Higher genus Abelian functions associated with algebraic curves

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Symmetry in Nonlinear Mathematical Physics
National Academy of Sciences of Ukraine,
Kyiv, June 25th 2009

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Outline

1. Background and motivation
2. Abelian functions associated to the (4,5)-curve
   - The cyclic tetragonal curve of genus six
   - The sigma-function expansion
   - New results for the (4,5)-case
3. Additional new results of interest
   - Bilinear and quadratic relations
   - Reductions of the Benney moment equations

Outline

The Weierstrass $\wp$-function

Recall the classic elliptic $\wp$-function by Weierstrass.

Karl Weierstrass
1815-1897

- We can define using the auxiliary σ-function,

$$\wp(u) = -\frac{d^2}{du^2}\ln[\sigma(u)].$$

- The function satisfies key differential equations,

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$
$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2.$$

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General (n,s)-curves

Let (n, s) be coprime with $n < s$. Define general (n,s)-curves as

$$y^n - x^s - \sum_{\alpha,\beta} \mu_{[ns-\alpha n-\beta s]} x^\alpha y^\beta \qquad \mu_j \text{ constants},$$

where $\alpha, \beta \in \mathbb{Z}$ with $\alpha \in (0, s-1), \beta \in (0, n-1)$ and $\alpha n + \beta s < ns$. These have genus $g = \frac{1}{2}(n-1)(s-1)$.

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They have a simpler subclass of cyclic (n, s)-curves

$$y^n = x^s + \lambda_{s-1} x^{s-1} + \ldots + \lambda_1 x + \lambda_0$$

Kleinian $\wp$-functions

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Define the Kleinian $\wp$-functions as the second log derivatives of the multivariate σ-function, $\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, \dots, u_g)$

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\mathbf{u}), \quad i \le j \in \{1, 2, \dots, g\}$$

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We can extend this notation to higher order derivatives

$$\wp_{ijk} = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \ln \sigma(\mathbf{u}) \qquad i \le j \le k \in \{1, 2, \dots, g\}$$

$$\wp_{ijkl} = -\frac{\partial^4}{\partial u_i \partial u_j \partial u_k \partial u_l} \ln \sigma(\mathbf{u}) \qquad i \le j \le k \le l \in \{1, 2, \dots, g\}$$

etc. They are all Abelian functions.

Kleinian $\wp$-functions: Examples

- Imposing this notation on the elliptic case would show

$$\wp \equiv \wp_{11}$$
$$\wp' \equiv \wp_{111}$$
$$\wp'' \equiv \wp_{1111}$$

- A curve with $g = 3$ has 6 $\wp_{ij}$ and 10 $\wp_{ijk}$:

$$\{\wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}\}$$
$$\{\wp_{111}, \wp_{112}, \wp_{113}, \wp_{122}, \wp_{123}, \wp_{133}, \wp_{222}, \wp_{223}, \wp_{233}, \wp_{333}\}$$

Review of higher genus work

n=2, s=3: elliptic curves

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n=3: trigonal curves

- Considerable work has been published by authors including Baldwin, Buchstaber, Eilbeck, Enolski, Gibbons, Leykin, Matsutani, Onishi and Previato.
- Recent work has focussed on a matrix formulation of the differential equations satisfied by $\wp$-functions.

Outline

The cyclic (4,5)-curve

n=4: tetragonal curves

We have considered the cyclic (4,5)-curve, which has genus $g = 6$.

$$y^4 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

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For every (n, s)-curve we can define a set of Sato weights that render all equations in the theory homogeneous.

- For the $(4, 5)$ curve they are given by

x	y	u_1	u_2	u_3	u_4	u_5	u_6	λ_4	λ_3	λ_2	λ_1	λ_0
-4	-5	11	7	6	3	2	1	-4	-8	-12	-16	-20

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Example: $\mathrm{wt}(\lambda_3 x^3) = -8 + 3(-4) = -20$

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The sigma-function expansion I

We want to construct a power series expansion for $\sigma(\boldsymbol{u})$.

