Generalized IW-contractions of Lie algebras

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Joint work with Roman Popovych

Popovych D.R. and Popovych R.O., Equivalence of diagonal contractions to generalized IW-contractions with integer exponents, *Linear Algebra Appl.*, 2009, V.431, 1096–1104, arXiv:0812.4667.

Popovych D.R. and Popovych R.O., Lowest dimensional example on non-universality of generalized Inonu-Wigner contractions, 2008, arXiv:0812.1705, 12 pp.

We prove that there exists just one pair of complex four-dimensional Lie algebras such that a well-defined contraction among them is not equivalent to a generalized IW-contraction (or to a one-parametric subgroup degeneration in conventional algebraic terms). Over the field of real numbers, this pair of algebras is split into two pairs with the same contracted algebra. The example we constructed demonstrates that even in the dimension four generalized IW-contractions are not sufficient for realizing all possible contractions, and this is the lowest dimension in which generalized IW-contractions are not universal. Moreover, this is also the first example of nonexistence of generalized IW-contraction for the case when the contracted algebra is not characteristically nilpotent and, therefore, admits nontrivial diagonal derivations. The lower bound (equal to three) of nonnegative integer parameter exponents which are sufficient to realize all generalized IW-contractions of four-dimensional Lie algebras is found. We also present a simple and rigorous proof of the known claim that any diagonal contraction (e.g., a generalized IW-contraction) is equivalent to a generalized IW-contraction with integer parameter powers.

Introduction

The concept of the Lie algebra contraction was introduced by Segal (1951) via limiting processes of bases. It became well known thanks to the papers by Inönü and Wigner (1953, 1954) who invented the so-called *Inönü–Wigner contractions* (*IW-contractions*). A rigorous general definition of contraction, based on limiting processes of Lie brackets, was given by Saletan (1961). He also studied the entire class of one-parametric contractions whose matrices are first-order polynomials with respect to contraction parameters. IW-contractions form a special subclass in the class of *Saletan contractions*.

Another extension of the class of IW-contractions was introduced by Doebner and Melsheimer (1967). They used contraction matrices which become diagonal after choosing suitable bases in the initial and contracted algebras, with diagonal elements being real powers of the contraction parameters. (In fact, integer exponents are sufficient.) In the modern physical literature, such contractions are usually called *generalized Inönü–Wigner contractions* although a number of other names (*p-contractions*, *Doebner–Melsheimer contractions* and *singular IW-contractions*) were previously used. In algebraic papers, similar contractions are called *one-parametric* *subgroup degenerations* (in a similar fashion, general contractions are extended to degenerations which are defined for Lie algebras over an arbitrary field in terms of the orbit closures with respect to the Zariski topology). Note that in fact a one-parametric subgroup degeneration is associated with a one-parametric matrix group only upon choosing special bases in the corresponding initial and contracted algebras. Unfortunately, this fact is often ignored.

Basic notions

The notion of contraction is defined for arbitrary fields in terms of orbit closures in the variety of Lie algebras. Let V be an n-dimensional vector space over a field $\mathbb{F}$, $n < \infty$, and $\mathcal{L}_n = \mathcal{L}_n(\mathbb{F})$ denote the set of all possible Lie brackets on V. We identify $\mu \in \mathcal{L}_n$ with the corresponding Lie algebra $\mathfrak{g} = (V, \mu)$. $\mathcal{L}_n$ is an algebraic subset of the variety $V^* \otimes V^* \otimes V$ of bilinear maps from $V \times V$ to V. Indeed, under setting a basis $\{e_1, \ldots, e_n\}$ of V there is the one-to-one correspondence between $\mathcal{L}_n$ and

$$\mathcal{C}_n = \{(c_{ij}^k) \in \mathbb{F}^{n^3} \mid c_{ij}^k + c_{ji}^k = 0,\ c_{ij}^{i'} c_{i'k}^{k'} + c_{ki}^{i'} c_{i'j}^{k'} + c_{jk}^{i'} c_{i'i}^{k'} = 0\},$$

