$. We use a Dirac type construction for the calculation, defining functions on $\bar{\mathcal{M}}_\xi$ as certain G_ξ-invariant functions on the submanifold $\mathcal{M}_\xi$. Then one can calculate the Poisson bracket $\{\cdot,\cdot\}_\xi$ of such a function corresponding to the symplectic structure (2.4) as an ordinary Poisson bracket on $T^*(M)$, arbitrarily extending these functions from the submanifold $\mathcal{M}_\xi \subset T^*(M)$ to a neighborhood $U(\mathcal{M}_\xi) \subset T^*(M)$. It is obvious that two extensions of a given function to the neighborhood $U(\mathcal{M}_\xi)$ of this type differ by a function that vanishes on the submanifold $\mathcal{M}_\xi \subset T^*(M)$. The difference between the corresponding Hamiltonian fields of these two different extensions to $U(\mathcal{M}_\xi)$ is completely controlled by the conditions of the following lemma (see also [1, 15, 18, 17, 23]).

Lemma 2.1. *Suppose that a function $f : U(\mathcal{M}_\xi) \rightarrow \mathbb{R}$ is smooth and vanishes on $\mathcal{M}_\xi \subset T^*(M)$, i.e., $f|_{\mathcal{M}_\xi} = 0$. Then, at every point $(u_\xi, v_\xi) \in \mathcal{M}_\xi$ the corresponding Hamiltonian vector field $K_f \in T(U(\mathcal{M}_\xi))$ is tangent to the orbit $Or(G; (u_\xi, v_\xi))$.*

As a corollary of Lemma 2.1, we obtain an algorithm for computing the reduced Poisson bracket $\{\cdot,\cdot\}_{\xi}^{r}$ on the space $\bar{\mathcal{M}}_{\xi}$ according to definition (2.10). Namely, we choose two functions defined on $\bar{\mathcal{M}}_{\xi}$ and invariant under the action of the subgroup G_{ξ} and arbitrarily smoothly extend them to a certain open domain $U(\mathcal{M}_{\xi}) \subset T^{*}(M)$. Then we determine the corresponding Hamiltonian vector fields on $T^{*}(M)$ and project them onto the space tangent to $\mathcal{M}_{\xi}$, adding, if necessary, the corresponding vectors tangent to the orbit $Or(G)$. It is easy to see that the projections obtained depend on the chosen extensions to the domain $U(\mathcal{M}_{\xi}) \subset T^{*}(M)$. As a result, we establish that the reduced Poisson bracket $\{\cdot,\cdot\}_{\xi}^{r}$ is uniquely defined via the restriction of the initial Poisson bracket upon $\mathcal{M}_{\xi} \subset T^{*}(M)$, and one can readily verify that the submanifold $M_{\xi} \subset T^{*}(M)$ is defined by a collection of relations of the type

$$H_{a_s} = \xi_s, \qquad \xi_s := <\xi, a_s>, \tag{2.11}$$

where $a_s \in \mathcal{G}, s = \overline{1, dimG}$, is a certain basis of the Lie algebra $\mathcal{G}$. By virtue of the nondegeneracy of the restriction and the functional independence of the basis functions (2.11), it is obvious that the reduced Poisson bracket $\{\cdot,\cdot\}_{\xi}^{r}$ is [1, 15, 17] nondegenerate on $\bar{\mathcal{M}}_{\xi}$. Consequently, we establish that the dimension of the reduced space $\bar{\mathcal{M}}_{\xi}$ is even. Taking into account that the element $\xi \in \mathcal{G}^{*}$ is regular and the dimension of the Lie algebra of the stabilizer $\mathcal{G}_{\xi}$ is equal to $dim\ G_{\xi}$, we easily establish that $dim\ \bar{\mathcal{M}}_{\xi} = dim\ T^{*}(M) - 2dim\ \mathcal{G}_{\xi}$. Since, by construction, $dim\ T^{*}(M) = 2n$, we conclude that the dimension of the reduced space $\bar{\mathcal{M}}_{\xi}$ is even.

In order completely verify the correctness of the algorithm, it is necessary to establish the existence of the corresponding projections of Hamiltonian vector fields onto the tangent space $T(\mathcal{M}_{\xi})$. The following result [22] solves this problem.

Theorem 2.2. *At every point $(u_{\xi}, v_{\xi}) \in \mathcal{M}_{\xi}$, one can choose a vector $V_f \in T(Or(G))$ such that $K_f(u_{\xi}, v_{\xi}) + V_f(u_{\xi}, v_{\xi}) \in T_{(u_{\xi},v_{\xi})}(\mathcal{M}_{\xi})$. Furthermore, the vector $V_f \in T(Or(G))$ is uniquely determined up to a vector tangent to the orbit $Or(G_{\xi})$.*

Now assume that two functions $f_1, f_2 \in D(\mathcal{M}_{\xi})$ are G_{ξ}-invariant. Then their reduced Poisson bracket $\{f_1, f_2\}_{\xi}^{r}$ on $\bar{\mathcal{M}}_{\xi}$ is defined according to the rule:

$$\{f_1, f_2\}_{\xi}^{r} := -\omega^{(2)}(K_{f_1} + V_{f_1}, K_{f_2} + V_{f_2}) = \{f_1, f_2\} + \omega^{(2)}(V_{f_1}, V_{f_2}), \tag{2.12}$$

where we have used the following identities on $\mathcal{M}_{\xi} \subset T^{*}(M)$:

$$\omega^{(2)}(K_{f_1} + V_{f_1}, V_{f_2}) = 0 = \omega^{(2)}(K_{f_2} + V_{f_2}, V_{f_1}), \tag{2.13}$$

which follow immediately from

$$\omega^{(2)}(K_f + V_f, K_a) = 0 \tag{2.14}$$

for all $a \in \mathcal{G}_{\xi}$ and $f \in D(\mathcal{M}_{\xi})$ on $\mathcal{M}_{\xi}$. With regard to (2.13), relation (2.12) takes the form

$$\{f_1, f_2\}_{\xi}^{r} = \{f_1, f_2\} + \frac{1}{2}(V_{f_1} f_2 - V_{f_2} f_1), \tag{2.15}$$

for arbitrary smooth extensions $f_1, f_2 \in D(\mathcal{M}_{\xi})$ of G_{ξ}-invariant functions, as defined above on the domain $U(\mathcal{M}_{\xi})$. Thus, as a consequence of (2.2), one has the following [1, 9, 15] theorem of Dirac type.

