Invariants of Lie Algebras via Moving Frames

Vyacheslav BOYKO †**, Jiri PATERA** ‡ **and Roman POPOVYCH** †

† Institute of Mathematics of NAS of Ukraine,
3 Tereshchenkivs'ka Str., Kyiv-4, 01601 Ukraine
E-mail: boyko@imath.kiev.ua, rop@imath.kiev.ua

‡ Centre de Recherches Mathématiques, Université de Montréal,
C.P. 6128 succursale Centre-ville, Montréal (Québec), H3C 3J7 Canada
E-mail: patera@CRM.UMontreal.CA

Abstract

A purely algebraic algorithm for computation of invariants (generalized Casimir operators) of Lie algebras by means of moving frames is discussed. Results on the application of the method to computation of invariants of low-dimensional Lie algebras and series of solvable Lie algebras restricted only by a required structure of the nilradical are reviewed.

1 Introduction

The invariants of Lie algebras are one of their defining characteristics. They have numerous applications in different fields of mathematics and physics, in which Lie algebras arise (representation theory, integrability of Hamiltonian differential equations, quantum numbers etc). In particular, the polynomial invariants of a Lie algebra exhaust its set of Casimir operators, i.e., the center of its universal enveloping algebra. This is why non-polynomial invariants are also called generalized Casimir operators, and the usual Casimir operators are seen as 'specific' generalized Casimir operators. Since the structure of invariants strongly depends on the structure of the algebra and the classification of all (finite-dimensional) Lie algebras is an inherently difficult problem (actually unsolvable[1]), it seems to be impossible to elaborate a complete theory for generalized Casimir operators in the general case. Moreover, if the classification of a class of Lie algebras is known, then the invariants of such algebras can be described exhaustively. These problems have already been solved for the semi-simple and low-dimensional Lie algebras, and also for the physically relevant Lie algebras of fixed dimensions.

The standard method of construction of generalized Casimir operators consists of integration of overdetermined systems of first-order linear partial differential equations. It turns out to be rather cumbersome calculations, once the dimension of Lie algebra is not one of the lowest few. Alternative methods use matrix representations of Lie algebras. They are not much easier and are valid for a limited class of representations.

In our recent papers [3, 4, 5, 6, 7] we have developed the purely algebraic algorithm for computation of invariants (generalized Casimir operators) of Lie algebras. The suggested approach is simpler and generally valid. It extends to our problem the exploitation of the Cartan's method of moving frames in Fels–Olver version [9]. (For modern development of the moving frames method and more references see also [14, 15].)

[1]The problem of classification of Lie algebras is *wild* since it includes, as a subproblem, the problem on reduction of pairs of matrices to a canonical form [10]. For a detailed review on classification of Lie algebras we refer to [17].

2 Preliminaries

Consider a Lie algebra $\mathfrak{g}$ of dimension $\dim\mathfrak{g} = n < \infty$ over the complex or real field $\mathbb{F}$ (either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$) and the corresponding connected Lie group G. Let $\mathfrak{g}^*$ be the dual space of the vector space $\mathfrak{g}$. The map $\mathrm{Ad}^*\colon G \to \mathrm{GL}(\mathfrak{g}^*)$ defined for any $g \in G$ by the relation

$$\langle \mathrm{Ad}^*_g x, u\rangle = \langle x, \mathrm{Ad}_{g^{-1}}u\rangle \quad \text{for all } x \in \mathfrak{g}^* \text{ and } u \in \mathfrak{g}$$

is called the *coadjoint representation* of the Lie group G. Here $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the usual adjoint representation of G in $\mathfrak{g}$, and the image Ad_G of G under Ad is the inner automorphism group $\mathrm{Int}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The image of G under Ad^* is a subgroup of $\mathrm{GL}(\mathfrak{g}^*)$ and is denoted by Ad^*_G.

The maximal dimension of orbits of Ad^*_G is called the *rank of the coadjoint representation* of G (and $\mathfrak{g}$) and denoted by $\mathrm{rank}\mathrm{Ad}^*_G$. It is a basis independent characteristic of the algebra $\mathfrak{g}$. Orbits of this dimension are called regular ones.

