Potential Conservation Laws

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Joint work with Michael Kunzinger

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Generalizing results the recent paper by Bluman, Cheviakov and Ivanova [*J. Math. Phys.*, 2006, V.47, 113505], we prove that in the case of two independent variables local conservation laws of potential systems have characteristics depending only on local variables if and only if they are induced by local conservation laws of the corresponding initial systems of differential equations. Therefore, characteristics of pure potential conservation laws have to essentially depend on potential variables that gives a criterion for the selection of such conservation laws. Moreover, we present extensions to multi-dimensional standard and gauged potential systems, Abelian and general coverings and general foliated systems of differential equations. An example illustrating possible applications of these results is given. A special version of the Hadamard lemma for fiber bundles and the notions of weighted jet spaces are proposed as new tools for the investigation of potential conservation laws.

In a recent paper by Bluman, Cheviakov and Ivanova (2006) a remarkable result on potential conservation laws was obtained. Namely, it was shown that for an arbitrary system of differential equations a conservation law of a potential system with a characteristic which depends only on the independent variables is induced by a local conservation law of the initial system. We show that this theorem admits a significant generalization and that, moreover, a converse statement is true as well. The possibility of deriving this result is suggested by recalling the rule of transforming conservation laws under point transformations between systems of differential equations. The application of a hodograph-type transformation to a characteristic which exclusively depends on the independent variables may result in a characteristic including dependent variables. Generally, characteristics of induced conservation laws of potential systems can depend on derivatives of unknown functions of the initial system, and systems of other kinds related to standard potential systems (systems determining Abelian or general coverings, gauged potential systems, general foliated systems) can be investigated in the same framework.

Basic properties of conservation laws

Let $\mathcal{L}$ be a system $L(x,u_{(\rho)})=0$ of l differential equations $L^1=0,\ \dots,\ L^l=0$ for m unknown functions $u=(u^1,\dots,u^m)$ of n independent variables $x=(x_1,\dots,x_n)$. Here $u_{(\rho)}$ denotes the set of all the derivatives of the functions u with respect to x of order no greater than ρ, including u as the derivative of order zero. We identify $L_{(k)}$ with the corresponding system of algebraic equations in the k-th order jet space $J^k(x|u)$ and associate it with the manifold $\mathcal{L}_{(k)}$ determined by this system.

Here $J^k(x|u)$ is with the independent variables x and the dependent variables u. A smooth function defined on a subset of $J^k(x|u)$ for some k, i.e., depending on x and a finite number of derivatives of u, will be called a differential function of u. The notation $H[u]$ means that H is a differential function of u.

Definition. A *conserved vector* of the system $\mathcal{L}$ is an n-tuple $F = (F^1[u], \dots, F^n[u])$ for which the total divergence $\operatorname{Div} F := D_iF^i$ vanishes for all solutions of $\mathcal{L}$, i.e., $\operatorname{Div}F\big|_{\mathcal{L}}= 0$.

$D_i = D_{x_i} = D_i = \partial_{x_i} + u^a_{\alpha+\delta_i}\partial_{u^a_\alpha}$ is the operator of total differentiation with respect to the variable x_i We use the summation convention for repeated indices and consider any function as its zero-order derivative. The indices i and j run from 1 to n, the index a runs from 1 to m, and the index s from 1 to p unless otherwise stated.

Definition. A conserved vector F is called *trivial* if $F^i = \hat{F}^i + \check{F}^i$ where $\hat{F}^i$ and $\check{F}^i$ are, like F^i, differential functions of u, $\hat{F}^i$ vanishes on the solutions of $\mathcal{L}$ and the n-tuple $\check{F} = (\check{F}^1, \dots, \check{F}^n)$ is a null divergence (i.e. its divergence vanishes identically). Two conserved vectors F and F' are called *equivalent* if the tuple $F' - F$ is a trivial conserved vector.

$\operatorname{CV}(\mathcal{L})$ is the space of conserved vectors of the system $\mathcal{L}$.
$\operatorname{CV}_0(\mathcal{L})$ is the space of trivial conserved vectors of $\mathcal{L}$.
$\operatorname{CL}(\mathcal{L}) = \operatorname{CV}(\mathcal{L})/\operatorname{CV}_0(\mathcal{L})$ is called *the space of (local) conservation laws* of $\mathcal{L}$.

The equality $\operatorname{Div} F = \lambda^\mu L^\mu$ and the l-tuple $\lambda = (\lambda^1[u], \dots, \lambda^l[u])$ are called the *characteristic form* and the *characteristic* of the conservation law containing the conserved vector F, respectively.
! They are introduced only for totally nondegenerate systems.