- It is known that $\sigma(\boldsymbol{u})$ has weight

$$\frac{1}{24}(n^2-1)(s^2-1) = 15.$$

- The expansion will be multivariate in the variables $\boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5, u_6)$ [+ve weight] and may depend on the curve coefficients, $\{\lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_0\}$ [-ve weight].

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Hence we can write the expansion as

$$\sigma(\boldsymbol{u}) = C_{15} + C_{19} + C_{23} + \ldots + C_{15+4n} + \ldots$$

where each C_k has weight k in the u_i and $(15-k)$ in the λ_j.

The sigma-function expansion II

We can find the C_k in turn as follows:

1. Identify the possible terms — those with correct weight.
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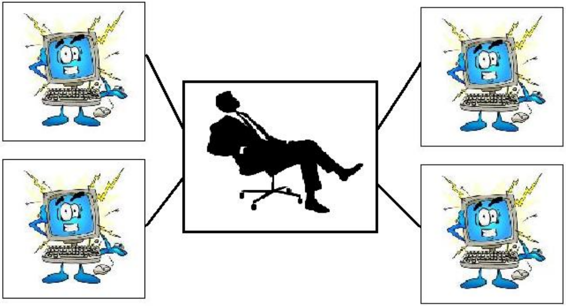
Latter polynomials were very large and represent significant computation challenges.

Polynomial	# terms
C_{19}	50
C_{35}	2193
C_{51}	28359
C_{59}	81832

The sigma-function expansion III

- We used Distributed Maple to run computations in parallel on a cluster of machines in our department.
- Available for free at

  `www.risc.uni-linz.ac.at/software/distmaple/`



New results for the (4,5)-curve I

New results we have derived for the (4,5)-curve include:

- A basis for Abelian functions associated to the (4,5)-curve, with poles of order at most 2.
- A solution to the Jacobi Inversion Problem for this case.
- A two-term addition formula similar to that found in lower genus cases.

$$\frac{\sigma(\mathbf{u}+\mathbf{v})\sigma(\mathbf{u}-\mathbf{v})}{\sigma(\mathbf{u})^2\sigma(\mathbf{v})^2} = f(\boldsymbol{u},\boldsymbol{v}) - f(\boldsymbol{v},\boldsymbol{u})$$

where $f(\boldsymbol{u},\boldsymbol{v})$ is a polynomial of Abelian functions.

$$f(\boldsymbol{u},\boldsymbol{v}) = -6\wp_{55}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})\lambda_4^2\lambda_1 + 4\wp_{44}(\boldsymbol{v})\wp_{46}(\boldsymbol{u})\lambda_4\lambda_1 + \dots$$

New results for the (4,5)-curve II

- A complete set of PDEs that express the 4-index $\wp$-functions, using Abelian functions of order at most 2.

$$
\begin{aligned}
\textbf{(4)} \quad & \wp_{6666} = 6\wp_{66}^2 - 3\wp_{55} + 4\wp_{46} \\
\textbf{(5)} \quad & \wp_{5666} = 6\wp_{56}\wp_{66} - 2\wp_{45} \\
& \vdots \\
\textbf{(20)} \quad & \wp_{2336} = 4\wp_{23}\wp_{36} + 2\wp_{26}\wp_{33} + 8\wp_{16}\lambda_3 - 2\wp_{55}\lambda_1 \\
& \qquad + 2\wp_{35}\lambda_2 + 8\wp_{16}\wp_{26} - 2\wp_{1356} + 4\wp_{13}\wp_{56} \\
& \qquad + 4\wp_{15}\wp_{36} + 4\wp_{16}\wp_{35} - 2\wp_{1266} + 4\wp_{12}\wp_{66} \\
& \vdots
\end{aligned}
$$

These are generalisations of the elliptic PDE:

$$\wp''(u) = 6\wp(u)^2 - \frac{1}{2}g_2$$

New results for the (4,5)-curve III

- We demonstrate that the function

$$W(x, y, t) = \wp_{66}(x, y, t, u_4, u_5, u_6)$$

gives a solution to the KP-equation,

$$\left[W_{xxx} - 12WW_x - 4W_t\right]_x + 3W_{yy} = 0.$$

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$$\left[W_{xxx} - 12WW_x - 4W_t\right]_x + 3W_{yy} = 0.$$

These results and other results were published earlier this year:

M. England and J.C. Eilbeck
Abelian functions associated with a cyclic tetragonal curve of genus six.
J. Phys. A: Math. Theor. 42 (2009) 095210.