A function $F \in C^\infty(\Omega)$, where Ω is a domain in $\mathfrak{g}^*$, is called a (global in Ω) *invariant* of Ad^*_G if $F(\mathrm{Ad}^*_g x) = F(x)$ for all $g \in G$ and $x \in \Omega$ such that $\mathrm{Ad}^*_g x \in \Omega$. The set of invariants of Ad^*_G on Ω is denoted by $\mathrm{Inv}(\mathrm{Ad}^*_G)$ without an explicit indication of the domain Ω. Let below Ω is a neighborhood of a point from a regular orbit. It can always be chosen in such a way that the group Ad^*_G acts regularly on Ω. Then the maximal number $N_\mathfrak{g}$ of functionally independent invariants in $\mathrm{Inv}(\mathrm{Ad}^*_G)$ coincides with the codimension of the regular orbits of Ad^*_G, i.e., it is given by the difference $N_\mathfrak{g} = \dim\mathfrak{g} - \mathrm{rank}\,\mathrm{Ad}^*_G$.

To calculate the invariants explicitly, one should fix a basis $\mathcal{E} = \{e_1, \dots, e_n\}$ of the algebra $\mathfrak{g}$. It leads to fixing the dual basis $\mathcal{E}^* = \{e_1^*, \dots, e_n^*\}$ in the dual space $\mathfrak{g}^*$ and to the identification of $\mathrm{Int}(\mathfrak{g})$ and Ad^*_G with the associated matrix groups. The basis elements $e_1, \dots, e_n$ satisfy the commutation relations $[e_i, e_j] = c_{ij}^k e_k$, where c_{ij}^k are components of the tensor of structure constants of $\mathfrak{g}$ in the basis $\mathcal{E}$. Here and in what follows the indices i, j and k run from 1 to n and the summation convention over repeated indices is used. Let $x \to \check{x} = (x_1, \dots, x_n)$ be the coordinates in $\mathfrak{g}^*$ associated with $\mathcal{E}^*$.

It is well known that there exists a bijection between elements of the universal enveloping algebra (i.e., *Casimir operators*) of $\mathfrak{g}$ and polynomial invariants of $\mathfrak{g}$ (which can be assumed defined globally on $\mathfrak{g}^*$). See, e.g., [1]. Such a bijection is established, e.g., by the symmetrization operator Sym which acts on monomials by the formula

$$\mathrm{Sym}(e_{i_1}\cdots e_{i_r}) = \frac{1}{r!}\sum_{\sigma\in S_r} e_{i_{\sigma_1}}\cdots e_{i_{\sigma_r}},$$

where $i_1, \dots, i_r$ take values from 1 to n, $r \in \mathbb{N}$. The symbol S_r denotes the permutation group consisting of r elements. The symmetrization also can be correctly defined for rational invariants [1]. If $\mathrm{Int}(\mathrm{Ad}^*_G)$ has no a functional basis consisting of only rational invariants, the correctness of the symmetrization needs an additional investigation for each fixed algebra $\mathfrak{g}$ since general results on this subject do not exist. After symmetrized, elements from $\mathrm{Int}(\mathrm{Ad}^*_G)$ are naturally called invariants or *generalized Casimir operators* of $\mathfrak{g}$. The set of invariants of $\mathfrak{g}$ is denoted by $\mathrm{Inv}(\mathfrak{g})$.

Functionally independent invariants $F^l(x_1, \dots, x_n)$, $l = 1, \dots, N_\mathfrak{g}$, forms a *functional basis* (*fundamental invariant*) of $\mathrm{Inv}(\mathrm{Ad}^*_G)$ since any element from $\mathrm{Inv}(\mathrm{Ad}^*_G)$ can be (uniquely) represented as a function of these invariants. Accordingly the set of $\operatorname{Sym} F^l(e_1,\dots,e_n)$, $l=1,\dots,N_{\mathfrak g}$, is called a basis of $\operatorname{Inv}(\mathfrak g)$.