Theorem 2.3. *The reduced Poisson bracket of two functions on the quotient space $\bar{\mathcal{M}}_{\xi} = \mathcal{M}_{\xi}/G_{\xi}$ is determined with the use of arbitrary smooth extensions of them to functions on an open neighborhood $U(\mathcal{M}_{\xi})$ according to the Dirac-type formula (2.15).*

2.2. **Symplectic reduction on principal fiber bundles with a connection.** We begin by reviewing reduction theory for Hamiltonian systems with symmetry on principle fiber bundles. As the material is partially available in [4, 16], we shall provide only a sketch here using notation that is to be employed in the sequel.

Let G denote a Lie group with the unity element $e \in G$ and $\mathcal{G} \simeq T_e(G)$ be its Lie algebra. Consider a principal fiber bundle $\pi : (M, \varphi) \to N$ with the structure group G and base manifold N, on which the Lie group G acts via a mapping $\varphi : M \times G \to M$. In particular, for each $g \in G$ there is a group diffeomorphism $\varphi_g : M \to M$, generating for any fixed $u \in M$ the following induced mapping: $\hat{u} : G \to M$, where

$$\hat{u}(g) = \varphi_g(u). \tag{2.16}$$

This mapping induces a connection $\Gamma(\mathcal{A})$ on the principal fiber bundle $\pi : (M, \varphi) \to N$, where the morphism $\mathcal{A}$: $(T(M), \varphi_{g*}) \to (\mathcal{G}, Ad_{g^{-1}})$, such that for each $u \in M$ a mapping $\mathcal{A}(u) : T_u(M) \to \mathcal{G}$ is a left inverse of the mapping $\hat{u}_*(e) : \mathcal{G} \to T_u(M)$, that is

$$\mathcal{A}(u)\hat{u}_*(e) = 1. \tag{2.17}$$

As usual, we denote by $\varphi_g^* : T^*(M) \to T^*(M)$ the corresponding lift of the mapping $\varphi_g : M \to M$ at any $g \in G$. If $\alpha^{(1)} \in \Lambda^1(M)$ is the canonical G - invariant 1-form on M, the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(M))$, given by the expression

$$\omega^{(2)} := d\lambda(\alpha^{(1)}) = d\ pr_M^* \alpha^{(1)}, \tag{2.18}$$

generates the corresponding momentum mapping $l : T^*(M) \to \mathcal{G}^*$, where

$$l \cdot \alpha^{(1)}(u) = \hat{u}^*(e)\alpha^{(1)}(u) \tag{2.19}$$

for all $u \in M$. We remark here that the principal fiber bundle structure $\pi : (M, \varphi) \to N$ entails in part the exactness of the following sequences of mappings:

$$0 \to \mathcal{G} \overset{\hat{u}_*(e)}{\to} T_u(M) \overset{\pi_*(u)}{\to} T_{\pi(u)}(N) \to 0, \tag{2.20}$$

that is

$$\pi_*(u)\hat{u}_*(e) = 0 = \hat{u}^*(e)\pi^*(u) \tag{2.21}$$

for all $u \in M$. Combining (2.21) with (2.17) and (2.19), one obtains the embedding:

$$[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in \text{range } \pi^*(u) \tag{2.22}$$

for the canonical 1-form $\alpha^{(1)} \in \Lambda^1(M)$ at $u \in M$. The expression (2.22) means of course, that

$$\hat{u}^*(e)[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0 \tag{2.23}$$

for all $u \in M$. As the mapping $\pi^*(u) : T^*(N) \to T^*(M)$ is injective for each $u \in M$, it has the unique inverse mapping $(p^*(u))^{-1}$defined on its image $\pi^*(u)T^*_{\pi(u)}(N) \subset T^*_u(M)$. Whence, for each $u \in M$ one can define a morphism $\pi_{\mathcal{A}} : (T^*(M), \varphi_g^*) \to T^*(N)$ as

$$\pi_{\mathcal{A}}(u) : \alpha^{(1)}(u) \to (\pi^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u). \tag{2.24}$$

It is easy to check using (2.24) that the diagram

$$\begin{array}{ccc} T^*(M) & \overset{\pi_{\mathcal{A}}}{\to} & T^*(N) \\ pr_M\downarrow & & \downarrow pr_N \\ M & \overset{\pi}{\to} & N \end{array} \tag{2.25}$$

is commutative.

Now suppose an element $\xi \in \mathcal{G}^*$ be G-invariant, that is $Ad^*_{g^{-1}}\xi = \xi$ for all $g \in G$. Let $\pi^\xi_{\mathcal{A}}$ denote the restriction of the mapping (2.24) upon the subset $\mathcal{M}_\xi := l^{-1}(\xi) \in T^*(M)$, that is $\pi^\xi_{\mathcal{A}} : \mathcal{M}_\xi \to T^*(N)$, where for all $u \in M$

$$\pi^\xi_{\mathcal{A}}(u) : l^{-1}(\xi) \to (\pi^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]l^{-1}(\xi). \tag{2.26}$$

The structure of the reduced phase space $\bar{\mathcal{M}}_\xi :=l^{-1}(\xi)/G$ can now be characterized by means of the following lemma.

Lemma 2.4. *The mapping $\pi^\xi_{\mathcal{A}}(u) : \mathcal{M}_\xi \to T^*(N)$, where $\mathcal{M}_\xi := l^{-1}(\xi)$ is a principal fiber G -bundle with the reduced space $\bar{\mathcal{M}}_\xi$, maps $\bar{\mathcal{M}}_\xi$ diffeomorphically onto $T^*(N)$.*

Denote by $< .,. >_{\mathcal{G}}$ the standard Ad-invariant non-degenerate scalar product on $\mathcal{G} \times \mathcal{G}$. The following characteristic theorem can be derived directly from Lemma 2.4 .