which is determined for any Lie bracket $\mu \in \mathcal{L}_n$ and any structure constant tuple $(c_{ij}^k) \in \mathcal{C}_n$ by the formula $\mu(e_i, e_j) = c_{ij}^k e_k$. Throughout the indices i, j, k, i', j' and k' run from 1 to n and the summation convention over repeated indices is used. $\mathcal{L}_n$ is called the *variety of n-dimensional Lie algebras (over the field $\mathbb{F}$)* or, more precisely, the variety of possible Lie brackets on V. The group $\mathrm{GL}(V)$ acts on $\mathcal{L}_n$ in the following way:

$$(U \cdot \mu)(x, y) = U^{-1}\big(\mu(Ux, Uy)\big) \quad \forall U \in \mathrm{GL}(V), \forall \mu \in \mathcal{L}_n, \forall x, y \in V.$$

(This is the right action conventional for the 'physical' contraction theory. In the algebraic literature, the left action defined by the formula $(U \cdot \mu)(x, y) = U\big(\mu(U^{-1}x, U^{-1}y)\big)$ is used that is not of fundamental importance.) Denote the orbit of $\mu \in \mathcal{L}_n$ under the action of $\mathrm{GL}(V)$ by $\mathcal{O}(\mu)$ and the closure of it with respect to the Zariski topology on $\mathcal{L}_n$ by $\overline{\mathcal{O}(\mu)}$.

Definition 1. The Lie algebra $\mathfrak{g}_0 = (V, \mu_0)$ is called a *contraction* (or *degeneration*) of the Lie algebra $\mathfrak{g} = (V, \mu)$ if $\mu_0 \in \overline{\mathcal{O}(\mu)}$. The contraction is *proper* if $\mu_0 \in \overline{\mathcal{O}(\mu)} \setminus \mathcal{O}(\mu)$. The contraction is *nontrivial* if $\mu_0 \not\equiv 0$.

In the case $\mathbb{F} = \mathbb{C}$ the orbit closures with respect to the Zariski topology coincide with the orbit closures with respect to the Euclidean topology and Definition 1 is reduced to the usual definition of contractions which is also suitable for the case $\mathbb{F} = \mathbb{R}$.

Definition 2. Consider a parameterized family of the Lie algebra $\mathfrak{g}_\varepsilon = (V, \mu_\varepsilon)$ isomorphic to $\mathfrak{g} = (V, \mu)$. The family of the new Lie brackets μ_ε, $\varepsilon \in (0, 1]$, is defined via the Lie bracket μ with a continuous function $U\colon (0,1] \to \mathrm{GL}(V)$ by the rule $\mu_\varepsilon(x,y) = U_\varepsilon{}^{-1}\mu(U_\varepsilon x, U_\varepsilon y)\ \forall\, x,y \in V$. If for any $x,y \in V$ there exists the limit

$$\lim_{\varepsilon\to+0} \mu_\varepsilon(x,y) = \lim_{\varepsilon\to+0} U_\varepsilon{}^{-1}\mu(U_\varepsilon x, U_\varepsilon y) =: \mu_0(x,y)$$

then μ_0 is a well-defined Lie bracket. The Lie algebra $\mathfrak{g}_0 = (V,\mu_0)$ is called a *one-parametric continuous contraction* (or simply a *contraction*) of the Lie algebra $\mathfrak{g}$. The procedure $\mathfrak{g} \to \mathfrak{g}_0$ providing $\mathfrak{g}_0$ from $\mathfrak{g}$ is also called a *contraction.*

If a basis of V is fixed, the operator U_ε is defined by the corresponding matrix $U_\varepsilon \in \mathrm{GL}_n(\mathbb{F})$ and Definition 2 can be reformulated in terms of structure constants. Let c_{ij}^k be the structure constants of the algebra $\mathfrak{g}$ in the fixed basis $\{e_1,\dots,e_n\}$. Then Definition 2 is equivalent to that the limit

$$\lim_{\varepsilon\to+0} (U_\varepsilon)^i_{i'}(U_\varepsilon)^j_{j'}(U_\varepsilon{}^{-1})^{k'}_k c^k_{ij} =: c^{k'}_{0,i'j'}$$

exists for all values of i', j' and k' and, therefore, $c^{k'}_{0,i'j'}$ are components of the well-defined structure constant tensor of a Lie algebra $\mathfrak{g}_0$. The parameter ε and the matrix-function U_ε are called a *contraction parameter* and a *contraction matrix*, respectively.