In framework of the infinitesimal approach any invariant $F(x_1,\dots,x_n)$ of Ad^*_G is a solution of the linear system of first-order partial differential equations [1, 2, 16] $X_iF=0$, i.e., $c^k_{ij}x_kF_{x_j}=0$, where $X_i=c^k_{ij}x_k\partial_{x_j}$ is the infinitesimal generator of the one-parameter group $\{\mathrm{Ad}^*_G(\exp\varepsilon e_i)\}$ corresponding to e_i. The mapping $e_i\to X_i$ gives a representation of the Lie algebra $\mathfrak g$.

3 The algorithm

Let $\mathcal G=\mathrm{Ad}^*_G\times\mathfrak g^*$ denote the trivial left principal Ad^*_G-bundle over $\mathfrak g^*$. The right regularization $\widehat R$ of the coadjoint action of G on $\mathfrak g^*$ is the diagonal action of Ad^*_G on $\mathcal G=\mathrm{Ad}^*_G\times\mathfrak g^*$. It is provided by the map

$$\widehat R_g(\mathrm{Ad}^*_h,x)=(\mathrm{Ad}^*_h\cdot\mathrm{Ad}^*_{g^{-1}},\mathrm{Ad}^*_gx),\quad g,h\in G,\quad x\in\mathfrak g^*.$$

The action $\widehat R$ on the bundle $\mathcal G=\mathrm{Ad}^*_G\times\mathfrak g^*$ is regular and free. We call $\widehat R$ the *lifted coadjoint action* of G. It projects back to the coadjoint action on $\mathfrak g^*$ via the Ad^*_G-equivariant projection $\pi_{\mathfrak g^*}\colon\mathcal G\to\mathfrak g^*$. Any *lifted invariant* of Ad^*_G is a (locally defined) smooth function from $\mathcal G$ to a manifold, which is invariant with respect to the lifted coadjoint action of G. The function $\mathcal I\colon\mathcal G\to\mathfrak g^*$ given by $\mathcal I=\mathcal I(\mathrm{Ad}^*_g,x)=\mathrm{Ad}^*_gx$ is the *fundamental lifted invariant* of Ad^*_G, i.e., $\mathcal I$ is a lifted invariant and any lifted invariant can be locally written as a function of $\mathcal I$ in a unique way. Using an arbitrary function $F(x)$ on $\mathfrak g^*$, we can produce the lifted invariant $F\circ\mathcal I$ of Ad^*_G by replacing x with $\mathcal I=\mathrm{Ad}^*_gx$ in the expression for F. Ordinary invariants are particular cases of lifted invariants, where one identifies any invariant formed as its composition with the standard projection $\pi_{\mathfrak g^*}$. Therefore, ordinary invariants are particular functional combinations of lifted ones that happen to be independent of the group parameters of Ad^*_G.

The essence of the normalization procedure by Fels and Olver can be presented in the form of on the following statement.

Proposition 1. Suppose that $\mathcal I=(\mathcal I_1,\dots,\mathcal I_n)$ is a fundamental lifted invariant, for the lifted invariants $\mathcal I_{j_1}$, …, $\mathcal I_{j_\rho}$ and some constants c_1, …, c_ρ the system $\mathcal I_{j_1}=c_1$, …, $\mathcal I_{j_\rho}=c_\rho$ is solvable with respect to the parameters θ_{k_1}, …, θ_{k_ρ} and substitution of the found values of $\theta_{k_1},\dots,\theta_{k_\rho}$ into the other lifted invariants results in $m=n-\rho$ expressions $\hat{\mathcal I}_l$, $l=1,\dots,m$, depending only on x's. Then $\rho=\operatorname{rank}\mathrm{Ad}^*_G$, $m=N_{\mathfrak g}$ and $\hat{\mathcal I}_1$, …, $\hat{\mathcal I}_m$ form a basis of $\operatorname{Inv}(\mathrm{Ad}^*_G)$.

The *algebraic algorithm* for finding invariants of the Lie algebra $\mathfrak g$ is briefly formulated in the following four steps.