Theorem 2.5. *Given a principal fiber G -bundle with a connection $\Gamma(\mathcal{A})$ and a G -invariant element $\xi \in \mathcal{G}^*$, then the connection $\Gamma(\mathcal{A})$ defines a symplectomorphism $\nu_\xi : \bar{\mathcal{M}}_\xi \to T^*(N)$ between the reduced phase space $\bar{\mathcal{M}}_\xi$ and cotangent bundle $T^*(N)$, where $l : T^*(M) \to \mathcal{G}^*$ is the natural momentum mapping for the group G -action on M. Moreover,*

$$(\pi^\xi_{\mathcal{A}})(d\ pr_N^*\beta^{(1)} + pr_N^*\ \Omega^{(2)}_\xi) = d\ pr_M^*\alpha^{(1)}\Big|_{l^{-1}(\xi)} \tag{2.27}$$

holds for the canonical 1-forms $\beta^{(1)} \in \Lambda^1(N)$ and $\alpha^{(1)} \in \Lambda^1(M)$, where $\Omega^{(2)}_\xi :=< \Omega^{(2)}, \xi >_{\mathcal{G}}$ is the ξ -component of the corresponding curvature form $\Omega^{(2)} \in \Lambda^{(2)}(N) \otimes \mathcal{G}$.

Remark 2.6. As the canonical 2-form $d\lambda(\alpha^{(1)}) = d\ pr_M^*\alpha^{(1)} \in \Lambda^{(2)}(T^*(M))$ is by definition G-invariant on $T^*(M)$, it is evident that its restriction to the G-invariant submanifold $\mathcal{M}_\xi \subset T^*(M)$ will be effectively defined only on the reduced space $\bar{\mathcal{M}}_\xi$ for which (2.27)is satisfied.

The following results are direct consequences of Theorem 2.5 that are useful for many applications [22, 16].

Theorem 2.7. *Let $\xi \in \mathcal{G}^*$ have the isotropy group G_ξ acting on the subset $\mathcal{M}_\xi \subset T^*(M)$ freely and properly, so that the reduced phase space $(\bar{\mathcal{M}}_\xi, \sigma_\xi^{(2)})$, where $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G_\xi$, has symplectic structure defined by*

$$\sigma_\xi^{(2)} := d\ pr_M^*\alpha^{(1)}\Big|_{l^{-1}(\xi)}. \tag{2.28}$$

If a principal fiber bundle $\pi : (M, \varphi) \to N$ has G_ξ as its structure group, then the reduced symplectic space $(\bar{\mathcal{M}}_\xi, \sigma_\xi^{(2)})$ is symplectomorphic to the cotangent space $(T^*(N), \omega_\xi^{(2)})$, where

$$\omega_\xi^{(2)} = d\ pr_N^*\beta^{(1)} + pr^* N\Omega_\xi^{(2)}, \tag{2.29}$$

and the corresponding symplectomorphism is of the form (2.27).

Theorem 2.8. *In order for two symplectic spaces $(\bar{\mathcal{M}}_\xi, \sigma_\xi^{(2)})$ and $(T^*(N), dpr_N^*\beta^{(1)})$ to be symplectomorphic, it is necessary and sufficient that the element $\xi \in \ker h$, where for the G-invariant element $\xi \in \mathcal{G}^*$ the mapping $h : \xi \to [\Omega_\xi^{(2)}] \in H^2(N; \mathbb{Z})$, where $H^2(N; \mathbb{Z})$ is the cohomology class of 2-forms on the manifold N.*

3. Symplectic analysis of Maxwell and Yang-Mills dynamical systems

Here we shall show how are approach can be applied to various dynamical systems of the Maxwell and Yang-Mills types.

3.1. Hamiltonian analysis of Maxwell's electromagnetic dynamical systems.

We take the Maxwell electromagnetic equations to be

$$\begin{aligned} \partial E/\partial t &= \nabla \times B - J, \quad \partial B/\partial t = -\nabla \times E, \\ &< \nabla, E >= \rho, \qquad < \nabla, B >= 0, \end{aligned} \tag{3.1}$$

on the cotangent phase space $T^*(N)$, with $N \subset T(D; \mathbb{E}^3)$ - the smooth manifold of smooth vector fields on an open domain $D \subset \mathbb{R}^3$ - all expressed in the light speed units. Here $(E, B) \in T^*(N)$, where the coordinates are the electric and magnetic fields, respectively, and $\rho : D \to \mathbb{R}$ and $J : D \to \mathbb{E}^3$ are, respectively, fixed charge density and current functions on the domain D, satisfying the equation of continuity

$$\partial\rho/\partial t + < \nabla, J >= 0 \tag{3.2}$$

for all $t \in \mathbb{R}$. Here, ∇ is the gradient operator with respect to a variable $x \in D$, $\times$ is the usual vector product in three-dimensional Euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, < \cdot, \cdot >)$, which is real three-space $\mathbb{R}^3$ endowed with the usual scalar product $< \cdot, \cdot >$.

With an eye toward framing equations (3.1) in the context of a reduced symplectic space, we define an appropriate configuration space $M \subset \mathcal{T}(D; \mathbb{E}^3)$ with a vector potential field coordinate $A \in M$. The cotangent space $T^*(M)$ may be identified with pairs $(A; Y) \in T^*(M)$, where $Y \in \mathcal{T}^*(D; \mathbb{E}^3)$ is a suitable vector field density in D. There exists the canonical symplectic form $\omega^{(2)} \in \Lambda^2(T^*(M))$ on $T^*(M)$, allowing, owing to the definition of the Liouville form

$$\lambda(\alpha^{(1)})(A; Y) = \int_D d^3x(< Y, dA >) := (Y, dA), \tag{3.3}$$

the canonical expression

$$\omega^{(2)} := d\lambda(\alpha^{(1)}) = d\ pr_M^*\alpha^{(1)} = (dY, \wedge dA), \tag{3.4}$$

where $\wedge$ is the usual exterior Product, d^3x denotes Lebesgue measure in the domain D, and $pr_M : T^*(M) \to M$ is the standard projection upon the base space M. Now we define a Hamiltonian function $\tilde{H} \in \mathcal{D}(T^*(M))$ as

$$\tilde{H}(A, Y) = 1/2[(Y, Y) + (\nabla \times A, \nabla \times A) + (< \nabla, A >, < \nabla, A >)], \tag{3.5}$$

to describe the Maxwell equations in vacuo, if the densities $\rho = 0$ and $J = 0$.In fact, owing to (3.4) one easily obtains from (3.5) that

$$\begin{aligned} \partial A/\partial t \quad &: \quad = \delta\tilde{H}/\delta Y = Y, \\ \partial Y/\partial t \quad &: \quad = -\delta\tilde{H}/\delta A = -\nabla \times B + \nabla < \nabla, A >, \end{aligned} \tag{3.6}$$

which are true wave equations in vacuo, where

$$B := \nabla \times A, \tag{3.7}$$

is the corresponding magnetic field. Now defining

$$E := -Y - \nabla W \tag{3.8}$$

for some function $W : D \to \mathbb{R}$ as the corresponding electric field, the system of equations (3.6) assumes, owing to definition (3.7), the form

$$\partial B/\partial t = -\nabla \times E, \quad \partial E/\partial t = \nabla \times B, \tag{3.9}$$

which are precisely the Maxwell equations in vacuo, if the Lorentz condition

$$\partial W/\partial t + < \nabla, A >= 0 \tag{3.10}$$

is imposed.

