Non-linear superposition for hyperbolic Burgers equation

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Symmetry in Nonlinear Mathematical Physics, Kiev'09

- Classical Burgers equation

$$u_t - u\,u_x - \kappa\,u_{xx} = 0$$

- Taking into account the memory effects (relaxation)

Hyperbolic modification of Burgers equation

$$\tau u_{tt} - \kappa u_{xx} + A\,u u_x + B\,u_t + H\,u_x = f(u) \qquad (1)$$

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I. Traveling-wave variables

$$u(x,\,t) = U(\xi) \qquad \xi = x + \mu\,t$$

Reduced equation

$$\Delta\,U''(\xi) + \Theta\,U'(\xi) + A\,U(\xi)\,U'(\xi) = f(U) \qquad (2)$$

$$\Delta = \tau\,\mu^2 - \kappa, \quad \Theta = H + B\,\mu$$

- Qualitative analysis
- Ansatz-based methods
- Approximated solutions
- Cole-Hopf transformation
- Painleve analysis

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- Classical Burgers equation:

$$u_t - u\,u_x - \kappa\,u_{xx} = 0. \tag{3}$$

- Cole-Hopf transformation:

$$u(x,t) = c\,(Log\,f(x,t))_x = c\,f_x(x,t)/f(x,t) \tag{4}$$

leads us to linear heat equation (for $c = -2\,\kappa$):

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- Linearity of heat equation

$$u = (Log(f_1 + f_2))_x = \frac{f_{1\,x} + f_{2\,x}}{f_1 + f_2} = \tag{6}$$

$$= \frac{\frac{f_{1\,x}}{f_1}\frac{f_1}{f_2} + \frac{f_{2\,x}}{f_2}}{\frac{f_1}{f_2} + 1}$$

New solution of Burgers equation

$$u = \frac{u_1\,Exp(h) + u_2}{Exp(h) + 1}, \tag{7}$$

where $u_i = \frac{f_{i\,x}}{f_i}$, $i = 1, 2$ are already known solutions and $h(x,t) = Log(f_1/f_2)$.

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GBE

$$\tau\, u_{tt} + A\, u u_x + B\, u_t + H\, u_x - \kappa\, u_{xx} = \lambda(u - m_1)(u - m_2)(u - m_3) \qquad (8)$$

Ansatz

$$u(x,t) = \frac{M(x,t)\, Exp(h(x,t)) + Q(x,t)}{Exp(h(x,t)) + 1} \qquad (9)$$

- $P = M - Q$

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- E1:

$$P^2(x,t) - 2\,\tau\,h_t^2(x,t) + A\,P(x,t)\,h_x(x,t) + 2\,\kappa\,h_x^2(x,t) = 0 \tag{10}$$

- E2:

$$P^3(x,t) - \lambda(m_1 + m_2 + m_3)\,P^2(x,t) + 3\,P^2(x,t)\,Q(x,t) + 2\,\tau\,h_t(x,t)\,P_t(x,t) - \tag{11}$$

$$-2\,\kappa\,h_x(x,t)\,P_x(x,t) + P(x,t)\,(B\,h_t(x,t) + \tau\,h_t^2(x,t) + \tau\,h_{t\,t}(x,t) + H\,h_x(x,t) +$$

$$+A\,Q(x,t)\,h_x(x,t) - \kappa\,h_x^2(x,t) - A\,P_x(x,t) - \kappa\,h_{x\,x}(x,t)) = 0$$

- Scaling transformation of E1

$$\sqrt{2\,\tau}\,h_t(x,t) - P(x,t) - \sqrt{2\kappa}\,h_x(x,t) = 0 \tag{12}$$

$$h(x,t) = \frac{1}{\sqrt{2\,\tau}} \int_0^t P(\xi, x + \sqrt{\kappa/\tau}\,t - \sqrt{\kappa/\tau}\,\xi)d\xi + \Phi(x + \sqrt{\kappa/\tau}\,t) \tag{13}$$

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Definition

Let $M = \frac{M_1\, Exp(h_1)+Q_1}{Exp(h_1))+1}$, $Q = \frac{M_2\, Exp(h_2)+Q_2}{Exp(h_2)+1}$ satisfy GBE.
$M \sim Q$ if $u = \frac{M\, Exp(h)+Q}{Exp(h)+1}$ satisfies GBE.

Theorem

"$\sim$" is the equivalence relation.

Equivalence relation

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"$\sim$" is the equivalence relation.

Notation

Γ denotes the equivalence class of the stationary solution $u(x,t) = m_1$.

Definition

In Γ

$$M \circ_h Q := \frac{M Exp(h) + Q}{Exp(h) + 1}.$$

Lemma

Γ *is closed with respect to the operation "*$\circ_h$*".*

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Γ has the following properties:

- Any element of Γ is the unity for itself: $M \circ_h M = M$
- The operation is commutative: $M \circ_h Q = Q \circ_{-h} M$
- Association property: $M_1 \circ_{H_1} (Q_1 \circ_{-h_2} M_2) = (M_1 \circ_{h_1} Q_1) \circ_{H_2} M_2$

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Examples

- Let $m_1 = 0$, $M(x,t) = m_2$, $Q(x,t) = m_1 = 0$.
- We can construct the solution:

$$h(x,t) = \frac{(m_1 - m_2)\,x}{\sqrt{2}} + \phi(t + x\sqrt{\tau}), \tag{14}$$

where $\phi(t + x\sqrt{\tau})$ is arbitrary C^1 function.

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- For

$$\phi(t + x\sqrt{\tau}) = Sin(t + x\sqrt{\tau})$$

$$u(x,t) = m_2 - \frac{2(m_2 + m_3)}{1 + Exp(-\sqrt{2}\,(m_2 + m_3)\,x + Sin(t + x\sqrt{\tau}))} \tag{15}$$

- For

$$\phi(t + x\sqrt{\tau}) = Log\left(\frac{1 + Exp(t + x\sqrt{\tau})\,R}{1 + Exp(t + x\sqrt{\tau})}\right)$$

$$u(x,t) = \frac{m_2\,Exp(\omega_1)[1 + R\,Exp(\omega_2)]}{1 + Exp(\omega_1) + Exp(\omega_2) + R\,Exp(\omega_1 + \omega_2)}, \tag{16}$$

where $\omega_1 = -\frac{m_2\,x}{\sqrt{2}},\ \ \omega_2 = t + x\sqrt{\tau}$

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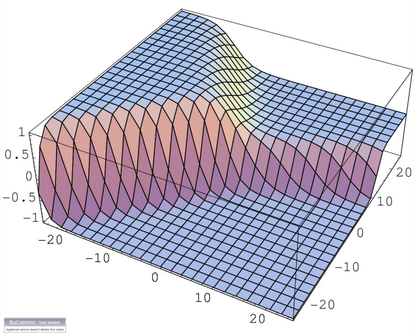
Solutions


1
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-0.5
-1
-20
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10
20
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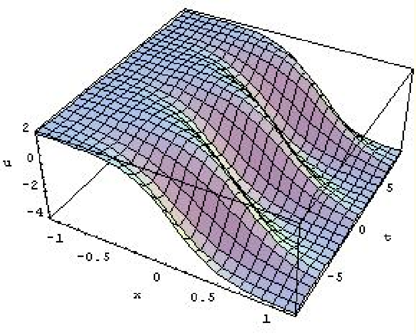
-20
ReaConverter - trial version


2
u 0
-2
-4
-1
-0.5
0
x
0.5
1
5
0
t
-5

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