1. *Construction of the generic matrix $B(\theta)$ of* Ad^*_G. $B(\theta)$ is the matrix of an inner automorphism of the Lie algebra $\mathfrak g$ in the given basis e_1, …, e_n, $\theta=(\theta_1,\dots,\theta_r)$ is a complete tuple of group parameters (coordinates) of $\operatorname{Int}(\mathfrak g)$, and $r=\dim\mathrm{Ad}^*_G=\dim\operatorname{Int}(\mathfrak g)=n-\dim\mathrm Z(\mathfrak g)$, where $\mathrm Z(\mathfrak g)$ is the center of $\mathfrak g$.

2. *Representation of the fundamental lifted invariant.* The explicit form of the fundamental lifted invariant $\mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_n)$ of Ad^*_G in the chosen coordinates $(\theta, \check{x})$ in $\mathrm{Ad}^*_G \times \mathfrak{g}^*$ is $\mathcal{I} = \check{x} \cdot B(\theta)$, i.e.,

$$(\mathcal{I}_1, \ldots, \mathcal{I}_n) = (x_1, \ldots, x_n) \cdot B(\theta_1, \ldots, \theta_r).$$

3. *Elimination of parameters by normalization.* We choose the maximum possible number ρ of lifted invariants $\mathcal{I}_{j_1}$, …, $\mathcal{I}_{j_\rho}$, constants c_1, …, c_ρ and group parameters θ_{k_1}, …, θ_{k_ρ} such that the equations $\mathcal{I}_{j_1} = c_1$, …, $\mathcal{I}_{j_\rho} = c_\rho$ are solvable with respect to θ_{k_1}, …, θ_{k_ρ}. After substituting the found values of θ_{k_1}, …, θ_{k_ρ} into the other lifted invariants, we obtain $N_\mathfrak{g} = n - \rho$ expressions $F^l(x_1, \ldots, x_n)$ without θ's.

4. *Symmetrization.* The functions $F^l(x_1, \ldots, x_n)$ necessarily form a basis of $\mathrm{Inv}(\mathrm{Ad}^*_G)$. They are symmetrized to $\mathrm{Sym}\, F^l(e_1, \ldots, e_n)$. It is the desired basis of $\mathrm{Inv}(\mathfrak{g})$.

Our experience on the calculation of invariants of a wide range of Lie algebras shows that the version of the algebraic method, which is based on Proposition 1, is most effective. In particular, it provides finding the cardinality of the invariant basis in the process of construction of the invariants. The algorithm can in fact involve different kinds of coordinate in the inner automorphism groups (the first canonical, the second canonical or special one) and different techniques of elimination of parameters (empiric techniques, with additional combining of lifted invariants, using a floating system of normalization equations etc).

Let us underline that the search of invariants of a Lie algebra $\mathfrak{g}$, which has been done by solving a linear system of first-order partial differential equations under the conventional infinitesimal approach, is replaced here by the construction of the matrix $B(\theta)$ of inner automorphisms and by excluding the parameters θ from the fundamental lifted invariant $\mathcal{I} = \check{x} \cdot B(\theta)$ in some way.

4 Illustrative example

The four-dimensional solvable Lie algebra $\mathfrak{g}^b_{4.8}$ has the following non-zero commutation relations

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = (1 + b)e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = be_3, \quad |b| \leq 1.$$

Its nilradical is three-dimensional and isomorphic to the Weil–Heisenberg algebra $\mathfrak{g}_{3.1}$. (Here we use the notations of low-dimensional Lie algebras according to Mubarakzyanov's classification [11].)

We construct a presentation of the inner automorphism matrix $B(\theta)$ of the Lie algebra $\mathfrak{g}$, involving second canonical coordinates on Ad_G as group parameters θ. The matrices $\hat{\mathrm{ad}}_{e_i}$, $i = 1, \ldots, 4$, of the adjoint representation of the basis elements e_1, e_2, e_3

and e_4 respectively have the form

$$\begin{pmatrix} 0 & 0 & 0 & 1+b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1-b & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The inner automorphisms of $\mathfrak{g}_{4.8}^b$ are then described by the triangular matrix

$$\begin{aligned} B(\theta) &= \prod_{i=1}^{3} \exp(\theta_i \hat{\mathrm{ad}}_{e_i}) \cdot \exp(-\theta_4 \hat{\mathrm{ad}}_{e_4}) \\ &= \begin{pmatrix} e^{(1+b)\theta_4} & -\theta_3 e^{\theta_4} & \theta_2 e^{b\theta_4} & b\theta_2\theta_3 + (1+b)\theta_1 \\ 0 & e^{\theta_4} & 0 & \theta_2 \\ 0 & 0 & e^{b\theta_4} & b\theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, a functional basis of lifted invariants is formed by