Since definition (3.8) was essentially imposed rather than arising naturally from the Hamiltonian approach and our equations are valid only for a vacuum, we shall try to improve upon these matters by employing the reduction approach devised in Section 2. Namely, we start with the Hamiltonian (3.5) and observe that it is invariant with respect to the abelian symmetry group $G := \exp \mathcal{G}$, where $\mathcal{G} \simeq C^{(1)}(D; \mathbb{R})$, acting on the base manifold M naturally lifted to $T^*(M)$: for any $\psi \in \mathcal{G}$ and $(A, Y) \in T^*(M)$

$$\varphi_\psi(A) := A + \nabla\psi, \qquad \varphi_\psi(Y) = Y. \tag{3.11}$$

The 1-form (3.3) under the transformation (3.11) also is invariant since

$$\begin{gathered} \varphi_\psi^* \lambda(\alpha^{(1)})(A, Y) = (Y, dA + \nabla d\psi) = \\ = (Y, dA) - (< \nabla, Y >, d\psi) = \lambda(\alpha^{(1)})(A, Y), \end{gathered} \tag{3.12}$$

where we made use of the condition $d\psi \simeq 0$ in $\Lambda^1(T^*(M))$ for any $\psi \in \mathcal{G}$. Thus, the corresponding momentum mapping (2.19) is given as

$$l(A, Y) = - < \nabla, Y > \tag{3.13}$$

for all $(A, Y) \in T^*(M)$. If $\rho \in \mathcal{G}^*$ is fixed, one can define the reduced phase space $\bar{\mathcal{M}}_\rho := l^{-1}(\rho)/G$ since the isotropy group $G_\rho = G$, owing to its commutativity and the condition (3.11). Now consider a principal fiber bundle $\pi : M \to N$ with the abelian structure group G and a base manifold N taken as

$$N := \{B \in \mathcal{T}(D; \mathbb{E}^3) : \quad < \nabla, \ B >= 0, \quad < \nabla, E(S) >= \rho\}, \tag{3.14}$$

where

$$\pi(A) = B = \nabla \times A. \tag{3.15}$$

We can construct a connection 1-form $\mathcal{A} \in \Lambda^1(M) \otimes \mathcal{G}$ on this bundle, such that for all $A \in M$,

$$\mathcal{A}(A) \cdot \hat{A}_*(l) = 1, \quad d < \mathcal{A}(A), \rho >_{\mathcal{G}} = \Omega_\rho^{(2)}(A) \in H^2(M; \mathbb{Z}), \tag{3.16}$$

where $\mathcal{A}(A) \in \Lambda^1(M)$ is a differential 1-form, which we choose as

$$\mathcal{A}(A) := -(W, d < \nabla, A >), \tag{3.17}$$

where $W \in C^{(1)}(D; \mathbb{R})$ is a scalar function, as yet not defined. As a result, the Liouville form (3.3) transforms into

$$\lambda(\tilde{\alpha}_\rho^{(1)}) := (Y, dA) - (W, d < \nabla, A >) = (Y + \nabla W, dA) := (\tilde{Y}, \ dA), \ \tilde{Y} := Y + \nabla W, \tag{3.18}$$

giving rise to the corresponding canonical symplectic structure on $T^*(M)$ as

$$\tilde{\omega}_\rho^{(2)} := d\lambda(\tilde{\alpha}_\rho^{(1)}) = (d\tilde{Y}, \wedge dA). \tag{3.19}$$

Accordingly the Hamiltonian function (3.5), as a function on $T^*(M)$, transforms into

$$\tilde{H}_\rho(A, \tilde{Y}) = 1/2[(\tilde{Y}, \tilde{Y}) + (\nabla \times A, \nabla \times A) + (< \nabla, A >, < \nabla, A >)], \tag{3.20}$$

coinciding with the well-known Dirac-Fock-Podolsky [5, 10] Hamiltonian expression. The corresponding Hamiltonian equations on the cotangent space $T^*(M)$, namely

$$\begin{aligned} \partial A/\partial t &: = \delta \tilde{H}/\delta \tilde{Y} = \tilde{Y}, \quad \tilde{Y} := -E - \nabla W, \\ \partial \tilde{Y}/\partial t &: = -\delta \tilde{H}/\delta A = -\nabla \times (\nabla \times A) + \nabla < \nabla, A >, \end{aligned}$$

describe true wave processes, related to the Maxwell equations in the vacuo, except for the external charge and current density conditions. In particular, from (3.20) we obtain

$$\partial^2 A/\partial t^2 - \nabla^2 A = 0 \Longrightarrow \partial E/\partial t + \nabla(\partial W/\partial t \ + \ < \nabla, A >) = -\nabla \times B, \tag{3.21}$$

giving rise to the true vector potential wave equation, but the Faraday induction law is satisfies if one additionally imposes the Lorentz condition (3.10).

To remedy this situation, we will apply to this symplectic space the reduction technique devised in Section 2. Namely, it follows from Theorem 2.7 that above cotangent manifold $T^*(N)$ is symplectomorphic to the corresponding reduced phase space $\bar{\mathcal{M}}_\rho$, that is

$$\bar{\mathcal{M}}_\rho \simeq \{(B; S) \in T^*(N) : < \nabla, E(S) >= \rho, \quad < \nabla, B >= 0\} \tag{3.22}$$

with the reduced canonical symplectic 2-form

$$\omega_\rho^{(2)}(B, S) = (dB, \wedge dS) = d\lambda(\alpha_\rho^{(1)})(B, S), \quad \lambda(\alpha_\rho^{(1)})(B, S) := -(S, dB), \tag{3.23}$$

where we define

$$\nabla \times S + F + \nabla W = -\tilde{Y} := E + \nabla W, \quad < \nabla, F >:= \rho, \tag{3.24}$$