Foliated systems of differential equations

Let $\bar{\mathcal{L}}$ be a system $\bar{L}(x, u_{(\bar\rho)}, v_{(\bar\rho)}) = 0$ of $\bar{l}$ differential equations $\bar{L}^1 = 0, \dots, \bar{L}^{\bar{l}} = 0$ for $m+p$ unknown functions $u = (u^1, \dots, u^m)$ and $v = (v^1, \dots, v^p)$ of n independent variables $x = (x_1, \dots, x_n)$. Let $\mathcal{L}$ be a system $L(x, u_{(\rho)}) = 0$ of l differential equations $L^1 = 0, \dots, L^l = 0$ for only m unknown functions u.
For each $k \in \mathbb{N} \cup \{0\}$ we consider the projection

$$\varpi_k\colon J^k(x|u, v) \to J^k(x|u)\colon \quad \varpi_k(x, u_{(k)}, v_{(k)}) = (x, u_{(k)}).$$

Any differential function $G = G[u]\colon J^k(x|u) \to \mathbb{R}$ is naturally associated with its pullback $G[u, v] \circ \varpi_k\colon J^k(x|u, v) \to \mathbb{R}$ under ϖ_k: $G \circ \varpi_k(x, u_{(k)}, v_{(k)}) = G(x, u_{(k)})$. It is also possible to consider the projection $\varpi\colon J^\infty(x|u, v) \to J^\infty(x|u)$ whose restriction to $J^k(x|u, v)$ coincides with ϖ_k and which induces pullbacks of differential functions of u of arbitrary (finite) order.

Definition. The system $\bar{\mathcal{L}}$ is called a *foliated system* over the *base system* $\mathcal{L}$ if both the systems $\mathcal{L}$ and $\bar{\mathcal{L}}$ are totally nondegenerate and $\varpi_k(\bar{\mathcal{L}}_{(k)}) = \mathcal{L}_{(k)}$ for any $k \in \mathbb{N}$. *Notation:* $\varpi\bar{\mathcal{L}} = \mathcal{L}$.

The system $\bar{\mathcal{L}}$ is foliated over the system $\mathcal{L}$ if and only if (the pullback of) each equation of $\mathcal{L}$ is a differential consequence of $\bar{\mathcal{L}}$ and for any local solution $u = u^0(x)$ of $\mathcal{L}$ there exist a local solution of the system $\bar{L}|_{u=u^0} = 0$ in v. The foliation also implies that any differential consequence of $\bar{\mathcal{L}}$ which does not involve the functions v is (the pullback of) a differential consequence of $\mathcal{L}$. In terms of solution sets, the strip $u = u^0(x)$, where $u^0(x)$ is a fixed solution of $\mathcal{L}$, is the solution set of the system $\bar{L}(x, u^0_{(\bar{\rho})}, v_{(\bar{\rho})}) = 0$.

Definition. The system $\bar{\mathcal{L}}$ is called a *strongly foliated system* over the base system $\mathcal{L}$ if $\bar{\mathcal{L}}$ is foliated over $\mathcal{L}$ and each of the equations minimally representing $\mathcal{L}$ can be included in a minimal set of equations forming $\bar{\mathcal{L}}$.

If $\bar{\mathcal{L}}$ is foliated over $\mathcal{L}$, we will assume that the maximally possible number $\hat{l}$ of equations of $\mathcal{L}$ is included in the minimal equation set forming and canonically representing $\bar{\mathcal{L}}$. Without loss of generality we can additionally assume that these equations are the first $\hat{l}$ equations in both of these systems. Such a representation of $\bar{\mathcal{L}}$ and $\mathcal{L}$ will be called a *canonical foliation* of $\bar{\mathcal{L}}$ over $\mathcal{L}$. The foliation is strong if and only if $\hat{l} = l$.

Definition. We say that a conservation law $\bar{\mathcal{F}}$ of $\bar{\mathcal{L}}$ is a pullback, with respect to ϖ, of a conservation law $\mathcal{F}$ of $\mathcal{L}$ (i.e., $\bar{\mathcal{F}} = \varpi^*\mathcal{F}$) or, in other words, is *induced* by this conservation law if there exists a conserved vector $\bar{F} \in \bar{\mathcal{F}}$ which is the pullback of a conserved vector $F \in \mathcal{F}$.

Definition. Let $\bar{\mathcal{L}}$ be canonically foliated over $\mathcal{L}$. A tuple

$$\lambda = (\lambda^1[u,v], \dots, \lambda^{l+\bar{l}-\hat{l}}[u,v])$$

is called an *extended characteristic* of a conservation law $\bar{\mathcal{F}}$ of $\bar{\mathcal{L}}$ if some conserved vector $\bar{F} \in \bar{\mathcal{F}}$ satisfies the condition

$$D_i\bar{F}^i = \sum_{\mu=1}^{l} \lambda^\mu L^\mu + \sum_{\nu=1}^{\bar{l}-\hat{l}} \lambda^{l+\nu} \bar{L}^{\hat{l}+\nu}. \tag{1}$$

Definition. We say that a usual or extended characteristic of $\bar{\mathcal{L}}$ is *induced* by a characteristic of $\mathcal{L}$ if the tuple of the characteristic components associated with the pullbacks of the equations of $\mathcal{L}$ is the pullback of the characteristic of $\mathcal{L}$ and the other characteristic components vanish.

