The Adapted Ordering Method for Lie Algebras and Superalgebras and their Generalizations

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In 1998 the Adapted Ordering Method (A.O.M.) was developed, by M. Dörrzapf and B. Gato-Rivera, for the study of the representation theory of the superconformal algebras in 2-d.

The idea originated, in rudimentary form, from a procedure due to A. Kent in 1991 to study the analytically continued Virasoro algebra, yielding 'generalised' Verma modules, where he constructed 'generalised' singular vectors in terms of analytically continued Virasoro operators. This analytical continuation is not necessary, however, for the A.O.M., nor is it necessary to construct singular vectors in order to apply it.

Later on, in 2004, B. Gato-Rivera tried to generalize this method so that it could be applied to other algebras, but only at the end of 2007 all the details were fixed. As a result, the present version of the A.O.M. (J. Phys A: Math. Theor., 2008) can be applied to many Lie algebras and superalgebras and their generalizations, provided they can be triangulated.

Motivation

The A.O.M. Allows:

- **To determine the maximal dimension for a given type of space of singular vectors. (Singular vectors are highest weight null vectors).**

- **To rule out the existence of possible types of singular vectors (if the max. dim = 0 for the corresponding spaces).**

- **To identify all singular vectors by only a few coefficients.**

- **To obtain easily product expressions of singular vector operators to obtain secondary singular vectors.**

- **To set the basis for constructing embedding diagrams.**

- **To spot subsingular vectors. (Subsingular vectors are null vectors which become singular after the quotient of the Verma module by a submodule).**

What is the A.O.M. ?

The underlying idea is the concept of ADAPTED ORDERINGS for all the possible terms of the 'would be' singular vectors with weights $\{w_i\}$:

- **First, one has to find a suitable total ordering for all the possible terms of the corresponding weights $\{w_i\}$. That is, we need a criterion to decide which of two given terms, with the same weights, is the bigger one.**

  Example: For the Virasoro algebra, level 2 (level = conformal weight):

  $$L_2 > L_1 L_1 \qquad \textbf{or} \qquad L_1 L_1 > L_2 \ \textbf{?}$$

- **Second, the ordering will be called ADAPTED to a subset of terms $C^A_{\{w_i\}}$, that belongs to the total set of terms $C_{\{w_i\}}$ with weights $\{w_i\}$, provided some conditions are met (see later).**

- **Third, the complement of the subset $C^A_{\{w_i\}}$ is the ORDERING KERNEL, $C^K_{\{w_i\}} = C_{\{w_i\}}/C^A_{\{w_i\}}$, which plays a crucial rôle, as we will see next.**

One needs to find a suitable, clever ordering in order to obtain the smallest possible kernels $C^K_{\{w_i\}}$ because:

- **The sizes of the kernels $C^K_{\{w_i\}}$ put un upper limit on the dimensions of the corresponding spaces of singular vectors with weights $\{w_i\}$.**

- **The coefficients with respect to the terms of the ordering kernel $C^K_{\{w_i\}}$ uniquely identify a singular vector $\Psi_{\{w_i\}}$. Since the size of the ordering kernels are in general small, it turns out that just a few coefficients (one, two, ...) completely determine a singular vector no matter its size.**

- **As a consequence, one can find easily product expressions for descendants singular vectors, setting the basis to construct embedding diagrams.**

These statements result from the following three theorems:

- **Theorem 1:** **If the ordering kernel $C^K_{\{w_i\}}$ has n elements, then there are at most n linearly independent singular vectors $\Psi_{\{w_i\}}$ with weights $\{w_i\}$.**

- **Theorem 2:** **If the ordering kernel $C^K_{\{w_i\}} = \emptyset$, then there are no singular vectors with weights $\{w_i\}$.**