for some fixed vector mapping $F \in C^{(1)}(D; \mathbb{E}^3)$, depending on the imposed external charge and current density conditions. The result (3.23) follows right away upon substituting the expression for the electric field $E = \nabla \times S + F$ into the symplectic structure (3.19), and taking into account the fact that $dF = 0$ in $\Lambda^1(M)$.Whence, the Hamiltonian function (3.20) reduces to the symbolic form

$$\begin{aligned} H_\rho(B, S) &= 1/2[(B, B) + (\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + \\ &\quad +(\ < \ \nabla, (\nabla \times)^{-1} B >, < \nabla, (\nabla \times)^{-1} B >)], \end{aligned} \tag{3.25}$$

where "$(\nabla\times)^{-1}$" is the corresponding inverse curl-operation, mapping [21] the divergence-free subspace $C^{(1)}_{\text{div}}(D; \mathbb{E}^3) \subset C^{(1)}(D; \mathbb{E}^3)$ into itself. Now it follows from (3.25) that the Maxwell equations (3.1) become a canonical Hamiltonian system on the reduced phase space $T^*(N)$, endowed with the canonical symplectic structure (3.23) and the modified Hamiltonian function (3.25). More precisely, one obtains easily that

$$\begin{aligned} \partial S/\partial t &: = \delta H/\delta B = B - (\nabla \times)^{-1} \nabla < \nabla, (\nabla \times)^{-1} B >, \\ \partial B/\partial t &: = -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E, \end{aligned} \tag{3.26}$$

where we made use of the definition $E = \nabla \times S + F$ and the elementary identity $\nabla \times \nabla = 0$. Thus, the second equation of (3.26) coincides with the second Maxwell equation of (3.1) in the classical form

$$\partial B/\partial t = -\nabla \times E.$$

Moreover, owing to (3.26), from (3.24) one obtains via the differentiation with respect to $t \in \mathbb{R}$ that

$$\begin{aligned} \partial E/\partial t &= \partial F/\partial t + \nabla \times \partial S/\partial t = \\ &= \partial F/\partial t + \nabla \times B, \end{aligned} \tag{3.27}$$

as well as, owing to (3.2),

$$< \nabla, \partial F/\partial t >= \partial \rho/\partial t = - < \nabla, J > . \tag{3.28}$$

Now we can write down from (3.28) that, up to non-essential curl-terms $\nabla \times (\cdot)$, the following relationship

$$\partial F/\partial t = -J \tag{3.29}$$

holds. In fact, the current vector $J \in C^{(1)}(D;\mathbb{E}^3)$, owing to the equation of continuity (3.2), is defined up to curl-terms $\nabla \times (\cdot)$ which can be included in the definition of the right-hand side of (3.29). Then upon substitution of (3.29) into (3.27), we obtain the first Maxwell equation of (3.1):

$$\partial E/\partial t = \nabla \times B - J, \tag{3.30}$$

which is naturally supplemented with the external charge and current densities conditions

$$\begin{gathered} <\nabla, B>=0, \quad <\nabla, E>=\rho, \\ \partial\rho/\partial t+<\nabla, J>=0, \end{gathered} \tag{3.31}$$

in virtue of the equation of continuity (3.2) and definition (3.22).

As for the wave equations related to the Hamiltonian system (3.26), we find that the electric field E is recovered from the second equation as

$$E := -\partial A/\partial t - \nabla W, \tag{3.32}$$

where $W \in C^{(1)}(D;\mathbb{R})$ is a smooth function that depends on the vector field $A \in M$. To determine this dependence, we substitute (3.29) into equation (3.30) taking into account that $B = \nabla \times A$, which yields

$$\partial^2 A/\partial t^2 - \nabla(\partial W/\partial t+<\nabla, A>) = \nabla^2 A + J. \tag{3.33}$$

With the above, if we now impose the Lorentz condition (3.10), we obtain from (3.33) the corresponding true wave equations in the space-time, taking into account the external charge and current density conditions (3.31).

Notwithstanding our progress so far, the problem of fulfilling the Lorentz constraint (3.10) naturally within the canonical Hamiltonian formalism still remains to be completely solved. To this end, we are compelled to analyze the structure of the Liouville 1-form (3.18) for the Maxwell equations on a slightly extended functional manifold $M \times L$. As the first step, we rewrite the 1-from (3.18) as

$$\begin{aligned} \lambda(\tilde{\alpha}_\rho^{(1)}) \quad &: \quad = (\tilde{Y}, dA) = (Y + \nabla W, dA) = (Y, dA) + \\ +(W, -d \quad &< \quad \nabla, A>) := (Y, dA) + (W, d\eta), \end{aligned} \tag{3.34}$$

where

$$\eta := -<\nabla, A> . \tag{3.35}$$

Considering now the elements $(Y, A; \eta, W) \in T^*(M \times L)$ as new independent canonical variables on the extended cotangent phase space $T^*(M \times L)$, where $L := C^{(1)}(D;\mathbb{R})$, we can rewrite the symplectic structure (3.19) in the following canonical form

$$\tilde{\omega}_\rho^{(2)} := d\lambda(\tilde{\alpha}_\rho^{(1)}) = (dY, \wedge dA) + (dW, \wedge d\eta). \tag{3.36}$$

In view of the Hamiltonian function (3.20), we obtain the expression

$$H(A, Y; \eta, W) = 1/2[(Y - \nabla W, Y - \nabla W) + (\nabla \times A, \nabla \times A) + (\eta, \eta)], \tag{3.37}$$

with respect to which the corresponding Hamiltonian equations take the form

$$\begin{aligned} \partial A/\partial t \quad &: \quad = \delta H/\delta Y = Y - \nabla W, \quad Y := -E, \\ \partial Y/\partial t \quad &: \quad = -\delta H/\delta A = -\nabla \times (\nabla \times A), \\ \partial \eta/\partial t \quad &: \quad = \delta H/\delta W =<\nabla, Y - \nabla W>, \\ \partial W/\partial t \quad &: \quad = -\delta H/\delta \eta = -\eta. \end{aligned} \tag{3.38}$$

From (3.38), we readily compute that

$$\begin{aligned} \partial B/\partial t + \nabla \times E \quad &= \quad 0, \quad \partial^2 W/\partial t^2 - \nabla^2 W =<\nabla, E>, \\ \partial E/\partial t - \nabla \times B \quad &= \quad 0, \quad \partial^2 A/\partial t^2 - \nabla^2 A = -\nabla(\partial W/\partial t+<\nabla, A>). \end{aligned} \tag{3.39}$$