$$\begin{aligned} \mathcal{I}_1 &= e^{(1+b)\theta_4} x_1, \\ \mathcal{I}_2 &= e^{\theta_4}(-\theta_3 x_1 + x_2), \\ \mathcal{I}_3 &= e^{b\theta_4}(\theta_2 x_1 + x_3), \\ \mathcal{I}_4 &= (b\theta_2\theta_3 + (1+b)\theta_1)x_1 + \theta_2 x_2 + b\theta_3 x_3 + x_4. \end{aligned}$$

Further the cases $b = -1$ and $b \neq -1$ should be considered separately.

There are no invariants in case $b \neq -1$ since in view of Proposition 1 the number of functionally independent invariants is equal to zero. Indeed, the system $\mathcal{I}_1 = 1$, $\mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}_4 = 0$ is solvable with respect to the whole set of the parameters θ.

It is obvious that in the case $b = -1$ the element e_1 generating the center $Z(\mathfrak{g}_{4.8}^{-1})$ is an invariant. (The corresponding lifted invariant $\mathcal{I}_1 = x_1$ does not depend on the parameters θ.) Another invariant is easily found via combining the lifted invariants: $\mathcal{I}_1\mathcal{I}_4 - \mathcal{I}_2\mathcal{I}_3 = x_1x_4 - x_2x_3$. After the symmetrization procedure we obtain the following polynomial basis of the invariant set of this algebra

$$e_1, \quad e_1e_4 - \frac{e_2e_3 + e_3e_2}{2}.$$

The second basis invariant can be also constructed by the normalization technique. We solve the equations $\mathcal{I}_2 = \mathcal{I}_3 = 0$ with respect to the parameters θ_2 and θ_3 and substitute the expressions for them into the lifted invariant $\mathcal{I}_4$. The obtained expression $x_4 - x_2x_3/x_1$ does not contain the parameters θ and, therefore, is an invariant of the coadjoint representation. For the basis of invariants to be polynomial, we multiply this invariant by the invariant x_1.

It is the technique that is applied below for the general case of the Lie algebras under consideration.

Note that in the above example the symmetrization procedure can be assumed trivial since the symmetrized invariant $e_1e_4 - \frac{1}{2}(e_2e_3 + e_3e_2)$ differs from the non-symmetrized version $e_1e_4 - e_2e_3$ (resp. $e_1e_4 - e_3e_2$) on the invariant $\frac{1}{2}e_1$ (resp. $-\frac{1}{2}e_1$). If we take the rational invariant $e_4 - e_2e_3/e_1$ (resp. $e_4 - e_3e_2/e_1$), the symmetrization is equivalent to the addition of the constant $\frac{1}{2}$ (resp. $-\frac{1}{2}$).

Invariants of $\mathfrak{g}^b_{4.8}$ were first described in [16] within the framework of the infinitesimal approach.

5 Review of obtained results

Using the moving frames approach, we recalculated invariant bases and, in a number of cases, enhanced their representation for the following Lie algebras (in additional brackets we cite the papers where invariants bases of the same algebras were computed by the infinitesimal method):

- the complex and real Lie algebras up to dimension 6 [3] ([8, 12, 16]);
- the complex and real Lie algebras with Abelian nilradicals of codimension one [4] ([18]);
- the complex indecomposable solvable Lie algebras with the nilradicals isomorphic to $\mathfrak{J}^n_0$, $n = 3, 4, \ldots$ (the nonzero commutation relations between the basis elements $e_1, \ldots, e_n$ of $\mathfrak{J}^n_0$ are exhausted by $[e_k, e_n] = e_{k-1}$, $k = 2, \ldots, n-1$) [4] ([13]);
- the nilpotent Lie algebra $\mathfrak{t}_0(n)$ of $n \times n$ strictly upper triangular matrices [4, 5] ([20]);
- the solvable Lie algebra $\mathfrak{t}(n)$ of $n \times n$ upper triangular matrices and the solvable Lie algebras $\mathfrak{st}(n)$ of $n \times n$ special upper triangular matrices [5, 6, 7] ([20]);
- the solvable Lie algebras with nilradicals isomorphic to $\mathfrak{t}_0(n)$ and diagonal nilindependent elements [5, 6, 7] ([20]).