The contraction $\mathfrak{g} \to \mathfrak{g}_0$ is called *trivial* if $\mathfrak{g}_0$ is Abelian and *improper* if $\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}$.

Definition 3. The contractions $\mathfrak{g} \to \mathfrak{g}_0$ and $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_0$ are called *(weakly) equivalent* if the algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_0$ are isomorphic to $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively.

Using the weak equivalence concentrates one's attention on existence and results of contractions and neglects differences in ways of contractions.

Generalized IW-contractions

Generalized Inönü–Wigner contractions is defined as a specific way for realizations of general contractions.

Definition 4. The contraction $\mathfrak{g} \to \mathfrak{g}_0$ (over $\mathbb{C}$ or $\mathbb{R}$) is called *a generalized Inönü–Wigner contraction* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon = \mathrm{diag}(\varepsilon^{\alpha_1},\dots,\varepsilon^{\alpha_n})$ for some $\alpha_1,\dots,\alpha_n \in \mathbb{R}$. The tuple of exponents $(\alpha_1,\dots,\alpha_n)$ is called the *signature* of the generalized IW-contraction $\mathfrak{g} \to \mathfrak{g}_0$.

In fact, the signature of a generalized IW-contraction $\mathfrak{C}$ is defined up to a positive multiplier since the reparametrization $\varepsilon = \tilde{\varepsilon}^\beta$, where $\beta > 0$, leads to a generalized IW-contraction strongly equivalent to $\mathfrak{C}$.

Due to the possibility of changing bases in the initial and contracted algebras, we can set A and P equal to the unit matrix. This is appropriate for some theoretical considerations but not for working with specific Lie algebras. For $U_\varepsilon = \mathrm{diag}(\varepsilon^{\alpha_1}, \dots, \varepsilon^{\alpha_n})$ the structure constants of the resulting algebra $\mathfrak{g}_0$ are calculated by the formula $c^k_{0,ij} = \lim_{\varepsilon\to+0} c^k_{ij}\,\varepsilon^{\alpha_i+\alpha_j-\alpha_k}$ with no summation with respect to the repeated indices. Therefore, the constraints $\alpha_i+\alpha_j \geqslant \alpha_k$ if $c^k_{ij} \neq 0$ are necessary and sufficient for the existence of the well-defined generalized IW-contraction with the contraction matrix U_ε, and $c^k_{0,ij} = c^k_{ij}$ if $\alpha_i+\alpha_j = \alpha_k$ and $c^k_{0,ij} = 0$ otherwise. This obviously implies that the conditions of existence of generalized IW-contractions and the structure of contracted algebras can be reformulated in the basis-independent terms of gradings of contracted algebras associated with filtrations on initial algebras. In particular, the contracted algebra $\mathfrak{g}_0$ has to admit a derivation whose matrix is diagonalizable to $\mathrm{diag}(\alpha_1, \dots, \alpha_n)$.

Theorem 1. *Any generalized IW-contraction is equivalent to a generalized IW-contraction with an integer signature (and the same associated constant matrices).*

The found proof of Theorem 1 gives a *constructive way* for finding an integer signature via solving the system S, e.g., by the Gaussian elimination. The similar remark is true for the proof of Theorem 2.

There exists a class of contractions, which is wider than the class of generalized IW-contractions and, at the same time, any contraction from this class is equivalent to a generalized IW-contraction involving only integer parameter powers. Similar to generalized IW-contractions, this class is singled out by restrictions on contraction matrices instead of restrictions on algebra structure.

Definition 5. The contraction $\mathfrak{g} \to \mathfrak{g}_0$ (over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) is called *diagonal* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon = \mathrm{diag}(f_1(\varepsilon), \dots, f_n(\varepsilon))$ for some continuous functions $f_i\colon (0,1] \to \mathbb{F}\backslash\{0\}$.