Outline

B-Functions and bilinear relations

We used pole analysis on the $\wp$-function definition to derive the 4 functions that sum $\wp_{ij}\wp_{ijk}$ to have poles of order at most 3.

$$\begin{aligned} B^A_{ijklm} &= \wp_{ij}\wp_{klm} + \tfrac{1}{3}\wp_{jk}\wp_{ilm} + \tfrac{1}{3}\wp_{jl}\wp_{ikm} + \tfrac{1}{3}\wp_{jm}\wp_{ikl} \\ \vdots \quad & \quad - \tfrac{2}{3}\wp_{kl}\wp_{ijm} - \tfrac{2}{3}\wp_{km}\wp_{ijl} - \tfrac{2}{3}\wp_{lm}\wp_{ijk}. \end{aligned}$$

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- Using these *B-functions* we can construct a basis for the odd Abelian functions with poles of order at most 3.
- This allows us to find sets of bilinear relations.

$$\textbf{(-6)} \quad 0 = -\wp_{555} + 2\wp_{456} + 2\wp_{566}\wp_{66} - 2\wp_{56}\wp_{666},$$

$$\textbf{(-7)} \quad 0 = 2\wp_{455} - 2\wp_{446} - 2\wp_{466}\wp_{66} + 2\wp_{666}\lambda_4 + 2\wp_{46}\wp_{666}$$

- We derived complete sets for the cyclic (3,4) and (3,5)-cases.

Generalising the quadratic PDE

- Bilinear relations can be employed to find generalisations of the fundamental elliptic differential equation.

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3$$

We seek PDEs to express the product of two 3-index $\wp$-functions, using Abelian functions of order at most 3.

Generalising the quadratic PDE

- Bilinear relations can be employed to find generalisations of the fundamental elliptic differential equation.

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  We seek PDEs to express the product of two 3-index $\wp$-functions, using Abelian functions of order at most 3.

- Many can be obtained by differentiating bilinear relations, or multiplying them by 3-index $\wp$-functions.
- Alternative methods need to be employed to complete the set.

Quadratic 3-index relations

- From the $\wp$-function definition we can match the poles of order 5 and 6 in $\wp_{ijk}\wp_{lmn}$ with a polynomial of $\wp_{ij}^3$. This significantly simplifies the derivation of quadratic 3-index identities using the σ-function expansion.
- We have completed the sets for the cyclic (3,4)-case and cyclic (3,5)-case. We have the beginnings of a set for the (4,5)-case.

$$\textbf{(-6)} \qquad \wp_{666}^2 = 4\wp_{66}^3 - 7\wp_{56}^2 + 4\wp_{46}\wp_{66} - 8\wp_{55}\wp_{66} - 4\wp_{66}\lambda_4 + 4\wp_{44} + 2\wp_{5566},$$

$$\textbf{(-7)} \qquad \wp_{566}\wp_{666} = 4\wp_{66}^2\wp_{56} + 2\wp_{46}\wp_{56} - \wp_{55}\wp_{56} - 2\wp_{45}\wp_{66} + 2\wp_{36},$$

$$\vdots$$

Reductions of the Benney moment equations

- Kleinian $\wp$-functions have been used in the solution of problems which involve the integral of a meromorphic Abelian differential on an algebraic curves.
- One such class of problems involves reductions of the Benney Moment equations.

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- Kleinian $\wp$-functions have been used in the solution of problems which involve the integral of a meromorphic Abelian differential on an algebraic curves.
- One such class of problems involves reductions of the Benney Moment equations.
- We have recently considered the reduction that leads to the cyclic tetragonal surface of genus six...

M. England and J. Gibbons
A genus six cyclic tetragonal reduction of the Benney equations.
ArXiv 0903.5203

The End

Thanks for listening.

Further Information

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