Note that earlier only conjectures on invariants of two latter families of Lie algebras were known. Moreover, for the last family the conjecture was formulated only for the particular case of a single nilindependent element. Here we present the exhaustive statement on invariants of this series of Lie algebras, obtained in [7].

Consider the solvable Lie algebra $\mathfrak{t}_\gamma(n)$ with the nilradical $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ isomorphic to $\mathfrak{t}_0(n)$ and s nilindependent element f_p, $p = 1, \ldots, s$, which act on elements of the nilradical in the way as the diagonal matrices $\Gamma_p = \mathrm{diag}(\gamma_{p1}, \ldots, \gamma_{pn})$ act on strictly triangular matrices. The matrices Γ_p, $p = 1, \ldots, s$, and the unity matrix are linear independent since otherwise $\mathrm{NR}(\mathfrak{t}_\gamma(n)) \neq \mathfrak{t}_0(n)$. The parameter matrix $\gamma = (\gamma_{pi})$ is defined up to nonsingular $s \times s$ matrix multiplier and homogeneous shift in rows. In other words, the algebras $\mathfrak{t}_\gamma(n)$ and $\mathfrak{t}_{\gamma'}(n)$ are isomorphic iff there exist $\lambda \in \mathrm{M}_{s,s}(\mathbb{F})$, $\det\lambda \neq 0$, and $\mu \in \mathbb{F}^s$ such that

$$\gamma'_{pi} = \sum_{p'=1}^{s} \lambda_{pp'}\gamma_{p'i} + \mu_p, \quad p = 1, \ldots, s, \ i = 1, \ldots, n.$$

The parameter matrix γ and γ' are assumed equivalent. Up to the equivalence the additional condition $\operatorname{Tr}\Gamma_p = \sum_i \gamma_{pi} = 0$ can be imposed on the algebra parameters. Therefore, the algebra $\mathfrak{t}_\gamma(n)$ is naturally embedded into $\mathfrak{st}(n)$ as a (mega)ideal under identification of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ with $\mathfrak{t}_0(n)$ and of f_p with Γ_p.

We choose the union of the canonical basis of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ and the s-element set $\{f_p, p = 1,\dots,s\}$ as the canonical basis of $\mathfrak{t}_\gamma(n)$. In the basis of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ we use 'matrix' enumeration of basis elements e_{ij}, $i<j$, with the 'increasing' pair of indices similarly to the canonical basis $\{E^n_{ij},\ i<j\}$ of the isomorphic matrix algebra $\mathfrak{t}_0(n)$.

Hereafter E^n_{ij} (for the fixed values i and j) denotes the $n\times n$ matrix $(\delta_{ii'}\delta_{jj'})$ with i' and j' running the numbers of rows and column correspondingly, i.e., the $n\times n$ matrix with the unit on the cross of the i-th row and the j-th column and the zero otherwise. The indices i, j, k and l run at most from 1 to n. Only additional constraints on the indices are indicated. The subscript p runs from 1 to s, the subscript q runs from 1 to s'. The summation convention over repeated indices p and q is used unless otherwise stated. The number s is in the range $0,\dots,n-1$. In the case $s=0$ we assume $\gamma=0$, and all terms with the subscript p should be omitted from consideration. The value s' ($s'<s$) is defined below.

Thus, the basis elements $e_{ij}\sim E^n_{ij}$, $i<j$, and $f_p\sim\sum_i\gamma_{pi}E^n_{ii}$ satisfy the commutation relations $[e_{ij},e_{i'j'}]=\delta_{i'j}e_{ij'}-\delta_{ij'}e_{i'j}$, $[f_p,e_{ij}]=(\gamma_{pi}-\gamma_{pj})e_{ij}$, where δ_{ij} is the Kronecker delta.