- **Theorem 3:** **If two singular vectors $\Psi^1_{\{w_i\}}$ and $\Psi^2_{\{w_i\}}$ have the same coefficients with respect to the terms of the ordering kernel, then they are identical $\Psi^1_{\{w_i\}} = \Psi^2_{\{w_i\}}$.**

Some Observations

- **The maximal possible dimension n for a given space of singular vectors does not imply that all the singular vectors of the corresponding type are n-dimensional.**

- **If the maximal dimension for a given space of singular vectors is zero, then such 'would be' singular vectors do not exist. This is a very practical result for some algebras since it allows to discard the existence of many (or most) types of singular vectors.**

- **Are there any prescriptions in order to construct the most suitable orderings with the smallest kernels? No, there are no general prescriptions or recipes as the orderings depend entirely on the given algebras.**

- **The way to proceed is a matter of trial and error: one constructs a total ordering first, then one computes the kernel and decides whether this kernel is small enough. In the case it is not, then one constructs a second ordering and repeats the procedure until one finds a suitable ordering.**

- **It may also happen, for a given algebra, that this procedure does not give any useful information because all the total orderings one can construct are adapted only to the empty subset, in which case the ordering kernel is the whole set of terms:** $C^K_{\{w_i\}} = C_{\{w_i\}}$.

Technical Details

Let $\mathcal{A}$ denote a Lie algebra or superalgebra with a triangular decomposition: $\mathcal{A} = \mathcal{A}^- \oplus \mathcal{H}_\mathcal{A} \oplus \mathcal{A}^+$, where $\mathcal{A}^-$ is the set of *creation operators*, **$\mathcal{A}^+$ is the set of** *annihilation operators*, **and $\mathcal{H}_\mathcal{A}$ is the** *Cartan subalgebra*. **In general, an eigenvector with respect to the Cartan subalgebra with** *relative weights* **given by the set $\{w_i\}$, in particular a singular vector $\Psi_{\{w_i\}}$, can be expressed as a sum of products of creation operators with total weights $\{w_i\}$ acting on a h.w. vector with weights $\{\Delta_i\}$:**

$$\Psi_{\{w_i\}} \;=\; \sum_{m_1,m_2,....\in N_0} k_{a_{-1}^{m_1},a_{-2}^{m_2},.....} X_{\{w_i\}}^{a_{-1}^{m_1},a_{-2}^{m_2},.....} |\{\Delta_i\}\rangle \qquad (1)$$

where $X_{\{w_i\}}^{a_{-1}^{m_1},a_{-2}^{m_2},.....}$ are the products of the creation operators: $a_{-1}^{m_1} a_{-2}^{m_2}......$, with total weights $\{w_i\}$, which will be denoted simply as *terms*, **and $k_{a_{-1}^{m_1},a_{-2}^{m_2},.....}$ are coefficients which depend on the given term.**

Observe that the weights of $\Psi_{\{w_i\}}$ are given by $\{w_i + \Delta_i\}$.

Let us define the set $C_{\{w_i\}}$ as the set of all the terms with weights $\{w_i\}$:

$$C_{\{w_i\}} \;=\; \{X_{\{w_i\}}^{a_{-1}^{m_1}, a_{-2}^{m_2}, ...}, m_1, m_2, ... \in N_0\}\,, \tag{2}$$

and let $\mathcal{O}$ denote a total ordering on $C_{\{w_i\}}$.

We define an Adapted Ordering on $C_{\{w_i\}}$ as follows:

A total ordering $\mathcal{O}$ on $C_{\{w_i\}}$ is called adapted to the subset $C^A_{\{w_i\}}$ in the Verma module $V_{\{\Delta_i\}}$ if for any element $X_0 \in C^A_{\{w_i\}}$ at least one annihilation operator Γ exists for which $\Gamma\, X_0|\{\Delta_i\}\rangle$ contains a non-trivial term $\tilde{X}$: $\Gamma\, X_0|\{\Delta_i\}\rangle \;=\; (k_{\tilde{X}}\tilde{X} + ...)\,|\{\Delta_i\}\rangle$, which is absent, however, for all $\Gamma\, X|\{\Delta_i\}\rangle$, where X is any term $X \in C_{\{w_i\}}$ which is $\mathcal{O}$-*larger* than X_0, that is, such that $X > X_0$.