It is evident that these equations describe Maxwell's equations in the vacuo, without taking into account both the external charge and current density relationships (3.31) and the Lorentz condition (3.10). Our next step is to apply the reduction technique devised in Section 2 to the symplectic structure (3.36) . Whence we find that under the transformations (3.24), the corresponding reduced manifold $\bar{\mathcal{M}}_\rho$ becomes endowed with the symplectic structure

$$\bar{\omega}_\rho^{(2)} := (dB, \wedge dS) + (dW, \wedge d\eta), \tag{3.40}$$

and the Hamiltonian (3.37) assumes the form

$$H(S,B;\eta,W)=1/2[(\nabla\times S+F+\nabla W,\nabla\times S+F+\nabla W)+(B,B)+(\eta,\eta)]. \tag{3.41}$$

The Hamiltonian equations for H are

$$\begin{aligned} \partial S/\partial t\ &:\ =\delta H/\delta B=B, \qquad \partial W/\partial t:=-\delta H/\delta\eta=-\eta,\\ \partial B/\partial t\ &:\ =-\delta H/\delta S=-\nabla\times(\nabla\times S+F+\nabla W)=-\nabla\times E,\\ \partial\eta/\partial t\ &:\ =\delta H/\delta W=-<\nabla,\nabla\times S+F+\nabla W>=-<\nabla,E>-\Delta W, \end{aligned} \tag{3.42}$$

which coincide under the constraint (3.24) completely with Maxwell equations (3.1), describing true space-time processes and taking into account, *a priori,* both the imposed external charge and current density relationships (3.31) and the Lorentz condition (3.10),thus solving the problem mentioned in [5, 10]. Indeed, it is easy to obtain from (3.42) that

$$\begin{aligned} \partial^2W/\partial t^2-\Delta W &= \rho, \qquad \partial W/\partial t+<\nabla,A>=0,\\ \nabla\times B &= J+\partial E/\partial t, \qquad \partial B/\partial t=-\nabla\times E, \end{aligned} \tag{3.43}$$

Hence, using (3.43) and (3.31), one can easily calculate [13, 12] the magnetic wave equation

$$\partial^2A/\partial t^2-\Delta A=J, \tag{3.44}$$

supplementing the suitable wave equation on the scalar potential $W\in L$, thereby completing the calculations. Thus, we have proved the desired result; namely,

Proposition 3.1. *The electromagnetic Maxwell equations (3.1) together with Lorentz condition (3.10) are equivalent to the Hamiltonian system (3.42) with respect to the canonical symplectic structure (3.40) and Hamiltonian function (3.41), which, respectively, reduce to the electromagnetic equations (3.43) and (3.44) under the external charge and current density relationships (3.31).*

The above result can be used for developing an alternative quantization procedure of Maxwell's equations, as it circumvents the related quantum operator compatibility problems discussed in detail in [5, 6, 10]. We hope to consider this aspect of the quantization problem in a future investigation.

Remark 3.2. If one to considers the motion of a charged point particle under a Maxwell field, it is convenient to introduce a trivial fiber bundle structure π: $M\to N$, such that $M=N\times G$, $N:=D\subset\mathbb{R}^3$ and $G:=\mathbb{R}/\{0\}$ is the corresponding (abelian) structure Lie group. An analysis similar to the above gives rise to the reduced (on the space $\bar{\mathcal{M}}_\xi:=l^{-1}(\xi)/G\simeq T^*(N)$, $\xi\in\mathcal{G}$) symplectic structure

$$\omega^{(2)}(q,p)=<dp,\wedge dq>+d<\mathcal{A}(q,g),\xi>_{\mathcal{G}},$$

where $\mathcal{A}(q,g):=<A(q),dq>+g^{-1}dg$ is a suitable connection 1-form on the phase space M, with $(q,p)\in T^*(N)$ and $g\in G$. The corresponding canonical Poisson brackets on $T^*(N)$ are easily found to be

$$\{q^i,q^j\}=0,\quad \{p_j,q^i\}=\delta^i_j,\qquad \{p_i,p_j\}=F_{ji}(q) \tag{3.45}$$

for all $(q,p)\in T^*(N)$. If one introduces a new momentum variable $\tilde p:=p+A(q)$ on $T^*(N)\ni(q,p)$, it is easy to verify that $\omega^{(2)}_\xi\to\tilde\omega^{(2)}_\xi:=<d\tilde p,\wedge dq>$, which gives rise to the following Poisson brackets [20, 23, 22]:

$$\{q^i,q^j\}=0,\qquad \{\tilde p_j,q^i\}=\delta^i_j,\qquad \{\tilde p_i,\tilde p_j\}=0, \tag{3.46}$$

where $i,j=\overline{1,3}$, iff for all $i,j,k=\overline{1,3}$ the standard Maxwell field equations are satisfied on $N:$

$$\partial F_{ij}/\partial q_k+\partial F_{jk}/\partial q_i+\partial F_{ki}/\partial q_j=0 \tag{3.47}$$

with the curvature tensor $F_{ij}(q):=\partial A_j/\partial q^i-\partial A_i/\partial q^j$, $i,j=\overline{1,3}$, $q\in N$.

It is not difficult to see that the above approach permits a natural generalization for non-abelian structure Lie groups, yielding a description of Yang-Mills field equations within our reduction formulation. We proceed to such an extension in the next subsection.

3.2. **Hamiltonian analysis of Yang-Mills dynamical systems.** As above, we start by defining a phase space M of a particle moving under a Yang-Mills field in a region $D \subset \mathbb{R}^3$ with $M := D\times G$, where G is a (not in general semisimple) Lie group, acting on M from the right. Over the space M one can define quite naturally a connection $\Gamma(\mathcal{A})$ by consider the trivial principal fiber bundle $\pi : M \to N$, where $N := D$, with the structure group G. Namely, if $g \in G$, $q \in N$, then a connection 1-form on $M \ni (q, g)$ can be expressed [4, 15, 14, 19] as

$$\mathcal{A}(q; g) := g^{-1}(d + \sum_{i=1}^{n} a_i A^{(i)}(q))g, \tag{3.48}$$

where $\{a_i \in \mathcal{G} : i = \overline{1,n}\}$ is a basis for the Lie algebra $\mathcal{G}$ of the Lie group G, and $A_i : D \to \Lambda^1(D)$, $i = \overline{1,n}$, are the Yang-Mills fields on the physical space $D \subset \mathbb{R}^3$.