The Lie algebra $\mathfrak{t}_\gamma(n)$ can be considered as the Lie algebra of the Lie subgroup $T_\gamma(n) = \{B\in T(n) \mid \exists\,\varepsilon_p\in\mathbb{F}\colon b_{ii}=e^{\gamma_{pi}\varepsilon_p}\}$ of the Lie group $T(n)$ of non-singular upper triangular $n\times n$ matrices.

Below $A^{i_1,i_2}_{j_1,j_2}$, where $i_1\leqslant i_2$, $j_1\leqslant j_2$, denotes the submatrix $(a_{ij})^{i=i_1,\dots,i_2}_{j=j_1,\dots,j_2}$ of a matrix $A=(a_{ij})$. The conjugate value of k with respect to n is denoted by $\varkappa$, i.e., $\varkappa = n-k+1$. The standard notation $|A| = \det A$ is used.

Proposition 2. Up to the equivalence relation on algebra parameters, the following conditions can be assumed satisfied for some $s'\in\{0,\dots,\min(s,[n/2])\}$ and k_q, $q=1,\dots,s'$, $1\leqslant k_1<k_2<\dots<k_{s'}\leqslant[n/2]$:

$$\begin{aligned}&\gamma_{qk}=\gamma_{q\varkappa},\ k<k_q,\quad \gamma_{q\varkappa_q}-\gamma_{qk_q}=1,\quad \gamma_{pk_q}=\gamma_{p\varkappa_q},\ p\neq q,\quad q=1,\dots,s',\\ &\gamma_{pk}=\gamma_{p\varkappa},\ p>s',\ k=1,\dots,[n/2].\end{aligned}$$

We will say that the parameter matrix γ has a *reduced form* if it satisfies the conditions of Proposition 2.

Theorem 1. *Let the parameter matrix γ have a reduced form. A basis of* $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ *is formed by the expressions*

$$|\mathcal{E}^{1,k}_{\varkappa,n}|\prod_{q=1}^{s'}|\mathcal{E}^{1,k_q}_{\varkappa_q,n}|^{\alpha_{qk}},\quad k\in\{1,\dots,[n/2]\}\setminus\{k_1,\dots,k_{s'}\},$$

$$f_p+\sum_{k=1}^{[\frac{n}{2}]}\frac{(-1)^{k+1}}{|\mathcal{E}^{1,k}_{\varkappa,n}|}(\gamma_{pk}-\gamma_{p,k+1})\sum_{k<i<\varkappa}\begin{vmatrix}\mathcal{E}^{1,k}_{i,i} & \mathcal{E}^{1,k}_{\varkappa,n}\\ 0 & \mathcal{E}^{i,i}_{\varkappa,n}\end{vmatrix},\quad p=s'+1,\dots,s,$$

where $\varkappa := n - k + 1$, $\mathcal{E}^{i_1,i_2}_{j_1,j_2}$, $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, *denotes the matrix* $(e_{ij})^{i=i_1,\dots,i_2}_{j=j_1,\dots,j_2}$ *and*

$$\alpha_{qk} := -\sum_{k'=1}^{k} (\gamma_{q\varkappa'} - \gamma_{qk'}).$$

We use the short 'non-symmetrized' form for basis invariants, where it is uniformly assumed that in all monomials elements of $\mathcal{E}^{1,k}_{i,i}$ is placed before (or after) elements of $\mathcal{E}^{i,i}_{\varkappa,n}$.

6 Conclusion

The main advantage of the proposed method is in that it is purely algebraic. Unlike the conventional infinitesimal method, it eliminates the need to solve systems of partial differential equations, replaced in our approach by the construction of the matrix $B(\theta)$ of inner automorphisms and by excluding the parameters θ from the fundamental lifted invariant $\mathcal{I} = \check{x} \cdot B(\theta)$ in some way.

The efficient exploitation of the method imposes certain constraints on the choice of bases of the Lie algebras. See, e.g., Proposition 2 and Theorem 1. That then automatically yields simpler expressions for the invariants. In some cases the simplification is considerable.