Example: Let us consider the Virasoro algebra V:

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{C}{12}(m^3 - m)\delta_{m+n,0}\,, \quad [C, L_m] = 0\,, \quad m, n \in \mathbf{Z} \tag{3}$$

where C can be taken to be constant . V can be written in its triangular decomposition: $V = V^- \oplus \mathcal{H}_V \oplus V^+$, where:

V^- = **span** $\{L_{-m} : m \in \mathbf{N}\}$ **is the set of** *creation operators*,

V^+ =**span** $\{L_m : m \in \mathbf{N}\}$ **is the set of** *annihilation operators*,

and the *Cartan subalgebra* **is given by** $\mathcal{H}_V$ =**span** $\{L_0, C\}$.

For elements of V that are eigenvectors of L_0 with respect to the adjoint representation the L_0-eigenvalue is usually called the *level* l. **The** *terms* **are given by the products of the form** $L_{-p_I} \ldots L_{-p_1}$, $p_q \in \mathbf{N}$ **for** $q = 1, \ldots, I$, $I \in \mathbf{N}$, **with level** $l = \sum_{q=1}^{I} p_q$.

On can find an adapted ordering such that, at each level l the ordering kernel is given by the single element $C_l^K = \{L_{-1}^l\}$.

Let us consider the set of terms at level 3, $C_3 = \{L_{-1}^3, L_{-2}L_{-1}, L_{-3}\}$.

One finds the total ordering $L_{-1}^3 < L_{-2}L_{-1} < L_{-3}$, which is adapted to $C_3^A = \{L_{-2}L_{-1}, L_{-3}\}$ with the ordering kernel $C_3^K = \{L_{-1}^3\}$.

To see this one has to compute the action of the annihilation operators $\Gamma \in \{L_1, L_2, L_3\}$ on the three terms. In fact, the action of L_1 already reveals the structure of C_3^A, as $L_1L_{-2}L_{-1}|\Delta\rangle$ contains the term L_{-1}^2 that is absent in $L_1L_{-3}|\Delta\rangle$. The action of the three annihilation operators on $L_{-1}^3|\Delta\rangle$, however, produce terms that are also created by the action of these operators on $L_{-2}L_{-1}|\Delta\rangle$ and/or $L_{-3}|\Delta\rangle$.

Final Remarks

The Adapted Ordering Method has been applied so far to some infinite-dimensional algebras: the four $N = 2$ superconformal algebras (Neveu-Schwarz, Ramond, topological and twisted), and the $N = 1$ Ramond and Virasoro algebras, allowing to prove several conjectured results as well as to obtain many new results. For example, this method allowed to discover subsingular vectors and two-dimensional spaces of singular vectors for the twisted $N = 2$ algebra and also for the Ramond $N = 1$ algebra.

However, the A.O.M. follows only from the definition of Adapted Ordering plus the three theorems above, which are proven. There is nothing in the definition of Adapted Ordering, neither in the theorems, that restricts the application of this method to infinite-dimensional algebras.

For the same reason, it seems clear that the Adapted Ordering Method should be useful also for generalized Lie algebras and superalgebras such as affine Kac-Moody algebras, non-linear W-algebras, superconformal W-algebras, loop Lie algebras, Borcherds algebras, F-Lie algebras for $F > 2$ ($F = 1$ are Lie algebras and $F = 2$ are Lie superalgebras), etc.

I am convinced therefore that this method should be of very much help for the study of the representation theory of many algebras, in particular the $N > 2$ superconformal algebras, and some (at least) of the generalized Lie algebras and superalgebras mentioned above.