Now one defines the natural left invariant Liouville form on M as

$$\lambda(\alpha^{(1)})(q; g) :=< p, dq > + < y, g^{-1}dg >_{\mathcal{G}}, \tag{3.49}$$

where $y \in T^*(G)$ and $< \cdot, \cdot >_{\mathcal{G}}$ denotes as before the usual Ad-invariant nondegenerate bilinear form on $\mathcal{G}^* \times \mathcal{G}$, and it is clear that $g^{-1}dg \in \Lambda^1(G) \otimes \mathcal{G}$. The main assumption we need to proceed is that the connection 1-form is compatible with the Lie group G action on M. The means that

$$R_h^* \mathcal{A}(q; g) = Ad_{h^{-1}} \mathcal{A}(q; g) \tag{3.50}$$

is satisfied for all $(q, g) \in M$ and $h \in G$, where $R_h : G \to G$ is the right translation by an element $h \in G$ on the Lie group G.

Having gathered all preliminary elements needed for the reduction Theorem 2.7 to be applied to our model, we now suppose that the Lie group G canonical action on M is naturally lifted to the cotangent space $T^*(M)$ endowed, owing to (3.3), with the G-invariant canonical symplectic structure

$$\begin{aligned} \omega^{(2)}(q, p; g, y) \quad : &= d\, pr_M^* \alpha^{(1)}(q, p; g, y) =< dp, \wedge dq > + \\ + &< dy, \wedge g^{-1}dg >_{\mathcal{G}} + < ydg^{-1}, \wedge dg >_{\mathcal{G}} \end{aligned} \tag{3.51}$$

for all $(q, p; g, y) \in T^*(M)$. Choose an element $\xi \in \mathcal{G}^*$ and assume that its isotropy subgroup $G_\xi = G$, that is $Ad_h^* \xi = \xi$ for all $h \in G$. In the general case such an element $\xi \in \mathcal{G}^*$ cannot exist unless it is trivial, $\xi = 0$, as it happens, for instance, in the case of the Lie group $G = SL_2(\mathbb{R})$. Then one can construct the reduced phase space $l^{-1}(\xi)/G$ symplectomorphic to $(T^*(N), \omega_\xi^{(2)})$, where it follows from (2.27) that for any $(q, p) \in T^*(N)$,

$$\begin{aligned} \omega_\xi^{(2)}(q, p) &= < dp, \wedge dq > + < \Omega^{(2)}(q), \xi >_{\mathcal{G}} = \\ &= < dp, \wedge dq > + \sum_{s=1}^{n} \sum_{i,j=1}^{3} e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j. \end{aligned} \tag{3.52}$$

In the above we have expanded the element $\xi = \sum_{i=1}^n e_i a^i \in \mathcal{G}^*$ with respect to the bi-orthogonal basis $\{a^i \in \mathcal{G}^*, a_j \in \mathcal{G} : < a^i, a_j >_{\mathcal{G}} = \delta^i_j, i, j = \overline{1,n}\}$, with constant coefficients $e_i \in \mathbb{R}$, $i = \overline{1,3}$. We also denoted by $F_{ij}^{(s)}(q)$, $i, j = \overline{1,3}$, $s = \overline{1,n}$, the corresponding curvature 2-form $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$ components, that is

$$\Omega^{(2)}(q) := \sum_{s=1}^{n} \sum_{i,j=1}^{3} a_s\, F_{ij}^{(s)}(q) dq^i \wedge dq^j \tag{3.53}$$

for any point $q \in N$. Summarizing the calculations above, we have the following result.

Theorem 3.3. *Suppose the Yang-Mills field (3.48) on the fiber bundle $\pi : M \to N$ with $M = D \times G$ is invariant with respect to the Lie group G action $G \times M \to M$. Suppose also that an element $\xi \in G^*$ is chosen so that $Ad_G^* \xi = \xi$. Then for the naturally constructed momentum mapping $l : T^*(M) \to G^*$ (which is equivariant), the reduced phase space $l^{-1}(\xi)/G \simeq T^*(N)$ is endowed with the symplectic structure (3.52), having the component-wise Poisson brackets form*

$$\{p_i, q^j\}_\xi = \delta_i^j, \quad \{q^i, q^j\}_\xi = 0, \quad \{p_i, p_j\}_\xi = \sum_{s=1}^{n} e_s F_{ji}^{(s)}(q) \tag{3.54}$$

for all $i, j = \overline{1,3}$ and $(q, p) \in T^*(N)$.

The corresponding extended Poisson bracket on the whole cotangent space $T^*(M)$ comprises, owing to (3.11), the following set of Poisson relationships:

$$\begin{aligned} \{y_s, y_k\}_\xi &= \sum_{r=1}^{n} c_{sk}^r\, y_r, \qquad \{p_i, q^j\}_\xi = \delta_i^j\ , \\ \{y_s, p_j\}_\xi &= 0 = \{q^i, q^j\}, \ \ \{p_i, p_j\}_\xi = \sum_{s=1}^{n} y_s\, F_{ji}^{(s)}(q), \end{aligned} \tag{3.55}$$

where $i, j = \overline{1,n}$, $c_{sk}^r \in \mathbb{R}$, $s, k, r = \overline{1,m}$, are the structure constants of the Lie algebra $\mathcal{G}$, and we made use of the expansion $A^{(s)}(q) = \sum_{j=1}^n A_j^{(s)}(q)\ dq^j$ as well as introducing alternative fixed values $e_i := y_i$, $i = \overline{1,n}$. The result (3.55) follows readily by making the shift in the expression (3.51) defined as $\sigma^{(2)} \to \sigma_{ext}^{(2)}$, where $\sigma_{ext}^{(2)} := \sigma^{(2)}\big|_{\mathcal{A}_0 \to \mathcal{A}}$, $\mathcal{A}_0(g) := g^{-1}dg$, $g \in G$. With this, the invariance properties of the connection $\Gamma(\mathcal{A})$ imply that

$$\sigma_{ext}^{(2)}(q, p; u, y) =< dp, \wedge dq > +d < y(g), Ad_{g^{-1}}\mathcal{A}(q; e) >_{\mathcal{G}}=$$

$$=< dp, \wedge dq > + < d\ Ad_{g^{-1}}^* y(g), \wedge \mathcal{A}(q; e) >_{\mathcal{G}}=< dp, \wedge dq > + \sum_{s=1}^{m} dy_s \wedge du^s +$$