Possibilities on the usage of the approach and directions for further investigation were outlined in our previous papers [3, 4, 5, 6, 7]. Recently advantages of the moving frames approach for computation of generalized Casimir operators were demonstrated in [19] with a new series of solvable Lie algebras. The problem on optimal ways of applications of this approach to unsolvable Lie algebras is still open.

References

[1] Abellanas L. and Martinez Alonso L., A general setting for Casimir invariants, *J. Math. Phys.*, 1975, V.16, 1580–1584.

[2] Beltrametti E.G. and Blasi A., On the number of Casimir operators associated with any Lie group, *Phys. Lett.*, 1966, V.20, 62–64.

[3] Boyko V., Patera J. and Popovych R., Computation of invariants of Lie algebras by means of moving frames, *J. Phys. A: Math. Gen.*, 2006, V.39, 5749–5762, arXiv:math-ph/0602046.

[4] Boyko V., Patera J. and Popovych R., Invariants of Lie algebras with fixed structure of nilradicals, *J. Phys. A: Math. Teor.*, 2007, V.40, 113–130, arXiv:math-ph/0606045.

[5] Boyko V., Patera J. and Popovych R., Invariants of triangular Lie algebras, *J. Phys. A: Math. Teor.*, 2007, V.40, 7557–7572, arXiv:0704.0937.

[6] Boyko V., Patera J. and Popovych R., Invariants of triangular Lie algebras with one nilindependent diagonal element, *J. Phys. A: Math. Teor.*, 2007, V.40, 9783–9792, arXiv:0705.2394.

[7] Boyko V., Patera J. and Popovych R., Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements, *Linear Algebra Appl.*, 2008, V.428, 834–854, arXiv:0706.2465.

[8] Campoamor-Stursberg R., Some remarks concerning the invariants of rank one solvable real Lie algebras, *Algebra Colloq.*, 2005, V.12, 497–518.

[9] Fels M. and Olver P., Moving coframes: I. A practical algorithm, *Acta Appl. Math.*, 1998, V.51, 161–213; II. Regularization and theoretical foundations, *Acta Appl. Math.*, 1999, V.55, 127–208.

[10] Kirillov A.A., Elements of the Theory of Representations, Grundlehren der Mathematischen Wissenschaften, Band 220, Berlin – New York, Springer-Verlag, 1976.

[11] Mubarakzyanov G.M., On solvable Lie algebras, *Izv. Vys. Ucheb. Zaved. Matematika*, 1963, N 1 (32), 114–123 (in Russian).

[12] Ndogmo J.C., Invariants of solvable Lie algebras of dimension six, *J. Phys. A: Math. Gen.*, 2000, V.33, 2273–2287.

[13] Ndogmo J.C. and Winternitz P., Generalized Casimir operators of solvable Lie algebras with Abelian nilradicals, *J. Phys. A: Math. Gen.*, 1994, V.27, 2787–2800.

[14] Olver P.J., Lectures on moving frames, Preprint, University of Minnesota, 2008.

[15] Olver P.J. and Pohjanpelto J., Moving frames for Lie pseudo-groups, *Canadian J. Math.*, 2008, V.60, 1336–1386.

[16] Patera J., Sharp R.T., Winternitz P. and Zassenhaus H., Invariants of real low dimension Lie algebras, *J. Math. Phys.*, 1976, V.17, 986–994.

[17] Popovych R., Boyko V., Nesterenko M. and Lutfullin M. Realizations of real low-dimensional Lie algebras, *J. Phys. A: Math. Gen.*, 2003, V.36, 7337–7360; see arXiv:math-ph/0301029v7 for the revised and extended version.

[18] Snobl L. and Winternitz P., A class of solvable Lie algebras and their Casimir invariants, *J. Phys. A: Math. Gen.*, 2005, V.38, 2687–2700, arXiv:math-ph/0411023.

[19] Snobl L. and Winternitz P., All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency $n-1$, *J. Phys. A: Math. Theor.*, 2009, V.42, 105201, 16 pages, arXiv:0809.3259.

[20] Tremblay S. and Winternitz P., Invariants of the nilpotent and solvable triangular Lie algebras, *J. Phys. A: Math. Gen.*, 2001, V.34, 9085–9099, arXiv:0709.3116.