1 Potential structures

Two-dimensional potential systems

$$\mathcal{L} \longrightarrow \begin{array}{l} p \text{ linearly independent local conservation laws} \\ \text{with conserved vectors } (F^s, G^s),\ s = 1, \dots, p \end{array} \longrightarrow \begin{array}{l} v^s_x = F^s[u], \\ v^s_t = -G^s[u] \end{array}$$

Multi-dimensional potential systems

$$\mathcal{L} \longrightarrow \begin{array}{l} p \text{ linearly independent local conservation laws with} \\ \text{conserved vectors } G^s = (G^{s1}, \dots, G^{sn}),\ s = 1, \dots, p \end{array} \longrightarrow v^{sij}_j = G^{si}$$

! Here each conserved vector corresponds to a number of potentials.

Gauged potential systems

A system $\mathcal{L}_\mathrm{g}$ of differential equations with the independent variables x and the dependent variables u and v is called a *gauge* on the potentials v^{sij} if any differential consequence of the coupled system $\mathcal{L}_\mathrm{gp} = \mathcal{L}_\mathrm{p} \cap \mathcal{L}_\mathrm{g}$, which does not involve the potentials v^{sij}, is a differential consequence of the initial system $\mathcal{L}$. The coupled system $\mathcal{L}_\mathrm{gp}$ is called a *gauged potential system.* The gauge $\mathcal{L}_\mathrm{g}$ is called *weak* if a minimal set of equations generating all the differential consequences of $\mathcal{L}_\mathrm{p}$ is contained in a minimal set representing the coupled system $\mathcal{L}_\mathrm{gp}$ called a *weakly gauged potential system.*

Abelian coverings

$$\mathcal{L} \longrightarrow v^s_i = G^{si}[u] \quad \text{where } (D_j G^{si} - D_i G^{sj})\big|_{\mathcal{L}} = 0.$$

! Defining a potential in the framework of Abelian coverings involves $\frac{1}{2}n(n-1)$ conserved vectors of a special form.

General coverings (pseudo-potentials)

$$\mathcal{L} \longrightarrow v^s_i = G^{si}[u|v] \quad \text{where } (\hat{D}_j G^{si} - \hat{D}_i G^{sj})\big|_{\mathcal{L}} = 0.$$

The notation $G[u|v]$ means that G is a differential function of u and v, depending on x, v and derivatives of u (there are no derivatives of v of orders greater than 0!). $\hat{D}_i = \partial_{x_i} + u^a_{\alpha,i}\partial_{u^a_\alpha} + G^{si}[u|v]\partial_{v^s}$.

The notion of characteristics for potential structures

To correctly introduce characteristics for conservation laws of potential structure, it is necessary to modify the notion of total nondegeneracy of systems of differential equations in two directions.

1. Taking into account of differential consequences.

By $L_{(k)}$ we will denote a *maximal set* of algebraically independent differential consequences of $\mathcal{L}$ that have, as differential equations, orders not greater than k. We identify $L_{(k)}$ with the corresponding system of algebraic equations in $J^k(x|u)$ and associate it with the manifold $\mathcal{L}_{(k)}$ determined by this system.

For the manifold $\mathcal{L}_{(k)}$ to actually represent the system $\mathcal{L}$ of differential equations, the $\mathcal{L}$ have to be *locally solvable* in each point of $\mathcal{L}_{(k)}$. For the application of the Hadamard lemma to differential functions vanishing on the manifold $\mathcal{L}_{(k)}$, we need the system $L_{(k)}$ to be, as a system of algebraic equations defined in the jet space $J^k(x|u)$, *of maximal rank* in each point of $\mathcal{L}_{(k)}$. If for any k the system $\mathcal{L}$ satisfies both these conditions then it is called *totally nondegenerate*.