$$+ \sum_{j=1}^{n} \sum_{s=1}^{m} A_j^{(s)}(q) dy_s \wedge dq - < Ad_{g^{-1}}^* y(g), \mathcal{A}(q, e) \wedge \mathcal{A}(q, e) >_{\mathcal{G}} +$$

$$+ \sum_{k \geq s=1}^{m} \sum_{l=1}^{m} y_l\ c_{sk}^l\ du^k \wedge du^s + \sum_{s=1}^{n} \sum_{i \geq j=1}^{3} y_s F_{ij}^{(s)}(q) dq^i \wedge dq^j, \tag{3.56}$$

where the coordinates of $(q, p; u, y) \in T^*(M)$ are defined as follows: $\mathcal{A}_0(e) := \sum_{s=1}^m du^i\ a_i$, and $Ad_{g^{-1}}^* y(g) = y(e) := \sum_{s=1}^m y_s\ a^s$ for any element $g \in G$. This leads immediately to the Poisson brackets (2.8) plus additional brackets connected with conjugated sets of variables $\{u^s \in \mathbb{R} : s = \overline{1,m}\} \in \mathcal{G}^*$ and $\{y_s \in \mathbb{R} : s = \overline{1,m}\} \in \mathcal{G}$:

$$\{y_s, u^k\}_\xi = \delta_s^k, \quad \{u^k, q^j\}_\xi = 0, \quad \{p_j, u^s\}_\xi = A_j^{(s)}(q), \quad \{u^s, u^k\}_\xi = 0, \tag{3.57}$$

where $j = \overline{1,n}$, $k, s = \overline{1,m}$, and $q \in N$.

Note here that the transition from the symplectic structure $\sigma^{(2)}$ on $T^*(N)$ to its extension $\sigma_{ext}^{(2)}$ on $T^*(M)$ suggested above just consists formally in adding an exact part to the symplectic structure $\sigma^{(2)}$, which transforms it into equivalent one. Looking now at the expressions (3.56), one can infer immediately that an element $\xi := \sum_{s=1}^m e_s a^s \in \mathcal{G}^*$ will be invariant with respect to the Ad^*-action of the Lie group G iff

$$\{y_s, y_k\}_\xi\big|_{y_s = e_s} = \sum_{r=1}^{m} c_{sk}^r\ e_r\ = 0 \tag{3.58}$$

identically for all $s, k = \overline{1,m}$, $j = \overline{1,n}$ and $q \in N$. In this and only this case does the reduction scheme elaborated above go through.

Returning our attention to the expression (3.57), one can easily derive the exact shifted expression

$$\omega_{ext}^{(2)}(q, p; u, y) = \omega^{(2)}(q, p + \sum_{s=1}^{n} y_s\ A^{(s)}(q)\ ; u, y), \tag{3.59}$$

on the phase space $T^*(M) \ni (q, p; u, y)$, where we abbreviated for brevity $< A^{(s)}(q), dq >$ as $\sum_{j=1}^n A_j^{(s)}(q)\ dq^j$. Expressions like (3.59) were discussed within a somewhat different context in [20, 23], which also provide a good background for the infinite-dimensional generalization of the symplectic structure techniques. Having observed from (3.59) that the simple change of variables

$$\tilde{p} := p + \sum_{s=1}^{m} y_s\ A^{(s)}(q) \tag{3.60}$$

in the cotangent space $T^*(N)$ recasts our symplectic structure (3.56) into the old canonical form (3.51), one obtains that the following new set of canonical Poisson brackets on $T^*(M) \ni (q, \tilde{p}; u, y)$:

$$\begin{aligned} \{y_s, y_k\}_\xi &= \sum_{r=1}^{n} c_{sk}^{r}\, y_r, \quad \{\tilde{p}_i, \tilde{p}_j\}_\xi = 0, \quad \{\tilde{p}_i, q^j\} = \delta_i^j, \\ \{y_s, q^j\}_\xi &= 0 \ = \{q^i, q^j\}_\xi,\ \{u^s, u^k\}_\xi = 0, \quad \{y_s, \tilde{p}_j\}_\xi = 0, \\ \{u^s, q^i\}_\xi &= 0, \qquad \{y_s, u^k\}_\xi = \delta_s^k, \qquad \{u^s, \tilde{p}_j\}_\xi = 0, \end{aligned} \tag{3.61}$$

where $k, s = \overline{1, m}$ and $i, j = \overline{1, n}$, holds iff the nonabelian Yang-Mills field equations

$$\begin{aligned} &\partial F_{ij}^{(s)}/\partial q^l + \partial F_{jl}^{(s)}/\partial q^i + \partial F_{li}^{(s)}/\partial q^j + \\ &+ \sum_{k,r=1}^{m} c_{kr}^{s} (F_{ij}^{(k)} A_l^{(r)} + F_{jl}^{(k)} A_i^{(r)} + F_{li}^{(k)} A_j^{(r)}) = 0 \end{aligned} \tag{3.62}$$

are fulfilled for all $s = \overline{1, m}$ and $i, j, l = \overline{1, n}$ on the base manifold N. This effect of complete reduction of gauge Yang-Mills variables from the symplectic structure (3.56) is known in literature [20] as the principle of minimal interaction and has proven to be quite useful for studying different interacting systems as in [21, 24]. We plan to continue the study of the geometric properties of reduced symplectic structures connected with such interesting infinite-dimensional coupled dynamical systems as those of Yang-Mills-Vlasov, Yang-Mills-Bogolubov and Yang-Mills-Josephson types [21, 24], as well as their relationships with associated principal fiber bundles endowed with canonical connection structures.

4. Acknowledgments

The authors are cordially thankful to the Abdus Salam International Centre for Theoretical Physics in Trieste, Italy, for the hospitality during their 2007-2008 research scholarships. A.P. is especially grateful to Profs. P.I. Holod (Kyiv, UKMA), J.M. Stakhira and I.M. Bolesta (Lviv, LNU) for fruitful discussions, useful comments and remarks. The authors also express their sincere appreciation to the referees whose insightful remarks, suggestions and valuable comments were instrumental to improving the manuscript. Last but not least, thanks go to Prof. D.L. Blackmore (NJIT) for his help in editing the article and academician Prof. A.A. Logunov (Moscow, IJP) for his interest in the work.

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