Theorem 2. *Any diagonal contraction is equivalent to a generalized IW-contraction with an integer signature.*

In other words, Theorem 2 says that generalized IW-contractions are universal in the class of diagonal contractions.

Corollary 1. *Any diagonal contraction whose matrix possesses a finite limit at* $\varepsilon \to +0$ *is equivalent to a generalized IW-contraction with nonnegative integer exponents.*

Note 1. Other additional restrictions on exponents of generalized IW-contractions which are equivalent to diagonal contractions with certain properties can be set in a similar way. In particular, it obviously follows from the proof of Theorem 2 that for any fixed j the jth exponent can be chosen nonnegative (negative) if there exists a finite (infinite) limit of f_j at $\varepsilon \to +0$.

Lowest dimensional example on non-universality of generalized IW-contractions

For a long time it was not known whether any continuous one-parametric contraction can be represented by a generalized IW-contraction. As all continuous contractions arising in the physical literature enjoy this property, it was even claimed that this is true for an arbitrary continuous one-parametric contraction.

The first crucial advance in tackling this problem was made by Burde (1999,2005) where examples of contractions to characteristically nilpotent Lie algebras were constructed for all dimensions not less that seven. Since each proper generalized IW-contraction induces a proper grading for the contracted algebra and each characteristically nilpotent Lie algebra possesses only nilpotent derivations and hence has no proper gradings, *any contraction to characteristically nilpotent Lie algebras is obviously inequivalent to a generalized IW-contraction.* Unfortunately, these examples are not yet well-known to the physical community.

Contractions of low-dimensional Lie algebras were studied by a number of scientists (e.g., Agaoka (1999, 2002), Burde (2005), Burde&Steinhoff (1999), Lauret (2003), Nesterenko&Popovych (2006), Weimar-Woods (1991). Thus, it was shown by Nesterenko&Popovych (2006) that each contraction of complex three-dimensional Lie algebras is equivalent to a simple IW-contraction. Any contraction of real three-dimensional Lie algebras is realized by a generalized IW-contraction with nonnegative powers of the contraction parameter which are not greater than two. Moreover, only the contraction of so(3) to the Heisenberg algebra is inequivalent to a simple IW-contraction. The same result for continuous one-parametric contractions of real three-dimensional Lie algebras was also claimed by Weimar-Woods (1991) but contractions within parameterized series of algebras were not explicitly discussed. All possible contractions of three-dimensional Lie algebras were realized by generalized IW-contractions much earlier. Therefore, the problem was to prove that there are no other contractions of three-dimensional Lie algebras. For the complex case, it made in a rigorous way in by Burde&Steinhoff (1999).

Almost all contractions of four-dimensional Lie algebras were realized in Nesterenko&Popovych (2006) via generalized IW-contractions. For the real case, the exceptions were the contractions

$$A_{4.10} \to A_{4.1}, \quad 2A_{2.1} \to A_{4.1}, \quad 2A_{2.1} \to A_1 \oplus A_{3.2}, \quad A_{4.10} \to A_1 \oplus A_{3.2},$$

where the above Lie algebras have the following nonzero commutation relations:

$$\begin{aligned}
2A_{2.1}\colon &\quad [e_1, e_2] = e_1,\ [e_3, e_4] = e_3;\\
A_1 \oplus A_{3.2}\colon &\quad [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3;\\
A_{4.1}\colon &\quad [e_2, e_4] = e_1,\ [e_3, e_4] = e_2;\\
A_{4.10}\colon &\quad [e_1, e_3] = e_1,\ [e_2, e_3] = e_2,\ [e_1, e_4] = -e_2,\ [e_2, e_4] = e_1.
\end{aligned}$$

Since the complexifications of the algebras $2A_{2.1}$ and $A_{4.10}$ are isomorphic, this gives only two exceptions for the complex case: $2\mathfrak{g}_{2.1} \to \mathfrak{g}_{4.1}$ and $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$. Here $\mathfrak{g}_{\dots}$ denotes the complexification of the algebra $A_{\dots}$.

Recently Campoamor-Stursberg (2008) found that in fact both contractions to $A_{4.1}$ are equivalent to generalized IW-contractions. As remarked by Nesterenko, the matrix proposed by Campoamor-Stursberg for the contraction $2A_{2.1} \to A_{4.1}$ can be optimized via lowering the maximal parameter exponent.