! This definition slightly differs from that given in [Olver,1993]. Namely, conditions on prolongations of the system $\mathcal{L}$ is replaced by conditions on maximal sets of its algebraically independent differential consequences

2. Generalizing the notion of order.

It is useful to introduce the more general notion of *weight* of differential variables instead of the order, which takes into account the structure of the system of differential equations under consideration. Namely, for each variable of the infinite-order jet space $J^\infty(x|u)$ (being the inverse limit of the jet space tower $\{J^k(x|u), k \in \mathbb{N} \cup \{0\}\}$ with respect to the canonical projections $\pi^k\colon J^k(x|u) \to J^{k-1}(x|u)$, $k \in \mathbb{N}$) we define its weight ϱ by the rule:

$$\varrho(x_i) = 0, \quad \varrho(u^a_\alpha) = \varrho_a + |\alpha|.$$

The weights $\varrho(u^a) = \varrho_a$ are defined on the basis of the structure of the system $\mathcal{L}$. (In subsequent sections we will provide concrete examples on how to specify the ϱ_a initially.) In what follows u^a_α stands for the variable in $J^\infty(x|u)$, corresponding to the derivative $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$, $\alpha = (\alpha_1,\dots,\alpha_n)$ is an arbitrary multiindex, $\alpha_i \in \mathbb{N}\cup\{0\}$, $|\alpha| := \alpha_1+\dots+\alpha_n$. If $\varrho_a = 0$ then the weight of u^a_α obviously coincides with the usual derivative order $|\alpha|$. We include in the *weighted jet space* $J^k_\varrho(x|u)$ the variables whose weight is not greater than k. The infinite-order jet space $J^\infty(x|u)$ is the inverse limit of the weighted jet space tower $\{J^k_\varrho(x|u),\ k \in \mathbb{N}\cup\{0\}\}$ with respect to the canonical projections $\pi^k_\varrho\colon J^k_\varrho(x|u) \to J^{k-1}_\varrho(x|u)$, $k \in \mathbb{N}$.

The technique of working with weights does not differ from the order technique and so a number of analogous notions can be introduced. Thus, the weight $\varrho(H)$ of any differential function $H[u]$ equals the maximal weight of variables explicitly appearing in H. The weight of the equation $H[u] = 0$ equals $\varrho(H)$. A complete set of independent differential consequences of the system $\mathcal{L}$ which have weights not greater than k and the associated manifold in $J^k_\varrho(x|u)$ are denoted by the symbols $L_{[k]} = L_{[k],\varrho}$ and $\mathcal{L}_{[k]} = \mathcal{L}_{[k],\varrho}$, respectively. The system $\mathcal{L}$ is called *totally nondegenerate with respect to the weight* ϱ if for any $k \in \mathbb{N}$ it is locally solvable in each point of $\mathcal{L}_{[k]}$ and the algebraic system $L_{[k]}$ is *of maximal rank* in each point of $\mathcal{L}_{[k]}$. The Hadamard lemma can be applied, in the conventional way, to differential functions defined in $J^k_\varrho(x|u)$ and vanishing on $\mathcal{L}_{[k]}$.

Extention of a weight to a potential structure

2D pot. system: $\varrho(v^s_\alpha)=\max\bigl(0,\varrho(F^s)-1,\varrho(G^s)-1\bigr)+|\alpha|$

Abelian covering: $\varrho(v^s_\alpha)=\max\bigl(0,\varrho(G^{s1})-1,\dots,\varrho(G^{sn})-1\bigr)+|\alpha|$

Standard multiD pot. system: $\varrho(v^{sij}_\alpha)=\max\bigl(0,\varrho(G^{s1})-1,\dots,\varrho(G^{sn})-1\bigr)+|\alpha|$

General covering: $\varrho(v^s_\alpha)=\max\bigl(0,\varrho(G^{si})-1,\ s=1,\dots,p,\ i=1,\dots,n\bigr)+|\alpha|$

Lemma. *The system $\mathcal{L}$ is totally nondegenerate with respect to a weight if and only if the potential system $\mathcal{L}_{\rm p}$ is totally nondegenerate with respect to this weight extended to the derivatives of the potentials.*

The main theorem

Theorem. *The following statements on a conservation law of a two-dimensional potential system (resp. a system determining an Abelian covering, resp. a multi-dimensional standard potential system without gauges) are equivalent if the corresponding initial system is totally nondegenerate:*

1) the conservation law is induced by a conservation law of the initial system;

2) it contains a conserved vector which does not depend on potentials;

3) some of its extended characteristics are induced by characteristics of the initial system;

4) it possesses a characteristic not depending on potentials.

The equivalence of the first three statements is also true for conservation laws of general foliated systems, including multi-dimensional gauged potential systems and covering systems.

Proposition. *A conservation law of a system determining an Abelian covering (resp. a potential system in the two-dimensional case) is not induced by a conservation law of the corresponding initial system if and only if it is associated with a completely reduced characteristic depending on potentials.*

Here, a characteristic of a system of differential equations is called completely reduced if it does not depend on the derivatives of the unknown functions, which are assumed to be constrained to the solution set of the system. In particular, any completely reduced characteristic of a system determining an Abelian covering does not depend on the derivatives of potentials of orders greater than 0 since they are constrained due to differential consequences of the potential part of the system. Any conservation law possesses a completely reduced characteristic since expressing the constrained variables via the unconstrained ones in a characteristic results in an equivalent characteristic.