We first proved the fact that the contraction $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ is not equivalent to a generalized IW-contraction. As all other contractions relating complex four-dimensional Lie algebras were already realized as generalized IW-contractions, we can state the following results.

Theorem 3. *There exists a unique contraction among complex four-dimensional Lie algebras (namely, $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$) which is not equivalent to a generalized Inönü–Wigner contraction.*

Corollary 2. *There exist precisely two contractions among real four-dimensional Lie algebras (namely, $2A_{2.1} \to A_1 \oplus A_{3.2}$ and $A_{4.10} \to A_1 \oplus A_{3.2}$) which cannot be realized as generalized Inönü–Wigner contractions.*

Theorem 4. *Any generalized Inönü–Wigner contraction among complex or real four-dimensional Lie algebras is equivalent to the one including only nonnegative integer parameter exponents which are not greater than three. This upper bound is exact, i.e., it cannot be totally decreased for all four-dimensional Lie algebras.*

Discussion of technique applied

The proof of Theorem 3 has a number of special features which, when combined, form a technique applicable to a wide range of similar problems. For this reason we decided to list them below.

1. All necessary criteria for general contractions do not work for the study of generalized IW-contractions since the contraction is known to exist and, therefore, the necessary criteria are definitely satisfied. The problem is to determine whether the contraction can be realized in a special way and this requires other tools.
2. There exists a simple criterion stating that a contraction is not equivalent to a generalized IW-contraction if the contracted algebra admits improper gradings only. In contrast with the contractions to characteristically nilpotent Lie algebras, this criterion is not applicable to the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ since the latter has nontrivial diagonal derivations and therefore possesses proper gradings.

3. In the canonical basis, the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ has a two-dimensional algebra of diagonal derivations. Therefore, we have to consider a number of different gradings for the contracted algebra. The study of the gradings aims at resolving a twofold challenge—to obtain possible values of parameter exponents and to understand the structure of constant components of contraction matrices. Thus, the structure of derivations of the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ implies that only signatures of the form $(\beta, \alpha, \alpha, 0)$ are admissible.
4. Further restrictions on parameter exponents follow from the absence of simple IW-contractions from $2\mathfrak{g}_{2.1}$ to $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$. Up to positive multipliers, any signature associated with a simple IW-contraction consists of zeros and units. Hence we have the condition $0 \neq \alpha \neq \beta \neq 0$.
5. The matrix P in the representation $U_\varepsilon = AW_\varepsilon P$ of the contraction matrix U_ε is determined up to changes of basis within graded components and up to automorphisms of the contracted algebra. Since in the case under consideration the matrix P provides an isomorphism among gradings, we can set P equal to the unit matrix.
6. A significant part of subcases for parameter exponents can be ignored as the associated systems of equations for entries of the matrix A are extensions of their counterparts for other subcases and hence the inconsistency of the former systems is immediate from that of the latter ones.
7. Using the scaling automorphisms of the contracted (or initial) algebra, we set $\det A = 1$ to simplify the entries of the inverse matrix A^{-1}.
8. We consider each tuple of parameter exponents for which the corresponding system of algebraic equations for entries of the matrix A is minimal. This nonlinear system is represented in a specific form that allows us to apply methods of solving *linear* systems of algebraic equations. In particular, we try, wherever possible, to avoid writing out the entries of the inverse matrix $B = A^{-1}$ in terms of entries of the matrix A.

It is now known that all contractions of three-dimensional complex (resp. real) Lie algebras can be realized via generalized IW-contractions and that this is not true for the dimension four and the dimensions greater than six. Similar results for dimensions one and two are trivial. The problem of universality of generalized IW-contractions for five- and six-dimensional Lie algebras is still open. It is expected that for these dimensions the answer and the approach to this problem will be similar to those used in the dimension four.

Since generalized IW-contractions are not universal in the whole set of Lie algebras, the following question is natural and important: In which classes of Lie algebras closed under contractions any contraction is equivalent to a generalized IW-contraction?